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PRESERVERS OF PSEUDO SPECTRA OF OPERATOR JORDAN TRIPLE PRODUCTS

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Abstract. Let $\mathscr{H}$ be an infinite-dimensional complex Hilbert space and let $\mathscr{L}(\mathscr{H})$ be the algebra of all bounded linear operators on $\mathscr{H}$. For $\varepsilon > 0$ and $T \in \mathscr{L}(\mathscr{H})$, let $r_\varepsilon(T)$ denote the ε-pseudo spectral radius of T. We characterize surjective maps ϕ on $\mathscr{L}(\mathscr{H})$ which satisfy

$$r_\varepsilon(\phi(T)\phi(S)\phi(T)) = r_\varepsilon(TST)$$

for all $T, S \in \mathscr{L}(\mathscr{H})$. As application, mappings from $\mathscr{L}(X)$ onto itself that preserve the pseudo spectrum of Jordan triple product of operators are described. We also obtain analogous results for the finite-dimensional case, without the surjectivity assumption on ϕ.

1. Introduction

Throughout this paper, $\mathscr{H}$ will denote a Hilbert space over the complex field $\mathbb{C}$ and $\mathscr{L}(\mathscr{H})$ will denote the algebra of all bounded linear operators on $\mathscr{H}$ with identity operator I. For $T \in \mathscr{L}(\mathscr{H})$ we write T^* for its adjoint, $\sigma(T)$ for its spectrum, and $\|T\|$ the (spectral) norm of T. For $\varepsilon > 0$, the ε-pseudo spectrum of T, $\sigma_\varepsilon(T)$, is defined by

$$\sigma_\varepsilon(T) := \cup\{\sigma(T+E) : E \in \mathscr{L}(\mathscr{H}), \|E\| < \varepsilon\},$$

and coincides with the set

$$\{z \in \mathbb{C} : \|(z-A)^{-1}\| > \varepsilon^{-1}\}$$

with the convention that $\|(z-A)^{-1}\| = \infty$ if $z \in \sigma(T)$. Unlike the spectrum, which is a purely algebraic concept, the ε-pseudo spectrum depends on the norm. The ε-pseudo spectral radius of T, $r_\varepsilon(T)$, is given by

$$r_\varepsilon(T) := \sup\{|z| : z \in \sigma_\varepsilon(T)\}.$$

Pseudo spectra are a useful tool for analyzing operators, furnishing a lot of information about the algebraic and geometric properties of operators and matrices. They

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play a very natural role in numerical computations, especially in those involving spectral perturbations. The book [19] give an extensive account of the pseudo spectra, as well as investigations and applications in numerous fields.

Recently, general preserver problems with respect to various algebraic operations on $\mathscr{M}_n$, the algebra of $n\times n$ complex matrices, or on operator algebras, attracted a lot of attention of researchers in the fields; see for instance [1, 2, 3, 4, 5, 6, 10, 14, 15, 16, 18, 21] and the references therein. On the subject focused on the structures of nonlinear transformations on $\mathscr{M}_n$ or on $\mathscr{L}(\mathscr{H})$ that respect the pseudo spectra of certain algebraic operations, we mention: [8] where the authors studied mappings on $\mathscr{M}_n$ that preserve the pseudo spectrum of different kind of binary operations on matrices, [11] concerned with the Hilbert space setting, and in [12] general preserver problems that to do with preservers of pseudo spectra of matrix Lie products are considered, and investigation of several extensions of these results were obtained. Linear preservers of pseudo spectrum have also been studied in a recent paper by Kumar and Kulkarni [17].

This paper follows the same path of studies by considering general pseudo spectra preservers, and characterizes mappings on $\mathscr{L}(\mathscr{H})$ that preserve the ε-pseudo spectra of triple Jordan product of operators. In the next section, we characterize surjective map ϕ on $\mathscr{L}(\mathscr{H})$ that preserves the ε-pseudo spectral radius of triple Jordan products in a sense that

$$r_\varepsilon(\phi(T)\phi(S)\phi(T)) = r_\varepsilon(TST) \quad (T,S\in\mathscr{L}(\mathscr{H})).$$

It is shown that such a map ϕ has a nice structure. Precisely, ϕ has the form $T\mapsto \xi(T)UTU^*$ or $T\mapsto \xi(T)UTU^*$ for some unitary or conjugate unitary operator U on $\mathscr{H}$ and a (general) functional ξ from $\mathscr{L}(\mathscr{H})$ into the unit circle $\mathbb{T}$ of the field $\mathbb{C}$. We also give analogous result for the finite-dimensional case, and we classify transformations on $\mathscr{M}_n$ that preserverve the pseudo spectral radius of Jordan triple product of matrices, without sujectivity assumption on them. As a consequence, we describe, in the last section, mappings on $\mathscr{L}(\mathscr{H})$ or on the matrix spaces that preserve the ε-pseudo spectrum of Jordan triple product of operators. We prove that such transformations are of standard forms up a third root of unity. Our study can also be viewed as a pseudo spectra versions of the study of [6, 21] that have to do with preservers of eigenvalues of matrices or of peripheral spectrum of operators, and of the main result of [1] where the author characterized maps ϕ on $\mathscr{M}_n$ preserving the local spectrum of Jordan triple product of matrices at a fixed nonzero vector. It should be pointed out that our approach uses some arguments which are influenced by ideas from Cui and Hou [9], Cui et al. [11], and Dobovišek et al. [13].

2. Pseudo spectral radius preservers

We first fix some notation. The inner product on $\mathscr{H}$ will be denoted by $\langle .,.\rangle$. For $x, f\in\mathscr{H}$, as usual we denote by $x\otimes f$ the rank at most one operator on $\mathscr{H}$ given by $z\mapsto\langle z, f\rangle x$, and all at most rank one operators in $\mathscr{L}(\mathscr{H})$ can be written into this form. For an operator $T\in\mathscr{L}(\mathscr{H})$ we will denote by T^{tr} the transpose of T relative to an arbitrary but fixed orthogonal basis of $\mathscr{H}$. For a subset σ of $\mathbb{C}$ we will denote by $\overline{\sigma}$

the complex conjugation set of σ, and for $\varepsilon > 0$ and $z \in \mathbb{C}$ we will denote by $D(z, \varepsilon)$ the open disc of $\mathbb{C}$ centered at z and of radius ε.

Before stating the main results of this section, we collect some lemmas needed in what follows. The first one summarizes some properties of the pseudo spectrum; see [19].

LEMMA 1. *For $\varepsilon > 0$ and $T \in \mathscr{L}(\mathscr{H})$, the following statements hold.*

(i) $\sigma(T) + D(0, \varepsilon) \subseteq \sigma_\varepsilon(T)$.

(ii) For every nonzero $c \in \mathbb{C}$, $\sigma_\varepsilon(cT) = c\sigma_{\frac{\varepsilon}{|c|}}(T)$.

(iii) For every $T \in \mathscr{L}(\mathscr{H})$, $\sigma_\varepsilon(T^{tr}) = \sigma_\varepsilon(T)$ and $\sigma_\varepsilon(T^*) = \overline{\sigma_\varepsilon(T)}$.

(iv) For every unitary operator $U \in \mathscr{L}(\mathscr{H})$, $\sigma_\varepsilon(UTU^*) = \sigma_\varepsilon(T)$.

The second lemma, quoted from [11, Proposition 2.5], will be the backbone of the proof of our main results. It identifies the ε-pseudo spectral radius of rank one operators.

LEMMA 2. *Let $\varepsilon > 0$ and $x, f \in \mathscr{H}$ be arbitrary. Then*

$$r_\varepsilon(x \otimes f) = \frac{1}{2}(\sqrt{|\langle x, f\rangle|^2 + 4\varepsilon^2 + 4\varepsilon\|x\|\|f\|} + |\langle x, f\rangle|).$$

The third and fourth lemmas, established in [13], characterizes mappings that preserve zero Jordan triple product of operators or matrices.

LEMMA 3. *Let $\mathscr{L}(X)$ be the algebra of all bounded linear operator on an infinite-dimensional complex Banach space X, and let ϕ be a surjective map from $\mathscr{L}(X)$ into it self. If ϕ satisfies*

$$\phi(T)\phi(S)\phi(T) = 0 \Longleftrightarrow TST = 0 \quad (T, S \in \mathscr{L}(X)),$$

then there exists is a functional $\ell : \mathscr{L}(\mathscr{H}) \to \mathbb{C} \setminus \{0\}$ and either there is a bounded invertible linear or conjugate linear operator A on X such that $\phi(T) = \ell(T)ATA^{-1}$ for all $T \in \mathscr{L}(X)$, or X is reflexive and there is a bounded invertible linear or conjugate linear operator A from X^*, the dual of X, into X such that $\phi(T) = \ell(T)AT^*A^{-1}$ for all $T \in \mathscr{L}(X)$.

LEMMA 4. *Let $n \geqslant 3$ and let ϕ be a map from $\mathscr{M}_n$ into itself. If ϕ satisfies*

$$\phi(T)\phi(S)\phi(T) = 0 \Longleftrightarrow TST = 0 \quad (T, S \in \mathscr{M}_n),$$

then there is a functional $\ell : \mathscr{M}_n \to \mathbb{C} \setminus \{0\}$, an invertible matrix $A \in \mathscr{M}_n$, and a field automorphism η of $\mathbb{C}$ such that either $\phi(T) = \ell(T)AT^\eta A^{-1}$ or $\phi(T) = \ell(T)A(T^\eta)^{tr}A^{-1}$ for all $T \in \mathscr{M}_n$. Here T^η denotes the matrix obtained from T by applying η to every entry of it.

The fifth one characterizes nontrivial projections and self-adjoint operators through their pseudo spectral properties; see Cui et al. [11, Theorem 2.2 and Corollary 2.3].

LEMMA 5. *Let $\varepsilon > 0$, $T \in \mathscr{L}(\mathscr{H})$, $a \in \mathbb{C}$, and $t \in \mathbb{R}$. Then the following assertions hold.*

(i) $e^{it}T$ is self-adjoint if and only if $\sigma_\varepsilon(T) \subseteq \{z \in \mathbb{C} : |Im(e^{it}z)| < \varepsilon\}$.

(ii) There exists a nontrivial projection $P \in \mathscr{L}(\mathscr{H})$ such that $T = aP$ if and only if $\sigma_\varepsilon(T) = D(0,\varepsilon) \cup D(a,\varepsilon)$.

Let us review some more notation that we will need in the sequel. Arbitrarily fix an orthogonal basis $\{e_i\}_{i\in I}$ of $\mathscr{H}$. For a vector $x = \sum_{i\in I} x_i e_i \in \mathscr{H}$, a ring homomorphism $\eta : \mathbb{C} \to \mathbb{C}$, and a matrix $T = (t_{ij}) \in \mathscr{M}_n$, let $\overline{x} = \sum_{i\in I} \overline{x}_i e_i$, $\eta(x) = \sum_{i\in I} \eta(x_i) e_i$, and $T^\eta = (\eta(t_{ij}))$. Note that there are very large number of automorphisms of $\mathbb{C}$, but the identity and the conjugation maps are the only continuous automorphisms; see [20].

We now have collected all the necessary ingredients and are therefore in a position to state and prove the main results of this section. The following theorem is one of the purposes of this paper. It characterizes nonlinear maps on $\mathscr{M}_n$ that preserve the pseudo spectral radius of Jordan triple product of matrices.

THEOREM 1. *Let $n \geqslant 3$ and $\varepsilon > 0$. A map ϕ from $\mathscr{M}_n$ into itself satisfies*

$$r_\varepsilon(\phi(T)\phi(S)\phi(T)) = r_\varepsilon(TST) \quad (T,S \in \mathscr{M}_n) \tag{1}$$

if and only if there exist a functional $\ell : \mathscr{L}(\mathscr{H}) \to \mathbb{T}$ and a unitary matrix $U \in \mathscr{M}_n$ such that $\phi(T) = \ell(T)UT^\sharp U^*$ for all $T \in \mathscr{M}_n$, where $T^\sharp$ stands for T, or T^{tr}, or T^*, or $\overline{T}$, the complex conjugation of T.

Proof. Checking the 'if' part is straightforward, so we will only deal with the 'only if' part. So assume that

$$r_\varepsilon(\phi(T)\phi(S)\phi(T)) = r_\varepsilon(TST)$$

for all $T,S \in \mathscr{L}(X)$.

First, we claim that the map ϕ preserves zero Jordan triple product of matrices in both directions, i.e., for any $T,S \in \mathscr{M}_n$, $\phi(T)\phi(S)\phi(T) = 0$ if and only if $TST = 0$. Indeed, assume that $\phi(T)\phi(S)\phi(T) = 0$, and note that $r_\varepsilon(TST) = r_\varepsilon(0) = \varepsilon$. From Lemmas 1 and 5, TST is a self-adjoint operator satisfying

$$D(z,\varepsilon) \subseteq \sigma_\varepsilon(TST) \subseteq D(0,\varepsilon)$$

for every $z \in \sigma(TST)$. This yields that $\sigma(TST) = \{0\}$, and consequently $TST = 0$; which proves the necessity condition. The sufficiency condition is dealt with similarly, and the claim is proved. Therefore, by Lemma 4, there exist is a functional $\ell : \mathscr{M}_n \to \mathbb{C} \setminus \{0\}$, an invertible matrix $U \in \mathscr{M}_n$, and an automorphism $\eta : \mathbb{C} \to \mathbb{C}$ such that either

$$\phi(T) = \ell(T)UT^\eta U^{-1} \text{ or } \phi(T) = \ell(T)U(T^\eta)^{tr}U^{-1}$$

for every $T \in \mathscr{M}_n$. Assume firstly that $\phi(T) = \ell(T)UT^\eta U^{-1}$ for every $T \in \mathscr{M}_n$. We divide the proof of it into several steps. $\square$

STEP 1. $r_\varepsilon(\phi(T)) = r_\varepsilon(T)$ for every $T \in \mathscr{M}_n$.

Proof. Note that, by the equality (1), we have

$$1+\varepsilon = r_\varepsilon(I) = r_\varepsilon(\phi(I)^3) = r_\varepsilon(\ell(I)^3 I) = |\ell(I)|^3 + \varepsilon.$$

This implies that $|\ell(I)| = 1$, and so by taking into account Lemma 1, we get

$$\begin{aligned} r_\varepsilon(T) &= r_\varepsilon(\phi(I)\phi(T)\phi(I)) \\ &= r_\varepsilon(\ell(I)^2\ell(T)UT^\eta U^{-1}) \\ &= r_\varepsilon(\ell(T)UT^\eta U^{-1}) \\ &= r_\varepsilon(\phi(T)) \end{aligned}$$

for all $T \in \mathscr{M}_n$. □

STEP 2. The matrix U can be chosen as a unitary matrix

Proof. Let $U = V|U|$ be the polar decomposition of U, and note that $|U| > 0$ and $r_\varepsilon(VTV^*) = r_\varepsilon(T)$ for every $T \in \mathscr{M}_n$. Replacing ϕ by the mapping $T \mapsto V^*\phi(T)V$, we may assume that $U > 0$. Thus, by taking into account the fact that

$$(x \otimes f)^\eta = \eta(x) \otimes \overline{\eta(\overline{f})}$$

for any vectors $x, f \in \mathbb{C}^n$, one gets

$$\phi(x \otimes f) = \ell(x \otimes f)U(\eta(x)) \otimes U^{-1}(\overline{\eta(\overline{f})}) \tag{2}$$

for every rank one matrix $x \otimes f \in \mathscr{M}_n$. We shall apply the technique from [9, Proof of Assertion 1]. To use it, we must prove first that

$$\|U(\eta(x))\|\|U^{-1}(\overline{\eta(\overline{x})})\| = 1 \tag{3}$$

for every vector x in $\mathbb{C}^n$.

Note that, for any $x, f \in \mathbb{C}^n$, we have

$$\langle U(\eta(x)), U^{-1}(\overline{\eta(\overline{f})})\rangle = \langle \eta(x), \overline{\eta(\overline{f})}\rangle = \eta(\langle x, f\rangle),$$

and in particular

$$\langle U(\eta(x)), U^{-1}(\overline{\eta(\overline{f})})\rangle = 0 \text{ if and only if } \langle x, f\rangle = 0.$$

Thus, for any orthogonal unit vectors $x, f \in \mathbb{C}^n$, the above step together with Lemma 2 allow to get that

$$\begin{aligned} \varepsilon^2 + \varepsilon|\ell(x\otimes f)|\|U(\eta(x))\|\|U^{-1}(\overline{\eta(\overline{f})})\| &= r_\varepsilon(\phi(x\otimes f))^2 \\ &= r_\varepsilon(x\otimes f)^2 \\ &= \varepsilon^2+\varepsilon, \end{aligned}$$

and so

$$|\ell(x\otimes f)| = \frac{1}{\|U(\eta(x))\|\|U^{-1}(\overline{\eta(\overline{f})})\|} \tag{4}$$

for any unit vectors x and f in $\mathbb{C}^n$ with $\langle x,f\rangle = 0$. On the other hand, for any unit vectors $x,f\in\mathbb{C}^n$, we have

$$r_\varepsilon(\phi(x\otimes f)\phi(f\otimes x)\phi(x\otimes f)) = r_\varepsilon(x\otimes f) = r_\varepsilon(\phi(x\otimes f)).$$

This implies that

$$r_\varepsilon(\ell(x\otimes f)^2\ell(f\otimes x)U(\eta(x))\otimes U^{-1}(\overline{\eta(\overline{f})})) = r_\varepsilon(\ell(x\otimes f)U(\eta(x))\otimes U^{-1}(\overline{\eta(\overline{f})})),$$

from which we infer that

$$|\ell(x\otimes f)||\ell(f\otimes x)| = 1 \tag{5}$$

for any unit vectors x and f in $\mathbb{C}^n$ with $\langle x,f\rangle = 0$.

By combining (4) and (5) we have, in fact,

$$\|U(\eta(x))\|\|U^{-1}(\overline{\eta(\overline{x})})\| = \frac{1}{\|U(\eta(f))\|\|U^{-1}(\overline{\eta(\overline{f})})\|} \tag{6}$$

for any unit vectors x and f in $\mathbb{C}^n$ with $\langle x,f\rangle = 0$.

Now, let us establish that there exits a positive constant c such that

$$\|U(\eta(x))\|\|U^{-1}(\overline{\eta(\overline{x})})\| = c \tag{7}$$

for every unit vector x in $\mathbb{C}^n$. To do so, pick arbitrary unit vectors x_1 and x_2 in $\mathbb{C}^n$, and let $f\in\mathbb{C}^n$ be a unit vector such that $\langle x_1,f\rangle = \langle x_2,f\rangle = 0$. By taking into account (6), we have

$$\begin{aligned}\|U(\eta(x_1))\|\|U^{-1}(\overline{\eta(\overline{x_1})})\| &= \frac{1}{\|U(\eta(f))\|\|U^{-1}(\overline{\eta(\overline{f})})\|}\\ &= \|U(\eta(x_2))\|\|U^{-1}(\overline{\eta(\overline{x_2})})\|,\end{aligned}$$

and then, (7) holds true. Moreover, the equation (6) tells us that $c=1$, which shows that the equation (3) hods true, too.

Thus, the same argument than the one used in the end of the proof of [9, Assertion 1] allows to get that $U=\lambda I$ for some $\lambda>0$. Dividing by λ if necessary, we may assume that $\lambda=1$, and the required conclusion in the step follows. □

STEP 3. η is either the identity or the conjugation.

Proof. For $z\in\mathbb{C}$, $e_1=(1,0,0,\ldots,0)^{tr}$ and $e_2=(0,1,0,\ldots,0)^{tr}$, let $B=e_1\otimes(e_1+\overline{z}e_2)$ and $C=e_1\otimes(e_1+e_2)$. It easy to check that $BCB=B$, and so, by Steps 1 and 2, we have

$$r_\varepsilon(\ell(B)B^\eta) = r_\varepsilon(\ell(B)^2\ell(C)B^\eta C^\eta B^\eta) = r_\varepsilon(\ell(B)^2\ell(C)B^\eta).$$

This implies that

$$r_\varepsilon(\ell(B)\eta(e_1)\otimes(\eta(e_1)+\overline{\eta(z)}\eta(e_2))) = r_\varepsilon(\ell(B)^2\ell(C)\eta(e_1)\otimes(\eta(e_1)+\overline{\eta(z)}\eta(e_2)));$$

which yields that $|\ell(B)^2\ell(C)| = |\ell(B)|$. Thus, $|\ell(B)\ell(C)| = 1$. By similarity, we have

$$r_\varepsilon(C) = r_\varepsilon(\ell(C)C^\eta) = r_\varepsilon(\ell(C)C),$$

implying that $|\ell(C)| = 1$. Consequently, $|\ell(B)| = 1$ for every $z \in \mathbb{C}$. Therefore

$$r_\varepsilon(e_1\otimes(e_1+\bar{z}e_2)) = r_\varepsilon(B) = r_\varepsilon(\ell(B)B^\eta) = r_\varepsilon(B^\eta) = r_\varepsilon(e_1\otimes(e_1+\overline{\eta(z)}e_2)),$$

and so, by Lemma 2, one gets

$$\|e_1+\overline{\eta(z)}e_2)\| = \|e_1+\bar{z}e_2\|.$$

This finally yields that $|\eta(z)| = |z|$ for all $z \in \mathbb{C}$, and consequently η is either the identity or the conjugation as required. □

STEP 4. The map ℓ can be chosen so that $|\ell(T)| = 1$ for every $T \in \mathscr{M}_n$.

Proof. First, we assert that

$$|\ell(x\otimes f)| = 1 \tag{8}$$

for any $x, f \in \mathbb{C}^n$ with $\langle x, f\rangle = 0$. Indeed, by Steps 1 and 2 together with the equality (2), we have $r_\varepsilon(x\otimes f) = r_\varepsilon(\ell(x\otimes f)\eta(x)\otimes\overline{\eta(\overline{f})})$ for all $x, f \in \mathbb{C}^n$. Thus, for any orthogonal vectors $x, f \in \mathbb{C}^n$, the above step and Lemma 2 tell us that

$$\|x\|\|f\| = |\ell(x\otimes f)|\|\eta(x)\|\|\eta(\overline{f})\| = |\ell(x\otimes f)|\|x\|\|f\|;$$

which yields that $|\ell(x\otimes f)| = 1$ as asserted.

Now, Let $T \in \mathscr{M}_n$ be an arbitrary matrix. If T is a scalar multiple of the identity, i.e., there exists a scalar λ such that $T = \lambda I$, then the fact that $r_\varepsilon(\ell(\lambda I)\eta(\lambda)I) = r_\varepsilon(\lambda I)$ implies that $|\ell(T)| = |\ell(\lambda I)| = 1$. If T is not a scalar multiple of the identity, then there exists a nonzero vector $x \in \mathbb{C}^n$ such that x and Tx are linearly independent. Choose a vector $f \in \mathbb{C}^n$ so that $\langle x, f\rangle = 0$ and $\langle Tx, f\rangle \neq 0$, and note that by taking into account (8), we have

$$\begin{aligned} r_\varepsilon(\langle Tx, f\rangle x\otimes f) &= r_\varepsilon((x\otimes f)T(x\otimes f)) \\ &= r_\varepsilon(\phi(x\otimes f)\phi(T)\phi(x\otimes f)) \\ &= r_\varepsilon(\ell(x\otimes f)^2\ell(T)\langle T^\eta(\eta(x)), \overline{\eta(\overline{f})}\rangle\eta(x)\otimes\overline{\eta(\overline{f})}) \\ &= r_\varepsilon(\ell(T)\langle T^\eta(\eta(x)), \overline{\eta(\overline{f})}\rangle\eta(x)\otimes\overline{\eta(\overline{f})}). \end{aligned}$$

Using Lemma 2 and Step 3, we infer that

$$\begin{aligned} |\langle Tx, f\rangle|\|x\|\|f\| &= |\ell(T)||\langle T^\eta(\eta(x)), \overline{\eta(\overline{f})}\rangle|\|\eta(x)\|\|\eta(\overline{f}))\| \\ &= |\ell(T)||\langle Tx, f\rangle|\|x\|\|f\|. \end{aligned}$$

This entails that $|\ell(T)| = 1$ in this case too, and the step is proved.

Thus, in the case when ϕ has the first form, the theorem follows from Steps 2 and 4. It remains to consider the case when ϕ has the form $T \mapsto \ell(T)U(T^{\eta})^{tr}U^{-1}$. Set $\chi(T) := \phi(T^{tr})$ and $\ell'(T) := \ell(T^{tr})$ for every $T \in \mathscr{M}_n$, and note, by Lemma 1, the map χ satisfies

$$r_{\varepsilon}(TST) = r_{\varepsilon}(T^{tr}S^{tr}T^{tr}) = r_{\varepsilon}(\chi(T)\chi(S)\chi(T)).$$

Moreover, upon replacing T by T^{tr}, we have $\chi(T) = \ell'(T)UT^{\eta}U^{-1}$ for every matrix $T \in \mathscr{M}_n$. So, by what has been shown above, the matrix U can be chosen as a unitary matrix and $|\ell'(T)| = 1$ for every $T \in \mathscr{M}_n$; which yields the desired conclusion in this case too. The proof of the theorem is therefore complete. $\square$

The next theorem, furnishing the pseudo spectral radius version of [21, Teorem 2.1], extends the above theorem to the infinite-dimensional case but at the price of the additional assumption that ϕ is surjective.

THEOREM 2. *Let $\mathscr{H}$ be an infinite-dimensional complex Hilbert space and $\varepsilon > 0$. A surjective map ϕ from $\mathscr{L}(\mathscr{H})$ into itself satisfies*

$$r_{\varepsilon}(\phi(T)\phi(S)\phi(T)) = r_{\varepsilon}(TST) \quad (T,S \in \mathscr{L}(\mathscr{H}))$$

if and only if there is a functional $\ell : \mathscr{L}(\mathscr{H}) \to \mathbb{T}$ and a linear or conjugate linear unitary operator $U \in \mathscr{L}(\mathscr{H})$ such that either $\phi(T) = \ell(T)UTU^*$ or $\phi(T) = \ell(T)UT^*U^*$ for all $T \in \mathscr{L}(\mathscr{H})$. Here $\mathbb{T}$ denotes the unit circle of the complex field $\mathbb{C}$.

Proof. As the sufficiency condition is obvious, we only need to prove the necessity. So, assume that

$$r_{\varepsilon}(\phi(T)\phi(S)\phi(T)) = r_{\varepsilon}(TST)$$

for all $T,S \in \mathscr{L}(X)$. Similar argument as in the proof of Theorem 1 allows to get that the map ϕ preserves zero Jordan triple product of operators in both directions. Therefore, from Lemma 3, there exist is a functional $\ell : \mathscr{L}(\mathscr{H}) \to \mathbb{C} \setminus \{0\}$ and a bounded invertible linear or conjugate linear operator U on $\mathscr{H}$ such that either $\phi(T) = \ell(T)UTU^{-1}$, or $\phi(T) = \ell(T)UT^*U^{-1}$ for all $T \in \mathscr{L}(\mathscr{H})$.

Assume firstly that $\phi(T) = \ell(T)UTU^{-1}$ for every $T \in \mathscr{L}(\mathscr{H})$. Arbitrarily fix an orthogonal basis $\{e_i\}_{i \in I}$ of $\mathscr{H}$, and for $x = \sum_{i \in I} x_i e_i$ let $J(x) = \sum_{i \in I} \overline{x}_i e_i$. Clearly, $J^2 = J$ and for any vectors $x, f \in \mathscr{H}$, we have

$$J(x \otimes f)J = J(x) \otimes J(f) = \eta(x) \otimes \overline{\eta(\overline{f})}$$

where $\eta : \mathbb{C} \to \mathbb{C}$ is the conjugation. If U is conjugate linear, then let $V = UJ$ and note that $V \in \mathscr{L}(\mathscr{H})$ and

$$\phi(x \otimes f) = \ell(x \otimes f)VJ(x \otimes f)JV^{-1} = \ell(x \otimes f)V(\eta(x) \otimes \overline{\eta(\overline{f})})V^{-1}$$

for all vectors $x, f \in \mathscr{H}$. So, in both cases, i.e., U is linear or conjugate linear, ϕ has the form

$$\phi(x \otimes f) = \ell(x \otimes f)V(\eta(x) \otimes \overline{\eta(\overline{f})})V^{-1}$$

for every rank one operator $x \otimes f \in \mathscr{L}(\mathscr{H})$, where V is a bounded linear operator on $\mathscr{H}$ and η is either the identity or the conjugation.

Now, by inspecting the proof of Theorem 1, with no extra efforts, one can see that same approach used there allows to get that the operator V can be chosen as a unitary operator and $|\ell(T)| = 1$ for all $T \in \mathscr{L}(\mathscr{H})$. Consequently, U can be chosen as a unitary or conjugate unitary operator. The only thing that should be observed is that the conclusion "U can be chosen as a scalar multiple of the identity" in Step 2 remains true when using $\mathscr{H}$ instead of $\mathbb{C}^n$ in (7). This is, in fact, a consequence of [9, Lemma 2.4]. This concludes the proof of the theorem in the case when ϕ has the first form.

In the remainder case when ϕ has the form $T \mapsto \ell(T)UT^*U^{-1}$, set $\chi(T) := \phi(T^*)$ and $\ell'(T) := \ell(T^*)$ for every $T \in \mathscr{L}(\mathscr{H})$, and note the map χ satisfies

$$r_\varepsilon(TST) = r_\varepsilon(T^*S^*T^*) = r_\varepsilon(\chi(T)\chi(S)\chi(T))$$

for all $T, S \in \mathscr{L}(X)$. Moreover, upon replacing T by T^*, we have $\chi(T) = \ell'(T)UTU^{-1}$ for every $T \in \mathscr{L}(\mathscr{H})$. Thus, by what has been shown above, U can be chosen as a unitary or conjugate unitary operator and $|\ell'(T)| = 1$ for every $T \in \mathscr{H}$; which yields the desired conclusion in this case, too. This achieves the proof of the theorem. $\square$

3. Pseudo spectrum preservers

This section is devoted to derive some consequences of the main results of this paper. Theses consequences concern nonlinear preservers of pseudo spectrum of operators. The first one describes maps from $\mathscr{L}(\mathscr{H})$ onto itself preserving the pseudo spectrum of Jordan triple product of operators.

THEOREM 3. *Let $\mathscr{H}$ be an infinite-dimensional complex Hilbert space and $\varepsilon > 0$. A surjective map ϕ from $\mathscr{L}(\mathscr{H})$ into itself satisfies*

$$\sigma_\varepsilon(\phi(T)\phi(S)\phi(T)) = \sigma_\varepsilon(TST) \quad (T, S \in \mathscr{L}(\mathscr{H})) \tag{9}$$

if and only if there are a third root of unity c and a unitary operator $U \in \mathscr{L}(\mathscr{H})$ such that either $\phi(T) = cUTU^*$ or $\phi(T) = cUT^{tr}U^*$ for all $T \in \mathscr{L}(\mathscr{H})$. Where T^{tr} is the transpose of T with respect to an arbitrary but fixed orthogonal basis of $\mathscr{H}$.

Proof. We only need to check the necessity condition. So, assume that ϕ satisfies (9), and note that the map ϕ preserves the the ε-pseudo spectral radius of Jordan triple product of operators. By Theorem 2, there exist a functional $\ell : \mathscr{L}(\mathscr{H}) \to \mathbb{T}$ and a unitary or conjugate unitary operator $U \in \mathscr{L}(\mathscr{H})$ such that either $\phi(T) = \ell(T)UTU^*$ or $\phi(T) = \ell(T)UT^*U^*$ for all $T \in \mathscr{L}(\mathscr{H})$.

Assume firstly that $\phi(T) = \ell(T)UTU^*$ for all $T \in \mathscr{L}(\mathscr{H})$. We claim that there is a third root of unity c such that

$$\ell(x \otimes x) = c$$

for every unit vector $x \in \mathscr{H}$. Fix an arbitrary unit vector $x \in \mathscr{H}$, and suppose that U is conjugate unitary. Using Lemma 5, one gets

$$\begin{aligned} D(0,\varepsilon) \cup D(1,\varepsilon) &= \sigma_\varepsilon(x \otimes x) \\ &= \sigma_\varepsilon(\ell(x \otimes x)^3 U(x \otimes x)U^*)) \\ &= \overline{\sigma_\varepsilon(\overline{\ell(x \otimes x)^3} x \otimes x)} \\ &= D(0,\varepsilon) \cup D(\ell(x \otimes x)^3, \varepsilon), \end{aligned}$$

and consequently $\ell(x \otimes x)^3 = 1$. Now, let $x, y \in \mathscr{H}$ be arbitrary unit vectors, and let $z \in \mathscr{H}$ such that $\langle x, z\rangle \neq 0 \neq \langle y, z\rangle$. We have

$$\begin{aligned} \sigma_\varepsilon(|\langle x,z\rangle|^2 x \otimes x) &= \sigma_\varepsilon((x \otimes x)(z \otimes z)(x \otimes x)) \\ &= \sigma_\varepsilon(\ell(x \otimes x)^2 \ell(z \otimes z) U(|\langle x,z\rangle|^2 x \otimes x)U^*)) \\ &= \overline{\sigma_\varepsilon(\overline{\ell(x \otimes x)^2 \ell(z \otimes z)} |\langle x,z\rangle|^2 x \otimes x)}, \end{aligned}$$

and so $|\langle x,z\rangle|^2 = \ell(x \otimes x)^2 \ell(z \otimes z) |\langle x,z\rangle|^2$. This implies that $\ell(x \otimes x) = \ell(z \otimes z)$ since $\ell(x \otimes x)^3 = 1$. Similarly, we have $\ell(y \otimes y) = \ell(z \otimes z)$, and so $\ell(x \otimes x) = \ell(y \otimes y) = c$, where c is a positive constant. The case when U is unitary is dealt with similarly, and the claim is proved.

Next, let us prove that the case when U is conjugate unitary cannot occurs. Suppose on the contrary that U is conjugate unitary. Let $T \in \mathscr{L}(\mathscr{H})$ be an arbitrary non self-adjoint operator, and let $x \in \mathscr{H}$ be a unit vector so that $\langle Tx, x\rangle \neq 0$. The fact that

$$\sigma_\varepsilon((x \otimes x)T(x \otimes x)) = \sigma_\varepsilon(\ell(x \otimes x)^2 \ell(T) \overline{\langle Tx, x\rangle} U(x \otimes x)U^*))$$

implies that $\langle Tx, x\rangle = \ell(x \otimes x)^2 \ell(T) \overline{\langle Tx, x\rangle}$, and so

$$\ell(T) = \frac{c\langle Tx, x\rangle}{\overline{\langle Tx, x\rangle}}$$

since $c^3 = 1$. In particular, for $T \in \mathscr{L}(\mathscr{H})$ for which there exit nonzero vectors x and y in $\mathscr{H}$ such that $Tx = x$ and $Ty = iy$, we have

$$c = \frac{c\langle Tx, x\rangle}{\overline{\langle Tx, x\rangle}} = \ell(T) = \frac{c\langle Ty, y\rangle}{\overline{\langle Ty, y\rangle}} = -c,$$

a contradiction.

Now, let $T \in \mathscr{L}(\mathscr{H})$ be a nonzero operator, and let $x \in \mathscr{H}$ so that $\langle Tx, x\rangle \neq 0$. Since U is a unitary operator, similar argument as above allows to get that

$$\ell(T) = \frac{c\langle Tx, x\rangle}{\langle Tx, x\rangle} = c;$$

which yields the desired conclusion in the theorem in the case when ϕ has the first form.

In the remainder case when ϕ has the form $T \mapsto \ell(T)UT^*U^*$, similar discussion just as above allows to get that the case when U is unitary cannot occurs and the map ℓ can be chosen so that $\ell(T) = c$ for all $T \in \mathscr{L}(\mathscr{H})$, where c is a third root of unity. Consequently, U is conjugate unitary. Arbitrarily fix an orthogonal basis $\{e_i\}_{i\in I}$ of $\mathscr{H}$, and for $x = \sum_{i\in I} x_i e_i$ let $J(x) = \sum_{i\in I} \overline{x}_i e_i$. Clearly, $JT^*J = T^{tr}$, where T^{tr} is the transpose of T for an arbitrary but fixed orthogonal basis of $\mathscr{H}$. Let $V = UJ$, and note that V is a unitary operator and

$$\phi(T) = cV(JT^*J)V^* = cVT^{tr}V^*$$

for every $T \in \mathscr{L}(\mathscr{H})$. This yields the desired conclusion in this case too, and concludes the proof. $\square$

The second consequence, furnishing a pseudo spectrum version of [6, Theorem 2.1] and the main result of [1], refine the above result in the finite dimensional case. We do not require the surjective assumption on ϕ.

THEOREM 4. *Let $n \geqslant 3$ and $\varepsilon > 0$. A map ϕ from $\mathscr{M}_n$ into itself satisfies*

$$\sigma_\varepsilon(\phi(T)\phi(S)\phi(T)) = \sigma_\varepsilon(TST) \quad (T,S \in \mathscr{M}_n)$$

if and only if there are a third root of unity c and a unitary matrix $U \in \mathscr{M}_n$ such that either $\phi(T) = cUTU^*$ or $\phi(T) = cUT^{tr}U^*$ for all $T \in \mathscr{M}_n$.

Proof. Assume that $\sigma_\varepsilon(\phi(T)\phi(S)\phi(T)) = \sigma_\varepsilon(TST)$ for any $T,S \in \mathscr{M}_n$, and note that the map ϕ preserves the the ε-pseudo spectral radius of Jordan triple product of matrices. By Theorem 1, there exist a functional $\ell : \mathscr{L}(\mathscr{H}) \to \mathbb{T}$ and a unitary matrix $U \in \mathscr{L}(\mathscr{H})$ such that $\phi(T) = \ell(T)UT^\sharp U^*$ for all $T \in \mathscr{L}(\mathscr{H})$, where $T^\sharp$ stands for T, or T^{tr}, or T^*, or $\overline{T}$, the complex conjugation of T. Similar argument as in the proof of Theorem 3 allows to get that the cases when ϕ has the forms $T \mapsto \ell(T)UT^*U^*$ or $T \mapsto \ell(T)U\overline{T}U^*$ cannot occur, and in the remainder cases the map ℓ can be chosen so that $\ell(T) = c$ for all $T \in \mathscr{L}(\mathscr{H})$, where c is a third root of unity. This yields the desired conclusion in the necessity condition.

As the sufficiency condition is clear, the proof is therefore complete. $\square$

We close this paper by the following natural problem which suggests itself.

PROBLEM. Can the Hilbert space $\mathscr{H}$ be replaced by a general Banach space in Theorems 2 and 3 of this paper?

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