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MATRICES WITH TOTALLY POSITIVE POWERS AND THEIR GENERALIZATIONS

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Abstract. In this paper, eventually totally positive matrices (i.e. matrices all whose powers starting at some point are totally positive) are studied. We present a new approach to eventual total positivity which is based on the theory of eventually positive matrices. We mainly focus on the spectral properties of such matrices. We also study eventually J-sign-symmetric matrices and matrices, whose powers are P-matrices.

1. Introduction

In the 1940's Gantmacher and Krein described spectral properties of strictly totally positive matrices (i.e. matrices for which all minors are positive). Among these results they proved sufficient criteria for a matrix $\mathbf{A}$ to have a strictly totally positive power $\mathbf{A}^k$ for some positive even integer k (see [12]).

Then the theory of eventually positive matrices (i.e. matrices $\mathbf{A}$ such that $\mathbf{A}^k$ is (entry-wise) positive for all $k \geqslant k_0$, for some positive integer k_0) was developed (see [10], [18], [14]). Such matrices were characterized by their spectral properties, similar to the properties of positive matrices (i.e the largest in absolute value eigenvalue is positive, and the corresponding eigenvectors of both $\mathbf{A}$ and the transpose of $\mathbf{A}$ can be chosen to be positive). Thus the theory of positive matrices was extended to matrices with some negative entries. Different aspects of eventual positivity were studied in [2]-[7], [16]. Total positivity of Hadamard powers of matrices as well as continuous real powers of totally positive matrices were studied in [8].

In this paper, a new approach (through eventual positivity) for the matrices with totally positive powers, described by Gantmacher and Krein is provided. Using the theory of eventual positivity, we give a necessary and sufficient characterization of eventually totally positive matrices (i.e. matrices $\mathbf{A}$ such that $\mathbf{A}^k$ is strictly totally positive for all $k \geqslant k_0$, for some positive integer k_0). Then, we analyze a certain cone-theoretic generalization of strictly totally positive matrices and study some properties of eventually

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P-matrices (i.e. matrices $\mathbf{A}$ such that $\mathbf{A}^k$ has positive principal minors for all $k \geqslant k_0$, for some positive integer k_0).

The paper is organized as follows. We first collect definitions and statements on the Perron–Frobenius property and eventually positive matrices. We provide a certain generalization of the Perron–Frobenius property which characterizes the class of matrices, similar to eventually positive matrices. In section 2, we define eventually totally positive matrices and consider the property of matrix eigenvalues and eigenvectors which is equivalent to eventual total positivity. We provide examples to show how the theory of totally positive matrices is extended to the matrices with some negative minors. Section 3 deals with similarity transformations preserving eventual total positivity and related properties. In section 4, we describe a cone-theoretic generalization of the class of eventually totally positive matrices, namely eventually totally J-sign-symmetric matrices. In section 5, we analyze the structure of eventually P-matrices with positive distinct spectra.

2. Eventually positive matrices and their generalizations

Here, as usual, let $\rho(A)$ denote the spectral radius of a matrix $\mathbf{A}$. An *eigenfunctional* of a matrix $\mathbf{A}$ corresponding to an eigenvalue λ is defined as an eigenvector of $\mathbf{A}^T$ (the transpose of $\mathbf{A}$), corresponding to the same eigenvalue. Given a vector $x \in \mathbb{R}^n$ with the coordinates $(x^1, \ldots, x^n)$, we define the *signature vector*, Sign(x), as follows

$$\mathrm{Sign}(x) := (\mathrm{sgn}(x^1), \ldots, \mathrm{sgn}(x^n))^T.$$

For a column vector $x = (x^1, \ldots, x^n)^T$ and a row vector $y = (y_1, \ldots, y_n)$, their tensor product $x \otimes y$ is defined as the following $n \times n$ matrix:

$$x \otimes y = \begin{pmatrix} x_1y_1 & \ldots & x_1y_n \\ \ldots & \ldots & \ldots \\ x_ny_1 & \ldots & x_ny_n \end{pmatrix}.$$

Let us recall some related definitions (see, for example, [5], [6], [14], [16]).

DEFINITION 1. For a real $n \times n$ matrix $\mathbf{A}$, an eigenvalue λ of $\mathbf{A}$ is called a *dominant eigenvalue* if $|\lambda| = \rho(A)$. In addition, λ is called a *strictly dominant eigenvalue*, if $|\lambda| > |\mu|$ for any other eigenvalue μ of $\mathbf{A}$.

DEFINITION 2. An $n \times n$ matrix $\mathbf{A}$ is said to have the *Perron–Frobenius property* if $\mathbf{A}$ has a positive dominant eigenvalue with a corresponding nonnegative eigenvector. A matrix $\mathbf{A}$ is said to have the *strong Perron–Frobenius property* if $\mathbf{A}$ has a unique positive simple strictly dominant eigenvalue a the corresponding positive eigenvector.

Following Johnson and Tarazaga (see [14]), we denote PF a class of all matrices $\mathbf{A}$ which have the strong Perron–Frobenius property together with their transpose $\mathbf{A}^T$.

Let us give the following generalization of the strong Perron–Frobenius property.

DEFINITION 3. A matrix $\mathbf{A}$ is said to have the *signature equality property* if it satisfies the following conditions:

1. $\mathbf{A}$ has a unique simple strictly dominant eigenvalue $\lambda > 0$ with a corresponding eigenvector x and eigenfunctional x^*;

2. Both x and x^* have no zero coordinates, and $\mathrm{Sign}(x) = \mathrm{Sign}(x^*)$.

If a matrix $\mathbf{A}$ has the strong Perron–Frobenius property, then it obviously has the signature equality property. For an analogue of the Perron–Frobenius property, we have the following definition.

DEFINITION 4. A matrix $\mathbf{A}$ is said to have the *weak signature equality property* if it satisfies the following conditions:

1. $\mathbf{A}$ has a dominant eigenvalue $\lambda > 0$ with a corresponding eigenvector x and eigenfunctional x^*;

2. The inequalities $x^i(x^*)^i \geqslant 0$, $i = 1, \ldots, n$ hold for the coordinates $(x^1, \ldots, x^n)$ of the eigenvector x and $((x^*)^1, \ldots, (x^*)^n)$ of the eigenfunctional x^*.

To show a link between the peripheral spectrum of a matrix and the asymptotic limit of matrix powers, we need the following lemma (a similar statement can be found in [11]).

LEMMA 1. *Let an $n \times n$ matrix $\mathbf{A}$ have a unique simple strictly dominant eigenvalue $\lambda_1 = \rho(A)$ with a corresponding eigenvector x_1 and eigenfunctional x_1^*. Then the following approximation holds:*

$$\frac{1}{\rho(A)^k}\mathbf{A}^k \to x_1 \otimes x_1^* \qquad \textit{as} \quad k \to \infty.$$

Proof. The proof follows the reasoning of Johnson and Tarazaga (see [14], p. 328, proof of Theorem 1). Let us write the Jordan decomposition of $\mathbf{A}$:

$$\mathbf{A} = \mathbf{S}\Lambda\mathbf{S}^{-1},$$

where Λ is the Jordan canonical form of $\mathbf{A}$. In this case, we have

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \Lambda' \end{pmatrix} = \rho(A)\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\rho(A)}\Lambda' \end{pmatrix}.$$

The columns of the matrix $\mathbf{S}$ are the eigenvectors of $\mathbf{A}$ and the rows of of the matrix $\mathbf{S}^{-1}$ are the eigenfunctionals of $\mathbf{A}$. So the first column of the matrix $\mathbf{S}$ coincides with the first eigenvector x_1 and the first row of $\mathbf{S}^{-1}$ coincides with the first eigenfunctional x_1^*. Thus we can write $\mathbf{S} = \begin{pmatrix} x_1 & S' \end{pmatrix}$ and $\mathbf{S}^{-1} = \begin{pmatrix} x_1^* \\ (S^{-1})' \end{pmatrix}$. (Here S' is an $n \times (n-1)$ matrix and $(S^{-1})'$ is an $(n-1) \times n$ matrix.) Since $\mathbf{A}$ and $\mathbf{A}^k$ share the same eigenvectors and eigenfunctionals, we obtain:

$$\mathbf{A}^k = \mathbf{S}\Lambda^k\mathbf{S}^{-1}, \qquad k = 2, 3, \ldots.$$

Thus
$$\mathbf{A}^k = \rho^k(A)\begin{pmatrix} x_1 & S' \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\rho^k(A)}(\Lambda')^k \end{pmatrix}\begin{pmatrix} x_1^* \\ (S^{-1})' \end{pmatrix}.$$

Since $\rho(\frac{1}{\rho(A)}\Lambda') < 1$, we have $\frac{1}{\rho^k(A)}(\Lambda')^k \to 0$ as $k \to \infty$. In coordinates, this means

$$\frac{1}{\rho^k(A)}\mathbf{A}^k =$$

$$\begin{pmatrix} x_1^1 & s'_{11} & \dots & s'_{1n-1} \\ x_1^2 & s'_{21} & \dots & s'_{2n-1} \\ \dots & \dots & \dots & \dots \\ x_1^n & s'_{nn-1} & \dots & s'_{nn-1} \end{pmatrix}\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & \varepsilon_{11} & \dots & \varepsilon_{1n-1} \\ 0 & \dots & \dots & \dots \\ 0 & \varepsilon_{n-11} & \dots & \varepsilon_{n-1n-1} \end{pmatrix}\begin{pmatrix} (x_1^*)^1 & (x_1^*)^2 & \dots & (x_1^*)^n \\ (s^{-1})'_{11} & (s^{-1})'_{12} & \dots & (s^{-1})'_{1n} \\ \dots & \dots & \dots & \dots \\ (s^{-1})'_{n-11} & (s^{-1})'_{n-12} & \dots & (s^{-1})'_{n-1n} \end{pmatrix}$$
$$= \begin{pmatrix} x_1^1(x_1^*)^1 & x_1^1(x_1^*)^2 & \dots & x_1^1(x_1^*)^n \\ x_1^2(x_1^*)^1 & x_1^2(x_1^*)^2 & \dots & x_1^2(x_1^*)^n \\ \dots & \dots & \dots & \dots \\ x_1^n(x_1^*)^1 & x_1^n(x_1^*)^2 & \dots & x_1^n(x_1^*)^n \end{pmatrix} + \begin{pmatrix} \varepsilon'_{11} & \varepsilon'_{12} & \dots & \varepsilon'_{1n} \\ \varepsilon'_{21} & \varepsilon'_{22} & \dots & \varepsilon'_{2n} \\ \dots & \dots & \dots & \dots \\ \varepsilon'_{n1} & \varepsilon'_{n2} & \dots & \varepsilon'_{nn} \end{pmatrix}.$$

Hence $\dfrac{1}{\rho^k}\mathbf{A}^k \to x_1 \otimes x_1^*$ as $k \to \infty$. $\square$

Let us recall the following definition introduced in [10].

DEFINITION 5. A real $n \times n$ real matrix $\mathbf{A}$ is called *eventually positive (EP)* (respectively, *eventually nonnegative (EN)*) if there exists a positive integer k_0 such that $\mathbf{A}^k > 0$ (respectively, $\mathbf{A}^k \geqslant 0$) for all $k \geqslant k_0$. For an EP matrix, the least such k_0 is called the *power index* of $\mathbf{A}$.

The following statement was proved for eventually positive matrices (see [14], p. 328, Theorem 1).

THEOREM 1. (Johnson, Tarazaga) *Let $\mathbf{A} \in \mathbb{R}^{n\times n}$. Then the following statements are equivalent.*

1. *Both of the matrices $\mathbf{A}$ and $\mathbf{A}^T$ have the strong Perron–Frobenius property.*
2. *The matrix $\mathbf{A}$ is eventually positive.*
3. *There is a positive integer k such that $\mathbf{A}^k > 0$ and $\mathbf{A}^{k+1} > 0$.*

A weaker statement holds for eventually nonnegative matrices (see [16], p. 136, Theorem 2.3).

THEOREM 2. (Noutsos) *Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ be an eventually nonnegative matrix that is not nilpotent. Then both matrices $\mathbf{A}$ and $\mathbf{A}^T$ have the Perron–Frobenius property.*

Note, that the converse to Theorem 2 may not hold. For a counterexample, see [6], p. 394, Example 2.5.

The following generalization of nonnegative matrices was introduced in [15].

Let J be any subset of $[n] := \{1,2,\dots,n\}$ and let $J^c := [n] \setminus J$. Then

$$[n] \times [n] = (J \times J) \cup (J^c \times J^c) \cup (J \times J^c) \cup (J^c \times J)$$

is a partition of $[n] \times [n]$ into four pairwise disjoint subsets.

DEFINITION 6. A matrix $\mathbf{A} = \{a_{ij}\}_{i,j=1}^n$ is called *J-sign-symmetric (JS)* if

$$a_{ij} \geqslant 0 \quad \text{on} \quad (J \times J) \cup (J^c \times J^c);$$

and

$$a_{ij} \leqslant 0 \quad \text{on} \quad (J \times J^c) \cup (J^c \times J).$$

A matrix $\mathbf{A} = \{a_{ij}\}_{i,j=1}^n$ is called *strictly J-sign-symmetric (SJS)* if

$$a_{ij} > 0 \quad \text{on} \quad (J \times J) \cup (J^c \times J^c);$$

and

$$a_{ij} < 0 \quad \text{on} \quad (J \times J^c) \cup (J^c \times J).$$

Let us recall the following properties of JS matrices (see, for example, [15]).

1. A matrix $\mathbf{A}$ is JS (SJS) if and only if $\mathbf{A}$ can be represented as follows:

$$\mathbf{A} = \mathbf{D}\widetilde{\mathbf{A}}\mathbf{D}^{-1}, \tag{1}$$

   where $\widetilde{\mathbf{A}}$ is a nonnegative (respectively, positive) matrix, $\mathbf{D}$ is a nonsingular diagonal matrix.

2. The spectral radius $\rho(A)$ of an SJS matrix $\mathbf{A}$ is a simple positive eigenvalue of $\mathbf{A}$, strictly larger than the absolute value of any other eigenvalue of $\mathbf{A}$. The eigenvector x_1 and the eigenfunctional x_1^*, corresponding to the eigenvalue $\lambda_1 = \rho(A)$ may be chosen to satisfy the inequalities $\mathbf{D}x_1 > 0$, $\mathbf{D}x_1^* > 0$ entrywise, where $\mathbf{D}$ is an invertible diagonal matrix from (1).

The proof of Property 2 immediately follows from the Perron theorem (see, for example, [3], p. 27, Theorem 1.4).

Now let us examine a more general class of matrices, which includes eventually nonnegative matrices.

DEFINITION 7. A real $n \times n$ matrix $\mathbf{A}$ is called *eventually strictly J–sign-symmetric (ESJS)* if there exists a positive integer k_0 such that $\mathbf{A}^k$ is SJS for all $k \geqslant k_0$. A real $n \times n$ matrix $\mathbf{A}$ is called *eventually J–sign-symmetric (EJS)* if there exists a positive integer k_0 such that $\mathbf{A}^k$ is JS for all $k \geqslant k_0$.

Let us recall that the *sign pattern* $\mathrm{Sign}(\mathbf{A})$ of a real $n \times n$ matrix $\mathbf{A} = \{a_{ij}\}_{i,j=1}^n$ is defined by the equalities:

$$\mathrm{Sign}(\mathbf{A}) = \{s_{ij}\}_{i,j=1}^n,$$

where $s_{ij} = \mathrm{sgn}(a_{ij})$ (see, for example, [7]).

We now state the following property satisfied by all ESJS matrices.

LEMMA 2. *A matrix $\mathbf{A}$ is ESJS if and only if $\mathbf{A} = \mathbf{D}\widetilde{\mathbf{A}}\mathbf{D}^{-1}$, where $\widetilde{\mathbf{A}}$ is an EP matrix, $\mathbf{D}$ is a nonsingular diagonal matrix.*

Proof. ($\Leftarrow$) Let $\mathbf{A} = \mathbf{D}\widetilde{\mathbf{A}}\mathbf{D}^{-1}$, where $\widetilde{\mathbf{A}}$ is an EP matrix. Then there is a positive integer k_0 such that $(\widetilde{\mathbf{A}})^k$ is positive for all $k \geqslant k_0$. Thus

$$\mathbf{A}^k = (\mathbf{D}\widetilde{\mathbf{A}}\mathbf{D}^{-1})^k = \mathbf{D}(\widetilde{\mathbf{A}})^k\mathbf{D}^{-1}.$$

Since $(\widetilde{\mathbf{A}})^k$ is positive, we obtain by Property 1 of SJS matrices that $\mathbf{A}^k$ is SJS for all $k \geqslant k_0$.

($\Rightarrow$) Let $\mathbf{A}$ be ESJS. Since $\mathbf{A}^{k_0}$ is SJS for some positive integer k_0, $\rho(A^{k_0})$ is a positive simple strictly dominant eigenvalue of $\mathbf{A}^{k_0}$ by Property 2 of SJS matrices. Since all the eigenvalues of $\mathbf{A}^{k_0}$ are powers of the eigenvalues of $\mathbf{A}$, we conclude there is an eigenvalue λ of $\mathbf{A}$ such that $\lambda^{k_0} = \rho(A^{k_0}) > 0$. Since $\rho(A^{k_0})$ is simple and strictly dominant, λ is also simple and strictly dominant. Since $\mathbf{A}$ is real, λ is also real (positive or negative). Applying the same reasoning to $\mathbf{A}^{k_0+1}$ and taking into account that either k_0 or k_0+1 must be odd, we conclude that $\lambda > 0$. Thus the conditions of Lemma 1 hold. Applying Lemma 1, we obtain that $\frac{1}{\rho(A)^k}\mathbf{A}^k \to x_1 \otimes x_1^*$ as $k \to \infty$, where x_1 and x_1^* are the eigenvector and the eigenfunctional corresponding to the simple strictly dominant eigenvalue λ. Thus there is a positive integer k_1 such that $\mathrm{Sign}(\mathbf{A}^k) = \mathrm{Sign}(x_1 \otimes x_1^*)$ for all $k \geqslant k_1$.

Since $\mathbf{A}^k$ is SJS starting from $k = k_0$ and $\mathbf{A}$ and $\mathbf{A}^k$ share the same eigenvectors we have by Property 2 of SJS matrices that $\mathbf{D}x_1 > 0$ and $\mathbf{D}x_1^* > 0$ for some invertible diagonal matrix $\mathbf{D}$. This implies the equality $\mathrm{Sign}(x_1) = \mathrm{Sign}(x_1^*)$. Examine $\mathbf{D} = \mathrm{diag}[\mathrm{sgn}(x_1^1),\ \mathrm{sgn}(x_1^2),\ \ldots,\ \mathrm{sgn}(x_1^n)]$ and put $\widetilde{\mathbf{A}} = \mathbf{D}^{-1}\mathbf{A}\mathbf{D}$. In this case, it is easily verified $\mathbf{D}^{-1}\mathrm{Sign}(x_1 \otimes x_1^*)\mathbf{D}$ is positive. We obtain

$$\mathrm{Sign}(\mathbf{D}^{-1}\mathbf{A}\mathbf{D})^k = \mathbf{D}^{-1}\mathrm{Sign}(\mathbf{A}^k)\mathbf{D} = \mathbf{D}^{-1}\mathrm{Sign}(x_1 \otimes x_1^*)\mathbf{D}$$

for $k \geqslant k_1$. This equality shows that $(\mathbf{D}^{-1}\mathbf{A}\mathbf{D})^k$ is positive for all $k \geqslant k_1$. Thus $\mathbf{A} = \mathbf{D}\widetilde{\mathbf{A}}\mathbf{D}^{-1}$, where $\widetilde{\mathbf{A}} = \mathbf{D}^{-1}\mathbf{A}\mathbf{D}$ is an EP matrix. $\square$

The following statement generalizes results of Johnson and Tarazaga [14].

THEOREM 3. *Let $\mathbf{A} \in \mathbb{R}^{n\times n}$. Then the following statements are equivalent.*

1. *The matrix $\mathbf{A}$ has the signature equality property.*
2. *The matrix $\mathbf{A}$ is eventually SJS.*
3. *There is a positive integer k such that both $\mathbf{A}^k$ and $\mathbf{A}^{k+1}$ are SJS.*

Proof. (1) $\Rightarrow$ (2) Suppose $\mathbf{A}$ have a unique simple strictly dominant eigenvalue $\lambda_1 > 0$ with a corresponding eigenvector x_1 and eigenfunctional x_1^*. Let, in addition, both x_1 and x_1^* have no zero coordinates, and the equality $\mathrm{Sign}(x_1) = \mathrm{Sign}(x_1^*)$ holds. By Lemma 1, there is a positive integer k_0 such that

$$\mathrm{Sign}(\mathbf{A}^k) = \mathrm{Sign}(x_1 \otimes x_1^*)$$

for all $k \geqslant k_0$.

Let us organize a partition of $[n]$ as follows. We put $J := \{i \in [n] : x_1^i > 0\}$. In this case, $J^c = [n] \setminus J = \{i \in [n] : x_1^i < 0\}$. Since $\mathrm{sgn}(x_1^i) = \mathrm{sgn}((x_1^*)^i)$ for all $i = 1, \ldots, n$ and all of the coordinates are nonzero, we have

$$x_1^i (x_1^*)^j > 0 \qquad \text{if} \quad i, j \in J \quad \text{or} \quad i, j \in J^c;$$

$$x_1^i (x_1^*)^j < 0 \qquad \text{if} \quad i \in J, j \in J^c \quad \text{or} \quad i \in J^c, j \in J.$$

Thus the matrix $x_1 \otimes x_1^*$ is SJS according to Definition 6.

$(2) \Rightarrow (3)$ Since $\mathbf{A}$ is eventually SJS, we obtain by Lemma 2 that $\mathbf{A} = \mathbf{D}\widetilde{\mathbf{A}}\mathbf{D}^{-1}$, where $\widetilde{\mathbf{A}}$ is an EP matrix, $\mathbf{D}$ is a nonsingular diagonal matrix. Thus there is a positive integer k_0 such that $\widetilde{\mathbf{A}}^k$ is positive for all $k \geqslant k_0$. Considering the equalities

$$\mathbf{A}^{k_0} = \mathbf{D}\widetilde{\mathbf{A}}^{k_0}\mathbf{D}^{-1}$$

and

$$\mathbf{A}^{k_0+1} = \mathbf{D}\widetilde{\mathbf{A}}^{k_0+1}\mathbf{D}^{-1}$$

we obtain that

$$\mathrm{sgn}((a^{k_0})_{ij}) = \mathrm{sgn}(\mathrm{d_{ii}})\mathrm{sgn}((\widetilde{a}^{k_0})_{ij})\mathrm{sgn}\left(\frac{1}{d_{jj}}\right);$$

$$\mathrm{sgn}((a^{k_0+1})_{ij}) = \mathrm{sgn}(\mathrm{d_{ii}})\mathrm{sgn}((\widetilde{a}^{k_0+1})_{ij})\mathrm{sgn}\left(\frac{1}{d_{jj}}\right),$$

where $(a^k)_{ij}$, $(\widetilde{a}^k)_{ij}$ denote the entries of the matrices $\mathbf{A}^k$ and $\widetilde{\mathbf{A}}^k$, respectively, for $k = k_0,\ k_0+1$. Since $\mathrm{sgn}((\widetilde{a}^{k_0})_{ij}) = \mathrm{sgn}((\widetilde{a}^{k_0+1})_{ij}) = +1$ for all $i, j = 1, \ldots, n$, we have $\mathrm{Sign}(\mathbf{A}^{k_0}) = \mathrm{Sign}(\mathbf{A}^{k_0+1})$.

$(3) \Rightarrow (1)$ Suppose both the matrices $\mathbf{A}^k$ and $\mathbf{A}^{k+1}$ are SJS for some positive integer k. Since $\mathbf{A}^k$ is SJS, $\rho(A^k)$ is an eigenvalue of $\mathbf{A}^k$ (by Property 2), and there must be an eigenvalue λ of $\mathbf{A}$ such that $\lambda^k = \rho(A^k)$. Since $\rho(A^k)$ is a positive simple strictly dominant eigenvalue of $\mathbf{A}^k$, λ is a real simple strictly dominant eigenvalue of $\mathbf{A}$ (positive or negative). Applying the same reasoning for $\mathbf{A}^{k+1}$ and taking into account that one of the integers k and $k+1$ must be odd, we have $\lambda > 0$.

Applying Property 2 again to $\mathbf{A}^k$, we get that the corresponding to $\rho(A^k)$ eigenvector x_1 and eigenfunctional x_1^* satisfy the equality $\mathrm{Sign}(x_1) = \mathrm{Sign}(x_1^*)$. Observing that x_1 and x_1^* are also an eigenvector and eigenfunctional of $\mathbf{A}$, corresponding to the eigenvalue λ we complete the proof. $\square$

3. Eventually totally positive matrices

Let us recall the following definitions and notations.

DEFINITION 8. Let $e_1, \ldots, e_n$ be an arbitrary basis in $\mathbb{R}^n$ and let $x_1, \ldots, x_j$ $(2 \leqslant j \leqslant n)$ be any vectors in $\mathbb{R}^n$ defined by their coordinates: $x_i = (x_i^1, \ldots, x_i^n)$,

$i = 1, \ldots, j$. Then the vector $x_1 \wedge \ldots \wedge x_j \in \mathbb{R}^{\binom{n}{j}}$ (here $\binom{n}{j} = \frac{n!}{j!(n-j)!}$) with coordinates of the form

$$(x_1 \wedge \ldots \wedge x_j)^{N(\alpha)} := \begin{vmatrix} x_1^{i_1} & \ldots & x_j^{i_1} \\ \ldots & \ldots & \ldots \\ x_1^{i_j} & \ldots & x_j^{i_j} \end{vmatrix},$$

where $N(\alpha)$ is the number of the set of indices $\alpha = (i_1, \ldots, i_j) \subseteq [n]$ in the lexicographic ordering ($1 \leqslant N(\alpha) \leqslant \binom{n}{j}$), is called an *exterior product* of $x_1, \ldots, x_j$.

We consider the jth exterior power $\wedge^j \mathbb{R}^n$ of the space $\mathbb{R}^n$ as the space $\mathbb{R}^{\binom{n}{j}}$. The set of all of exterior products of the form $e_{i_1} \wedge \ldots \wedge e_{i_j}$, where $1 \leqslant i_1 < \ldots < i_j \leqslant n$ forms a canonical basis in $\wedge^j \mathbb{R}^n$ (see [13]).

DEFINITION 9. Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator. Then its *jth exterior power* $\wedge^j A$ is defined as on operator on $\wedge^j \mathbb{R}^n$ acting by the rule

$$(\wedge^j A)(x_1 \wedge \ldots \wedge x_j) = Ax_1 \wedge \ldots \wedge Ax_j.$$

If $\mathbf{A} = \{a_{ij}\}_{i,j=1}^n$ is the matrix of A in a basis $e_1, \ldots, e_n$, then the matrix of $\wedge^j A$ in the basis $\{e_{i_1} \wedge \ldots \wedge e_{i_j}\}$, where $1 \leqslant i_1 < \ldots < i_j \leqslant n$, equals the jth compound matrix $\mathbf{A}^{(j)}$ of the initial matrix $\mathbf{A}$. Here the jth compound matrix $\mathbf{A}^{(j)}$ consists of all the minors of the jth order $A\begin{pmatrix} i_1 & \ldots & i_j \\ k_1 & \ldots & k_j \end{pmatrix}$, where $1 \leqslant i_1 < \ldots < i_j \leqslant n$, $1 \leqslant k_1 < \ldots < k_j \leqslant n$, of the initial $n \times n$ matrix $\mathbf{A}$, listed in the lexicographic order (see, for example, [17]).

It is easy to see that $\mathbf{A}^{(1)} = \mathbf{A}$ and $\mathbf{A}^{(n)}$ is one-dimensional and coincides with $\det(\mathbf{A})$.

The following properties of the compound matrices will be used later (see, for example, [12]).

1. Let $\mathbf{A}$, $\mathbf{B}$ be $n \times n$ matrices. Then $(\mathbf{AB})^{(j)} = \mathbf{A}^{(j)}\mathbf{B}^{(j)}$ for $j = 1, \ldots, n$ (the Cauchy–Binet formula).

2. The j-th compound $\mathbf{A}^{(j)}$ of an invertible matrix $\mathbf{A}$ is invertible and the following equality holds: $(\mathbf{A}^{(j)})^{-1} = (\mathbf{A}^{-1})^{(j)}$, $j = 1, \ldots, n$ (the Jacobi formula).

Let us recall the statement concerning the eigenvalues of the exterior power of an operator (see, for example, [17], p. 132).

THEOREM 4. (Kronecker) *Let $\{\lambda_i\}_{i=1}^n$ be the set of all eigenvalues of an $n \times n$ matrix $\mathbf{A}$, repeated according to multiplicity. Then all the possible products of the form $\{\lambda_{i_1} \ldots \lambda_{i_j}\}$, where $1 \leqslant i_1 < \ldots < i_j \leqslant n$, forms the set of all the possible eigenvalues of the jth compound matrix $\mathbf{A}^{(j)}$, repeated according to multiplicity. If $x_{i_1}, \ldots, x_{i_j}$ are linearly independent eigenvectors of $\mathbf{A}$, corresponding to the eigenvalues $\lambda_{i_1}, \ldots, \lambda_{i_j}$ $(1 \leqslant i_1 < \ldots < i_j \leqslant n)$ respectively, then their exterior product $x_{i_1} \wedge \ldots \wedge x_{i_j}$ is an eigenvector of $\mathbf{A}^{(j)}$, corresponding to the eigenvalue $\lambda_{i_1} \ldots \lambda_{i_j}$.*

As usual, we denote $S^-(x)$ the number of sign changes in the sequence of the coordinates $(x_1, \ldots, x_n)$ of the vector x with zero coordinates discarded, and $S^+(x)$ the maximum number of sign changes in the sequence $(x_1, \ldots, x_n)$, where zero coordinates are arbitrarily assigned values ± 1.

The following definition was given in [12].

DEFINITION 10. A system of nonzero vectors $\{x_1, \ldots, x_n\}$, $x_i \in \mathbb{R}^n$, is called a *Markov system* or *an oscillating system* if any linear combination $x = \sum_{i=1}^{j} c_i x_i$ satisfies the inequality

$$S^+(x) \leqslant j-1,$$

whenever $1 \leqslant j \leqslant n$, $c_i \in \mathbb{R}$, $\sum_{i=1}^{j} c_i^2 \neq 0$.

DEFINITION 11. A real $n \times n$ matrix $\mathbf{A}$ is said to have the *Gantmacher–Krein property* if $\mathbf{A}$ has n positive simple eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$ with the Markov system of corresponding eigenvectors $\{x_1, \ldots, x_n\}$.

Let us denote by GK the class of all matrices which possess the Gantmacher–Krein property together with their transposes.

The following lemma is proved in [1] (see [1], p. 198, Lemma 5.1).

LEMMA 3. (Ando) *Let $\{x_1, \ldots, x_j\}$ be real vectors from $\mathbb{R}^n$ $(j < n)$. In order that*

$$S^+\Big(\sum_{i=1}^{j} c_i x_i\Big) \leqslant j-1,$$

whenever $c_i \in \mathbb{R}$, $\sum_{i=1}^{j} c_i^2 \neq 0$, it is necessary and sufficient that $x_1 \wedge \ldots \wedge x_j$ be strictly positive or strictly negative.

Now we prove the following result.

LEMMA 4. *Let $\mathbf{A} \in \mathbb{R}^{n\times n}$. Then the following are equivalent.*

1. *The matrix $\mathbf{A}$ has the Gantmacher–Krein property.*
2. *The jth compound matrix $\mathbf{A}^{(j)}$ has the strong Perron–Frobenius property for all $j = 1, \ldots, n$.*

Proof. $(1) \Rightarrow (2)$. Assume that $\mathbf{A}$ has the Gantmacher–Krein property, i.e. $\mathbf{A}$ has n positive simple eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$, $\lambda_1 > \lambda_2 > \ldots > \lambda_n > 0$, and corresponding eigenvectors $\{x_1, \ldots, x_n\}$ form a Markov system. Let us consider the jth compound matrix $\mathbf{A}^{(j)}$. Applying the Kronecker theorem (Theorem 4) to $\mathbf{A}^{(j)}$ we obtain that $\rho(A^{(j)}) = \lambda_1 \ldots \lambda_j$ is a positive simple strictly dominant eigenvalue of $\mathbf{A}^{(j)}$ with a corresponding eigenvector $x_1 \wedge \ldots \wedge x_j$. Since the eigenvectors $x_1, \ldots, x_n$ form a Markov system, any linear combination $x = \sum\limits_{i=1}^{j} c_i x_i$ satisfies the inequality

$$S^+(x) \leqslant j-1,$$

whenever $1 \leqslant j \leqslant n$, $c_i \in \mathbb{R}$, $\sum\limits_{i=1}^{j} c_i^2 \neq 0$. Thus, by Ando's lemma (Lemma 3), the eigenvector $x_1 \wedge \ldots \wedge x_j$ may be chosen to be strictly positive.

$(2) \Rightarrow (1)$. The proof of the statement that all the eigenvalues $\lambda_1, \ldots, \lambda_n$ of $\mathbf{A}$ are positive, simple and different in absolute value of each other is identical to the argument found in [12] or [17], p. 130, the proof of Theorem 5.3. List the eigenvalues of the matrix $\mathbf{A}$ in descending order of their absolute values (taking into account their multiplicities):

$$|\lambda_1| \geqslant |\lambda_2| \geqslant |\lambda_3| \geqslant \ldots \geqslant |\lambda_n|.$$

Since the matrix $\mathbf{A}$ has the strong Perron–Frobenius property, $\lambda_1 = \rho(\mathbf{A}) > 0$ is a simple positive eigenvalue of $\mathbf{A}$, different in absolute value from the remaining eigenvalues. Examine the second compound matrix $\mathbf{A}^{(2)}$ which also has the strong Perron–Frobenius property we get: $\rho(\mathbf{A}^{(2)}) > 0$ is a simple positive eigenvalue of $\mathbf{A}^{(2)}$, different in absolute value from the remaining eigenvalues. By the Kronecker theorem, all eigenvalues of $\mathbf{A}^{(2)}$ are of the form $\lambda_{i_1}\lambda_{i_2}$ where $1 \leqslant i_1 < i_2 \leqslant n$. Therefore $\rho(\mathbf{A}^{(2)}) > 0$ can be represented in the form of the product $\lambda_{i_1}\lambda_{i_2}$ with some values of the indices i_1, i_2, $i_1 < i_2$. The facts that the eigenvalues are listed in a descending order and there is only one eigenvalue on the spectral circle $|\lambda| = \rho(\mathbf{A})$ imply that $\rho(\mathbf{A}^{(2)}) = \lambda_1\lambda_2 = \rho(\mathbf{A})\lambda_2$. Therefore $\lambda_2 = \frac{\rho(\mathbf{A}^{(2)})}{\rho(\mathbf{A})} > 0$.

Repeating the same reasoning for $\mathbf{A}^{(j)}$, $j = 3, \ldots, n$, we obtain the relations:

$$\lambda_j = \frac{\rho(\mathbf{A}^{(j)})}{\rho(\mathbf{A}^{(j-1)})} > 0,$$

where $j = 3, \ldots, n$. The simplicity of the eigenvalues λ_j for every j also follows from the above relations and the simplicity of $\rho(\mathbf{A}^{(j)})$.

Applying Ando's lemma (Lemma 3) to all the exterior products of the form $x_1 \wedge \ldots \wedge x_j$ (they are all positive), we obtain that the eigenvectors $\{x_1, \ldots, x_n\}$ form a Markov system. $\square$

Let us recall some well-known definitions (see, for example, [11]).

DEFINITION 12. A real $n \times n$ matrix $\mathbf{A}$ is called *totally positive (TP)* if $\mathbf{A}$ is nonnegative and its jth compound matrix $\mathbf{A}^{(j)}$ is also nonnegative for all $j = 2, \ldots, n$.

A real $n \times n$ matrix $\mathbf{A}$ is called *strictly totally positive (STP)* if $\mathbf{A}$ is positive and its jth compound matrix $\mathbf{A}^{(j)}$ is also positive for all $j = 2, \ldots, n$.

DEFINITION 13. A real $n \times n$ matrix $\mathbf{A}$ is called *oscillatory* if it is TP and there is a positive integer k such that $\mathbf{A}^k$ is STP.

Obviously, every STP matrix is oscillatory.

The following statement holds for STP matrices (see [12] or [17], p. 130, Theorem 5.3)

THEOREM 5. (Gantmacher, Krein) *Let an $n \times n$ matrix $\mathbf{A}$ be STP. Then all the eigenvalues of $\mathbf{A}$ are positive and simple:*

$$\rho(A) = \lambda_1 > \lambda_2 > \ldots > \lambda_n > 0.$$

The first eigenvector corresponding to the maximal eigenvalue λ_1 is strictly positive and the jth eigenvector x_j corresponding to the jth in absolute value eigenvalue λ_j has exactly $j-1$ changes of sign. Moreover, the following inequalities hold:

$$q - 1 \leqslant S^-(\sum_{i=q}^{p} c_i x_i) \leqslant S^+(\sum_{i=q}^{p} c_i x_i) \leqslant p - 1$$

for each $1 \leqslant q \leqslant p \leqslant n$ and $\sum_{i=q}^{p} c_i^2 \neq 0$.

Now let us introduce the following generalization of the class of STP matrices.

DEFINITION 14. A real $n \times n$ matrix $\mathbf{A}$ is called *eventually strictly totally positive (ESTP)* if there is a positive integer k_0 such that for all $k \geqslant k_0$ $\mathbf{A}^k$ is STP. Here the minimal value of k_0 is called the *power index* of a ESTP matrix $\mathbf{A}$.

It follows from the given above definition that *an oscillatory matrix is ESTP*. But the class of ESTP matrices also includes matrices with some negative entries and some negative minors.

Note that *if $\mathbf{A}$ and $\mathbf{B}$ are ESTP (ETP) matrices and $\mathbf{AB} = \mathbf{BA}$, then $\mathbf{AB}$ is also ESTP (ETP)*. However, if $\mathbf{AB} \neq \mathbf{BA}$ the above property may not hold. Moreover, $\mathbf{AB}$ might be ESTP (ETP) yet neither $\mathbf{A}$ nor $\mathbf{B}$ is so.

The following theorem was proved in [12].

THEOREM 6. *Let an $n \times n$ matrix $\mathbf{A}$ have n different in absolute value nonzero eigenvalues $\lambda_1, \ldots, \lambda_n$:*

$$|\lambda_1| > |\lambda_2| > \ldots > |\lambda_n| > 0.$$

and the eigenvectors $x_1, \ldots, x_n$ and $x_1^*, \ldots, x_n^*$ of $\mathbf{A}$ and $\mathbf{A}^T$, respectively, form two Markov systems. Then there is a positive integer k such that $\mathbf{A}^k$ is strictly totally positive.

The proof of Theorem 6 given by Gantmacher and Krein implies that in the case when some of the eigenvalues λ_i, $i = 1, \ldots, n$, are negative, the value k is necessarily even.

The following corollary concerns oscillatory matrices (see [12]).

COROLLARY 1. *If a totally nonnegative matrix $\mathbf{A}$ satisfies the following conditions*

1. *all the eigenvalues of* $\mathbf{A}$ *are positive and simple;*
2. *all the eigenvectors of* $\mathbf{A}$ *and* $\mathbf{A}^T$ *forms two Markov systems,*

then $\mathbf{A}$ *is oscillatory.*

Now let us prove the main result, which characterizes properties of ESTP matrices.

THEOREM 7. *Let* $\mathbf{A} \in \mathbb{R}^{n\times n}$. *Then the following statements are equivalent.*

1. *Both of the matrices* $\mathbf{A}$ *and* $\mathbf{A}^T$ *have the Gantmacher–Krein property.*
2. *For every* j, $j = 1, \ldots, n$, *both the* j*th compound matrix* $\mathbf{A}^{(j)}$ *and its transpose* $(\mathbf{A}^{(j)})^T$ *have the strong Perron–Frobenius property.*
3. *For every* j, $j = 1, \ldots, n$, *the* j*th compound matrix* $\mathbf{A}^{(j)}$ *is eventually positive.*
4. *The matrix* $\mathbf{A}$ *is eventually strictly totally positive.*

Proof. $(1) \Rightarrow (2)$. Applying Lemma 4 to the matrix $\mathbf{A}$, we obtain that the jth compound matrix $\mathbf{A}^{(j)}$ has the strong Perron–Frobenius property for every j, $j = 1, \ldots, n$. Applying Lemma 4 to $\mathbf{A}^T$ we obtain that $(\mathbf{A}^T)^{(j)}$ has the strong Perron–Frobenius property for every j, $j = 1, \ldots, n$. Observing that $(\mathbf{A}^T)^{(j)} = (\mathbf{A}^{(j)})^T$ for every j, $j = 1, \ldots, n$, we complete the proof.

$(2) \Rightarrow (3)$. It is sufficient to apply Theorem 1.

$(3) \Rightarrow (4)$. Since the compound matrices $\mathbf{A}^{(j)}$ are eventually positive for all $j = 1, \ldots, n$, we can find the power index k_j such that $(\mathbf{A}^{(j)})^k$ is positive for all positive integers $k \geqslant k_j$. Fix $k_0 = \max_j(k_j)$ and examine $\mathbf{A}^k$, for $k \geqslant k_0$. Applying the Cauchy–Binet formula, we obtain that

$$(\mathbf{A}^k)^{(j)} = (\mathbf{A}^{(j)})^k,$$

for all $j = 1, \ldots, n$, and $(\mathbf{A}^{(j)})^k$ is positive since $k \geqslant k_0 \geqslant k_j$. Thus $\mathbf{A}^k$ is STP for all $k \geqslant k_0$ and $\mathbf{A}$ is ESTP.

$(4) \Rightarrow (1)$. Since the matrix $\mathbf{A}$ is eventually strictly totally positive, we can find a power index k_0 such that $\mathbf{A}^k$ is STP for all $k \geqslant k_0$. Applying Theorem 5 to $\mathbf{A}^k$, we obtain that all the eigenvalues $\mu_1, \ldots, \mu_n$ of $\mathbf{A}^k$ are positive, simple and distinct:

$$\mu_1 > \mu_2 > \ldots > \mu_n > 0.$$

The corresponding eigenvectors $(x_1, \ldots, x_n)$ form a Markov system. Since all eigenvalues of $\mathbf{A}^k$ are just powers of the eigenvalues of $\mathbf{A}$, there is an eigenvalue λ_i of $\mathbf{A}$ such that $\lambda_i^k = \mu_i$, $i = 1, \ldots, n$. Thus we obtain that all the eigenvalues of $\mathbf{A}$ are simple, real (positive or negative) and different in absolute value from each other. Applying the same reasoning to $\mathbf{A}^{k+1}$ and observing that either k or $k+1$ must be odd, we obtain that all the eigenvalues of $\mathbf{A}$ are positive. Since $\mathbf{A}$ and $\mathbf{A}^k$ share the same eigenvectors, the corresponding eigenvectors of $\mathbf{A}$ form a Markov system. Now let us examine the transpose matrix $\mathbf{A}^T$. It is easy to see that $\mathbf{A}^T$ is also eventually strictly totally positive. Applying the same reasoning to $\mathbf{A}^T$ we may deduce that the eigenvectors of $\mathbf{A}^T$ also form a Markov system. □

COROLLARY 2. *Let an $n \times n$ matrix $\mathbf{A}$ have a Gantmacher–Krein property (be ESTP). Then $\mathbf{A}+\alpha\mathbf{I}$ also has the Gantmacher–Krein property (respectively, is ESTP), whenever $\alpha > 0$.*

Proof. For the proof, it is enough to observe that if $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $\mathbf{A}$, then $\lambda_1+\alpha, \ldots, \lambda_n+\alpha$ are the eigenvalues of $\mathbf{A}+\alpha\mathbf{I}$ with the same systems of the corresponding eigenvectors and eigenfunctionals. $\square$

EXAMPLE 1. Let us consider the matrix

$$\mathbf{A} = \begin{pmatrix} 10 & 2 & 2 \\ 3 & 2 & 1 \\ 7 & 4 & 6 \end{pmatrix}.$$

In this case, we have

$$\mathbf{A}^{(2)} = \begin{pmatrix} 14 & 4 & -2 \\ 26 & 46 & 4 \\ -2 & 11 & 8 \end{pmatrix}.$$

$$\mathbf{A}^{(3)} = 54.$$

Since

$$(\mathbf{A}^{(2)})^3 = \begin{pmatrix} 9980 & 10936 & 40 \\ 80264 & 112156 & 7264 \\ 218400 & 29156 & 2756 \end{pmatrix}$$

and

$$(\mathbf{A}^{(2)})^4 = \begin{pmatrix} 423976 & 543416 & 24104 \\ 4025224 & 5560136 & 346208 \\ 1010144 & 1445092 & 101872 \end{pmatrix}$$

are positive, we apply Theorem 1 and conclude that $\mathbf{A}^{(2)}$ is eventually positive. Thus $\mathbf{A}$ is ESTP (by Theorem 7).

EXAMPLE 2. Let

$$\mathbf{A} = \begin{pmatrix} 8 & 4 & 1 \\ 4 & 10 & 3 \\ -3 & 5 & 9 \end{pmatrix}.$$

In this case we have

$$\mathbf{A}^{(2)} = \begin{pmatrix} 64 & 20 & 2 \\ 52 & 75 & 31 \\ 50 & 45 & 75 \end{pmatrix};$$

$$\mathbf{A}^{(3)} = \det A = 470.$$

Since $\mathbf{A}^k > 0$ for $k = 5$ and $k = 6$, we apply Theorem 1 and Theorem 7 and obtain that $\mathbf{A}$ is ESTP. However, the eigenvalues of the 2×2 submatrix $\widehat{\mathbf{A}} = \begin{pmatrix} 8 & 1 \\ -3 & 9 \end{pmatrix}$, obtained from $\mathbf{A}$ by deleting the second row and column, are both complex: $\lambda_1 = \frac{17+i\sqrt{11}}{2}$ and $\lambda_2 = \frac{17-i\sqrt{11}}{2}$.

So let us note that *a principal submatrix (i.e. obtained from the initial matrix by deleting rows and columns with the same indices) of an ESTP matrix may not be ESTP*.

4. Similarity transformations preserving the Gantmacher–Krein property and being in GK

Let us recall some more definitions concerning matrix classes (see [3], [11]).

DEFINITION 15. A matrix $\mathbf{S}$ is called *monotone* if it is invertible and $\mathbf{S}^{-1}$ is nonnegative.

DEFINITION 16. Let $\mathbf{A} = \{a_{ij}\}_{i,j=1}^{n}$. A matrix $\mathbf{A}$ is called *sign-alternating* if the matrix $\mathbf{A}^{*}$ with the entries

$$a_{ij}^{*} = (-1)^{i+j}a_{ij}, \qquad i,j = 1, \ldots, n$$

is nonnegative.

A matrix $\mathbf{A}$ is sign-alternating if and only if it can be written in the following form

$$\mathbf{A} = \mathbf{D}\widetilde{\mathbf{A}}\mathbf{D},$$

where $\widetilde{\mathbf{A}}$ is a nonnegative matrix, $\mathbf{D}$ is a diagonal matrix with the diagonal entries $d_{ii} = (-1)^{i+1}$, $i = 1, \ldots, n$.

DEFINITION 17. A real $n \times n$ matrix $\mathbf{A}$ is called *totally sign-alternating (TSA)*[1] if $\mathbf{A}^{*}$ is TP.

A matrix $\mathbf{A}$ is TSA if and only if it can be written in the following form $\mathbf{A} = \mathbf{D}\widetilde{\mathbf{A}}\mathbf{D}$, where $\widetilde{\mathbf{A}}$ is a TP matrix, $\mathbf{D}$ is a diagonal matrix with diagonal entries $d_{ii} = (-1)^{i+1}$, $i = 1, \ldots, n$.

The following properties of TSA matrices were stated in [12].

LEMMA 5. *Let $\mathbf{A}$ be an invertible matrix. Then*

1. *If one of the matrices $\mathbf{A}$ and $\mathbf{A}^{-1}$ is TP then the other is TSA.*

2. *The matrix $\mathbf{A}$ is TP if and only if the matrix $(\mathbf{A}^{*})^{-1}$ is also TP.*

The similarity matrices preserving the strong Perron–Frobenius property and the class PF are described in [4]. The following statements are proved in [4] (see [4], p. 41, Theorems 3.6 and 3.7).

THEOREM 8. *For any invertible matrix $\mathbf{S}$, the following statements are equivalent:*

1. *Either $\mathbf{S}$ or $-\mathbf{S}$ is monotone.*

[1] Such matrices are called sign-regular in [12].

2. $\mathbf{S}^{-1}\mathbf{AS}$ *has the strong Perron–Frobenius property for all matrices* $\mathbf{A}$ *having the strong Perron–Frobenius property.*

THEOREM 9. *For any invertible matrix* $\mathbf{S}$*, the following statements are equivalent:*

1. $\mathbf{S}$ *and* $\mathbf{S}^{-1}$ *are either both nonnegative or both nonpositive.*

2. $\mathbf{S}^{-1}\mathbf{AS} \in PF$ *for all matrices* $\mathbf{A} \in PF$.

Now we analyze which similarity matrices $\mathbf{S}$ preserve the Gantmacher–Krein property or being in GK.

THEOREM 10. *For any invertible matrix* $\mathbf{S}$*, the following statements are equivalent:*

1. *Either* $\mathbf{S}$ *or* $-\mathbf{S}$ *is TSA.*

2. $\mathbf{S}^{-1}\mathbf{AS}$ *has the Gantmacher–Krein property for all matrices* $\mathbf{A}$ *having the Gantmacher–Krein property.*

Proof. $(1) \Rightarrow (2)$. Suppose (1) holds. Assume without loss of generality that $\mathbf{S}$ is TSA. Then $\mathbf{S}^{-1}$ is TP and $(\mathbf{S}^{-1})^{(j)}$ is nonnegative for all $j = 1, \ldots, n$. The Jacobi formula $(\mathbf{S}^{-1})^{(j)} = (\mathbf{S}^{(j)})^{-1}$ shows that $\mathbf{S}^{(j)}$ is monotone for $j = 1, \ldots, n$.

Let $\mathbf{A}$ be an arbitrary $n \times n$ matrix having the Gantmacher–Krein property. Applying Lemma 4 to $\mathbf{A}$, we obtain that the jth compound matrix $\mathbf{A}^{(j)}$ has the strong Perron–Frobenius property for all $j = 1, \ldots, n$. Let us examine the matrix $\mathbf{S}^{-1}\mathbf{AS}$. The Cauchy–Binet formula implies the equality

$$(\mathbf{S}^{-1}\mathbf{AS})^{(j)} = (\mathbf{S}^{-1})^{(j)}\mathbf{A}^{(j)}\mathbf{S}^{(j)}.$$

Applying Theorem 8 to every $\mathbf{A}^{(j)}$, we obtain that $(\mathbf{S}^{-1})^{(j)}\mathbf{A}^{(j)}\mathbf{S}^{(j)}$ has the strong Perron–Frobenius property for all $j = 1, \ldots, n$. Applying Lemma 4 to $\mathbf{S}^{-1}\mathbf{AS}$ completes the proof.

$(2) \Rightarrow (1)$. Conversely, suppose (1) does not hold, i.e., both $\mathbf{S}$ and $-\mathbf{S}$ are not TSA. Then there is a positive integer j, $1 \leqslant j \leqslant n$ such that the jth compound matrix $(\mathbf{S}^{-1})^{(j)} = (\mathbf{S}^{(j)})^{-1}$ has a positive entry and a negative entry.

In this case, following the reasoning from [4], we can find a positive vector v such that $(\mathbf{S}^{(j)})^{-1}v$ has a positive entry and a negative entry. Consider the following two cases.

(a) The matrix $(\mathbf{S}^{(j)})^{-1}$ has a column (say, the lth column) with a positive entry and a negative entry. In this case, we take $v \in \mathbb{R}^{\binom{n}{j}}$ with the coordinates $v = (v^1, \ldots, v^{\binom{n}{j}})$, where $v^l = 1$, $v^i = \varepsilon_i > 0$, $i \neq l$, $1 \leqslant i \leqslant \binom{n}{j}$. Thus, $(\mathbf{S}^{(j)})^{-1}v$ has a positive entry and a negative entry, for sufficiently small values ε_i, starting from some point.

(b) Every nonzero column of $(\mathbf{S}^{(j)})^{-1}$ is either nonpositive or nonnegative (with at least one nonzero entry). Let us assume that the lth column is nonnegative and the mth column is nonpositive. Without loss of generality, we assume that l and m are the numbers in the lexicographic numeration of the sets of indices $(i_1, \ldots, i_{r-1}, i_{r+1}, \ldots, i_{j+1})$ and $(i_1, \ldots, i_{s-1}, i_{s+1}, \ldots, i_{j+1})$, respectively (here $1 \leqslant i_1 < \ldots < i_{j+1} < n$, $1 \leqslant r,\ s \leqslant j+1$, $r \neq s$). (Indeed, suppose that all nonzero entries of each two columns of $(\mathbf{S}^{(j)})^{-1}$ with the numbers as above are of the same sign (say, positive). In this case it is easy to see that the whole matrix $(\mathbf{S}^{(j)})^{-1}$ is nonnegative.) Let us consider $(\mathbf{S}^{(j)})^{-1}((1-\lambda)\widetilde{e}_l + \lambda\widetilde{e}_m)$, where $\widetilde{e}_l$, $\widetilde{e}_m$ are the lth and the mth basic vectors in $\mathbb{R}^{\binom{n}{j}}$ respectively, $\lambda \in [0,1]$. Note that $(\mathbf{S}^{(j)})^{-1}\widetilde{e}_l$ is nonnegative and $(\mathbf{S}^{(j)})^{-1}\widetilde{e}_m$ is nonpositive. Let λ_0 be the largest number in $[0,1]$ such that $(\mathbf{S}^{(j)})^{-1}((1-\lambda)\widetilde{e}_l + \lambda\widetilde{e}_m)$ is still nonnegative. Since all the columns of $(\mathbf{S}^{(j)})^{-1}$ are linearly independent, we obtain that the vector $(\mathbf{S}^{(j)})^{-1}((1-\lambda_0)\widetilde{e}_l + \lambda_0\widetilde{e}_m)$ is nonzero for any λ_0. Choose $\lambda_1 > \lambda_0$, sufficiently close to λ_0. Then $(\mathbf{S}^{(j)})^{-1}((1-\lambda_1)\widetilde{e}_l + \lambda_1\widetilde{e}_m)$ has a positive entry and a negative entry. Now let $v = (v^1, \ldots, v^{\binom{n}{j}})$ be the positive vector in $\mathbb{R}^{\binom{n}{j}}$ with $v^l = 1-\lambda_1$, $v^m = \lambda_1$ and $v^i = \varepsilon_i$, $1 \leqslant i \leqslant \binom{n}{j}$, $i \neq l, m$. Then $(\mathbf{S}^{(j)})^{-1}v$ has a positive entry and a negative entry, for sufficiently small ε_i, starting from some point.

Fix an arbitrary STP matrix $\mathbf{A}$. By Gantmacher–Krein theorem (Theorem 5), $\mathbf{A}$ has the Gantmacher–Krein property. Thus, applying Lemma 4, we obtain that the jth compound matrix $\mathbf{A}^{(j)}$ has the strong Perron–Frobenius property, that is, the first eigenvector $\varphi_j = (\varphi_j^1, \ldots, \varphi_j^{\binom{n}{j}})$ corresponding to the greatest in absolute value eigenvalue $\rho(\mathbf{A}^{(j)})$ may be chosen to be positive. Without loss the generality we may assume that $\min_p \varphi_j^p > 1$. Construct a positive diagonal matrix $\mathbf{D}$ as follows:

(a') For the case (a), let l be the number in the lexicographic numeration of the set of indices $(i_1, \ldots, i_j)$, $1 \leqslant i_1 < \ldots < i_j \leqslant n$. Then we set $\mathbf{D} = \mathrm{diag}\{d_{11}, \ldots, d_{nn}\}$, where

$$d_{kk} = \begin{cases} \varepsilon^{j-1}\varphi_j^l, & \text{if } k = i_1; \\ \frac{1}{\varepsilon}, & \text{if } k \in \{i_2, \ldots, i_j\}; \\ \frac{\max_p \varphi_j^p}{\varepsilon^2} & \text{if } k \in [n] \setminus \{i_1, i_2, \ldots, i_j\}. \end{cases}$$

(b') For (b), where l and m are the numbers in the lexicographic numeration of the sets of indices $(i_1, \ldots, i_{r-1}, i_{r+1}, \ldots, i_{j+1})$ and $(i_1, \ldots, i_{s-1}, i_{s+1}, \ldots, i_{j+1})$, respectively (here $1 \leqslant i_1 < \ldots < i_{j+1} < n$, $1 \leqslant r,\ s \leqslant j+1$, $r \neq s$), we set

$\mathbf{D} = \mathrm{diag}\{d_{11}, \ldots, d_{nn}\}$ as follows:

$$d_{kk} = \begin{cases} \frac{1}{(1-\lambda_1)\varepsilon^{j-1}}\varphi_j^l, & \text{if } k = i_s; \\ \frac{1}{\lambda_1\varepsilon^{j-1}}\varphi_j^m, & \text{if } k = i_r; \\ \varepsilon, & \text{if } k \in \{i_1, \ldots, i_{j+1}\} \setminus \{i_r, i_s\}; \\ \frac{\max_p \varphi_j^p}{\varepsilon^j} & \text{if } k \in [n] \setminus \{i_1, i_2, \ldots, i_{j+1}\}. \end{cases}$$

In both cases (a') and (b'), choose the value ε as follows:

$$0 < \varepsilon < \min_i \varepsilon_i,$$

where the values ε_i $\left(1 \leqslant i \leqslant \binom{n}{j}\right)$ are chosen in (a) and (b), respectively, such that $(\mathbf{S}^{(j)})^{-1}v$ has a positive entry and a negative entry.

Considering the matrix $\mathbf{B} = \mathbf{D}^{-1}\mathbf{AD}$, by the Cauchy–Binet formula we have that $\mathbf{B}$ is also STP, for any positive diagonal matrix $\mathbf{D}$. Then, applying the Gantmacher–Krein theorem (Theorem 5), we obtain that $\mathbf{B}$ has the Gantmacher–Krein property. Now let us show that the matrix $\mathbf{S}^{-1}\mathbf{BS}$ does not have the Gantmacher–Krein property. For this, it is enough to show that at least one of its compound matrices does not have the strong Perron–Frobenius property. Indeed, consider its jth compound matrix $(\mathbf{S}^{-1}\mathbf{BS})^{(j)} = (\mathbf{S}^{(j)})^{-1}\mathbf{B}^{(j)}\mathbf{S}^{(j)}$ (through the Cauchy–Binet and Jacobi formulae). It is easy to see, that the eigenvalues of $(\mathbf{S}^{(j)})^{-1}\mathbf{B}^{(j)}\mathbf{S}^{(j)}$ are the same that those of $\mathbf{B}^{(j)}$. Consider the eigenvector of $(\mathbf{S}^{(j)})^{-1}\mathbf{B}^{(j)}\mathbf{S}^{(j)}$ which corresponds to the greatest in absolute value eigenvalue $\rho(\mathbf{B}^{(j)})$. It is of the form $(\mathbf{S}^{(j)})^{-1}\psi_j$, where ψ_j is the eigenvector of $\mathbf{B}^{(j)}$ which corresponds to the same eigenvalue $\rho(\mathbf{B}^{(j)})$. Since $\mathbf{B} = \mathbf{D}^{-1}\mathbf{AD}$, we have the equality $\mathbf{B}^{(j)} = (\mathbf{D}^{(j)})^{-1}\mathbf{A}^{(j)}\mathbf{D}^{(j)}$ (through the Cauchy–Binet and Jacobi formulae). Thus the first eigenvector ψ_j of $\mathbf{B}^{(j)}$ is of the form $(\mathbf{D}^{(j)})^{-1}\varphi_j$, where φ_j is the first eigenvector of $\mathbf{A}^{(j)}$ corresponding to the greatest in absolute value eigenvalue $\rho(\mathbf{A}^{(j)})$. We derive the following equalities:

$$\psi_j = (\mathbf{D}^{(j)})^{-1}\varphi_j;$$

$$(\mathbf{S}^{(j)})^{-1}\psi_j = (\mathbf{S}^{(j)})^{-1}((\mathbf{D}^{(j)})^{-1}\varphi_j).$$

Now let us consider the entries of $\mathbf{D}^{-1}$. From the choice of ε, the vector $(\mathbf{S}^{(j)})^{-1}\psi_j$ has both positive entries and negative entries as well as the vector $(\mathbf{S}^{(j)})^{-1}v$. Thus $(\mathbf{S}^{-1}\mathbf{BS})^{(j)}$ does not have the strong Perron–Frobenius property and $\mathbf{S}^{-1}\mathbf{BS}$ does not have the Gantmacher–Krein property, which is a contradiction. □

THEOREM 11. *For any $n \times n$ invertible matrix $\mathbf{S}$, the following statements are equivalent:*

1. $\mathbf{S}$ *is positive (negative) diagonal matrix.*
2. $\mathbf{S}^{-1}\mathbf{AS} \in GK$ *for all matrices* $\mathbf{A} \in GK$.

Proof. $(1) \Rightarrow (2)$. Let $\mathbf{A}$ be an arbitrary matrix from GK. Applying Theorem 7 to $\mathbf{A}$, we obtain that the jth compound matrix $\mathbf{A}^{(j)}$ belongs to PF for every $j = 1, \ldots, n$. Let $\mathbf{S}$ be a positive diagonal matrix (otherwise we consider $-\mathbf{S}$). In this case, it is easy to see that $\mathbf{S}^{(j)}$ is also a positive diagonal matrix for every $j = 1, \ldots, n$. Applying the Cauchy–Binet and Jacobi formulae, we obtain the equality

$$(\mathbf{S}^{-1}\mathbf{A}\mathbf{S})^{(j)} = (\mathbf{S}^{(j)})^{-1}\mathbf{A}^{(j)}\mathbf{S}^{(j)}. \qquad (j = 1, \ldots, n)$$

Since $\mathbf{S}^{(j)}$ and $(\mathbf{S}^{(j)})^{-1}$ are both nonnegative, we apply Theorem 9 and obtain that the matrix $(\mathbf{S}^{(j)})^{-1}\mathbf{A}^{(j)}\mathbf{S}^{(j)}$ also belongs to PF for every $j = 1, \ldots, n$. Applying Theorem 7 to the matrix $\mathbf{S}^{-1}\mathbf{A}\mathbf{S}$ we set that $\mathbf{S}^{-1}\mathbf{A}\mathbf{S}$ belongs to GK.

$(2) \Rightarrow (1)$. It is enough to prove that if $\mathbf{S}$ and $\mathbf{S}^{-1}$ are both TP then $\mathbf{S}$ is a positive diagonal matrix. If $\mathbf{S}^{-1}$ is TP then using Lemma 5 we obtain that $\mathbf{S}$ is TSA. Thus we have that the matrices $\mathbf{S}$ and $\mathbf{S}^*$ are both TP. We show that this is possible only if $\mathbf{S}$ is positive diagonal, by using induction on n.

For $n = 2$, we have $\mathbf{S} = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} \geqslant 0$, $\mathbf{S}^* = \begin{pmatrix} s_{11} & -s_{12} \\ -s_{21} & s_{22} \end{pmatrix} \geqslant 0$ and $\det(\mathbf{S}) > 0$. Obviously, this is possible if and only if $s_{12} = s_{21} = 0$, $s_{11}, s_{22} > 0$. For $n = 2$, the statement holds. Suppose the statement holds for order $n-1$. For the case of order n, suppose that $\mathbf{S}$ and $\mathbf{S}^*$ are both $n \times n$ and TP. Consider $\widetilde{\mathbf{S}_1}$ and $\widetilde{\mathbf{S}_n}$ — two $(n-1)\times(n-1)$ principal submatrices of $\mathbf{S}$, obtained by deleting the first (respectively, the last) row and column. The following equalities hold for these submatrices:

$$(\widetilde{\mathbf{S}_n})^* = \widetilde{\mathbf{S}^*}_n;$$

$$(\widetilde{\mathbf{S}_1})^* = \widetilde{\mathbf{S}^*}_1.$$

These equalities and total positivity of $\mathbf{S}$ and $\mathbf{S}^*$ imply that all the matrices $\widetilde{\mathbf{S}_n}$, $\widetilde{\mathbf{S}_1}$, $(\widetilde{\mathbf{S}_n})^*$ and $(\widetilde{\mathbf{S}_1})^*$ are totally positive. Thus we can apply the induction hypothesis and obtain that $\widetilde{\mathbf{S}_n}$ and $\widetilde{\mathbf{S}_1}$ are both positive diagonal. This implies $s_{ii} > 0$ for all $i = 1, \ldots, n$ and all off-diagonal entries of $\mathbf{S}$, except, probably, s_{1n} and s_{n1} are equal to zero. We show that s_{1n} and s_{n1} are also equal to zero. If n is even then $s^*_{1n} = -s_{1n}$, $s^*_{n1} = -s_{n1}$ and the inequalities $s^*_{1n}, s^*_{n1}, s_{1n}, s_{n1} \geqslant 0$ imply $s_{n1} = s_{1n} = 0$. If n is odd, consider the minors $S\begin{pmatrix} 1 & 2 \\ 2 & n \end{pmatrix}$ and $S\begin{pmatrix} n-1 & n \\ 1 & n-1 \end{pmatrix}$. If $s_{n1} > 0$ or $s_{1n} > 0$, then we have one of the following estimates

$$S\begin{pmatrix} 1 & 2 \\ 2 & n \end{pmatrix} = s_{12}s_{2n} - s_{22}s_{1n} = -s_{22}s_{1n} < 0;$$

$$S\begin{pmatrix} n-1 & n \\ 1 & n-1 \end{pmatrix} = s_{n-1,1}s_{n,n-1} - s_{n-1,n-1}s_{n1} = -s_{n-1,n-1}s_{n1} < 0.$$

This contradicts the nonnegativity of $\mathbf{S}^{(2)}$. $\square$

5. Eventually STJS matrices

Now we give the following definitions.

DEFINITION 18. An $n \times n$ matrix $\mathbf{A}$ is said to have the *total signature equality (TSE) property* if $\mathbf{A}$ has n positive simple eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$ with the systems of the corresponding eigenvectors $\{x_1, \ldots, x_n\}$ and eigenfunctionals $\{x_1^*, \ldots, x_n^*\}$ satisfying the following conditions:

1. Both $x_1 \wedge \ldots \wedge x_j$ and $x_1^* \wedge \ldots \wedge x_j^*$ have no zero coordinates for all $j = 1, \ldots, n$.
2. $\mathrm{Sign}(x_1 \wedge \ldots \wedge x_j) = \mathrm{Sign}(x_1^* \wedge \ldots \wedge x_j^*)$ for all $j = 1, \ldots, n$.

It is obvious that if $\mathbf{A}$ has TSE property then so does $\mathbf{A}^T$.

Now let us present the following generalizations of total positivity (for the definition and examples, see also [15]).

DEFINITION 19. A $n \times n$ matrix $\mathbf{A}$ is called *totally J–sign-symmetric (TJS)*, if it is J–sign-symmetric, and its j-th compound matrices $\mathbf{A}^{(j)}$ are also J–sign-symmetric for every j $(j = 2, \ldots, n)$.

DEFINITION 20. A $n \times n$ matrix $\mathbf{A}$ is called *strictly totally J–sign-symmetric (STJS)*, if it is strictly J–sign-symmetric, and its j-th compound matrices $\mathbf{A}^{(j)}$ are also strictly J–sign-symmetric for every j $(j = 2, \ldots, n)$.

Let us give an example of an STJS matrix.

EXAMPLE 3. Take

$$\mathbf{A} = \begin{pmatrix} 5.6 & 1.2 & 0.7 & 0.5 \\ 6.6 & 6.2 & 4.1 & 8.1 \\ 4.4 & 4.4 & 3.5 & 8 \\ 1 & 3.8 & 3.4 & 9 \end{pmatrix}$$

In this case, we have

$$\mathbf{A}^{(2)} = \begin{pmatrix} 26.8 & 18.34 & 42.06 & 0.58 & 6.62 & 3.62 \\ 19.36 & 16.52 & 42.6 & 1.12 & 7.4 & 3.85 \\ 20.08 & 18.34 & 49.9 & 1.42 & 8.9 & 4.6 \\ 1.76 & 5.06 & 17.16 & 3.66 & 13.96 & 4.45 \\ 18.88 & 18.34 & 51.3 & 5.5 & 25.02 & 9.36 \\ 12.32 & 11.46 & 31.6 & 1.66 & 9.2 & 4.3 \end{pmatrix};$$

$$\mathbf{A}^{(3)} = \begin{pmatrix} 15.656 & 58.464 & 15.438 & -2.602 \\ 22.008 & 87.992 & 25.676 & -3.532 \\ 4.168 & 19.76 & 7.69 & -0.45 \\ -9.584 & -35.408 & -8.354 & 2.386 \end{pmatrix};$$

$$\mathbf{A}^{(4)} = \det(\mathbf{A}) = 3.3928.$$

The following statement describes the spectral properties of STJS matrices (see [15], p. 559, Theorem 31).

THEOREM 12. *Let an $n \times n$ matrix $\mathbf{A}$ be STJS. Then all the eigenvalues of $\mathbf{A}$ are positive and simple:*

$$\rho(A) = \lambda_1 > \lambda_2 > \ldots > \lambda_n > 0.$$

If $\mathbf{A}$ is STJS, then $\mathbf{A}^T$ is also STJS (see [15], p. 558, Proposition 29). In this case, we have $\mathrm{Sign}(\mathbf{A}^{(j)}) = \mathrm{Sign}((\mathbf{A}^{(j)})^T)$ for $j = 1, \ldots, n$. Hence is not difficult to see that *an STJS matrix $\mathbf{A}$ has the total signature equality property*.

Analogously with ESTP matrices, we introduce the following generalization of the class of STJS matrices.

DEFINITION 21. A real $n \times n$ matrix $\mathbf{A}$ is called *eventually strictly totally J-sign-symmetric (ESTJS)* if there is a positive integer k_0 such that $\mathbf{A}^k$ is STJS for all $k \geqslant k_0$.

The following result characterizes properties of ESTJS matrices.

THEOREM 13. *Let $\mathbf{A}$ be an $n \times n$ matrix. Then the following statements are equivalent.*

1. *The matrix $\mathbf{A}$ has the total signature equality property.*
2. *For every j, $j = 1, \ldots, n$, the jth compound matrix $\mathbf{A}^{(j)}$ has the signature equality property.*
3. *For every j, $j = 1, \ldots, n$, the jth compound matrix $\mathbf{A}^{(j)}$ is ESJS.*
4. *The matrix $\mathbf{A}$ is eventually STJS.*

Proof. $(1) \Rightarrow (2)$. The proof follows from the definition and the Kronecker theorem (Theorem 4), applied to $\mathbf{A}$ and $\mathbf{A}^T$.

$(2) \Rightarrow (3)$. The proof follows from Theorem 3, applied to each $\mathbf{A}^{(j)}$, $j = 1, \ldots, n$.

$(3) \Rightarrow (4)$. We duplicate the reasoning of the proof of Theorem 7, implication $(3) \Rightarrow (4)$.

$(4) \Rightarrow (1)$. We duplicate the reasoning of the proof of Theorem 7, implication $(4) \Rightarrow (1)$, replacing the Gantmacher–Krein theorem (Theorem 5) with Theorem 12. $\square$

6. Eventually P-matrices

Recall the following definition (see [9]).

DEFINITION 22. An $n \times n$ matrix $\mathbf{A}$ is called a *P-matrix* if all its principal minors are positive, i.e the inequality $A\begin{pmatrix} i_1 & \ldots & i_k \\ i_1 & \ldots & i_k \end{pmatrix} > 0$ holds for all sets of indices $(i_1, \ldots, i_k)$, $1 \leqslant i_1 < \ldots < i_k \leqslant n$, and all k, $1 \leqslant k \leqslant n$.

The corresponding eventual property of matrices is defined as follows.

DEFINITION 23. A matrix $\mathbf{A}$ is called an eventually *P-matrix* if there is a positive integer k_0 such that $\mathbf{A}^k$ is a P-matrix for all positive integers $k \geqslant k_0$.

Assuming the positivity and simplicity of the spectrum, we obtain the following result describing the structure of eventually P-matrices.

THEOREM 14. *Let $\mathbf{A}$ be an $n \times n$ eventually P-matrix whose eigenvalues are all positive, simple and distinct:*

$$\rho(A) = \lambda_1 > \lambda_2 > \ldots > \lambda_n.$$

Then $\mathbf{A}$ is ESTJS.

Proof. Consider the jth compound matrix $\mathbf{A}^{(j)}$, $j = 1, \ldots, n$. Applying the Kronecker theorem (Theorem 4) to $\mathbf{A}^{(j)}$ we obtain that $\rho(\mathbf{A}^{(j)}) = \lambda_1 \ldots \lambda_j$ is a positive simple strictly dominant eigenvalue of $\mathbf{A}^{(j)}$. Thus the conditions of Lemma 1 hold. Applying Lemma 1 to each $\mathbf{A}^{(j)}$, $j = 1, \ldots, n$, we obtain the approximation:

$$\frac{1}{\rho(\mathbf{A}^{(j)})^k}(\mathbf{A}^{(j)})^k \to \varphi_j \otimes \varphi_j^* \qquad \text{as} \quad k \to \infty,$$

where $\varphi_j = (\varphi_j^1, \ldots, \varphi_j^{\binom{n}{j}})$ and $\varphi_j^* = ((\varphi_j^*)^1, \ldots, (\varphi_j^*)^{\binom{n}{j}})$ are the eigenvector and the eigenfunctional of $\mathbf{A}^{(j)}$ corresponding to $\rho(\mathbf{A}^{(j)})$, respectively. Applying the Cauchy–Binet formula, we have

$$(\mathbf{A}^{(j)})^k = (\mathbf{A}^k)^{(j)}.$$

Since $\mathbf{A}$ is an eventually P-matrix, we have that $(\mathbf{A}^{(j)})^k$ has positive principal diagonal entries for sufficiently large k.

Let us consider the principal diagonal entries of the matrix $\varphi_j \otimes \varphi_j^*$. These are $\varphi_j^i (\varphi_j^*)^i$, $i = 1, \ldots, \binom{n}{j}$. So the following inequalities hold:

$$\varphi_j^i (\varphi_j^*)^i > 0, \qquad i = 1, \ldots, \binom{n}{j}.$$

It follows that $\mathrm{Sign}(\varphi_j) = \mathrm{Sign}(\varphi_j^*)$, i.e. $\mathbf{A}^{(j)}$ has the signature equality property. Applying Theorem 3, we have that $\mathbf{A}^{(j)}$ is eventually SJS for all $j = 1, \ldots, n$. Then, by Theorem 13 it follows that $\mathbf{A}$ is eventually STJS. $\square$

COROLLARY 3. *Any eventually P-matrix with a positive simple distinct spectrum has the total signature equality property.*

Proof. Follows from the above reasoning and Theorem 13. $\square$

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