Operators and Matrices
Volume 10, Number 1 (2016), 113–126 doi:10.7153/oam-10-08

POSITIVE DEFINITE SOLUTIONS OF CERTAIN NONLINEAR MATRIX EQUATIONS

Z. MOUSAVI, F. MIRZAPOUR AND M. S. MOSLEHIAN

(*Communicated by L. Chi-Kwong*)

Abstract. We investigate positive definite solutions of nonlinear matrix equations $X - f(\Phi(X)) = Q$ and $X - \sum_{i=1}^{m} f(\Phi_i(X)) = Q$, where Q is a positive definite matrix, Φ and Φ_i $(1 \leqslant i \leqslant m)$ are positive linear maps on $\mathbb{M}_n(\mathbb{C})$ and f is a nonnegative matrix monotone or matrix anti-monotone function on $[0,\infty)$. In this article, using appropriate inequalities and some fixed point results, we prove the existence of unique positive definite solutions for the mentioned above equations.

1. Introduction

We consider the positive solutions of the nonlinear matrix equations

$$X - f(\Phi(X)) = Q, \quad X - \sum_{i=1}^{m} f(\Phi_i(X)) = Q,$$

where Q is a positive $n \times n$ matrix, Φ and Φ_i $(1 \leqslant i \leqslant m)$ are positive linear maps on $\mathbb{M}_n(\mathbb{C})$ and f is a nonnegative matrix monotone or matrix anti-monotone function on $[0,\infty)$. Since 1990, this type of equations has been studied in initial form of $X + AX^{-1}A^* = Q$ under assumption that Q is positive semidefinite [1, 8]. This form of nonlinear matrix equations have many applications in analysis of networks, dynamic programming, control theory and statistics. A particular kind of this equation solved for an optimal interpolation theory problem [20]. The equation $X - AX^{-1}A^* = Q$ arises in the analysis of stationary Gaussian reciprocal processes over a finite interval [15] and the equation $X + \sum_{i=1}^{n} A_i X^{-1} B_i = Q$ solved for the analysis of certain Markov processes called Tree-Like stochastic processes [3]. Specific equations such as $X + A^*X^{-n}A = Q$ $(n \in \mathbb{N})$, $X - A^*X^qA = Q$ $(q \geqslant 1)$, $X - \sum_{i=1}^{n} A_i^* X^r A_i = Q$, $X - \sum_{i=1}^{n} A_i^* X^{\delta_i} A_i = Q$ and so on have been extensively studied in the literature [12, 4, 5, 9, 16]. Fixed point theorems play a crucial role in solving of these matrix equations [10].

Some researchers focused on theoretical results involving the existence of positive solutions or the necessary and sufficient conditions of existence of positive solutions [17, 16, 8] and the others investigated numerical iterative methods and perturbation

Mathematics subject classification (2010): 15A24.
Keywords and phrases: Nonlinear matrix equation, matrix monotone, positive definite solution, positive linear map.

Paper OaM-10-08

analysis [12, 13, 21]. There are relatively few papers that deal with general matrix equations. El-Sayed and Ran studied the equation $X + A^* f(X)A = Q$, in the case that f is either matrix monotone or matrix anti-monotone [7]. In addition, Ran and Reurings derived sufficient conditions for the existence and the uniqueness of a positive definite solution of the same equation [19].

2. Preliminaries

Let $\mathbb{M}_n(\mathbb{C})$ be the $n \times n$ complex matrix algebra equipped with the usual operator norm $\|\cdot\|$. It is known that the strong operator topology and the norm topology on finite dimensional space $\mathbb{B}(\mathbb{C}^n) = \mathbb{M}_n(\mathbb{C})$ coincide. When we write $\lim_{m\to\infty} A_m = A$ we mean $\lim_{m\to\infty} \|A_m - A\| = 0$. We shall show the $n \times n$ identity matrix by I. A linear map $\Phi : \mathbb{M}_n(\mathbb{C}) \to \mathbb{M}_n(\mathbb{C})$ is said to be a positive map if $\Phi(A) \geqslant 0$ for all $A \geqslant 0$. In addition, Φ is normalized, if $\Phi(I) = I$. A typical positive linear map $\Phi : \mathbb{M}_n(\mathbb{C}) \to \mathbb{M}_n(\mathbb{C})$ is $\Phi(X) = \sum_{i=1}^m A_i^* X A_i$, where $A_i \in \mathbb{M}_n(\mathbb{C})$, $i = 1, \cdots, m$.

For an arbitrary $n \times n$ complex matrix A, the symbols $\lambda_1(A)$, $\lambda_n(A)$, $\sigma_1(A)$ and $\sigma_n(A)$ stand for the maximum eigenvalue, minimum eigenvalue, maximum singular value and minimum singular value, respectively. If A and B are Hermitian matrices and $A - B$ is positive semidefinite (positive definite, resp.), then we write $A \geqslant B$ $(A > B$, resp.). It is known that if A is a positive semidefinite matrix then $\lambda_n(A)I \leqslant A \leqslant \lambda_1(A)I$.

Suppose that B and C are two $n \times n$ Hermitian matrices and $B \leqslant C$ $(B < C$, resp.). The notation $[B,C]$ $((B,C)$, resp.) means the set of all Hermitian matrices such that $B \leqslant X \leqslant C$ $(B < X < C$, resp.). Throughout this paper, we denote the class of all Hermitian matrices and positive definite matrices by H_n and P_n, respectively. Let $J \subset \mathbb{R}$ be an interval. A real-valued continuous function f on J is said to be *matrix monotone (matrix anti-monotone)* if for all $A, B \in H_n$, $A \leqslant B$ implies that $f(A) \leqslant f(B)$ $(f(A) \geqslant f(B)$, resp.). It is known (see e.g. [14]) that a nonnegative matrix monotone function f has a representation of the form

$$f(t) = \int_0^\infty \frac{(1+s)t}{s+t} dm(s), \qquad t > 0,$$

where m is a positive measure on the half-line $[0, \infty)$. For example, the function $f(t) = \log t$ is a nonnegative matrix monotone function on $(1, +\infty)$. As usual a Hermitian matrix C with spectrum in the domain J of f is called a fixed point of f if $f(C) = C$. To achieve our result we employ the following known results.

LEMMA 2.1. [2] *If $A, B \in \mathbb{P}_n$ such that $A \geqslant B > 0$ $(A > B > 0)$, then*

(i) $A^\alpha \geqslant B^\alpha > 0$ $(A^\alpha > B^\alpha > 0)$ *for* $0 < \alpha \leqslant 1$

(ii) $0 < A^\alpha \leqslant B^\alpha$ $(0 < A^\alpha < B^\alpha)$ *for* $-1 \leqslant \alpha < 0$.

LEMMA 2.2. [11] *Let $A, B \in H_n$ and $A < B$. Let $g : [A, B] \to \mathbb{C}^{n\times n}$ be a matrix monotone function such that $A \leqslant g(A) \leqslant g(B) \leqslant B$. Then g has two fixed points X_+*

and X_- (maximum and minimum fixed point, resp.) in $[A,B]$ such that

$$X_+ = \lim_{n\to+\infty} B_n \quad \text{and} \quad X_- = \lim_{n\to+\infty} A_n,$$

where $A_n = g(A_{n-1})$, $A_0 = A$, $B_n = g(B_{n-1})$ and $B_0 = B$ for $n = 1,2,3,\cdots$. Moreover, if $X_+ = X_-$, then g has a unique fixed point X_+ (or X_-).

3. Results

We start our work with the following lemma which is interesting on its own right.

LEMMA 3.1. *Let f be a nonnegative matrix monotone function on $J = [0,\infty)$ and $A \in P_n$. If $\mu \geqslant 1$, then $f(\mu A) \leqslant \mu f(A)$. If $0 < \mu \leqslant 1$, then $f(\mu A) \geqslant \mu f(A)$.*

Proof. For any $\mu \geqslant 1$ and $s \geqslant 0$, we have $(sI+\mu A) \geqslant (sI+A)$ so $(sI+\mu A)^{-1} \leqslant (sI+A)^{-1}$. Since A commutate with $(sI+\mu A)^{-1}$, we have $A(sI+\mu A)^{-1} \leqslant A(sI+A)^{-1}$. Similarly, for any $0 < \mu \leqslant 1$, we deduce that $A(sI+\mu A)^{-1} \geqslant A(sI+A)^{-1}$. Hence

$$\begin{aligned} f(\mu A) &= \int_J (1+s)(\mu A)(s+\mu A)^{-1} dm(s) \\ &\begin{cases} \leqslant \mu \int_J (1+s)A(s+A)^{-1} dm(s) = \mu f(A) & \mu \geqslant 1 \\ \geqslant \mu \int_J (1+s)A(s+A)^{-1} dm(s) = \mu f(A) & 0 < \mu \leqslant 1. \end{cases} \end{aligned}$$ □

THEOREM 3.2. *Let f be a nonnegative matrix monotone function on $J = [0,\infty)$, $A \in P_n$ and $F(X) = Q + f(X)$, where $Q \in P_n$. Then for any $0 < \mu \leqslant 1$*

$$F(\mu A) \geqslant \mu(1+\omega(\mu))F(A),$$

where $\omega(\mu) = \frac{(1-\mu)\lambda_n(Q)}{\mu(\lambda_1(Q)+f(\lambda_1(A)))}$.

Proof. It is sufficient to show that $F(\mu A) - \mu(1+\omega(\mu))F(A) \geqslant 0$. By using Lemma 3.1, we have $f(\mu A) - \mu f(A) \geqslant 0$ for any $0 < \mu \leqslant 1$. In addition, A is a positive definite matrix, so $A \leqslant \lambda_1(A)I$, which implies $-f(A) \geqslant -f(\lambda_1(A)I)$. Let $\omega(\mu)$ be an arbitrary nonnegative function of μ. Then

$$\begin{aligned} F(\mu A) - \mu(1+\omega(\mu))F(A) &= Q + f(\mu A) - \mu(1+\omega(\mu))(Q+f(A)) \\ &= Q + (f(\mu A) - \mu f(A)) - \mu(1+\omega(\mu))Q - \mu\omega(\mu)f(A) \\ &\geqslant (1-\mu)Q - \mu\omega(\mu)Q - \mu\omega(\mu)f(A) \\ &\geqslant \{(1-\mu)\lambda_n(Q) - \mu\omega(\mu)\lambda_1(Q) - \mu\omega(\mu)f(\lambda_1(A))\}I \\ &\geqslant \{(1-\mu)\lambda_n(Q) - \mu\omega(\mu)[\lambda_1(Q) + f(\lambda_1(A))]\}I. \end{aligned}$$

Letting $\omega(\mu) = \frac{(1-\mu)\lambda_n(Q)}{\mu(\lambda_1(Q)+f(\lambda_1(A)))}$, we get $F(\mu A) \geqslant \mu(1+\omega(\mu))F(A)$.

Note that since $\mu > 0$, $(1-\mu) \geqslant 0$, $\lambda_n(Q) > 0$ and $f(\lambda_1(A)) \geqslant 0$, so $\omega(\mu) \geqslant 0$. □

COROLLARY 3.3. *Let f be a nonnegative matrix anti-monotone function on $J = [0,\infty)$, $A \in P_n$ and $F(X) = Q + f(X)$, where $Q \in P_n$. Then for any $0 < \mu \leqslant 1$*

$$F^2(\mu A) \geqslant \mu(1+\omega(\mu))F^2(A),$$

where $\omega(\mu) = \frac{(1-\mu)\lambda_n(Q)}{\mu(\lambda_1(Q)+f(f(\lambda_1(A))+\lambda_1(Q)))}$.

Proof. Since f is a matrix anti-monotone function, so $f(Q+f(X))$ is matrix monotone and $f(Q+f(A)) \leqslant f(\lambda_1(Q)I + f(\lambda_1(A))I)$. Hence $F^2(X) = Q + f(Q + f(X))$ satisfies the conditions of Theorem 3.2. $\square$

REMARK 3.4. Theorem 3.2 and Corollary (3.3) are general forms of Lemmas 2.3 and 3.1 in [5] with short proofs.

LEMMA 3.5. *Let f be a matrix anti-monotone function on J. Suppose that $A, B \in H_n$ with spectra in J and $A \leqslant f(B) \leqslant f(A) \leqslant B$. Then the following assertions hold:*

(i) f^2 has two fixed points X_+ and X_- in $[A,B]$, where $X_+ = \lim_{n\to+\infty} f^{2n}(B)$ and $X_- = \lim_{n\to+\infty} f^{2n}(A)$.

(ii) If $A \leqslant X \leqslant B$ is a fixed point of f, then $X_- \leqslant X \leqslant X_+$.

(iii) If $X_- = X_+ = \overline{X}$, then f has a unique fixed point such that

$$\overline{X} = \lim_{n\to+\infty} f^n(X_0),$$

where $A \leqslant X_0 \leqslant B$.

Proof. (i) Since $A \leqslant f(B) \leqslant f(A) \leqslant B$, therefore, $A \leqslant f(B) \leqslant f^2(A) \leqslant f^2(B) \leqslant f(A) \leqslant B$ implying that $A \leqslant f^2(A) \leqslant f^2(B) \leqslant B$. Hence, $g = f^2$ satisfies the conditions of Lemma 2.2. Consequently, f^2 has two fixed points X_+ and X_- in $[A,B]$, where $X_+ = \lim_{n\to+\infty} f^{2n}(B)$ and $X_- = \lim_{n\to+\infty} f^{2n}(A)$.

(ii) Let X be a matrix such that $A \leqslant X \leqslant B$ and $f(X) = X$. By acting repeatedly, we have $f^{2n}(A) \leqslant f^{2n}(X) = X \leqslant f^{2n}(B)$. Letting $n \to \infty$ implies that $X_- \leqslant X \leqslant X_+$.

(iii) If $X_- = X_+ = \overline{X}$, then $\overline{X} = \lim_{n\to\infty} f^{2n}(A) = \lim_{n\to\infty} f^{2n}(B)$. Moreover, if $X_0 \in [A,B]$, then for any $n = 0,1,2,\cdots$, we have

$$f^{2n}(A) \leqslant f^{2n}(X_0) \leqslant f^{2n}(B), \tag{3.1}$$

which gives

$$f^{2n}(A) \leqslant f^{2n+1}(B) \leqslant f^{2n+1}(X_0) \leqslant f^{2n+1}(A) \leqslant f^{2n}(B). \tag{3.2}$$

By (3.1) and (3.2), we get $\lim_{n\to+\infty} f^{2n}(X_0) = \lim_{n\to+\infty} f^{2n+1}(X_0) = \overline{X}$, hence, $\lim_{n\to+\infty} f^n(X_0) = \overline{X}$. Therefore, $\overline{X} = \lim_{n\to+\infty} f^n(X_0) = \lim_{n\to+\infty} f(f^{n-1}(X_0)) = f(\overline{X})$. $\square$

We recall the well-known Schauder fixed-point theorem as follows: Let C be a nonempty, compact and convex subset of a Banach space V. If $f : C \to C$ is continuous, then f has a fixed point (see [10]).

We are ready to state our first main result.

THEOREM 3.6. *Let f be a nonnegative matrix monotone function on $J=[0,\infty)$, let $\Phi:\mathbb{M}_n(\mathbb{C})\to\mathbb{M}_n(\mathbb{C})$ be a positive linear map and $g(X)=Q+f(\Phi(X))$, where X and Q are positive definite. If there exists $x_1>0$ such that $f(x_1)\leqslant\frac{x_1-\lambda_1(Q)}{\lambda_1(\Phi(I))}$, then the following assertions hold:*

(i) If $1\leqslant\lambda_1(\Phi(I))$, then $g(X)\leqslant\beta I$, where β is a positive solution of $x-\lambda_1(Q)=\lambda_1(\Phi(I))f(x)$;

(ii) If $0<\lambda_n(\Phi(I))\leqslant 1$, then $g(X)\geqslant\alpha I$, where α is a positive solution of $x-\lambda_n(Q)=\lambda_n(\Phi(I))f(x)$.

Moreover, if the conditions of (i) and (ii) are satisfied, then g has a unique positive definite fixed point $\overline{X}$ in $[\alpha I,\beta I]$ and $\{g^k(X_0)\}$ converges to $\overline{X}$ for any $\alpha I\leqslant X_0\leqslant\beta I$.

Proof. (i) First, we show that the real function $l(x)=b_1f(x)+b_2$ has a unique positive fixed point, where $b_1=\lambda_1(\Phi(I))$ and $b_2=\lambda_1(Q)$. Since f is a nonnegative matrix monotone function, so l is an increasing function such that $0<l(0)=\lambda_1(\Phi(I))f(0)+\lambda_1(Q)$. By the assumption, $f(x_1)\leqslant\frac{x_1-\lambda_1(Q)}{\lambda_1(\Phi(I))}$, so we have $l(x_1)=\lambda_1(\Phi(I))f(x_1)+\lambda_1(Q)\leqslant x_1$. Thus l satisfies the conditions of the Schauder fixed point theorem, so $l(x)$ has a positive fixed point in $[0,x_1]$.

Hence the equation $x-\lambda_1(Q)=\lambda_1(\Phi(I))f(x)$ has a positive solution β, i.e.

$$\beta-\lambda_1(Q)=\lambda_1(\Phi(I))f(\beta).$$

For $X\leqslant\beta I$,

$$Q+f(\Phi(X))\leqslant Q+f(\Phi(\beta I))=Q+f(\beta\Phi(I)).$$

It follows from Lemma 3.1 that

$$Q+f(\Phi(X))\leqslant Q+f(\lambda_1(\Phi(I))\beta I)\leqslant(\lambda_1(Q)+\lambda_1(\Phi(I))f(\beta))I=\beta I.$$

Therefore $g(X)\leqslant\beta I$.

(ii) Since $f(x_1)\leqslant\frac{x_1-\lambda_1(Q)}{\lambda_1(\Phi(I))}<\frac{x_1-\lambda_n(Q)}{\lambda_n(\Phi(I))}$, using an argument similar to (i), one can show that the equation $x-\lambda_n(Q)=\lambda_n(\Phi(I))f(x)$ has a positive solution α, i.e.

$$\alpha-\lambda_n(Q)=\lambda_n(\Phi(I))f(\alpha).$$

By the same way, there exists a positive number γ such that

$$\gamma-\lambda_1(Q)=\lambda_n(\Phi(I))f(\gamma).$$

Since $\lambda_n(Q)\leqslant\lambda_1(Q)$, we have $\alpha\leqslant\gamma$ and since $\lambda_n(\Phi(I))f(x)\leqslant\lambda_1(\Phi(I))f(x)$, we get $\gamma\leqslant\beta$, so $\alpha\leqslant\beta$. We consider $X\geqslant\alpha I$.

$$Q+f(\Phi(X))\geqslant Q+f(\Phi(\alpha I))=Q+f(\alpha\Phi(I)).$$

From Lemma 3.1 we deduce that

$$Q+f(\Phi(X))\geqslant(\lambda_n(Q)+\lambda_n(\Phi(I))f(\alpha))I=\alpha I.$$

Therefore $g(X) \geqslant \alpha I$.

Now we assume that the conditions of (i) and (ii) are satisfied. Hence, for any $\alpha I \leqslant X \leqslant \beta I$, we have

$$\alpha I \leqslant g(\alpha I) \leqslant g(X) \leqslant g(\beta I) \leqslant \beta I.$$

If g acts on the latter inequality repeatedly, we infer that the increasing sequence $\{g^n(\alpha)I\}$ and the decreasing sequence $\{g^n(\beta)I\}$ are bounded above to βI and bounded below to αI, respectively. Due to Lemma 2.2, $X_- = \lim_{n\to+\infty} g^n(\alpha I)$ and $X_+ = \lim_{n\to+\infty} g^n(\beta I)$ are fixed points of g and $X_- \leqslant X_+$.

In the sequel, we prove the uniqueness of the fixed point by the technique used in [5]. It is sufficient to show that $X_- \geqslant X_+$. For any $\alpha I \leqslant X \leqslant \beta I$, we have

$$X_- = g(X_-) \geqslant \alpha I = \frac{\alpha}{\beta}\beta I \geqslant \frac{\alpha}{\beta} g(X_+) = \frac{\alpha}{\beta} X_+.$$

Set $t_0 = \max\{t : X_- \geqslant tX_+\}$. Then $0 < t_0 < \infty$. We claim that $t_0 \geqslant 1$. In contrary assume that $0 < t_0 < 1$. Employing Theorem 3.2, we obtain

$$X_- = g(X_-) \geqslant g(t_0X_+) \geqslant [1+\omega(t_0)]t_0 g(X_+) = [1+\omega(t_0)]t_0 X_+,$$

but $[1+\omega(t_0)]t_0 > t_0$, which contradicts the maximality of t_0. Consequently, $t_0 \geqslant 1$, which gives $X_- \geqslant X_+$.

Assume that $\alpha I \leqslant X_0 \leqslant \beta I$. Since g is a matrix monotone function, it follows that for any $n = 0,1,2,\cdots$,

$$g^n(\alpha I) \leqslant g^n(X_0) \leqslant g^n(\beta I),$$

and

$$\lim_{n\to+\infty} g^n(\alpha I) = \lim_{n\to+\infty} g^n(\beta I) = \overline{X} = X_+ = X_-,$$

which implies $\lim_{n\to+\infty} g^n(X_0) = \overline{X}$. $\square$

Using the same strategy one can prove the next result.

PROPOSITION 3.7. *Let f be a nonnegative matrix monotone function on $J = [0,\infty)$. Let $\Phi_i : \mathbb{M}_n(\mathbb{C}) \to \mathbb{M}_n(\mathbb{C})$ $(1 \leqslant i \leqslant m)$ be positive linear maps and $g(X) = Q + \sum_{i=1}^m f(\Phi_i(X))$, where X and Q are positive definite. If there exists $x_1 > a$ such that $f(x_1) \leqslant \frac{x_1 - \lambda_1(Q)}{\sum_{i=1}^m \lambda_1(\Phi_i(I))}$, then the following assertions hold:*

(i) If $1 \leqslant \lambda_1(\Phi_i(I))$, $i = 1,2,\ldots,m$, then $g(X) \leqslant \beta I$, where β is a positive solution of $x - \lambda_1(Q) = \sum_{i=1}^m \lambda_1(\Phi_i(I))f(x)$;

(ii) If $0 < \lambda_n(\Phi_i(I)) \leqslant 1$, $i = 1,2,\ldots,m$, then $g(X) \geqslant \alpha I$, where α is a positive solution of $x - \lambda_n(Q) = \sum_{i=1}^m \lambda_n(\Phi_i(I))f(x)$.

Moreover, if the conditions of (i) and (ii) are satisfied, then g has a unique positive definite fixed point $\overline{X}$ in $[\alpha I, \beta I]$ and $\{g^k(X_0)\}$ converges to $\overline{X}$ for any $\alpha I \leqslant X_0 \leqslant \beta I$.

The next result reads as follows.

THEOREM 3.8. *Let f be a nonnegative matrix anti-monotone function on $J = [0,\infty)$, let $\Phi : \mathbb{M}_n(\mathbb{C}) \to \mathbb{M}_n(\mathbb{C})$ be a positive linear map and $h(X) = Q + f(\Phi(X))$, where X and Q are positive definite. If the pair (α,β) with $\alpha \leqslant \beta$ is a solution of the following system:*

$$\begin{cases} x = \lambda_n(Q) + \lambda_n(\Phi(I))f(y) \\ y = \lambda_1(Q) + \lambda_1(\Phi(I))f(x) \end{cases}$$

then the following assertions hold:

(i) If $1 \leqslant \lambda_1(\Phi(I))$, then $h(X) \leqslant \beta I$;

(ii) If $0 < \lambda_n(\Phi(I)) \leqslant 1$, then $h(X) \geqslant \alpha I$.

Moreover, if the conditions of (i) and (ii) are satisfied, then g has a unique positive definite fixed point $\overline{X}$ in $[\alpha I, \beta I]$ and $\{h^k(X_0)\}$ converges to $\overline{X}$ for any $\alpha I \leqslant X_0 \leqslant \beta I$.

Proof. We show that $\alpha I \leqslant h(X) \leqslant \beta I$ for any $\alpha I \leqslant X \leqslant \beta I$, while $1 \leqslant \lambda_1(\Phi(I))$ and $0 < \lambda_n(\Phi(I)) \leqslant 1$. By using Lemma 3.1, we have

$$\alpha I = (\lambda_n(Q) + \lambda_n(\Phi(I))f(\beta))I \leqslant Q + f(\Phi(X)) \leqslant (\lambda_1(Q) + \lambda_1(\Phi(I))f(\alpha))I = \beta I.$$

According to Lemma 3.5, h^2 has two positive fixed points X_- and X_+ such that $X_- \leqslant X_+$, $\lim_{n\to+\infty} h^{2n}(\alpha I) = X_-$ and $\lim_{n\to+\infty} h^{2n}(\beta I) = X_+$.

To show uniqueness, we need only to verify $X_- \geqslant X_+$. Since $Q \leqslant h(X)$, we have $Q \leqslant h^2(X) \leqslant h(Q)$. Now $h^2(X) \leqslant \lambda_1(h(Q))I$ leads to

$$\begin{aligned} X_- = h^2(X_-) &= Q + f(\Phi(Q) + \Phi(f(\Phi(X_-)))) \\ &\geqslant \lambda_n(Q)I = \frac{\lambda_n(Q)\lambda_1(h(Q))}{\lambda_1(h(Q))} I \\ &\geqslant \frac{\lambda_n(Q)}{\lambda_1(h(Q))} h^2(X_+) = \frac{\lambda_n(Q)}{\lambda_1(h(Q))} X_+. \end{aligned}$$

Set $t_0 = \max\{t : X_- \geqslant tX_+\}$. Obviously, $0 < t_0 < +\infty$. We claim that $t_0 \geqslant 1$. If $0 < t_0 < 1$, then, by Corollary 3.3, we have

$$X_- = h^2(X_-) \geqslant h^2(t_0X_+) \geqslant t_0[1+\omega(t_0)]h^2(X_+) = t_0[1+\omega(t_0)]X_+,$$

but $t_0[1+\omega(t_0)] > t_0$, which contradicts maximality of t_0. Hence, $t_0 \geqslant 1$ and $X_- \geqslant X_+$. Due to Lemma 3.5, $h(X)$ has a unique fixed point in $[\alpha I, \beta I]$ such that

$$\overline{X} = \lim_{n\to+\infty} h^n(X_0),$$

where $\alpha I \leqslant X_0 \leqslant \beta I$. $\square$

Utilizing the same reasoning as in the proof of the previous theorem one can show the next result.

COROLLARY 3.9. *Let f be a nonnegative matrix anti-monotone function on $J = [0,\infty)$, let $\Phi_i : \mathbb{M}_n(\mathbb{C}) \to \mathbb{M}_n(\mathbb{C})$ $(1 \leqslant i \leqslant m)$ be positive linear maps and $h(X) = Q + \sum_{i=1}^m f(\Phi_i(X))$, where X and Q are positive definite. If the pair (α,β) with $\alpha \leqslant \beta$ is a solution of the following system:*

$$\begin{cases} x = \lambda_n(Q) + \sum_{i=1}^m \lambda_n(\Phi_i(I)) f(y) \\ y = \lambda_1(Q) + \sum_{i=1}^m \lambda_1(\Phi_i(I)) f(x) \end{cases}$$

then the following assertions hold:

(i) If $1 \leqslant \lambda_1(\Phi(I))$, then $h(X) \leqslant \beta I$;

(ii) If $0 < \lambda_n(\Phi(I)) \leqslant 1$, then $h(X) \geqslant \alpha I$.

Moreover, if the conditions of (i) and (ii) are satisfied, then g has a unique positive definite fixed point $\overline{X}$ in $[\alpha I, \beta I]$ and $\{h^k(X_0)\}$ converges to $\overline{X}$ for any $\alpha I \leqslant X_0 \leqslant \beta I$.

4. Application for nonlinear operator equations

The following corollaries and examples illustrate our results. Note that a fixed point of function $g(X) = Q + f(\Phi(X))$ is the solution of the equation $X - f(\Phi(X)) = Q$.

We shall consider the following equation.

$$X - \Big(\sum_{i=1}^m A_i^* X A_i\Big)^r = Q, \tag{4.1}$$

where $Q \in P_n$ and $r \in [-1,1)$.

COROLLARY 4.1. *(i) Let $r \in (0,1)$ and $g(X) = Q + (\sum_{i=1}^m A_i^* X A_i)^r$. Suppose that $1 \leqslant \lambda_1(\sum_{i=1}^m A_i^* A_i)$ and $0 < \lambda_n(\sum_{i=1}^m A_i^* A_i) \leqslant 1$. Then Equation (4.1) has a unique positive definite solution $\overline{X}$ in $[\alpha I, \beta I]$, where α and β are the positive solutions of the equations $x - \lambda_n(Q) = \lambda_n(\sum_{i=1}^m A_i^* A_i) x^r$ and $x - \lambda_1(Q) = \lambda_1(\sum_{i=1}^m A_i^* A_i) x^r$, respectively. In addition, $\{g^k(X_0)\}$ converges to $\overline{X}$ for any $\alpha I \leqslant X_0 \leqslant \beta I$.*

(ii) Let $r \in [-1,0)$ and $h(X) = Q + (\sum_{i=1}^m A_i^* X A_i)^r$. Suppose that $1 \leqslant \lambda_1(\sum_{i=1}^m A_i^* A_i)$ and $0 < \lambda_n(\sum_{i=1}^m A_i^* A_i) \leqslant 1$. If the pair (α,β) with $\alpha \leqslant \beta$ is a solution of the following system:

$$\begin{cases} x = \lambda_n(Q) + \lambda_n(\sum_{i=1}^m A_i^* A_i) y^r \\ y = \lambda_1(Q) + \lambda_1(\sum_{i=1}^m A_i^* A_i) x^r, \end{cases}$$

then Equation (4.1) has a unique positive definite solution $\overline{X}$ in $[\alpha I, \beta I]$. In addition, $\{h^k(X_0)\}$ converges to $\overline{X}$ for any $\alpha I \leqslant X_0 \leqslant \beta I$.

Proof. (i) In Theorem 3.6, we set $\Phi(X) = \sum_{i=1}^m A_i^* X A_i$ and $f(x) = x^r$, where $r \in (0,1)$. By Lemma 2.1, f is a nonnegative matrix monotone function on $J = [0,\infty)$.

Since there exists a $x_1 > 0$ such that $x_1^r \leqslant \frac{x_1 - \lambda_1(Q)}{\lambda_1(\sum_{i=1}^m A_i^* A_i)}$, $(\lim_{x\to\infty} \frac{\lambda_1(\sum_{i=1}^m A_i^* A_i) x^r}{x - \lambda_1(Q)} = 0,\quad r \in (0,1))$ we get the result.

(ii) This result comes from Theorem 3.8, if we set $\Phi(X)=\sum_{i=1}^m A_i^*XA_i$ and $f(x)=x^r$, where $r\in[-1,0)$. In this case, f is a matrix anti-monotone function on $J=(0,\infty)$ by Lemma 2.1. □

One can specialize Equation (4.1) by putting $\Phi(X)=A^*XA$, where A is an isometry, i.e. $A^*A=I$ and $f(x)=x^{-1}$. So $\lambda_n(\Phi(I))=\lambda_1(\Phi(I))=1$ and $f(x)$ is a matrix nonnegative anti-monotone function on $(0,\infty)$, hence the conditions of corollary 4.1 are satisfied. It is sufficient to obtain the following system solution.

$$\begin{cases} x=\lambda_n(Q)+\frac{1}{y} \\ y=\lambda_1(Q)+\frac{1}{x}. \end{cases}$$

Assume that K is the correlation matrix of $A\in P_n$, i.e.

$$K=(a_{ij}/(a_{ii}a_{jj})^{\frac{1}{2}}) \qquad for \qquad A=(a_{ij})$$

The matrix K is positive definite. We define a map $\Phi(X)=K\circ X$, where $\circ$ denotes the Hadamard product of matrices. Then Φ is a normalized positive linear map ([18]). Now we shall consider the following equation.

$$X-(K\circ X)^r=Q, \tag{4.2}$$

where $Q\in P_n$, $r\in[-1,1)$.

COROLLARY 4.2. *(i) Let $r\in(0,1)$ and $g(X)=Q+(K\circ X)^r$. Then Equation (4.2) has a unique positive definite solution $\overline{X}$ in $[\alpha I,\beta I]$, where α and β are positive solutions of equations $x-\lambda_n(Q)=x^r$ and $x-\lambda_1(Q)=x^r$, respectively. In addition, $\{g^k(X_0)\}$ converges to $\overline{X}$ for any $\alpha I\leqslant X_0\leqslant\beta I$.*

(ii) Let $r\in[-1,0)$ and $h(X)=Q+(K\circ X)^r$. If the pair (α,β) with $\alpha\leqslant\beta$ is a solution of the following system:

$$\begin{cases} x=\lambda_n(Q)+y^r \\ y=\lambda_1(Q)+x^r \end{cases}$$

then Equation (4.2) has a unique positive definite solution $\overline{X}$ in $[\alpha I,\beta I]$. In addition, $\{h^k(X_0)\}$ converges to $\overline{X}$ for any $\alpha I\leqslant X_0\leqslant\beta I$.

Proof. Set $\Phi(X)=K\circ X$ and $f(x)=x^r$ in Theorem 3.6 and 3.8. Since Φ is a normalized positive map, so $\lambda_1(\Phi(I))=\lambda_n(\Phi(I))=1$ which satisfies the conditions of mentioned theorems. (Additionally, $f(x)$ satisfies the conditions of the (i) and (ii) by similar discussions as in Corollary 4.1.) □

COROLLARY 4.3. *Let $g(X)=Q+\sum_{i=1}^m\log(A_i^*XA_i)$, where $Q\in P_n$. Suppose that $\lambda_n(Q)>1$, $1\leqslant\sigma_1^2(A_i)$ and $0<\sigma_n^2(A_i)\leqslant 1$, for $i=1,\dots,m$. Then the matrix equation*

$$X-\sum_{i=1}^m\log(A_i^*XA_i)=Q$$

has a unique positive solution $\overline{X}$ in $[\alpha I, \beta I]$, where α and β are positive solutions of equations $x - \lambda_n(Q) = \sum_{i=1}^m \sigma_n^2(A_i)\log(x)$ and $x - \lambda_1(Q) = \sum_{i=1}^m \sigma_1^2(A_i)\log(x)$, respectively. Moreover, $\{g^k(X_0)\}$ converges to $\overline{X}$ for any $\alpha I \leqslant X_0 \leqslant \beta I$.

Proof. It is sufficient to put $f(x) = \log(x)$ and $\Phi_i(X) = A_i^* X A_i$, for $i = 1, \ldots, m$ in Proposition 3.7. Since $f(x) = \log(x)$ is a matrix monotone function on $(0, \infty)$ and there exists $x_1 > 1$ such that $\log(x_1) \leqslant \frac{x_1 - \lambda_1(Q)}{\sum_{i=1}^m \sigma_1^2(A_i)}$ $(\lambda_n(Q) > 1)$, so the conditions of Proposition 3.7 are met.

(Since $\lim_{x\to\infty} \frac{\sum_{i=1}^m \sigma_1^2(A_i)\log(x)}{x - \lambda_1(Q)} = 0$.) □

COROLLARY 4.4. *Let $g(X) = Q + U^* X(X^s + I)^{-1}U$, where $Q \in P_n$, $s \in (0,1]$ and U is a unitary matrix. Then the matrix equation*

$$X - U^* X(X^s + I)^{-1}U = Q$$

has a unique positive solution $\overline{X}$ in $[\alpha I, \beta I]$, where α and β are positive solutions of equations $x - \lambda_n(Q) = \frac{x}{x^s+1}$ and $x - \lambda_1(Q) = \frac{x}{x^s+1}$, respectively. Moreover, $\{g^k(X_0)\}$ converges to $\overline{X}$ for any $\alpha I \leqslant X_0 \leqslant \beta I$.

Proof. Put $f(x) = \frac{x}{x^s+1}$ and $\Phi(X) = U^* X U$ in Theorem 3.6. Note that since $\varphi(x) = 1 + x^s$, where $s \in (0,1]$, is matrix monotone, so is $f(x) = \frac{x}{\varphi(x)} = \frac{x}{x^s+1}$, cf. [18, Corollary 1.14]. In addition, since $\lim_{x\to\infty} \frac{x}{(x-\lambda_1(Q))(x^s+1)} = 0$, so there exists a number $x_1 > 0$ such that $f(x_1) < x_1 - \lambda_1(Q)$. □

5. Numerical experiments

We carry out numerical examples for computing a positive definite solution of equations $X - \log(A^*XA) = Q$, $X - (A^*XA)^{-1} = Q$ and $X - (K \circ X)^{\frac{1}{2}} = Q$ by MATLAB. All computations are presented with the first 6 digits and for the stopping condition of all algorithms we have chosen $\varepsilon = 10^{-8}$. We have used the methods described in Theorems 3.6 and 3.8. The CPU time needed by all the algorithms is negligible, since it is less than a second. In the following $\|\,.\,\|_\infty$ stands for infinity norm of matrices.

EXAMPLE 5.1. Consider the equation $X - \log(A^*XA) = Q$ with

$$A = \begin{pmatrix} 0.00171 & 0.1120 & 0.0400 \\ 0.0020 & 0.4720 & -0.0020 \\ -0.0040 & -0.0010 & 2.0100 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 3 & 2 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$

We have $\lambda_1(Q) = 3$, $\lambda_n(Q) = 2 > 1$, $0 < \sigma_n^2(A) = 0.000002 \leqslant 1$ and $1 \leqslant \sigma_1^2(A) = 4.041720$. We compute α and β by solving the following equations, respectively.

$$x - 2 = 0.000002\log(x), \quad x - 3 = 4.041720\log(x),$$

which gives $\alpha = 2.000001$ and $\beta = 13.527597$. So we begin with an initial matrix X_0 such that

$$(2.000001)I \leqslant X_0 \leqslant (13.527597)I.$$

We use the following iterative algorithm for calculating the solution.

Algorithm (a)

1. Set $X_0 = ((\mu)2.000001 + (1-\mu)13.527597)I$, where $\mu \in [0,1]$.
2. For $n = 0,1,2,\dots$, compute $X_{n+1} = Q + \log(A^*X_nA)$, until
$$\| Q - X_{n+1} + \log(A^*X_{n+1}A) \|_\infty < \varepsilon.$$
3. X_{n+1} provides an approximation to $\overline{X}$.

Table (a) reports, for different values of the parameter μ ($\mu = 0.2$, $\mu = 0.5$ and $\mu = 0.8$), the number of iterations which is needed to satisfy the stopping conditions i.e. $\| Q - X + \log(A^*XA) \|_\infty < \varepsilon$ and the relative error $R_n = \frac{\|X_{n+1}-X_n\|_\infty}{\|X_{n+1}\|_\infty}$, where X is the approximation provided by Algorithm (a).

Table (a)

μ	n	R_n
0.2	29	4.942802e-10
0.5	29	5.108141e-10
0.8	29	5.373146e-10

The solution is

$$\overline{X} \simeq \begin{pmatrix} -6.128080 - 0.924340i & -2.744315 - 1.969056i & -2.992607 + 2.831025i \\ -4.683328 - 2.039036i & 1.542492 - 1.771237i & 0.754397 + 1.610607i \\ -2.060743 + 2.818150i & 0.7094100 + 1.511913i & 4.981168 + 0.028113i \end{pmatrix}.$$

EXAMPLE 5.2. Consider the equation $X - (A^*XA)^{-1} = Q$ with

$$A = \begin{pmatrix} 1 & -0.2 \\ 0.1 & -0.6 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 3 & 2 \\ 2 & 4 \end{pmatrix}.$$

We have $\lambda_n(Q) = 1.438447$, $\lambda_1(Q) = 5.561553$, $0 < \sigma_n^2(A) = 0.304220 \leqslant 1$ and $1 \leqslant \sigma_1^2(A) = 1.105780$. By solving the following system, α and β are obtained.

$$\begin{cases} x = 1.438447 + 0.304220\frac{1}{y} \\ y = 5.561553 + 1.105780\frac{1}{x} \end{cases}$$

It gives $\alpha = 1.4866950$ and $\beta = 6.3053370$. So initial matrix X_0 is chosen such that

$$(1.4866950)I \leqslant X_0 \leqslant (6.3053370)I.$$

We use the following iterative algorithm.

Algorithm (b)

1. Set $X_0 = ((\mu)1.4866950 + (1-\mu)6.3053370)I$, where $\mu \in [0,1]$.
2. For $n = 0,1,2,\ldots$, compute $X_{n+1} = Q + (A^*X_nA)^{-1}$, until
$$\| Q - X_{n+1} + (A^*X_{n+1}A)^{-1} \|_\infty < \varepsilon.$$
3. X_{n+1} provides an approximation to $\overline{X}$.

Table (b) reports, for different values of the parameter μ, the number of iterations which is needed to satisfy the stopping conditions i.e. $\| Q - X + (A^*XA)^{-1} \|_\infty < \varepsilon$ and the relative error.

Table (b)

μ	n	R_n
0.2	13	7.532181e-11
0.5	12	4.531828e-10
0.8	13	5.742987e-10

The solution is $\overline{X} \simeq \begin{pmatrix} 3.681601 & 2.703509 \\ 2.703509 & 5.105746 \end{pmatrix}$.

EXAMPLE 5.3. Consider the equation $X - (K \circ X)^{\frac{1}{2}} = Q$. Let A be any diagonal positive definite matrix. Then K is the identity matrix. Assume $Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

Here $\lambda_n(Q) = 1$, $\lambda_1(Q) = 3$ and $\alpha = 2.618034$, $\beta = 5.302776$ which are the solutions of the equations $x - 1 = \sqrt{x}$ and $x - 3 = \sqrt{x}$, respectively. Therefore,
$$(2.618034)I \leqslant X_0 \leqslant (5.302776)I.$$

ALGORITHM (C).

1. Set $X_0 = ((\mu)2.618034 + (1-\mu)5.302776)I$, where $\mu \in [0,1]$.
2. For $n = 0,1,2,\ldots$, compute $X_{n+1} = Q + (K \circ X_n)^{\frac{1}{2}}$, until $\| Q - X_{n+1} + (K \circ X_{n+1})^{\frac{1}{2}} \|_\infty < \varepsilon$.
3. X_{n+1} provides an approximation to $\overline{X}$.

The result is presented in the following table.

Table (c)

μ	n	R_n
0.2	16	4.958179e-09
0.5	16	3.342531e-09
0.8	15	4.727529e-09

The solution is $\overline{X} \simeq \begin{pmatrix} 2.6180334 & 0 & 0 \\ 0 & 5.302776 & 0 \\ 0 & 0 & 4.000000 \end{pmatrix}$.

Comparing the examples, we conclude that the initial matrix influences on number of iteration and relative error but the difference of number of iteration is insignificant in most examples.

Acknowledgements. M. S. Moslehian (the corresponding author) was supported by a grant from Ferdowsi University of Mashhad (No. MP94334MOS).

REFERENCES

[1] W. N. ANDERSON, JR., T. D. MORLEY AND G. E. TRAPP, *Positive solutions to* $X = A - BX^{-1}B^*$, Linear Algebra Appl. **134** (1990), 53–62.

[2] R. BHATIA, *Matrix Analysis*, Grad. Texts in Math. **169**, Springer-Verlag, New York, 1997.

[3] D. A. BINI, G. LATOUCHEB AND B. MEINI, *Solving nonlinear matrix equations arising in Tree-Like stochastic processes*, Linear Algebra Appl. **366** (2003), 39–64.

[4] X. DUAN AND A. LIAO, *On the nonlinear matrix equation* $X + A^*X^{-q}A = Q (q \geqslant 1)$, Math. Comput. Modelling **49** (2009), 936–945.

[5] X. DUAN AND A. LIAO, *On Hermitian positive definite solution of the matrix equation* $X - \sum_{i=1}^{m} A_i^* X^r A_i = Q$, J. Comput Appl. Math. **229** (2009), no. 1, 27–36.

[6] X. DUAN, A. LIAO AND B. TANG, *On the nonlinear matrix equation* $X - \sum_{i=1}^{m} A_i^* X^{\delta_i} A_i = Q$, Linear Algebra Appl. **429** (2008), 110–121.

[7] S. M. EL-SAYED AND A. C. M. RAN, *On an iteration method for solving a class of nonlinear matrix equations*, Siam J. Matrix Anal. Appl. **23** (2001), no. 3, 632–645.

[8] J. C. ENGWERDA, A. C. M. RAN AND A. L. RIJKEBOER, *Necessary and sufficient conditions for the existence of a positive definite solution of the matrix equation* $X + A^*X^{-1}A = Q$, Linear Algebra Appl. **186** (1993), 255–275.

[9] F. O. FARID, M. S. MOSLEHIAN, Q.-W. WANG AND Z.-C. WU, *On the Hermitian solutions to a system of adjointable operator equations*, Linear Algebra Appl. **437** (2012), no. 7, 1854–1891.

[10] A. GRANAS AND J. DUGUNDJI, *Fixed Point Theory*, Springer-verlag, New York, 2003.

[11] D. GUO, *Nonlinear functional analysis*, Shangdong Sci. and Tech. press, Jinan, 2001.

[12] V. HASANOV AND I. IVANOV, *Solutions and perturbation estimates for the matrix equations* $X \pm A^*X^{-n}A = I$, Appl. Math. Comput. **156** (2004), 513–525.

[13] I. G. IVANOV, V. I. HASANOV AND F. UHLIG, *Improved methods and starting values to solve the matrix equations* $X \pm A^*X^{-1}A = I$ *iterativly*, Math. Comput. **74** (2004), 263–278.

[14] F. KUBO AND T. ANDO, *Means of positive linear operators*, Math. Ann. **246** (1980), 205–224.

[15] B. C. LEVY, R. FREZZA AND A. J. KRENER, *Modeling and estimation of discretetime Gaussian reciprocal processes*, IEEE Trans. Automat. Control **35** (1990), 1011–1023.

[16] Y. D. LIM, *Solving the nonlinear matrix equation* $X = Q + \sum_{i=1}^{m} m_i X^{\delta_i} M_i^*$ *via a contraction principle*, Linear Algebra Appl. **430** (2009), 1380–1383.

[17] B. MEINI, *Nonlinear matrix equations and structured linear algebra*, Linear Algebra Appl. **413** (2006), 440–457.

[18] J. PEČARIĆ, T. FURUTA, J. M. HOT AND Y. SEO, *Mond-Pečarić method in operator inequalities*, Element, Zagreb, 2005.

[19] A. C. M. RAN AND M. C. B. REURINGS, *On the nonlinear matrix equation* $X - A^*f(X)A = Q$*: solutions and perturbation theory*, Linear Algebra Appl. **346** (2002), 15–26.

[20] A. C. M. RAN AND M. C. B. REURINGS, *A nonlinear matrix equation connected to interpolation theory*, Linear Algebra Appl. **379** (2004), 289–302.

[21] A. C. M. RAN, M. C. B. REURINGS AND L. RODMAN, *A perturbation analysis for nonlinear self-adjoint operator equations*, SIAM J. Matrix Anal. Appl. **28** (2006), no. 1, 89–104.

[22] M. C. B. REURINGS, *Contractive maps on normed linear spaces and their applications to nonlinear matrix equations*, Linear Algebra Appl. **418** (2006), 292–311.

(Received January 1, 2015)

Z. Mousavi
Department of Mathematics
Faculty of Sciences
University of Zanjan
P. O. Box 45195-313, Zanjan, Iran
e-mail: zeinabmoosavi@znu.ac.ir

F. Mirzapour
Department of Mathematics
Faculty of Sciences
University of Zanjan
P. O. Box 45195-313, Zanjan, Iran
e-mail: f.mirza@znu.ac.ir

M. S. Moslehian
Department of Pure Mathematics
Center of Excellence in Analysis on Algebraic Structures (CEAAS)
Ferdowsi University of Mashhad
P. O. Box 1159, Mashhad 91775, Iran.
e-mail: moslehian@um.ac.ir,
moslehian@member.ams.org