Operators and Matrices
Volume 10, Number 1 (2016), 73–81 doi:10.7153/oam-10-06

FRAMES AND OPERATORS IN HILBERT C^*–MODULES

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(*Communicated by D. R. Larson*)

Abstract. In this paper we introduce the concepts of atomic systems for operators and K-frames in Hilbert C^*–modules and we establish some results.

1. Introduction

Frames for Hilbert spaces were introduced by Duffin and Schaeffer [4] as part of their research in non-harmonic Fourier series. A finite or countable sequence $\{f_n\}_{n\in I}$ is called a frame for a separable Hilbert space $\mathscr{H}$ if there exist constants $A,B>0$ such that

$$A\|f\|^2 \leqslant \sum_{n\in I}|\langle f,f_n\rangle|^2 \leqslant B\|f\|^2, \quad f\in\mathscr{H}. \tag{1.1}$$

The frames have many properties which make them very useful in applications. See [3].

Frank and Larson [6, 7] extended this concept for countably generated Hilbert C^*-modules.

Let A be a C^*-algebra and $\mathscr{H}$ be a left A-module. We assume that the linear operations of A and $\mathscr{H}$ are comparable, i.e. $\lambda(ax)=(\lambda a)x=a(\lambda x)$ for every $\lambda\in\mathbb{C}, a\in A$ and $x\in\mathscr{H}$. Recall that $\mathscr{H}$ is a pre-Hilbert A-module if there exists a sesquilinear mapping $\langle .,.\rangle : \mathscr{H}\times\mathscr{H}\to A$ with the properties

1. $\langle x,x\rangle \geqslant 0$; if $\langle x,x\rangle = 0$, then $x=0$ for every $x\in\mathscr{H}$.
2. $\langle x,y\rangle = \langle y,x\rangle^*$ for every $x,y\in\mathscr{H}$.
3. $\langle ax,y\rangle = a\langle x,y\rangle$ for every $a\in A$, $x,y\in\mathscr{H}$.
4. $\langle x+y,z\rangle = \langle x,z\rangle+\langle y,z\rangle$ for every $x,y,z\in\mathscr{H}$.

The map $x\mapsto\|x\| = \|\langle x,x\rangle\|^{\frac{1}{2}}$ defines a norm on $\mathscr{H}$. A pre-Hilbert A-module is called a Hilbert A-module if $\mathscr{H}$ is complete with respect to that norm. So $\mathscr{H}$ becomes the structure of a Banach A-module. A Hilbert A-module $\mathscr{H}$ is called countably generated if there exists a countable set $\{x_n\}_{n\in J}\subseteq\mathscr{H}$ such that the linear span (over $\mathbb{C}$ and A) of this set is norm-dense in $\mathscr{H}$.

Mathematics subject classification (2010): 42C15, 46L05, 46H25.
Keywords and phrases: Atomic system, K-frame, C^*-algebra, Hilbert C^*-module, Bessel sequence.

Paper OaM-10-06

Suppose that $\mathscr{H},\mathscr{K}$ are Hilbert A-modules over a C^*-algebra A. We define $L(\mathscr{H},\mathscr{K})$ to be the set of all maps $T:\mathscr{H}\to\mathscr{K}$ for which there is a map $T^*:\mathscr{K}\to\mathscr{H}$ such that

$$\langle Tx,y\rangle=\langle x,T^*y\rangle \quad x\in\mathscr{H},y\in\mathscr{K}.$$

It is easy to see that each $T\in L(\mathscr{H},\mathscr{K})$ is A-linear and bounded. $L(\mathscr{H},\mathscr{K})$ is called the set of adjointable maps from $\mathscr{H}$ to $\mathscr{K}$. We denote $L(\mathscr{H},\mathscr{H})$ by $L(\mathscr{H})$. In fact $L(\mathscr{H})$ is a C^*-algebra.

For basic results on Hilbert modules see [2, 14, 15].

Throughout the present paper we suppose that A is a unital C^*-algebra and $\mathscr{H}$ is a Hilbert A-module.

DEFINITION 1.1. Let $J\subseteq\mathbb{N}$ be a finite or countable index set. A sequence $\{f_n\}_{n\in J}$ of elements of $\mathscr{H}$ is said to be a *frame* if there exist two constants $C,D>0$ such that

$$C\langle x,x\rangle\leqslant\sum_{n\in J}\langle x,f_n\rangle\langle f_n,x\rangle\leqslant D\langle x,x\rangle,\quad x\in\mathscr{H}. \tag{1.2}$$

The constants C and D are called the *lower* and *upper frame bounds*, respectively. We consider *standard frames* for which the sum in the middle of (1.2) converges in norm for every $x\in\mathscr{H}$. A frame $\{f_n\}_{n\in J}$ is said to be a *tight frame* if $C=D$, and said to be a *Parseval frame* (or a *normalized tight frame*) if $C=D=1$. If just the right-hand inequality in (1.2) holds, we say that $\{f_n\}_{n\in J}$ is a *Bessel sequence* with a *Bessel bound* D.

It follows from the above definition that a sequence $\{f_n\}_{n\in J}$ is a normalized tight frame if and only if

$$\langle x,x\rangle=\sum_{n\in J}\langle x,f_n\rangle\langle f_n,x\rangle,\quad x\in\mathscr{H}.$$

Let $\{f_n\}_{n\in J}$ be a standard frame for $\mathscr{H}$. The *frame transform* for $\{f_n\}_{n\in J}$ is the map $T:\mathscr{H}\to\ell^2(A)$ defined by $Tx=\{\langle x,f_n\rangle\}_{n\in J}$, where $\ell^2(A)$ denotes a Hilbert A-module $\{\{a_j\}_{j\in J}:a_j\in A,\sum_j a_ja_j^* \text{ converges in norm}\}$ with pointwise operations and the inner product $\langle\{a_j\}_{j\in J},\{b_j\}_{j\in J}\rangle=\sum_{j\in J}a_jb_j^*$. The adjoint operator $T^*:\ell^2(A)\to\mathscr{H}$ is given by $T^*(\{c_j\}_{j\in J})=\sum_{j\in J}c_jf_j$ ([7], Theorem 4.4). By composing T and T^*, we obtain the *frame operator* $S:\mathscr{H}\to\mathscr{H}$ given by

$$Sx=T^*Tx=\sum_{n\in J}\langle x,f_n\rangle f_n,\quad x\in\mathscr{H}.$$

The frame operator is positive and invertible, also it is the unique operator in $L(\mathscr{H})$ such that the reconstruction formula

$$x=\sum_{n\in J}\langle x,S^{-1}f_n\rangle f_n=\sum_{n\in J}\langle x,f_n\rangle S^{-1}f_n,$$

holds for all $x\in\mathscr{H}$. It is easy to see that the sequence $\{S^{-1}f_n\}_{n\in J}$ is a frame for $\mathscr{H}$. The frame $\{S^{-1}f_n\}_{n\in J}$ is said to be the *canonical dual frame* of the frame $\{f_n\}_{n\in J}$.

There exists Hilbert C^*-modules admitting no frames (see [11]). The Kasparov Stabilisation Theorem [10] is used in [7] to prove that every countably generated Hilbert Module over a unital C^*-algebra admits frames. The following Proposition gives an equivalent definition of frames in Hilbert C^*–modules.

PROPOSITION 1.2. [12] *Let $\mathscr{H}$ be a finitely or countably generated Hilbert A-module and $\{f_n\}_{n\in J}$ be a sequence in $\mathscr{H}$. Then $\{f_n\}_{n\in J}$ is a frame of $\mathscr{H}$ with bounds C and D if and only if*

$$C\|x\|^2 \leqslant \Big\|\sum_{n\in J}\langle x,f_n\rangle\langle f_n,x\rangle\Big\| \leqslant D\|x\|^2,$$

for all $x\in\mathscr{H}$.

We recall that an element $v\in\mathscr{H}$ is said to be a *basic element* if $e=\langle v,v\rangle$ is a minimal projection in A; that is $eAe=\mathbb{C}e$. A system $\{v_i\}_{i\in J}$ of basic elements of $\mathscr{H}$ is said to be *orthonormal* if $\langle v_i,v_j\rangle=0$, for all $i\neq j$; moreover if this orthonormal system generates a dense submodule of $\mathscr{H}$, then we call it an *orthonormal basis* for $\mathscr{H}$.

We need the following results to prove our results.

THEOREM 1.3. [5] *Let $\mathscr{F},\mathscr{H},\mathscr{K}$ be Hilbert C^*-modules over a C^*-algebra A. Also let $S\in L(\mathscr{K},\mathscr{H})$ and $T\in L(\mathscr{F},\mathscr{H})$ with $\overline{R(T^*)}$ orthogonally complemented. The following statements are equivalent:*

1. *$SS^*\leqslant\lambda TT^*$ for some $\lambda>0$;*
2. *there exists $\mu>0$ such that $\|S^*z\|\leqslant\mu\|T^*z\|$ for all $z\in\mathscr{H}$;*
3. *there exists $D\in L(\mathscr{K},\mathscr{F})$ such that $S=TD$, i.e., $TX=S$ has a solution;*
4. *$R(S)\subseteq R(T)$.*

PROPOSITION 1.4. [12] *Let $\{f_n\}_{n\in J}$ be a sequence of a finitely or countably generated Hilbert C^*-module $\mathscr{H}$ over a unital C^*-algebra A. Then the following statements are mutually equivalent:*

1. *$\{f_n\}_{n\in J}$ is a Bessel sequence for $\mathscr{H}$ with bound D.*
2. $\left\|\sum_{n\in J}\langle x,f_n\rangle\langle f_n,x\rangle\right\|\leqslant D\|x\|^2,\quad x\in\mathscr{H}.$
3. *$\theta:\ell^2(A)\to\mathscr{H}$ defined by*

   $$\theta(\{c_n\}_{n\in J})=\sum_{n\in J}c_nf_n.$$

   is a well-defined bounded operator with $\|\theta\|\leqslant\sqrt{D}$.
4. *$T:\mathscr{H}\to\ell^2(A)$ defined by $Tx=\{\langle x,f_n\rangle\}_{n\in J}$ is adjointable and $T^*=\theta$.*

PROPOSITION 1.5. [15] *Let $\mathscr{H}$ be a Hilbert C^*-module. If $T \in L(\mathscr{H})$, then $\langle Tx, Tx\rangle \leqslant \|T\|^2\langle x,x\rangle$ for every $x \in \mathscr{H}$.*

PROPOSITION 1.6. [12] *Let B be a C^*-algebra and $\{a_n\}_{n\in J}$ a sequence in B. If $\sum_{n\in J} a_n b_n^*$ converges for all $\{b_n\}_{n\in J} \in \ell^2(B)$, then $\{a_n\}_{n\in J} \in \ell^2(B)$.*

In [8], L. Găvruţa, presented a generalization of frames, named K-frames, which allows to reconstruct elements from the range of a linear and bounded operator in a Hilbert space. She also introduced the concept of atomic system for operators and gave new results and properties of K-frames in Hilbert spaces. See also [9, 16].

In the present paper, we extend this results for frames in C^* Hilbert modules.

2. Atomic systems in Hilbert C^*-modules

Let $J \subseteq \mathbb{N}$ be a finite or countable index set.

DEFINITION 2.1. A sequence $\{f_n\}_{n\in J}$ of $\mathscr{H}$ is called an *atomic system* for $K \in L(\mathscr{H})$ if the following statements hold:

1. the series $\sum_{n\in J} c_n f_n$ converges for all $c = \{c_n\}_{n\in J} \in \ell^2(A)$;
2. there exists $C > 0$ such that for every $x \in \mathscr{H}$ there exists $\{a_{n,x}\}_{n\in J} \in \ell^2(A)$ such that $\sum_{n\in J} a_{n,x}a_{n,x}^* \leqslant C\langle x,x\rangle$ and $Kx = \sum_{n\in J} a_{n,x} f_n$.

PROPOSITION 2.2. *Let $\{f_n\}_{n\in J}$ be a sequence in $\mathscr{H}$ such that $\sum_{n\in J} c_n f_n$ converges for all $c = \{c_n\}_{n\in J} \in \ell^2(A)$. Then $\{f_n\}_{n\in J}$ is a Bessel sequence in $\mathscr{H}$.*

Proof. It is clear that $\sum_{n\in J} c_n\langle f_n, x\rangle$ converges for all $c = \{c_n\}_{n\in J} \in \ell^2(A)$ and all $x \in \mathscr{H}$. Hence $\{\langle x, f_n\rangle\}_{n\in J} \in \ell^2(A)$ by Proposition 1.6. Let us define $T : \ell^2(A) \to \mathscr{H}$ by $T(\{c_n\})_{n\in J} = \sum_{n\in J} c_n f_n$. Therefore T is bounded and the adjoint operator is given by
$$T^* : \mathscr{H} \to \ell^2(A), \quad T^*(x) = \{\langle x, f_n\rangle\}_{n\in J}.$$
Since T^* is bounded, we get that $\{f_n\}_{n\in J}$ is a Bessel sequence in $\mathscr{H}$. $\square$

PROPOSITION 2.3. *Let $\{f_n\}_{n\in J}$ be a sequence in $\mathscr{H}$. Then $\{f_n\}_{n\in J}$ is a Bessel sequence in $\mathscr{H}$ if and only if $\{\langle x, f_n\rangle\}_{n\in J} \in \ell^2(A)$, for all $x \in \mathscr{H}$.*

Proof. It is clear that if $\{f_n\}_{n\in J}$ is a Bessel sequence in $\mathscr{H}$, then $\{\langle x, f_n\rangle\}_{n\in J} \in \ell^2(A)$, for all $x \in \mathscr{H}$. The converse follows from the Uniform Boundedness Principle. $\square$

In the following, we suppose that $\mathscr{H}$ is finite or countable generated Hilbert C^*-module.

THEOREM 2.4. *If $K \in L(\mathscr{H})$, then there exists an atomic system for K.*

Proof. Let $\{x_n\}_{n\in J}$ be a standard normalized tight frame for $\mathscr{H}$. Since

$$x=\sum_{n\in J}\langle x,x_n\rangle x_n,\quad x\in\mathscr{H},$$

we have

$$Kx=\sum_{n\in J}\langle x,x_n\rangle Kx_n,\quad x\in\mathscr{H}.$$

For $x\in\mathscr{H}$, putting $a_{n,x}=\langle x,x_n\rangle$ and $f_n=Kx_n$ for all $n\in J$, we get

$$\begin{aligned}\sum_{n\in J}\langle x,f_n\rangle\langle f_n,x\rangle &= \sum_{n\in J}\langle x,Kx_n\rangle\langle Kx_n,x\rangle\\ &=\sum_{n\in J}\langle K^*x,x_n\rangle\langle x_n,K^*x\rangle=\langle K^*x,K^*x\rangle\\ &\leqslant \|K^*\|^2\langle x,x\rangle.\end{aligned}$$

Therefore $\{f_n\}_{n\in J}$ is a Bessel sequence for $\mathscr{H}$ and we conclude that the series $\sum_{n\in J}c_nf_n$ converges for all $c=\{c_n\}_{n\in J}\in\ell^2(A)$ by Proposition 1.4. We also have

$$\sum_{n\in J}a_{n,x}a_{n,x}^*=\sum_{n\in J}\langle x,x_n\rangle\langle x_n,x\rangle=\langle x,x\rangle,$$

which completes the proof. □

THEOREM 2.5. *Let $\{f_n\}_{n\in J}$ be a Bessel sequence for $\mathscr{H}$ and $K\in L(\mathscr{H})$. Suppose that $T\in L(\mathscr{H},\ell^2(A))$ is given by $T(x)=\{\langle x,f_n\rangle\}_{n\in J}$ and $\overline{R(T)}$ is orthogonally complemented. Then the following statements are equivalent:*

1. *$\{f_n\}_{n\in J}$ is an atomic system for K;*
2. *There exist $C,B>0$ such that*

$$C\|K^*x\|^2\leqslant\left\|\sum_{n\in J}\langle x,f_n\rangle\langle f_n,x\rangle\right\|\leqslant B\|x\|^2;$$

3. *There exists $D\in L(\mathscr{H},\ell^2(A))$ such that $K=T^*D$.*

Proof. $(1)\Rightarrow(2)$. For every $x\in\mathscr{H}$, we have

$$\|K^*x\|=\sup_{\|y\|=1}\|\langle y,K^*x\rangle\|=\sup_{\|y\|=1}\|\langle Ky,x\rangle\|.$$

Since $\{f_n\}_{n\in J}$ is an atomic system for K, there exists $M>0$ such that for every $y\in\mathscr{H}$ there exists $a_y=\{a_{n,y}\}_{n\in J}\in\ell^2(A)$ for which $\sum_{n\in J}a_{n,y}a_{n,y}^*\leqslant M\langle y,y\rangle$ and $Ky=$

$\sum_{n\in J} a_{n,y} f_n$. Therefore

$$\begin{aligned}
\|K^*x\|^2 &= \sup_{\|y\|=1} \|\langle Ky,x\rangle\|^2 = \sup_{\|y\|=1} \Big\|\langle \sum_{n\in J} a_{n,y} f_n , x\rangle\Big\|^2 = \sup_{\|y\|=1} \Big\|\sum_{n\in J} a_{n,y}\langle f_n, x\rangle\Big\|^2 \\
&\leqslant \sup_{\|y\|=1} \Big\|\sum_{n\in J} a_{n,y} a_{n,y}^*\Big\| \Big\|\sum_{n\in J} \langle x, f_n\rangle\langle f_n, x\rangle\Big\| \\
&\leqslant \sup_{\|y\|=1} M\|y\|^2 \Big\|\sum_{n\in J} \langle x, f_n\rangle\langle f_n, x\rangle\Big\| \\
&= M\Big\|\sum_{n\in J} \langle x, f_n\rangle\langle f_n, x\rangle\Big\|,
\end{aligned}$$

for every $x \in \mathscr{H}$. So that

$$\frac{1}{M}\|K^*x\|^2 \leqslant \Big\|\sum_{n\in J} \langle x, f_n\rangle\langle f_n, x\rangle\Big\|, \quad x \in \mathscr{H}.$$

Moreover, $\{f_n\}_{n\in J}$ is a Bessel sequence for $\mathscr{H}$. Hence (2) holds.

$(2) \Rightarrow (3)$ Since $\{f_n\}_{n\in J}$ is a Bessel sequence, we get $T^*e_n = f_n$, where $\{e_n\}_{n\in J}$ is the standard orthonormal basis for $\ell^2(A)$. Therefore

$$\begin{aligned}
C\|K^*x\|^2 &\leqslant \Big\|\sum_{n\in J} \langle x, f_n\rangle\langle f_n, x\rangle\Big\| = \Big\|\sum_{n\in J} \langle x, T^*e_n\rangle\langle T^*e_n, x\rangle\Big\| \\
&= \Big\|\sum_{n\in J} \langle Tx, e_n\rangle\langle e_n, Tx\rangle\Big\| = \|Tx\|^2, \quad x \in \mathscr{H}.
\end{aligned}$$

By Theorem 1.3, there exists operator $D \in L(\mathscr{H}, \ell^2(A))$ such that $K = T^*D$.

$(3) \Rightarrow (1)$ For every $x \in \mathscr{H}$, we have

$$Dx = \sum_{n\in J} \langle Dx, e_n\rangle e_n.$$

Therefore

$$T^*Dx = \sum_{n\in J} \langle Dx, e_n\rangle T^*e_n, \quad x \in \mathscr{H}.$$

Let $a_n = \langle Dx, e_n\rangle$, so for all $x \in \mathscr{H}$ we get

$$\sum_{n\in J} a_n a_n^* = \sum_{n\in J} \langle Dx, e_n\rangle\langle e_n, Dx\rangle = \langle Dx, Dx\rangle \leqslant \|D\|^2\langle x, x\rangle.$$

Since $\{f_n\}_{n\in J}$ is a Bessel sequence for $\mathscr{H}$, we obtain that $\{f_n\}_{n\in J}$ is an atomic system for K. $\square$

COROLLARY 2.6. *Let $\{f_n\}_{n\in J}$ be a frame for $\mathscr{H}$ with bounds $C,D > 0$ and $K \in L(\mathscr{H})$. Then $\{f_n\}_{n\in J}$ is an atomic system for K with bounds $\frac{1}{C^{-1}\|K\|^2}$ and D.*

Proof. Let S be the frame operator of $\{f_n\}_{n\in J}$. We prove that the condition (2) of Theorem 2.5 holds. Since $\{S^{-1}f_n\}_{n\in J}$ is a frame for $\mathscr{H}$ with bounds $D^{-1}, C^{-1} > 0$ and $x = \sum_{n\in J}\langle x, f_n\rangle S^{-1}f_n$ for all $x \in \mathscr{H}$, we get

$$\begin{aligned}
\|K^*x\|^2 &= \sup_{\|y\|=1} \|\langle K^*x, y\rangle\|^2 = \sup_{\|y\|=1} \Big\|\langle \sum_{n\in J}\langle x, f_n\rangle K^*S^{-1}f_n, y\rangle\Big\|^2 \\
&= \sup_{\|y\|=1} \Big\|\sum_{n\in J}\langle x, f_n\rangle\langle K^*S^{-1}f_n, y\rangle\Big\|^2 \\
&\leqslant \sup_{\|y\|=1} \Big\|\sum_{n\in J}\langle x, f_n\rangle\langle f_n, x\rangle\Big\|\Big\|\sum_{n\in J}\langle Ky, S^{-1}f_n\rangle\langle S^{-1}f_n, Ky\rangle\Big\| \\
&\leqslant \sup_{\|y\|=1} C^{-1}\Big\|\sum_{n\in J}\langle x, f_n\rangle\langle f_n, x\rangle\Big\|\|Ky\|^2 \\
&= C^{-1}\|K\|^2\Big\|\sum_{n\in J}\langle x, f_n\rangle\langle f_n, x\rangle\Big\|.
\end{aligned}$$

So

$$\frac{1}{C^{-1}\|K\|^2}\|K^*x\|^2 \leqslant \Big\|\sum_{n\in J}\langle x, f_n\rangle\langle f_n, x\rangle\Big\| \leqslant D\|x\|^2, \quad x \in \mathscr{H}.$$

Therefore $\{f_n\}_{n\in J}$ is an atomic system for K. $\square$

The converse of the above corollary holds when the operator K is onto.

COROLLARY 2.7. *Let $\{f_n\}_{n\in J}$ be an atomic system for K. If $K \in L(\mathscr{H})$ is onto, then $\{f_n\}_{n\in J}$ is a frame for $\mathscr{H}$.*

Proof. By Proposition 2.1 from [1], $K \in L(\mathscr{H})$ is surjective if and only if there is $M > 0$ such that

$$M\|x\| \leqslant \|K^*x\|, \quad x \in \mathscr{H}.$$

Since $\{f_n\}$ is an atomic system for K, by Theorem 2.5, there exsit $C, B > 0$ such that

$$C\|K^*x\|^2 \leqslant \Big\|\sum_{n\in J}\langle x, f_n\rangle\langle f_n, x\rangle\Big\| \leqslant B\|x\|^2, \quad x \in \mathscr{H}.$$

Therefore

$$M^2C\|x\|^2 \leqslant \Big\|\sum_{n\in J}\langle x, f_n\rangle\langle f_n, x\rangle\Big\| \leqslant B\|x\|^2,$$

for all $x \in \mathscr{H}$. $\square$

3. K-frames in Hilbert C^*-modules

DEFINITION 3.1. Let $J \subseteq \mathbb{N}$ be a finite or countable index set. A sequence $\{f_n\}_{n\in J}$ of elements in a Hilbert A-module $\mathscr{H}$ is said to be a *K-frame* $(K \in L(\mathscr{H}))$ if there exist constants $C, D > 0$ such that

$$C\langle K^*x, K^*x\rangle \leqslant \sum_{n\in J}\langle x, f_n\rangle\langle f_n, x\rangle \leqslant D\langle x, x\rangle, \quad x \in \mathscr{H}. \tag{3.1}$$

THEOREM 3.2. *Let $\{f_n\}_{n\in J}$ be a Bessel sequence for $\mathscr{H}$ and $K \in L(\mathscr{H})$. Suppose that $T \in L(\mathscr{H}, \ell^2(A))$ is given by $T(x) = \{\langle x, f_n\rangle\}_{n\in J}$ and $\overline{R(T)}$ is orthogonally complemented. Then $\{f_n\}_{n\in J}$ is a K-frame for $\mathscr{H}$ if and only if there exists a linear bounded operator $L : \ell^2(A) \to \mathscr{H}$ such that $Le_n = f_n$ and $R(K) \subseteq R(L)$, where $\{e_n\}_n$ is the orthonormal basis for $\ell^2(A)$.*

Proof. Suppose that (3.1) holds. Then $C\|K^*x\|^2 \leqslant \|Tx\|^2$ for all $x \in \mathscr{H}$. By Theorem 1.3, there exists $\lambda > 0$ such that

$$KK^* \leqslant \lambda T^*T.$$

Setting $T^* = L$, we get $KK^* \leqslant \lambda LL^*$ and therefore $R(K) \subseteq R(L)$.

Conversely, since $R(K) \subseteq R(L)$, by Theorem 1.3 there exists $\lambda > 0$ such that $KK^* \leqslant \lambda LL^*$. Therefore

$$\frac{1}{\lambda}\langle K^*x, K^*x\rangle \leqslant \langle L^*x, L^*x\rangle = \sum_{n\in J}\langle x, f_n\rangle\langle f_n, x\rangle, \quad x \in \mathscr{H}.$$

Hence $\{f_n\}_{n\in J}$ is a K-frame for $\mathscr{H}$. □

In the following theorem we offer a condition for getting a frame from a K-frame.

THEOREM 3.3. *Let $\{f_n\}_{n\in J}$ be a K-frame for $\mathscr{H}$ with bounds $C, D > 0$. If the operator K is surjective, then $\{f_n\}_{n\in J}$ is a frame for $\mathscr{H}$.*

Proof. By Proposition 2.1 from [1], $K \in L(\mathscr{H})$ is surjective if and only if there is $M > 0$ such that

$$M\langle x, x\rangle \leqslant \langle K^*x, K^*x\rangle, \quad x \in \mathscr{H}.$$

Since $\{f_n\}_{n\in J}$ is a K-frame, we get from (3.1)

$$MC\langle x, x\rangle \leqslant C\langle K^*x, K^*x\rangle \leqslant \sum_{n\in J}\langle x, f_n\rangle\langle f_n, x\rangle \leqslant D\langle x, x\rangle, \quad x \in \mathscr{H}. \quad \square$$

Acknowledgement. The authors would like to thank the referee for his (her) useful comments.

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(Received July 20, 2014)

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