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A SUBNORMAL TOEPLITZ COMPLETION

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Abstract. In this paper we deal with a subnormal Toeplitz completion problem: Complete the unspecified Toeplitz operators of the partial block Toeplitz matrix

$$G := \begin{bmatrix} U^{*p} & ? \\ ? & U^{*q} \end{bmatrix} \quad (p, q = 1, 2, \cdots)$$

to make G subnormal, where U is the shift on the Hardy space $H^2(\mathbb{T})$ of the unit circle $\mathbb{T}$.

1. Introduction

Given a partially specified operator matrix with some known entries, the problem of finding suitable operators to complete the given partial operator matrix so that the resulting matrix satisfies certain given properties is called a *completion problem*. Dilation problems are special cases of completion problems: in other words, the dilation of T is a completion of the partial operator matrix $\left[\begin{smallmatrix} T & ? \\ ? & ? \end{smallmatrix}\right]$. A *partial block Toeplitz matrix* is simply an $n \times n$ matrix some of whose entries are specified Toeplitz operators and whose remaining entries are unspecified. A *subnormal completion* of a partial operator matrix is a particular specification of the unspecified entries resulting in a subnormal operator. A *subnormal Toeplitz completion* of a partial block Toeplitz matrix is a subnormal completion whose unspecified entries are Toeplitz operators.

In [6], the following subnormal Toeplitz completion problem was considered:

PROBLEM A. Let U be the unilateral shift on H^2. Complete the unspecified Toeplitz entries of the partial block Toeplitz matrix $A := \left[\begin{smallmatrix} U^* & ? \\ ? & U^* \end{smallmatrix}\right]$ to make A subnormal.

In this paper we are interested in the following problem which is a more general version of Problem A:

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PROBLEM B. Complete the unspecified Toeplitz operators of the partial block Toeplitz matrix

$$G := \begin{bmatrix} U^{*p} & ? \\ ? & U^{*q} \end{bmatrix} \quad (p,q = 1,2,\cdots) \tag{1}$$

to make G subnormal.

The case $p = q = 1$ of (1) has been considered in [6]. In this paper we answer Problem B for the cases that the unknown entries are rational Toeplitz operators.

Throughout this paper, let $\mathscr{H}$ denote a separable complex Hilbert space and $\mathscr{B}(\mathscr{H})$ denote the set of all bounded linear operators acting on $\mathscr{H}$. For an operator $T \in \mathscr{B}(\mathscr{H})$, T^* denotes the adjoint of T. An operator $T \in \mathscr{B}(\mathscr{H})$ is said to be *normal* if $T^*T = TT^*$, *hyponormal* if its self-commutator $[T^*,T] \equiv T^*T - TT^*$ is positive semi-definite, and *subnormal* if there exists a Hilbert space $\mathscr{K}$ containing $\mathscr{H}$ and a normal operator N on $\mathscr{K}$ such that $N\mathscr{H} \subseteq \mathscr{H}$ and $T = N|_{\mathscr{H}}$.

On the other hand, in 1970, P.R. Halmos addressed a problem on the subnormality of Toeplitz operators T_φ on the Hardy space $H^2 \equiv H^2(\mathbb{T})$ of the unit circle $\mathbb{T}$ in the complex plane $\mathbb{C}$. This is the so-called Halmos' Problem 5, presented in his lectures, *Ten problems in Hilbert space* [12], [13]:

HALMOS' PROBLEM 5. Is every subnormal Toeplitz operator either normal or analytic?

In 1984, Halmos' Problem 5 was answered in the negative by C. Cowen and J. Long [4]. However, until now researchers have been unable to characterize subnormal Toeplitz operators T_φ in terms of their symbols φ. Thus we may ask:

$$\text{Which subnormal Toeplitz operators are normal or analytic?} \tag{2}$$

A function $\varphi \in L^\infty$ is said to be of bounded type if there are analytic functions $\psi_1, \psi_2 \in H^\infty$ such that $\varphi(z) = \frac{\psi_1(z)}{\psi_2(z)}$ for almost all $z \in \mathbb{T}$. Evidently, rational functions are of bounded type. In 1976, M.B. Abrahamse has shown that the answer to Halmos' question is affirmative for Toeplitz operators with bounded type symbols ([1]):

ABRAHAMSE'S THEOREM. ([1, Theorem]) *Let $\varphi \in L^\infty$ be such that φ or $\overline{\varphi}$ is of bounded type. If*

(i) T_φ is hyponormal;

(ii) $\ker[T_\varphi^*, T_\varphi]$ is invariant for T_φ,

then T_φ is normal or analytic.

Consequently, since $\ker[T^*,T]$ is invariant for every subnormal operator T, it follows that if $\varphi \in L^\infty$ is such that φ or $\overline{\varphi}$ is of bounded type, then every subnormal Toeplitz operator T_φ must be either normal or analytic.

We now review a few essential facts for (block) Toeplitz operators and (block) Hankel operators, and for that we will use [2], [8], [9], [15], and [16]. For $\mathscr{X}$ a Hilbert space, let $L^2_{\mathscr{X}} \equiv L^2_{\mathscr{X}}(\mathbb{T})$ be the Hilbert space of $\mathscr{X}$-valued norm square-integrable measurable functions on $\mathbb{T}$, and let $H^2_{\mathscr{X}} \equiv H^2_{\mathscr{X}}(\mathbb{T})$ and $H^\infty_{\mathscr{X}} \equiv H^\infty_{\mathscr{X}}(\mathbb{T})$ be the corresponding Hardy spaces. Let $M_{m\times n} \equiv M_{m\times n}(\mathbb{C})$ denote the set of $m\times n$ complex matrices and write $M_n := M_{n\times n}(\mathbb{C})$. If Φ is a matrix-valued function in $L^\infty_{M_n} \equiv L^\infty_{M_n}(\mathbb{T})$, then the (*block*) *Toeplitz operator* T_Φ and the (*block*) *Hankel operator* H_Φ on $H^2_{\mathbb{C}^n}$ are defined by

$$T_\Phi f := P(\Phi f) \quad \text{and} \quad H_\Phi f := JP^\perp(\Phi f) \quad (f \in H^2_{\mathbb{C}^n}), \tag{3}$$

where P and $P^\perp$ denote the orthogonal projections that map $L^2_{\mathbb{C}^n}$ onto $H^2_{\mathbb{C}^n}$ and $\left(H^2_{\mathbb{C}^n}\right)^\perp$, respectively, and J denotes the unitary operator from $L^2_{\mathbb{C}^n}$ to $L^2_{\mathbb{C}^n}$ given by $(Jg)(z) := \overline{z}I_n g(\overline{z})$ for $g \in L^2_{\mathbb{C}^n}$ ($I_n :=$ the $n\times n$ identity matrix). For $\Phi \in L^\infty_{M_{m\times n}}$, write

$$\widetilde{\Phi}(z) := \Phi^*(\overline{z}). \tag{4}$$

A matrix function $\Theta \in H^\infty_{M_{m\times n}}$ is called an *inner* function if Θ is isometric a.e. on $\mathbb{T}$. The following basic relations can be easily derived from the definition:

$$T^*_\Phi = T_{\Phi^*}, \quad H^*_\Phi = H_{\widetilde{\Phi}} \quad (\Phi \in L^\infty_{M_n}); \tag{5}$$

$$T_{\Phi\Psi} - T_\Phi T_\Psi = H^*_{\Phi^*} H_\Psi \quad (\Phi, \Psi \in L^\infty_{M_n}); \tag{6}$$

$$H_\Phi T_\Psi = H_{\Phi\Psi}, \quad H_{\Psi\Phi} = T^*_{\widetilde{\Psi}} H_\Phi \quad (\Phi \in L^\infty_{M_n}, \Psi \in H^\infty_{M_n}). \tag{7}$$

For a matrix-valued function $\Phi = [\phi_{ij}] \in L^\infty_{M_n}$, we say that Φ is of *bounded type* if each entry ϕ_{ij} is of bounded type and that Φ is *rational* if each entry ϕ_{ij} is a rational function. For a matrix-valued function $\Phi \in H^2_{M_{n\times r}}$, we say that $\Delta \in H^2_{M_{n\times m}}$ is a *left inner divisor* of Φ if Δ is an inner matrix function such that $\Phi = \Delta A$ for some $A \in H^2_{M_{m\times r}}$ ($m \leqslant n$). We also say that two matrix functions $\Phi \in H^2_{M_{n\times r}}$ and $\Psi \in H^2_{M_{n\times m}}$ are *left coprime* if the only common left inner divisor of both Φ and Ψ is a unitary constant and that $\Phi \in H^2_{M_{n\times r}}$ and $\Psi \in H^2_{M_{m\times r}}$ are *right coprime* if $\widetilde{\Phi}$ and $\widetilde{\Psi}$ are left coprime. Two matrix functions Φ and Ψ in $H^2_{M_n}$ are said to be *coprime* if they are both left and right coprime. We would remark that if $\Phi \in H^2_{M_n}$ is such that $\det\Phi \neq 0$, then any left inner divisor Δ of Φ is square, i.e., $\Delta \in H^2_{M_n}$. If $\Phi \in H^2_{M_n}$ is such that $\det\Phi \neq 0$ then we say that $\Delta \in H^2_{M_n}$ is a *right inner divisor* of Φ if $\widetilde{\Delta}$ is a left inner divisor of $\widetilde{\Phi}$ (cf. [10]).

In 1988, the hyponormality of Toeplitz operators T_φ was completely characterized in terms of their symbols φ via an elegant theorem of C. Cowen [3].

COWEN'S THEOREM. ([3], [14]) *If $\varphi \in L^\infty$, then T_φ is hyponormal if and only if there exists a function $k \in H^\infty$ such that $||k||_\infty \leqslant 1$ and $\varphi - k\overline{\varphi} \in H^\infty$.*

In 2006, Gu, Hendricks and Rutherford [11] extended Cowen's Theorem to block Toeplitz operators. For a matrix-valued function $\Phi = [\phi_{ij}] \in L^\infty_{M_n}$, we say that Φ is *normal* if Φ is normal a.e. on $\mathbb{T}$. Then we have:

LEMMA 1.1. (Hyponormality of Block Toeplitz Operators) [11] *For each $\Phi \in L^\infty_{M_n}$, let*

$$\mathscr{E}(\Phi) := \left\{ K \in H^\infty_{M_n} : ||K||_\infty \leqslant 1 \ \text{and} \ \Phi - K\Phi^* \in H^\infty_{M_n} \right\}.$$

Then T_Φ is hyponormal if and only if Φ is normal and $\mathscr{E}(\Phi)$ is nonempty.

On the other hand, we note that by (7), the kernel of a block Hankel operator H_Φ is an invariant subspace of the shift operator T_{zI_n} on $H^2_{\mathbb{C}^n}$. Thus if $\ker H_\Phi \neq \{0\}$ then by the Beurling-Lax-Halmos Theorem, $\ker H_\Phi = \Theta H^2_{\mathbb{C}^m}$ for some inner matrix function Θ. In general, Θ need not be a square matrix function. We nevertheless have:

LEMMA 1.2. ([11]) *For $\Phi \in L^\infty_{M_n}$, the following statements are equivalent:*

1. *Φ is of bounded type;*
2. *$\ker H_\Phi = \Theta H^2_{\mathbb{C}^n}$ for some square inner matrix function Θ;*
3. *$\Phi = A\Theta^*$, where $A \in H^\infty_{M_n}$ and A and Θ are right coprime.*

For an inner matrix function $\Theta \in H^2_{M_n}$, we write

$$\mathscr{H}_\Theta := H^2_{\mathbb{C}^n} \ominus \Theta H^2_{\mathbb{C}^n}.$$

For $\Phi \in L^\infty_{M_n}$ we write

$$\Phi_+ := P_n(\Phi) \in H^2_{M_n} \quad \text{and} \quad \Phi_- := \big(P_n^\perp(\Phi)\big)^* \in H^2_{M_n},$$

where P_n and $P_n^\perp$ denote the orthogonal projections from $L^2_{M_n}$ onto $H^2_{M_n}$ and $(H^2_{M_n})^\perp$, respectively. Thus, we can write $\Phi = \Phi_-^* + \Phi_+$. In view of Lemma 1.2, if $\Phi \in L^\infty_{M_n}$ is such that Φ and Φ^* are of bounded type then Φ_+ and Φ_- can be written in the form

$$\Phi_+ = \Theta_1 A^* \quad \text{and} \quad \Phi_- = \Theta_2 B^*, \tag{8}$$

where Θ_1 and Θ_2 are inner, $A, B \in H^2_{M_n}$, Θ_1 and A are right coprime, and Θ_2 and B are right coprime. In (8), $\Theta_1 A^*$ and $\Theta_2 B^*$ will be called *right coprime factorizations* of Φ_+ and Φ_-, respectively.

Recently, Abrahamse's Theorem for matrix-valued rational symbols was obtained in [5]. We shall say that an inner matrix function $\Theta \in H^\infty_{M_n}$ is *nonconstant diagonal-constant* if Θ is of the form θI_n, where θ is a nonconstant inner function. We then have:

THEOREM 1.3. (Abrahamse's Theorem for matrix-valued rational symbols) [5] *Let $\Phi \equiv \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$ be a matrix-valued rational function. Thus in view of (8), we may write*

$$\Phi_- = \Theta B^* \quad \text{(right coprime factorization).}$$

Assume that Θ has a nonconstant diagonal-constant inner divisor. If

(i) *T_Φ is hyponormal;*

(ii) *$\ker[T_\Phi^*, T_\Phi]$ is invariant for T_Φ,*

then T_Φ is normal. Hence in particular, if T_Φ is subnormal then T_Φ is normal.

In this paper we answer Problem B by the aid of Theorem 1.3. Section 2 devotes the proof of the main result.

2. A subnormal Toeplitz completion problem

In this section we give an answer to Problem B.
We begin with:

LEMMA 2.1. *Let*

$$\Phi := \begin{bmatrix} \overline{z}^p & \varphi \\ \psi & \overline{z}^q \end{bmatrix} \quad (\varphi, \psi \in L^\infty;\ p, q = 1, 2, \cdots)$$

be such that T_Φ *is hyponormal. Then* $p = q$.

Proof. If T_Φ is hyponormal then by Lemma 1.1, Φ is normal, i.e., $\Phi^*\Phi = \Phi\Phi^*$, which implies

$$(z^p - z^q)\varphi = (\overline{z}^p - \overline{z}^q)\overline{\psi}, \text{ so that } (z^p - z^q)(\varphi + \overline{z}^{p+q}\overline{\psi}) = 0;$$

thus, either either $\varphi + \overline{z}^{p+q}\overline{\psi} = 0$ or $z^p = z^q$, i.e., $p = q$. Assume to the contrary that $p \neq q$. Thus

$$\varphi = -\overline{z}^{p+q}\overline{\psi}. \tag{9}$$

By Lemma 1.1, there exists a matrix function $K \equiv \begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix} \in \mathscr{E}(\Phi)$, so that

$$\begin{bmatrix} \overline{z}^p & \varphi \\ \psi & \overline{z}^q \end{bmatrix} - \begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix} \begin{bmatrix} z^p & \overline{\psi} \\ \overline{\varphi} & z^q \end{bmatrix} \in H^2_{M_2}, \tag{10}$$

which implies

$$\varphi - k_1\overline{\psi} - k_2 z^q \in H^2 \quad \text{and} \quad \overline{z}^q - k_3\overline{\psi} - k_4 z^q \in H^2. \tag{11}$$

From the second statement of (11), we can see that $\psi_+ \neq 0$. Also from the first statement of (11) together with (9), we have $(\overline{z}^{p+q} + k_1)\overline{\psi} \in H^2$, so that $\overline{z}^{p+q}\psi_- + \overline{z}^{p+q}\overline{\psi_+} + k_1\overline{\psi_+} \in H^2$, which gives

$$P^\perp\left(\overline{z}^{p+q}\psi_-\right) + \overline{z}^{p+q}\overline{\psi_+} = -P^\perp\left(k_1\overline{\psi_+}\right). \tag{12}$$

We next show that

$$P^\perp\left(\overline{z}^{p+q}\psi_-\right) \neq 0. \tag{13}$$

To prove (13), we assume to the contrary that $P^\perp\left(\overline{z}^{p+q}\psi_-\right) = 0$. Then, by (12), $\overline{z}^{p+q}\overline{\psi_+} = -P^\perp\left(k_1\overline{\psi_+}\right)$. Thus since $||K||_\infty \leqslant 1$, and hence $||k_1||_\infty \leqslant 1$, we have

$$||\psi_+||_2 = ||\overline{z}^{p+q}\overline{\psi_+}||_2 = ||P^\perp\left(k_1\overline{\psi_+}\right)||_2 \leqslant ||k_1\overline{\psi_+}||_2 \leqslant ||\overline{\psi_+}||_2,$$

which implies $||k_1\overline{\psi_+}||_2 = ||\overline{\psi_+}||_2$, i.e.,

$$\int |k_1\psi_+|^2 \frac{d\theta}{2\pi} = \int |\psi_+|^2 \frac{d\theta}{2\pi}, \text{ and hence } \int (1 - |k_1|^2)|\psi_+|^2 \frac{d\theta}{2\pi} = 0.$$

But since $\psi_+ \neq 0$, it follows that $|k_1| = 1$ a.e. on $\mathbb{T}$. Since $||K||_\infty \leqslant 1$, we must have $k_3 = 0$. Then by the second statement of (11), we have $\overline{z}^q \in H^2$, a contradiction. This proves (13). Now, since $P^\perp(\overline{z}^{p+q}\psi_-) \perp \overline{z}^{p+q}\overline{\psi_+}$, it follows from (12) and (13) that

$$||\psi_+||_2 = ||\overline{z}^{p+q}\overline{\psi_+}||_2 < ||P^\perp(\overline{z}^{p+q}\psi_-) + \overline{z}^{p+q}\overline{\psi_+}||_2 = ||P^\perp(k_1\overline{\psi_+})||_2 \leqslant ||\psi_+||_2,$$

a contradiction. Therefore we must have $p = q$. □

In view of Lemma 2.1, in Problem B it suffices to consider the case

$$\Phi := \begin{bmatrix} \overline{z}^p & \varphi \\ \psi & \overline{z}^p \end{bmatrix} \quad (\varphi, \psi \in L^\infty \text{ are rational; } p = 1, 2, \cdots).$$

On the other hand, in view of the scalar-valued version of (8), a rational function $\varphi \equiv \overline{\varphi_-} + \varphi_+$ has the following *coprime* factorizations:

$$\varphi_- = \theta_0\overline{a} \quad \text{and} \quad \varphi_+ = \theta_2\overline{c},$$

where the θ_i are inner functions (in fact, finite Blaschke products), $a \in \mathscr{H}_{\theta_0}$ and $c \in \mathscr{H}_{z\theta_2}$. Thus if φ and ψ are rational functions, then we can write

$$\varphi_- = \theta_0\overline{a} \quad \text{and} \quad \psi_- = \theta_1\overline{b} \quad \text{(coprime factorizations).}$$

Let m and n be the multiplicities of zeros of a and b at the origin, respectively. Then φ_- and ψ_- have the following *coprime* factorizations:

$$\varphi_- \equiv \theta_0\overline{a} = z^m\theta_0'\overline{a} \quad \text{and} \quad \psi_- \equiv \theta_1\overline{b} = z^n\theta_1'\overline{b} \quad \text{(coprime factorizations),} \tag{14}$$

where θ_0' and θ_1' are finite Blaschke products and $(\theta_0'\theta_1')(0) \neq 0$.

LEMMA 2.2. *Let $\Phi \equiv \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$ be a matrix-valued rational function. Then in view of (14), we may write*

$$\Phi_- := \begin{bmatrix} z^p & z^n\theta_1'\overline{b} \\ z^m\theta_0'\overline{a} & z^p \end{bmatrix} \quad (p = 1, 2, \cdots),$$

where θ_0' and θ_1' are finite Blaschke products and $(\theta_0'\theta_1')(0) \neq 0$. If

$$\Phi_- = \Theta B^* \quad \textit{(right coprime factorization)},$$

then Θ has an inner divisor of the form zI_2, except in the following two cases:

(i) $m + n = 2p$ and $(ab)(0) = (\theta_0'\theta_1')(0)$;

(ii) $m + n \geqslant 2p$ and $mn = 0$.

Proof. By Lemma 1.2, $\ker H_{\Phi_-^*} = \Theta H^2_{\mathbb{C}^2}$. We observe that for $f, g \in H^2$,

$$\Phi_-^* \begin{bmatrix} f \\ g \end{bmatrix} \in H^2_{\mathbb{C}^2} \iff \begin{bmatrix} \overline{z}^p & \overline{z}^m\overline{\theta_0'}a \\ \overline{z}^n\overline{\theta_1'}b & \overline{z}^p \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} \in H^2_{\mathbb{C}^2},$$

which implies that if $\begin{bmatrix} f \\ g \end{bmatrix} \in \ker H_{\Phi_-^*}$, then

$$\overline{z}^p f + \overline{z}^m \overline{\theta_0'} a g \in H^2 \quad \text{and} \quad \overline{z}^n \overline{\theta_1'} b f + \overline{z}^p g \in H^2. \tag{15}$$

We split the proof into three cases.

Case 1 ($0 \leqslant m+n < 2p$): In this case, $n < p$ or $m < p$. Suppose $m < p$. If $\begin{bmatrix} f \\ g \end{bmatrix} \in \ker H_{\Phi_-^*}$, then by the first statement of (15), we have $\overline{z}^{p-m}\theta_0' f \equiv h \in H^2$. Thus $\theta_0' f = z^{p-m} h$ a.e. on $\mathbb{T}$ and hence $\theta_0' f = z^{p-m} h$. Since $\theta_0'(0) \neq 0$, we have $f = z^{p-m} f_1$ for some $f_1 \in H^2$. In turn, by the second statement of (15), $\overline{z}^{m+n-p}\overline{\theta_1'} b f_1 + \overline{z}^p g \in H^2$. Thus if $m+n-p \leqslant 0$, then $g = z^p g_1$ for some $g_1 \in H^2$; if instead, $m+n-p > 0$, then $\overline{z}^{2p-m-n}\theta_1' g \in H^2$, and hence $g = z^{2p-m-n} g_2$ for some $g_2 \in H^2$. We thus have

$$\Theta H^2_{\mathbb{C}^2} = \ker H_{\Phi_-^*} \subseteq \begin{bmatrix} z^{p-m} & 0 \\ 0 & z^p \end{bmatrix} H^2_{\mathbb{C}^2} \bigcap \begin{bmatrix} z^{p-m} & 0 \\ 0 & z^{2p-m-n} \end{bmatrix} H^2_{\mathbb{C}^2} \subseteq (zI_2) H^2_{\mathbb{C}^2},$$

which implies that zI_2 is an inner divisor of Θ (cf. [10, Corollary IX.2.2]).

If instead $n < p$, then the same argument shows that zI_2 is an inner divisor of Θ.

Case 2 ($m+n = 2p$, $mn \neq 0$ and $(ab)(0) \neq (\theta_0'\theta_1')(0)$):

(a) Suppose $m = n$. Then

$$\Phi_- = z^p \theta_0' \theta_1' \begin{bmatrix} \theta_0'\theta_1' & \theta_1' a \\ \theta_0' b & \theta_0'\theta_1' \end{bmatrix}^* \equiv \Theta_1 B_1^* = \Theta B^*.$$

Since by assumption, $\det B_1(0) = \big[\theta_0'\theta_1'(\theta_0'\theta_1' - ab)\big](0) \neq 0$, and hence $B_1(0)$ is invertible, it follows (cf. [7, Lemma 3.3]) that Θ has an inner divisor zI_2.

(b) Suppose $m \neq n$. Since $m+n = 2p$ and $mn \neq 0$, it follows that $0 < n < p$ or $0 < m < p$. Suppose $0 < n < p$. If $\begin{bmatrix} f \\ g \end{bmatrix} \in \ker H_{\Phi_-^*}$, then by the second statement of (15), we have $z^{p-n}\overline{\theta_1'} b f \in H^2$, and hence $f = \theta_1' f_1$ for some $f_1 \in H^2$. In turn, $\overline{z}^n b f_1 + \overline{z}^p g \in H^2$, so that $g = z^{p-n} g_1$ for some $g_1 \in H^2$. We claim that

$$f_1(0) = 0, \ \text{so that} \ f = z\theta_1' f_2 \ \text{for some} \ f_2 \in H^2. \tag{16}$$

By the first statement of (15), we have

$$\overline{z}^p \theta_1' f_1 + \overline{z}^p \overline{\theta_0'} a g_1 \in H^2, \quad \text{so that} \quad g_1 = \theta_0' g_2 \ \text{for some} \ g_2 \in H^2.$$

In turn, $\overline{z}^p \theta_1' f_1 + \overline{z}^p a g_2 \in H^2$, so that $\theta_1'(0) f_1(0) + a(0) g_2(0) = 0$, which gives

$$g_2(0) = -\frac{\theta_1'(0)}{a(0)} f_1(0). \tag{17}$$

Also, by the second statement of (15), $\overline{z}^n b f_1 + \overline{z}^n \theta_0' g_2 \in H^2$, so that $b(0) f_1(0) + \theta_0'(0) g_2(0) = 0$, which gives

$$g_2(0) = -\frac{b(0)}{\theta_0'(0)} f_1(0). \tag{18}$$

If $f_1(0) \neq 0$, then by (17) and (18), we have $(ab)(0) = (\theta_0'\theta_1')(0)$, which contradicts the case assumption. This proves (16). We thus have

$$\Theta H^2_{\mathbb{C}^2} = \ker H_{\Phi_-^*} \subseteq \begin{bmatrix} z & 0 \\ 0 & z^{p-n} \end{bmatrix} H^2_{\mathbb{C}^2},$$

which implies that zI_2 is an inner divisor of Θ since $p-n \geqslant 1$.

If instead $0 < m < p$, then the same argument gives that zI_2 is an inner divisor of Θ.

Case 3 $(m+n > 2p,\ mn \neq 0)$:

(a) Suppose $m \geqslant p+1$. If $\begin{bmatrix} f \\ g \end{bmatrix} \in \ker H_{\Phi_-^*}$, then by the first statement of (15), we have $g = z^{m-p}\theta_0' g_1$ for some $g_1 \in H^2$. In turn, by the second statement of (15), $\overline{z}^n \overline{\theta_1'} bf + \overline{z}^{2p-m}\,\theta_0' g_1 \in H^2$. Thus if $m \geqslant 2p$, then $f = z^n \theta_1' f_1$ for some $f_1 \in H^2$, and if instead $m < 2p$, then $\overline{z}^{n+m-2p}\,\overline{\theta_1'} bf \in H^2$, so that $f = z^{n+m-2p}\,\theta_1' f_2$ for some $f_2 \in H^2$. We thus have

$$\Theta H^2_{\mathbb{C}^2} = \ker H_{\Phi_-^*} \subseteq \begin{bmatrix} z^n & 0 \\ 0 & z^{m-p} \end{bmatrix} H^2_{\mathbb{C}^2} \bigcap \begin{bmatrix} z^{n+m-2p} & 0 \\ 0 & z^{m-p} \end{bmatrix} H^2_{\mathbb{C}^2} \subseteq (zI_2) H^2_{\mathbb{C}^2}$$

which implies that zI_2 is an inner divisor of Θ.

(b) Suppose $m < p+1$. Then $n \geqslant p+1$ and the same argument as in Case 3(a) gives that zI_2 is an inner divisor of Θ.

This completes the proof. □

We need two auxiliary lemmas for the proof of the main result.

LEMMA 2.3. *(Normality of Block Toeplitz Operators)* [11, Theorem 4.3] *Let* $\Phi \equiv \Phi_+ + \Phi_-^* \in L^\infty_{M_n}$ *be normal. If* $det\Phi_+ \neq 0$, *then*

$$T_\Phi \text{ is normal} \Longleftrightarrow \Phi_+ - \Phi_+(0) = \Phi_- U \ \text{ for some constant unitary matrix } U. \quad (19)$$

LEMMA 2.4. *Let* $\Phi \equiv \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$ *be a matrix-valued rational function of the form*

$$\Phi_+ = A^*\Delta_0\Delta \quad \text{and} \quad \Phi_- = B^*\Delta,$$

where $\Delta_0\Delta = \theta I_n$ *with an inner function* θ, *and* B *and* Δ *are left coprime. If* $K \in \mathscr{E}(\Phi)$, *then*

$$cl\ ranH_{A\Delta^*} \subseteq ker(I - T_{\widetilde{K}} T^*_{\widetilde{K}}).$$

Proof. This follows from a careful analysis for the proof of [7, STEP 1 and (16) in STEP 2 of the proof of Theorem 3.5], which shows that the proof does not employ the diagonal-*constant*-ness of Δ, but uses only the diagonal-constant-ness of $\Delta_0\Delta$. □

We are ready for:

THEOREM 2.5. *Let* $\varphi, \psi \in L^\infty$ *be rational and consider*

$$\Phi := \begin{bmatrix} \overline{z}^p & \varphi \\ \psi & \overline{z}^q \end{bmatrix} \quad (p,q = 1,2,\cdots).$$

In view of (14), we may write

$$\Phi_- := \begin{bmatrix} z^p & z^n\theta_1'\overline{b} \\ z^m\theta_0'\overline{a} & z^q \end{bmatrix} \quad (p,q = 1,2,\cdots),$$

where θ_0' *and* θ_1' *are finite Blaschke products and* $(\theta_0'\theta_1')(0) \neq 0$. *Then the following statements are equivalent.*

1. T_Φ *is normal;*
2. T_Φ *is subnormal;*
3. $p = q$ *and one of the following conditions holds:*

(i) $\varphi = e^{i\theta}z^p + \zeta$ *and* $\psi = e^{i\omega}\varphi$ $(\zeta \in \mathbb{C};\ \theta, \omega \in [0,2\pi))$;

(ii) $\varphi = a\overline{z}^p + e^{i\theta}\sqrt{1+|a|^2}z^p + \zeta$ *and* $\psi = -\frac{\overline{a}}{a}\varphi$ $(a, \zeta \in \mathbb{C},\ a \neq 0,\ |a| \neq 1,$ $\theta \in [0,2\pi))$*,*

except the case $m+n = 2p$, $mn \neq 0$ *and* $(ab)(0) = (\theta_0'\theta_1')(0)$.

Proof. Clearly (1) $\Rightarrow$ (2). Moreover, (3) $\Rightarrow$ (1) follows from a straightforward calculation.

(2) $\Rightarrow$ (3): By Lemma 2.1, we have $p = q$. Thus we may write

$$\Phi \equiv \begin{bmatrix} \overline{z}^p & \varphi \\ \psi & \overline{z}^p \end{bmatrix} \equiv \Phi_-^* + \Phi_+ = \begin{bmatrix} z^p & \psi_- \\ \varphi_- & z^p \end{bmatrix}^* + \begin{bmatrix} 0 & \varphi_+ \\ \psi_+ & 0 \end{bmatrix}$$

and assume that T_Φ is subnormal. Since by Lemma 1.1, Φ is normal, we have

$$|\varphi| = |\psi|. \tag{20}$$

and also there exists a function $K \equiv \begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix} \in H^\infty_{M_2}$ such that $\Phi_-^* - K\Phi_+^* \in H^\infty_{M_2}$, i.e.,

$$\begin{bmatrix} \overline{z}^p & \overline{\varphi_-} \\ \overline{\psi_-} & \overline{z}^p \end{bmatrix} - \begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix} \begin{bmatrix} 0 & \overline{\psi_+} \\ \overline{\varphi_+} & 0 \end{bmatrix} \in H^2_{M_2}, \tag{21}$$

which implies that $\varphi_+ \neq 0$ and $\psi_+ \neq 0$, and hence $\det \Phi_+ \neq 0$. We write

$$\varphi_- \equiv \theta_0\overline{a} = z^m\theta_0'\overline{a} \quad \text{and} \quad \psi_- \equiv \theta_1\overline{b} = z^n\theta_1'\overline{b} \quad \text{(coprime factorizations)}$$

$(m,n = 0,1,\cdots$ and $(\theta_0'\theta_1')(0) \neq 0)$.

Note that if T_Φ is normal, then since $\det\Phi_+ \neq 0$, it follows from Lemma 2.3 that $\Phi_+ - \Phi_- U = \Phi_+(0)$ for some constant unitary matrix $U \equiv \left[\begin{smallmatrix} c_1 & c_2 \\ c_3 & c_4 \end{smallmatrix}\right]$ so that we have

$$\Phi_+ - \Phi_- U = \Phi_+(0) \Longleftrightarrow \begin{bmatrix} 0 & \varphi_+ \\ \psi_+ & 0 \end{bmatrix} - \begin{bmatrix} z^p & \theta_1\overline{b} \\ \theta_0\overline{a} & z^p \end{bmatrix} \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} = \begin{bmatrix} 0 & \varphi_+(0) \\ \psi_+(0) & 0 \end{bmatrix}$$

$$\Longrightarrow \begin{cases} c_1 z^p + c_3\theta_1\overline{b} = 0 \\ c_4 z^p + c_2\theta_0\overline{a} = 0 \\ \varphi_+ = c_2 z^p + c_4\theta_1\overline{b} + \xi_1 \\ \psi_+ = c_3 z^p + c_1\theta_0\overline{a} + \xi_2 \end{cases} \qquad (\xi_1, \xi_2 \in \mathbb{C}). \tag{22}$$

We split the proof into four cases.

Case 1 $(m = n = 0)$: In this case, by Lemma 2.2 and Theorem 1.3 we can conclude that T_Φ is normal. Observe that if $h \in H^\infty$ has a coprime factorization $h \equiv \theta\overline{d}$, then for any nonzero $\beta_1, \beta_2 \in \mathbb{C}$,

$$\beta_1 H_{\overline{z}^p} = \beta_2 H_{\overline{h}} \implies \theta = z^p.$$

Thus by (22), we have $c_1 = c_4 = 0$. But since U is unitary, it follows that $|c_2| = |c_3| = 1$, and hence $\theta_1\overline{b}$ and $\theta_0\overline{a}$ are constants and hence zeros. Thus again by (22), we have

$$\varphi = \varphi_+ = e^{i\omega_1} z^p + \xi_1 \quad \text{and} \quad \psi = \psi_+ = e^{i\omega_2} z^p + \xi_2.$$

But since $|\varphi| = |\psi|$, it follows that

$$\varphi = e^{i\theta} z^p + \zeta \quad \text{and} \quad \psi = e^{i\omega}\varphi \quad (\zeta \in \mathbb{C};\ \theta, \omega \in [0, 2\pi)).$$

Case 2 $\big(m = n = p$ and $(ab)(0) \neq (\theta_0'\theta_1')(0)\big)$: In this case, by Lemma 2.2 and Theorem 1.3 we can conclude that T_Φ is normal. Again by (22), we have

$$a, b \in \mathbb{C}\ (a \neq 0,\ b \neq 0) \quad \text{and} \quad \theta_0 = \theta_1 = z^p \text{ (i.e., } \theta_0' = \theta_1' = 1),$$

so that

$$\varphi_+ = \alpha z^p + \beta_1 \quad \text{and} \quad \psi_+ = \gamma z^p + \beta_2 \quad (\alpha, \beta_1, \beta_2, \gamma \in \mathbb{C}).$$

Since T_Φ is normal, and hence $H^*_{\Phi^*_+} H_{\Phi^*_+} = H^*_{\Phi^*_-} H_{\Phi^*_-}$ (by (6)), we have

$$\begin{bmatrix} |\alpha|^2 H^2_{\overline{z}^p} & 0 \\ 0 & |\gamma|^2 H^2_{\overline{z}^p} \end{bmatrix} = \begin{bmatrix} (1+|b|^2) H^2_{\overline{z}^p} & (a+\overline{b}) H^2_{\overline{z}^p} \\ (\overline{a}+b) H^2_{\overline{z}^p} & (1+|a|^2) H^2_{\overline{z}^p} \end{bmatrix},$$

which implies that

$$\begin{cases} b = -\overline{a} \\ |\alpha|^2 = 1 + |b|^2 = 1 + |a|^2 = |\gamma|^2. \end{cases} \tag{23}$$

Since $ab \neq (\theta_0'\theta_1')(0)$, we have $1 \neq |ab| = |a|^2$, i.e., $|a| \neq 1$. We thus have

$$\varphi_+ = e^{i\theta_1}\sqrt{1+|a|^2}\, z^p + \beta_1 \quad \text{and} \quad \psi_+ = e^{i\theta_2}\sqrt{1+|a|^2}\, z^p + \beta_2 \quad (|a| \neq 0),$$

which implies that

$$\varphi = a\overline{z}^p + e^{i\theta_1}\sqrt{1+|a|^2}\,z^p + \beta_1 \quad \text{and} \quad \psi = -\overline{a}\,\overline{z}^p + e^{i\theta_2}\sqrt{1+|a|^2}\,z^p + \beta_2.$$

Since $|\varphi| = |\psi|$, a straightforward calculation shows that

$$\varphi = a\overline{z}^p + e^{i\theta}\sqrt{1+|a|^2}\,z^p + \zeta \quad \text{and} \quad \psi = -\frac{\overline{a}}{a}\varphi, \tag{24}$$

where $a, \zeta \in \mathbb{C}, a \neq 0,\ |a| \neq 1$, and $\theta \in [0, 2\pi)$.

Case 3 ((i) $0 < m+n < 2p$; or (ii) $m+n > 2p$ $(mn \neq 0)$; or (iii) $m+n = 2p$ $(m \neq n,\ mn \neq 0)$ and $(ab)(0) \neq (\theta_0'\theta_1')(0)$): In this case, by Lemma 2.2 and Theorem 1.3 we can conclude that T_Φ is normal. Observe that each sub-case implies that

$$\{m \neq p \text{ or } n \neq p\} \text{ and } \{m \neq 0 \text{ or } n \neq 0\}.$$

Thus by the first and the second equations of (22), we have

(a) $m \neq 0, p \implies c_2 = c_4 = 0$;

(b) $n \neq 0, p \implies c_1 = c_3 = 0$.

In each case we obtain a contradiction with the fact that U is unitary.

(c) $m \neq p,\ n \neq 0$ (in view of (a) we may assume $m = 0$ and $n \neq 0$) $\implies c_4 = 0$ and $c_1 \neq 0$;

(d) $m \neq 0,\ n \neq p$ (in view of (b) we may assume $m \neq 0$ and $n = 0$) $\implies c_1 = 0$ and $c_4 \neq 0$.

Therefore, in this case we get

$$\{c_1 = 0,\ c_4 \neq 0\} \text{ or } \{c_1 \neq 0,\ c_4 = 0\}.$$

In each case we obtain a contradiction with the fact that U is unitary. Thus Case 3 cannot occur.

Case 4 ($m+n \geqslant 2p$ and $mn = 0$): Since $mn = 0$, we may, without loss of generality, assume that $n = 0$. Then we can write

$$\varphi_- := \theta_0\overline{a} \equiv z^m\theta_0'\overline{a} \quad \text{and} \quad \psi_- := \theta_1\overline{b} \quad \text{(coprime factorizations)},$$

where θ_0' and θ_1 are finite Blaschke products with $\theta_0'(0) \neq 0$ and $\theta_1(0) \neq 0$. From (21), we can see that which implies that

$$\begin{cases} \overline{z}^p - k_2\overline{\varphi_+} \in H^2, & \overline{\theta}_1 b - k_4\overline{\varphi_+} \in H^2 \\ \overline{z}^p - k_3\overline{\psi_+} \in H^2, & \overline{\theta}_0 a - k_1\overline{\psi_+} \in H^2. \end{cases} \tag{25}$$

Thus the following Toeplitz operators are all hyponormal (by Cowen's Theorem):

$$T_{\overline{z}^p+\varphi_+},\ T_{\overline{\theta}_1 b+\varphi_+},\ T_{\overline{z}^p+\psi_+},\ T_{\overline{\theta}_0 a+\psi_+}. \tag{26}$$

By (26) and [7, Lemma 3.2], we can see that

$$\varphi_+ = z^p\theta_1\theta_3\overline{d} \quad \text{and} \quad \psi_+ = \theta_0\theta_2\overline{c} \ \text{(coprime factorizations)},$$

where θ_2 and θ_3 are finite Blaschke products. A straightforward calculation together with (25) shows that

$$k_3(0)=0 \quad \text{and} \quad k_4(0)=0. \tag{27}$$

Write

$$\theta_2 = z^{q_2}\theta_2' \quad \text{and} \quad \theta_3 = z^{q_3}\theta_3' \quad (\theta_2'(0)\neq 0,\ \theta_3'(0)\neq 0).$$

Then

$$\Phi_+ = \begin{bmatrix} 0 & z^p\theta_1\theta_3\overline{d} \\ \theta_0\theta_2\overline{c} & 0 \end{bmatrix} = \begin{bmatrix} 0 & z^{p+q_3}\theta_1\theta_3'\overline{d} \\ z^{m+q_2}\theta_0'\theta_2'\overline{c} & 0 \end{bmatrix}.$$

Note that

$$\widetilde{\Phi_-} = \begin{bmatrix} z^p & \widetilde{\theta}_0\overline{\widetilde{a}} \\ \widetilde{\theta}_1\overline{\widetilde{b}} & z^p \end{bmatrix}.$$

Write

$$\widetilde{\Phi_-^*} = \widetilde{B}\widetilde{\Delta}^* \quad \text{(right coprime factorization)}.$$

We observe that for $f,g\in H^2$,

$$\begin{aligned}
\begin{bmatrix} f \\ g \end{bmatrix} \in \ker H_{\widetilde{\Phi_-^*}} &\Longrightarrow \begin{bmatrix} \overline{z}^p & \overline{\widetilde{\theta}}_1\widetilde{b} \\ \overline{\widetilde{\theta}}_0\widetilde{a} & \overline{z}^p \end{bmatrix}\begin{bmatrix} f \\ g \end{bmatrix} \in H^2_{\mathbb{C}^2} \\
&\Longrightarrow \begin{cases} \overline{z}^p f + \overline{\widetilde{\theta}}_1\widetilde{b}g \in H^2 \\ \overline{z}^p g + \overline{z}^m\overline{\widetilde{\theta}_0'}\widetilde{a}f \in H^2 \end{cases} \\
&\Longrightarrow \begin{cases} f = z^{m-p}\widetilde{\theta}_0' f_1,\ g=\widetilde{\theta}_1 g_1 \quad (f_1,g_1\in H^2) \\ \overline{z}^p\widetilde{\theta}_1 g_1 + \overline{z}^p\widetilde{a}f_1 \in H^2 \end{cases}.
\end{aligned} \tag{28}$$

Thus if $\begin{bmatrix} f \\ g \end{bmatrix} \in \ker H_{\widetilde{\Phi_-^*}}$, then

$$f = z^{m-p}\widetilde{\theta}_0' f_1, \quad g=\widetilde{\theta}_1 g_1 \quad (f_1,g_1\in H^2) \quad \text{and} \quad \widetilde{\theta}_1 g_1 + \overline{z}^p\widetilde{a}f_1 \in zH^2. \tag{29}$$

Put

$$\widetilde{\Theta}_2 := \frac{1}{\sqrt{|\alpha|^2+1}}\begin{bmatrix} z^{m-p+1}\widetilde{\theta}_0' & \overline{\alpha}z^{m-p}\widetilde{\theta}_0' \\ \alpha z\widetilde{\theta}_1 & -\widetilde{\theta}_1 \end{bmatrix} \qquad \left(\alpha := \frac{\theta_1(0)}{a(0)}\right).$$

A straightforward calculation gives that

$$\widetilde{\Theta}_2 H^2_{\mathbb{C}^2} = \left\{ \begin{bmatrix} f \\ g \end{bmatrix} : f = z^{m-p}\widetilde{\theta}_0' f_1,\ g=\widetilde{\theta}_1 g_1\ (f_1,g_1\in H^2) \text{ and } \widetilde{\theta}_1 g_1 + \overline{z}^p\widetilde{a}f_1 \in zH^2 \right\}.$$

Thus we have

$$\ker H_{\widetilde{\Phi_-^*}} \equiv \widetilde{\Delta}H^2_{\mathbb{C}^2} \subseteq \widetilde{\Theta}_2 H^2_{\mathbb{C}^2} \subseteq \begin{bmatrix} z^{m-p}\widetilde{\theta}_0' & 0 \\ 0 & \widetilde{\theta}_1 \end{bmatrix} H^2_{\mathbb{C}^2} \equiv \widetilde{\Theta}_3 H^2_{\mathbb{C}^2}, \tag{30}$$

which says that $\widetilde{\Theta}_2$ and $\widetilde{\Theta}_3$ are left inner divisors of $\widetilde{\Delta}$ and hence Θ_2 and Θ_3 are right inner divisors of Δ. It thus follows from Lemma 2.4 that

$$\text{cl ran}\, H_{A\Theta_k^*} \subseteq \text{cl ran}\, H_{A\Delta^*} \subseteq \ker(I - T_{\widetilde{K}} T_{\widetilde{K}}^*) \quad (k = 2, 3). \tag{31}$$

Now we will show that $m + q_2 > p + q_3$. If $r := p + q_3 - m - q_2 \geqslant 0$, then we can write

$$\Phi_+ = \begin{bmatrix} 0 & z^{p+q_3}\theta_1\theta_3'\overline{d} \\ z^{m+q_2}\theta_0'\theta_2'\overline{c} & 0 \end{bmatrix} = z^{p+q_3}\theta_0'\theta_1\theta_2'\theta_3' \begin{bmatrix} 0 & z^r\theta_1\theta_3'c \\ \theta_0'\theta_2'd & 0 \end{bmatrix}^*.$$

By (31), we have

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \text{cl ran}\, H_{A\Theta_3^*} \subseteq \text{cl ran}\, H_{A\Delta^*} \in \ker(I - T_{\widetilde{K}} T_{\widetilde{K}}^*).$$

Thus it follows from (27) that

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = T_{\widetilde{K}} T_{\widetilde{K}}^* \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} T_{\widetilde{k}_1} & T_{\widetilde{k}_3} \\ T_{\widetilde{k}_2} & T_{\widetilde{k}_4} \end{bmatrix} \begin{bmatrix} T_{\overline{\widetilde{k}}_1} & T_{\overline{\widetilde{k}}_2} \\ T_{\overline{\widetilde{k}}_3} & T_{\overline{\widetilde{k}}_4} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \widetilde{k}_1 k_2(0) \\ \widetilde{k}_2 k_2(0) \end{bmatrix},$$

which implies that $k_1 = 0$ and, by (25), $\overline{\theta_0}a_0 \in H^2$, a contradiction. Therefore $m + q_2 > p + q_3$ and we can write

$$\Phi_+ = \begin{bmatrix} 0 & z^{p+q_3}\theta_1\theta_3'\overline{d} \\ z^{m+q_2}\theta_0'\theta_2'\overline{c} & 0 \end{bmatrix} = z^{m+q_2}\theta_0'\theta_1\theta_2'\theta_3' \begin{bmatrix} 0 & \theta_1\theta_3'c \\ z^r\theta_0'\theta_2'd & 0 \end{bmatrix}^*,$$

where $r := m + q_2 - p - q_3 > 0$. Thus we have

$$A\Theta_2 = \begin{bmatrix} \overline{z}(\overline{\alpha}\theta_3'c) & \theta_3'c \\ \overline{z}^{m-p-r+1}(\theta_2'd) & \overline{z}^{m-p-r}(\alpha\theta_2'd) \end{bmatrix}.$$

If $m - p - r + 1 \leqslant 0$, then it follows from (31) that

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \in \text{cl ran}\, H_{A\Theta_2^*} \subseteq \text{cl ran}\, H_{A\Delta^*} \in \ker(I - T_{\widetilde{K}} T_{\widetilde{K}}^*).$$

Thus it follows from (27) that

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = T_{\widetilde{K}} T_{\widetilde{K}}^* \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} T_{\widetilde{k}_1} & T_{\widetilde{k}_3} \\ T_{\widetilde{k}_2} & T_{\widetilde{k}_4} \end{bmatrix} \begin{bmatrix} T_{\overline{\widetilde{k}}_1} & T_{\overline{\widetilde{k}}_2} \\ T_{\overline{\widetilde{k}}_3} & T_{\overline{\widetilde{k}}_4} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \widetilde{k}_1 k_1(0) \\ \widetilde{k}_2 k_1(0) \end{bmatrix},$$

which implies that $k_2 = 0$ and, by (25), $\overline{z}^p \in H^2$, a contradiction. Thus $m - p - r + 1 > 0$ and hence it follows from (31) that

$$\begin{bmatrix} \beta \\ 1 \end{bmatrix} \in \text{cl ran}\, H_{A\Theta_2^*} \subseteq \text{cl ran}\, H_{A\Delta^*} \in \ker(I - T_{\widetilde{K}} T_{\widetilde{K}}^*) \quad \text{for some } \beta \in \mathbb{C}.$$

It thus follows from (27) that

$$\begin{aligned}\begin{bmatrix}\beta\\1\end{bmatrix} = T_{\widetilde{K}}T^*_{\widetilde{K}}\begin{bmatrix}\beta\\1\end{bmatrix} &= \begin{bmatrix}T_{\widetilde{k}_1} & T_{\widetilde{k}_3}\\ T_{\widetilde{k}_2} & T_{\widetilde{k}_4}\end{bmatrix}\begin{bmatrix}T_{\overline{\widetilde{k}}_1} & T_{\overline{\widetilde{k}}_2}\\ T_{\overline{\widetilde{k}}_3} & T_{\overline{\widetilde{k}}_4}\end{bmatrix}\begin{bmatrix}\beta\\1\end{bmatrix}\\ &= \begin{bmatrix}T_{\widetilde{k}_1} & T_{\widetilde{k}_3}\\ T_{\widetilde{k}_2} & T_{\widetilde{k}_4}\end{bmatrix}\begin{bmatrix}(\beta k_1(0)+k_2(0))\\ 0\end{bmatrix}\\ &= \begin{bmatrix}\widetilde{k}_1(\beta k_1(0)+k_2(0))\\ \widetilde{k}_2(\beta k_1(0)+k_2(0))\end{bmatrix},\end{aligned}$$

which implies that k_1 is a constant and k_2 is a nonzero constant. Again by (25),

$$\overline{z}^p - k_2\overline{\varphi_+} \in H^2 \Longrightarrow z^p\overline{\varphi_+} \in H^2 \Longrightarrow \overline{\theta_1\theta_3}d \in H^2,$$

which implies that $\theta_1\theta_3$ is a constant. Without loss of generality we may assume $\theta_1\theta_3 = 1$, and hence $\psi_- = 0$. Similarly, from (25), $\overline{\theta_0}a - k_1\overline{\psi_+} \in H^2$, i.e., $\overline{\theta_0}a - k_1\overline{\theta_0\theta_2}c \in H^2$ implies $k_1 \neq 0$ and $\theta_2 = 1$. By (20), $|\varphi| = |\psi|$, so we have

$$|z^p\overline{d} + \overline{\theta_0}a| = |\varphi_+ + \overline{\varphi_-}| = |\psi_+| = |\theta_0\overline{c}| \quad (\text{where } z^p\overline{d},\ \theta_0\overline{a} \text{ and } \theta_0\overline{c} \text{ are in } H^2),$$

which implies $z^p\theta_0(z^p\overline{d} + \overline{\theta_0}a)(\overline{z}^pd + \theta_0\overline{a}) = z^p\theta_0c\overline{c}$, so that

$$ad = z\Big((\theta_0\overline{c})z^{p-1}c - (\theta_0\overline{d})z^{p-1}d - (\theta_0\overline{a})(\theta_0\overline{d})z^{2p-1} - (\theta_0\overline{a})z^{p-1}a\Big). \tag{32}$$

But since $m \geqslant 2p$, it follows that $\theta_0\overline{d} = z^m\overline{d}\theta_0' = (z^p\overline{d})z^{m-p}\theta_0' \in H^2$. Thus (32) implies that $ad = zh$ for some $h \in H^2$, and hence $(ad)(0) = 0$, a contradiction. Therefore Case 4 cannot occur.

This completes the proof. □

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