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MULTILINEAR VARIANTS OF GROTHENDIECK'S COMPOSITION THEOREM

DUMITRU POPA

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Abstract. In this paper we prove two multilinear variants of Grothendieck's composition theorem. We give examples which show that other natural multilinear variants of Grothendieck's composition theorem are not necessarily true. In addition we prove a new multilinear variant of Grothendieck's composition theorem.

1. Introduction and notation

The famous Grothendieck composition theorem asserts that the composition of two 2-summing operators is nuclear, see [6, Theorem 5.31], [12, 17.6.4 Theorem]. D. Pérez-García and I. Villanueva in [11, Theorem 2.1] proven a multilinear variant of Grothendieck's composition theorem. In this paper we will prove two multilinear variant of Grothendieck's composition theorem, Theorems 3, 4 and that, some of possible multilinear variants of Grothendieck's composition theorem are not necessarily true, Proposition 2 and Proposition 3.

We start by fixing some needed notation. All spaces considered in this paper are Banach spaces and all operators act between Banach spaces. Let X be a Banach space and X^* its dual. For a finite system $(x_i)_{1\leqslant i\leqslant m}\subset X$ and $1\leqslant p<\infty$, we write $w_p\left((x_i)_{1\leqslant i\leqslant m}\right)$ to denote $\sup\limits_{\|x^*\|\leqslant 1}\left(\sum\limits_{i=1}^{m}|x^*(x_i)|^p\right)^{\frac{1}{p}}$.

Let $1\leqslant p<\infty$. A bounded linear operator $T:X\to Y$ is p-summing if there exists a constant $C>0$ such that, for every choice of a finite system $(x_i)_{1\leqslant i\leqslant m}\subset X$ the following relation holds $\left(\sum\limits_{i=1}^{m}\|T(x_i)\|^p\right)^{\frac{1}{p}}\leqslant Cw_p\left((x_i)_{1\leqslant i\leqslant m}\right)$. In this case, we define the p-summing norm of T by $\pi_p(T)=\min\{C : C \text{ as above}\}$ and we denote by $\Pi_p(X,Y)$ the class of p-summing linear operators, see [4, 6, 12].

One of the most successful generalization of the linear summing concept is the multiple summing operator. This concept was introduced by M. Matos in [8], and, independently, by F. Bombal, D. Pérez-García and I. Villanueva in [2], although the origin of this class goes back to [14].

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Let $1 \leqslant p < \infty$. A bounded k-linear operator $T : X_1 \times \cdots \times X_k \to Y$ is multiple p-summing if there exists a constant $C > 0$ such that, for every choice of finite systems $\left(x_{i_j}\right)_{1 \leqslant i_j \leqslant m_j} \subset X_j$ $(1 \leqslant j \leqslant k)$ the following relation holds

$$\left(\sum_{i_1,\ldots,i_k=1}^{m_1,\ldots,m_k} \left\| T\left(x_{i_1}^1,\ldots,x_{i_k}^k\right) \right\|^p \right)^{\frac{1}{p}} \leqslant C w_p\left(\left(x_{i_1}^1\right)_{1\leqslant i_1 \leqslant m_1}\right) \cdots w_p\left(\left(x_{i_k}^k\right)_{1\leqslant i_k \leqslant m_k}\right).$$

In this case, we define the multiple p-summing norm of T by $\pi_p^{mult}(T) = \min\{C : C$ as above$\}$ and we denote by $\Pi_p^{mult}(X_1,\ldots,X_k;Y)$ the class of multiple p-summing operators.

Let $H_1,\ldots,H_k$, H be Hilbert spaces. A bounded k-linear operator $T : H_1 \times \cdots \times H_k \to H$ is Hilbert-Schmidt if there is an orthonormal basis $\left(e_{i_j}^j\right)_{i_j \in I_j} \subset H_j$ $(1 \leqslant j \leqslant k)$ such that

$$\|T\|_{HS} = \left(\sum_{i_1 \in I_1,\ldots,i_k \in I_k} \left\| T\left(e_{i_1}^1,\ldots,e_{i_k}^k\right) \right\|^2 \right)^{\frac{1}{2}} < \infty.$$

The set of all Hilbert-Schmidt operators between $H_1 \times \cdots \times H_k$ and H is denoted by $HS(H_1,\ldots,H_k;H)$ and the Hilbert-Schmidt norm of T, by $\|T\|_{HS}$.

We use the well known fact that $HS(H_1,\ldots,H_k;H) = \Pi_2^{mult}(H_1,\ldots,H_k;H)$ with the equality of the norms; for the linear case, see [4, 6, 12] and for the multilinear case, see [8, 9].

A bounded k-linear operator $T : X_1 \times \cdots \times X_k \to Y$ is nuclear if there exists $\left(\psi_n^j\right)_{n\in\mathbb{N}} \subset X_j^*$ $(1 \leqslant j \leqslant k)$, $(y_n)_{n\in\mathbb{N}} \subset Y$ such that $\sum_{n=1}^{\infty} \left\|\psi_n^1\right\| \cdots \left\|\psi_n^k\right\| \|y_n\| < \infty$ and $T(x_1,\ldots,x_k) = \sum_{n=1}^{\infty} \psi_n^1(x_1) \cdots \psi_n^k(x_k) y_n$ for $(x_1,\ldots,x_k) \in X_1 \times \cdots \times X_k$. Such a representation is called a nuclear representation of T. In this case

$$\|T\|_{nuc} = \inf\left\{ \sum_{n=1}^{\infty} \left\|\psi_n^1\right\| \cdots \left\|\psi_n^n\right\| \|y_n\| \right\},$$

where the infimum is taken over all nuclear representations of T. This class is denoted by $(\mathscr{N}, \|\ \|_{nuc})$, see [7, Definition 1.26] and [3, page 123].

We observe that in the case $k = 1$ we get the well-known concept of nuclear (linear) operator, see [4, 6, 12].

If X is a Banach space by $I_X : X \to X$ we denote the identity operator on X, $I_X(x) = x$.

We will need the pointwise multiplication of two sequences of scalars i.e. if $a = (a_n)_{n\in\mathbb{N}}$, $b = (b_n)_{n\in\mathbb{N}}$ are two scalar sequences, we write $ab = (a_n b_n)_{n\in\mathbb{N}}$.

Throughout in this paper by a diagram of the form

$$X_1 \times \cdots \times X_k \overset{(A_1,\ldots,A_k)}{\longrightarrow} Y_1 \times \cdots \times Y_k \overset{T}{\to} Z$$

we understand that all $A_1 : X_1 \to Y_1$, ..., $A_k : X_k \to Y_k$ are bounded linear and T is bounded k-linear.

The notations and terminology used along the paper are standard in Banach space theory, as for instance in [4, 6, 12].

2. A first multilinear variant of Grothendieck's composition theorem

We need the following multilinear variant of Grothendieck's composition theorem proven by D. Pérez-García and I. Villanueva, see [11, Theorem 2.1].

THEOREM 1. *If in the diagram*

$$X_1 \times \cdots \times X_k \overset{(A_1,\ldots,A_k)}{\rightarrow} Y_1 \times \cdots \times Y_k \overset{T}{\rightarrow} Z$$

all A_1, ..., A_k *are* 2*-summing and* T *is multiple* 2*-summing, then* $T \circ (A_1,...,A_k)$ *is nuclear and* $\|T \circ (A_1,...,A_k)\|_{nuc} \leqslant \pi_2^{mult}(T)\,\pi_2(A_1)\cdots\pi_2(A_k)$.

For a bounded $(n+k)$-linear operator $T : X_1 \times \cdots \times X_n \times X_{n+1} \times \cdots \times X_{n+k} \to Y$, we consider the bounded k-linear operator associated to T as $\widehat{T} : X_{n+1} \times \cdots \times X_{n+k} \to \mathscr{L}(X_1,\ldots,X_n;Y)$,

$$\widehat{T}(x_{n+1},\ldots,x_{n+k})(x_1,\ldots,x_n) = T(x_1,\ldots,x_n,x_{n+1},\ldots,x_{n+k})$$

see [5]; the operator $\widehat{T}$ associated to T as above is, in the notation of [5], the operator T^C, where $C = \{1,2,\ldots,n\}$.

We also need, see [1, Theorem 1]

THEOREM 2. *Let* $H_1,\ldots,H_n,H$ *be Hilbert spaces,* $X_{n+1},\ldots,X_{n+k}$ *be Banach spaces,* $T : H_1 \times \cdots \times H_n \times X_{n+1} \times \cdots \times X_{n+k} \to H$ *a bounded* $(n+k)$*-linear operator. Then* T *is multiple* 2*-summing if and only if* $\widehat{T} : X_{n+1} \times \cdots \times X_{n+k} \to HS(H_1,\ldots,H_n;H)$ *is multiple* 2*-summing with respect to the Hilbert-Schmidt norm on* $HS(H_1,\ldots,H_n;H)$*. In addition*

$$\pi_2^{mult}\left(\widehat{T} : X_{n+1} \times \cdots \times X_{n+k} \to HS(H_1,\ldots,H_n;H)\right) = \pi_2^{mult}(T).$$

Next we prove a first multilinear variant of Grothendieck's composition theorem, compare to Theorem 1.

THEOREM 3. *(i) Let* H,K *be Hilbert spaces and* $X_1,\ldots,X_k$, $Y_1,\ldots,Y_k$, Z *Banach spaces. If in the diagram*

$$H \times X_1 \times \cdots \times X_k \overset{(I_H,A_1,\ldots,A_k)}{\rightarrow} H \times Y_1 \times \cdots \times Y_k \overset{T}{\rightarrow} K \overset{U}{\rightarrow} Z$$

all $A_1,\ldots,A_k$ *are* 2*-summing,* T *is multiple* 2*-summing and* U *is* 2*-summing, then the composition* $U \circ T \circ (I_H,A_1,\ldots,A_k)$ *is nuclear and*

$$\|U \circ T \circ (I_H,A_1,\ldots,A_k)\|_{nuc} \leqslant \pi_2(U)\,\pi_2^{mult}(T)\,\pi_2(A_1)\cdots\pi_2(A_k).$$

(ii) Let H,K be Hilbert spaces and X_0, $X_1,\dots,X_k$, $Y_1,\dots,Y_k$, Z Banach spaces. If in the diagram

$$X_0\times X_1\times\cdots\times X_k \overset{(A_0,A_1,\dots,A_k)}{\rightarrow} H\times Y_1\times\cdots\times Y_k \overset{T}{\rightarrow} K \overset{U}{\rightarrow} Z$$

A_0 is bounded linear, all $A_1,\dots,A_k$ are 2-summing, T is multiple 2-summing and U is 2-summing, then the composition $U\circ T\circ(A_0,A_1,\dots,A_k)$ is nuclear and

$$\|U\circ T\circ(A_0,A_1,\dots,A_k)\|_{nuc}\leqslant\pi_2(U)\,\pi_2^{mult}(T)\,\|A_0\|\,\pi_2(A_1)\cdots\pi_2(A_k).$$

Proof. (i). Write $S=U\circ T\circ(I_H,A_1,\dots,A_k)$ and consider the diagram

$$X_1\times\cdots\times X_k \overset{(A_1,\dots,A_k)}{\rightarrow} Y_1\times\cdots\times Y_k \overset{\widehat{T}}{\rightarrow} L(H,K) \overset{h_U}{\rightarrow} L(H,Z)$$

where $h_U(B)=U\circ B$. A simple calculation shows that we have the equality

$$\widehat{S}=h_U\circ\widehat{T}\circ(A_1,\dots,A_k).\ (1)$$

Since T is multiple 2-summing from Theorem 2, $Y_1\times\cdots\times Y_k\overset{\widehat{T}}{\rightarrow}HS(H,K)$ is multiple 2-summing with respect to the Hilbert-Schmidt norm on $HS(H,K)$ and

$$\pi_2^{mult}\left(\widehat{T}:Y_1\times\cdots\times Y_k\rightarrow HS(H,K)\right)=\pi_2^{mult}(T).$$

Further, since U is 2-summing, from Grothendieck's composition theorem, for each $B\in HS(H,K)=\Pi_2(H,K)$ we have $U\circ B\in\mathcal{N}(H,Z)$ and $\|U\circ B\|_{nuc}\leqslant\pi_2(U)\,\pi_2(B)$. This means that $HS(H,K)\overset{h_U}{\rightarrow}\mathcal{N}(H,Z)$ is bounded linear and $\|h_U:HS(H,K)\rightarrow\mathcal{N}(H,Z)\|\leqslant\pi_2(U)$. By the ideal property of multiple 2-summing operators, $Y_1\times\cdots\times Y_k\overset{h_U\circ\widehat{T}}{\rightarrow}\mathcal{N}(H,Z)$ is multiple 2-summing with respect to the nuclear norm on $\mathcal{N}(H,Y)$ and

$$\begin{aligned}&\pi_2^{mult}\left(h_U\circ\widehat{T}:Y_1\times\cdots\times Y_k\rightarrow\mathcal{N}(H,Z)\right)\\&\leqslant\pi_2^{mult}\left(\widehat{T}:Y_1\times\cdots\times Y_k\rightarrow HS(H,K)\right)\|h_U:HS(H,K)\rightarrow\mathcal{N}(H,Z)\|\\&\leqslant\pi_2^{mult}(T)\,\pi_2(U).\end{aligned}$$

Since all $A_1,\dots,A_k$ are 2-summing, from Theorem 1 we obtain that $X_1\times\cdots\times X_k\overset{h_U\circ\widehat{T}\circ(A_1,\dots,A_k)}{\rightarrow}\mathcal{N}(H,Z)$ is nuclear and

$$\begin{aligned}&\left\|h_U\circ\widehat{T}\circ(A_1,\dots,A_k):X_1\times\cdots\times X_k\rightarrow\mathcal{N}(H,Z)\right\|_{nuc}\\&\leqslant\pi_2^{mult}\left(h_U\circ\widehat{T}:Y_1\times\cdots\times Y_k\rightarrow\mathcal{N}(H,Z)\right)\pi_2(A_1)\cdots\pi_2(A_k).\end{aligned}$$

By equality (1) we get that $X_1\times\cdots\times X_k\overset{\widehat{S}}{\rightarrow}\mathcal{N}(H,Z)$ is nuclear. From Proposition 2 in [11] it follows that S is nuclear and $\|S\|_{nuc}\leqslant\pi_2(U)\,\pi_2^{mult}(T)\,\pi_2(A_1)\cdots\pi_2(A_k)$.

(ii). Since $U \circ T \circ (A_0, A_1, ..., A_k)$ has the factorization

$$X_0 \times X_1 \times \cdots \times X_k \overset{\left(A_0, I_{X_1}, ..., I_{X_k}\right)}{\longrightarrow} H \times X_1 \times \cdots \times X_k \overset{(I_H, A_1, ..., A_k)}{\longrightarrow} H \times Y_1 \times \cdots \times Y_k \overset{T}{\to} K \overset{U}{\to} Z$$

the statement follows from (i) and the ideal property of nuclear operators. □

3. The failure of some possible multilinear variants of Grothendieck's composition theorem

In the following if a is a sequence of scalars by M_a we denote the multiplication operator which is defined on a space of sequences or on a cartezian product of sequence spaces with values into another space of sequences. We write $M_a(x) = ax$ respectively $M_a(x_1, ..., x_n) = ax_1 \cdots x_n$. The following result is similar to that from [13, Proposition 2].

PROPOSITION 1. *Let $n \geqslant 2$ be a natural number. For $a \in l_\infty$ let $M_a : l_1 \times \underbrace{l_2 \times \cdots \times l_2}_{n-1\text{-times}} \to l_2$ be the multiplication operator defined by $M_a(x_1, ..., x_n) = ax_1 \cdots x_n$. Then M_a is nuclear if and only if $a \in l_1$.*

Proof. If M_a is nuclear, by definition, there exists $\left(\psi_k^1\right)_{k \in \mathbb{N}} \subset l_1^*$, $\left(\psi_k^j\right)_{k \in \mathbb{N}} \subset l_2^*$ $(2 \leqslant j \leqslant n)$, $(z_k)_{k \in \mathbb{N}} \subset l_2$ such that $\sum\limits_{k=1}^{\infty} \left\|\psi_k^1\right\| \cdots \left\|\psi_k^n\right\| \|z_k\| < \infty$ and $M_a(x_1, ..., x_n) = \sum\limits_{k=1}^{\infty} \psi_k^1(x_1) \cdots \psi_k^n(x_n) z_k$ for $(x_1, ..., x_n) \in l_1 \times l_2 \times \cdots \times l_2$. Since $n \geqslant 2$, by Riesz's theorem there exists $(y_k)_{k \in \mathbb{N}} \subset l_2$ such that for each $k \in \mathbb{N}$

$$\left\|\psi_k^2\right\| = \|y_k\| \text{ and } \psi_k^2(y) = \langle y, y_k \rangle \text{ for } y \in l_2.$$

Thus $M_a(x_1, ..., x_n) = \sum\limits_{k=1}^{\infty} \psi_k^1(x_1) \langle x_2, y_k \rangle \psi_k^3(x_3) \cdots \psi_k^n(x_n) z_k$ for $(x_1, ..., x_n) \in l_1 \times l_2 \times \cdots \times l_2$. Let $m \in \mathbb{N}$. Then

$$a_m = \langle M_a(e_m, ..., e_m), e_m \rangle = \sum_{k=1}^{\infty} \psi_k^1(e_m) \langle e_m, y_k \rangle \psi_k^3(e_m) \cdots \psi_k^n(e_m) \langle z_k, e_m \rangle$$

and

$$\begin{aligned} |a_m| &\leqslant \sum_{k=1}^{\infty} \left|\psi_k^1(e_m)\right| |\langle e_m, y_k \rangle| \left|\psi_k^3(e_m)\right| \cdots \left|\psi_k^n(e_m)\right| |\langle z_k, e_m \rangle| \\ &\leqslant \sum_{k=1}^{\infty} \left\|\psi_k^1\right\| |\langle e_m, y_k \rangle| \left\|\psi_k^3\right\| \cdots \left\|\psi_k^n\right\| |\langle z_k, e_m \rangle|. \end{aligned}$$

By Cauchy-Schwarz's and Bessel's inequalities

$$\begin{aligned}\sum_{m=1}^{\infty}|a_m| &\leqslant \sum_{k=1}^{\infty}\left\|\psi_k^1\right\|\left\|\psi_k^3\right\|\cdots\left\|\psi_k^n\right\|\left(\sum_{m=1}^{\infty}|\langle e_m,y_k\rangle||\langle z_k,e_m\rangle|\right)\\ &\leqslant \sum_{k=1}^{\infty}\left\|\psi_k^1\right\|\left\|\psi_k^3\right\|\cdots\left\|\psi_k^n\right\|\left(\sum_{m=1}^{\infty}|\langle e_m,y_k\rangle|^2\right)^{\frac{1}{2}}\left(\sum_{m=1}^{\infty}|\langle z_k,e_m\rangle|^2\right)^{\frac{1}{2}}\\ &\leqslant \sum_{k=1}^{\infty}\left\|\psi_k^1\right\|\|y_k\|\left\|\psi_k^3\right\|\cdots\left\|\psi_k^n\right\|\|z_k\|\\ &= \sum_{k=1}^{\infty}\left\|\psi_k^1\right\|\left\|\psi_k^2\right\|\cdots\left\|\psi_k^n\right\|\|z_k\|\end{aligned}$$

thus $a\in l_1$. If $a\in l_1$, then from the equality

$$M_a(x_1,\dots,x_n)=\sum_{k=1}^{\infty}a_k\langle x_1,e_k\rangle\cdots\langle x_n,e_k\rangle e_k \text{ for } (x_1,\dots,x_n)\in l_1\times l_2\times\cdots\times l_2$$

we get that M_a is nuclear. $\square$

Next we show that one of possible multilinear variants of Grothendieck's composition theorem is not necessarily true, compare Theorem 1 and Theorem 3.

PROPOSITION 2. *(i) Let $n\geqslant 2$ be a natural number, $a\in l_2$ and the diagram $l_1\times\underbrace{l_2\times\cdots\times l_2}_{n-1\text{-times}}\overset{(J,I_{l_2},\dots,I_{l_2})}{\longrightarrow} l_2\times\underbrace{l_2\times\cdots\times l_2}_{n-1\text{-times}}\overset{M_a}{\longrightarrow} l_2$. Then M_a is multiple 2-summing, $J:l_1\hookrightarrow l_2$ (the canonical inclusion) is 2-summing, but $M_a\circ\left(J,I_{l_2},\dots,I_{l_2}\right)$ is nuclear if and only if $a\in l_1$.*

(ii) Let $n\geqslant 2$ be a natural number, $a\in l_2$ and the diagram $l_1\times\underbrace{l_2\times\cdots\times l_2}_{n-1\text{-times}}\overset{P}{\longrightarrow} l_2\overset{M_a}{\longrightarrow} l_2$ where $P(x_1,\dots,x_n)=x_1\cdots x_n$. Then P is multiple 2-summing, M_a is 2-summing, but $M_a\circ P$ is nuclear if and only if $a\in l_1$.

Proof. (i) Since $a\in l_2$, M_a is Hilbert-Schmidt and hence M_a is multiple 2-summing. Now $M_a\circ\left(J,I_{l_2},\dots,I_{l_2}\right)=M_a:l_1\times\underbrace{l_2\times\cdots\times l_2}_{n-1\text{-times}}\to l_2$, which, by Proposition 1 is nuclear if and only if $a\in l_1$.

(ii) Since $\widehat{P}:l_1\to\Pi_2^{mult}(l_2,\dots,l_2;l_2)$ is 2-summing, we get that P is multiple 2-summing, (see [8, Proposition 2.5], [10, Proposition 2.5]). M_a is 2-summing since is Hilbert-Schmidt, $a\in l_2$. Also from the obvious equality $M_a\circ P=M_a:l_1\times\underbrace{l_2\times\cdots\times l_2}_{n-1\text{-times}}\to l_2$, by Proposition 1, $M_a\circ P$ is nuclear if and only if $a\in l_1$. $\square$

Suggested by Theorem 3 we states

QUESTION 1. *Let $n \geqslant 2$ be a natural number and $H_1,\dots,H_n,H,K$ Hilbert spaces and the diagram $H_1 \times \cdots \times H_n \xrightarrow{B} H \xrightarrow{A} K$. If B is Hilbert-Schmidt and A is Hilbert-Schmidt, then does it follow that $A \circ B$ is nuclear?*

and

QUESTION 2. *Let $n \geqslant 2$ be a natural number, k a natural number, $H_1,\dots,H_n,H$ Hilbert spaces, $X_{n+1},\dots,X_{n+k}$, $Y_{n+1},\dots,Y_{n+k}$ Banach spaces and the diagram*

$$H_1 \times \cdots \times H_n \times X_{n+1} \times \cdots \times X_{n+k} \overset{\left(I_{H_1},\dots,I_{H_n},A_{n+1},\dots,A_{n+k}\right)}{\longrightarrow}$$
$$H_1 \times \cdots \times H_n \times Y_{n+1} \times \cdots \times Y_{n+k} \xrightarrow{T} H \xrightarrow{U} Y$$

If all $A_{n+1},\dots,A_{n+k}$ are 2-summing, T is multiple 2-summing and U is 2-summing, then does it follow that the composition $U \circ T \circ (I_{H_1},\dots,I_{H_n},A_{n+1},\dots,A_{n+k})$ is nuclear and

$$\|U \circ T \circ (I_{H_1},\dots,I_{H_n},A_{n+1},\dots,A_{n+k})\|_{nuc}$$
$$\leqslant \pi_2(U)\,\pi_2^{mult}(T)\,\pi_2(A_{n+1})\cdots\pi_2(A_{n+k})?.$$

It is not hard to prove that if the answer to the Question 1 is **YES**, then (with the same technique as in the proof of Theorem 3) the answer to the Question 2 is also **YES**.

However, as we will prove in the sequel the answer to both Questions is **No**.

The fact that the answer to the Question 1 is **No** follows from the following example. It is very possible that such an example be well-known, but we were not able to find in the literature.

PROPOSITION 3. *Let $n \geqslant 2$ be a natural number, $(\lambda_i)_{i\in\mathbb{N}} \in l_2$ and x_0, $y_0 \in l_2$ with $\|x_0\| = \|y_0\| = 1$ and the diagram*

$$\underbrace{l_2 \times \cdots \times l_2}_{n\ times} \xrightarrow{B} l_2 \xrightarrow{A} l_2$$

where $B(x_1,\dots,x_n) = \sum_{i=1}^{\infty} \lambda_i \langle x_1,e_i\rangle \langle x_2,e_i\rangle \cdots \langle x_n,e_i\rangle x_0$ and $A(x) = \langle x,x_0\rangle y_0$. Then B is Hilbert-Schmidt and A is Hilbert-Schmidt, but $A \circ B$ is nuclear if and only if $(\lambda_i)_{i\in\mathbb{N}} \in l_1$.

Proof. We have $\sum_{i_1,\dots,i_n=1}^{\infty} \left\| B\left(e^1_{i_1},\dots,e^n_{i_n}\right)\right\|^2 = \sum_{i=1}^{\infty} |\lambda_i|^2 < \infty$, $(\lambda_i)_{i\in\mathbb{N}} \in l_2$ i.e. B is Hilbert-Schmidt. Also A is obvious Hilbert-Schmidt. Let us suppose that $A \circ B$ is nuclear. By definition and Riesz's theorem there exists $\left(x^j_i\right)_{i\in\mathbb{N}} \subset l_2$ $(1 \leqslant j \leqslant n)$, $(y_i)_{i\in\mathbb{N}} \subset l_2$ such that $\sum_{i=1}^{\infty} \left\|x^1_i\right\| \cdots \|x^n_i\| \, \|y_i\| < \infty$ and

$$(A \circ B)(x_1,\dots,x_n) = \sum_{i=1}^{\infty} \left\langle x_1,x^1_i\right\rangle \cdots \langle x_n,x^n_i\rangle y_i \text{ for } (x_1,\dots,x_n) \in l_2 \times \cdots \times l_2.$$

From the definition of A and B, $\|x_0\| = 1$ we get

$$\sum_{i=1}^{\infty} \lambda_i \langle x_1, e_i \rangle \langle x_2, e_i \rangle \cdots \langle x_n, e_i \rangle y_0 = \sum_{i=1}^{\infty} \langle x_1, x_i^1 \rangle \cdots \langle x_n, x_i^n \rangle y_i$$

and, by the continuity of the scalar product in the first variable

$$\sum_{i=1}^{\infty} \lambda_i \langle x_1, e_i \rangle \langle x_2, e_i \rangle \cdots \langle x_n, e_i \rangle \langle y_0, y_0 \rangle = \sum_{i=1}^{\infty} \langle x_1, x_i^1 \rangle \cdots \langle x_n, x_i^n \rangle \langle y_0, y_i \rangle$$

i.e. because $\|y_0\| = 1$

$$\begin{aligned} &\sum_{i=1}^{\infty} \lambda_i \langle x_1, e_i \rangle \langle x_2, e_i \rangle \cdots \langle x_n, e_i \rangle \\ &= \sum_{i=1}^{\infty} \langle x_1, x_i^1 \rangle \cdots \langle x_n, x_i^n \rangle \langle y_0, y_i \rangle \text{ for } (x_1, \ldots, x_n) \in l_2 \times \cdots \times l_2 \end{aligned} \qquad (*)$$

From this point the proof is similar to that of Proposition 1.

Let $m \in \mathbb{N}$. Taking in $(*)$ $x_1 = \cdots = x_n = e_m$ we get $\lambda_m = \sum_{i=1}^{\infty} \langle e_m, x_i^1 \rangle \cdots \langle e_m, x_i^n \rangle \langle y_0, y_i \rangle$. Since $n \geqslant 2$ we deduce

$$\begin{aligned} |\lambda_m| &\leqslant \sum_{i=1}^{\infty} \left|\langle e_m, x_i^1 \rangle\right| \cdots \left|\langle e_m, x_i^n \rangle\right| \left|\langle y_0, y_i \rangle\right| \\ &\leqslant \sum_{i=1}^{\infty} \left|\langle e_m, x_i^1 \rangle\right| \left|\langle e_m, x_i^2 \rangle\right| \cdot \left\|x_i^3\right\| \cdots \|x_i^n\| \, \|y_0\| \, \|y_i\| \end{aligned}$$

hence, by Cauchy-Schwartz's and Bessel's inequalities

$$\begin{aligned} \sum_{m=1}^{\infty} |\lambda_m| &\leqslant \sum_{i=1}^{\infty} \left\|x_i^3\right\| \cdots \|x_i^n\| \, \|y_i\| \left(\sum_{m=1}^{\infty} \left|\langle e_m, x_i^1 \rangle\right| \left|\langle e_m, x_i^2 \rangle\right| \right) \\ &\leqslant \sum_{i=1}^{\infty} \left\|x_i^3\right\| \cdots \|x_i^n\| \, \|y_i\| \left(\sum_{m=1}^{\infty} \left|\langle e_m, x_i^1 \rangle\right|^2 \right)^{\frac{1}{2}} \left(\sum_{m=1}^{\infty} \left|\langle e_m, x_i^2 \rangle\right|^2 \right)^{\frac{1}{2}} \\ &= \sum_{i=1}^{\infty} \left\|x_i^1\right\| \left\|x_i^2\right\| \left\|x_i^3\right\| \cdots \|x_i^n\| \, \|y_i\| < \infty \end{aligned}$$

thus $(\lambda_i)_{i \in \mathbb{N}} \in l_1$. If $(\lambda_i)_{i \in \mathbb{N}} \in l_1$, then from the equality

$$(A \circ B)(x_1, \ldots, x_n) = \sum_{i=1}^{\infty} \lambda_i \langle x_1, e_i \rangle \langle x_2, e_i \rangle \cdots \langle x_n, e_i \rangle y_0$$

it follows that $A \circ B$ is nuclear. $\square$

To prove that the answer to the Question 2 is **No** we need the following result.

PROPOSITION 4. *Let $n \geqslant 2$ be a natural number, k a natural number, $(\lambda_i)_{i\in\mathbb{N}} \in l_\infty$ and $T : \underbrace{l_2 \times \cdots \times l_2}_{n\text{-times}} \times \underbrace{l_1 \times \cdots \times l_1}_{k\text{-times}} \to \mathbb{K}$ defined by*

$$T(x_1,\ldots,x_n,x_{n+1},\ldots,x_{n+k}) = \sum_{i=1}^{\infty} \lambda_i \langle x_1, e_i\rangle \cdots \langle x_n, e_i\rangle \langle x_{n+1}, e_i\rangle \cdots \langle x_{n+k}, e_i\rangle .$$

Then T is nuclear if and only if $(\lambda_i)_{i\in\mathbb{N}} \in l_1$.

Proof. Let us suppose that T is nuclear. By definition and Riesz's theorem there exists $\left(x_i^j\right)_{i\in\mathbb{N}} \subset l_2$ $(1 \leqslant j \leqslant n)$, $\left(\psi_i^j\right)_{i\in\mathbb{N}} \subset l_1^*$ $(n+1 \leqslant j \leqslant n+k)$, $(\alpha_i)_{i\in\mathbb{N}} \subset \mathbb{K}$ such that $\sum_{i=1}^{\infty} \left\|x_i^1\right\| \cdots \|x_i^n\| \left\|\psi_i^{n+1}\right\| \cdots \left\|\psi_i^{n+k}\right\| |\alpha_i| < \infty$ and

$$\begin{aligned} &T(x_1,\ldots,x_n,x_{n+1},\ldots,x_{n+k}) \\ &= \sum_{i=1}^{\infty} \left\langle x_1, x_i^1\right\rangle \cdots \langle x_n, x_i^n\rangle\, \psi_i^{n+1}(x_{n+1}) \cdots \psi_i^{n+k}(x_{n+k})\, \alpha_i \end{aligned}$$

for $(x_1,\ldots,x_n,x_{n+1},\ldots,x_{n+k}) \in \underbrace{l_2 \times \cdots \times l_2}_{n\text{-times}} \times \underbrace{l_1 \times \cdots \times l_1}_{k\text{-times}}$ i.e.

$$\begin{aligned} &\sum_{i=1}^{\infty} \lambda_i \langle x_1, e_i\rangle \cdots \langle x_n, e_i\rangle \langle x_{n+1}, e_i\rangle \cdots \langle x_{n+k}, e_i\rangle \\ &= \sum_{i=1}^{\infty} \left\langle x_1, x_i^1\right\rangle \cdots \langle x_n, x_i^n\rangle\, \psi_i^{n+1}(x_{n+1}) \cdots \psi_i^{n+k}(x_{n+k})\, \alpha_i \end{aligned}$$

$$\text{for } (x_1,\ldots,x_n,x_{n+1},\ldots,x_{n+k}) \in \underbrace{l_2 \times \cdots \times l_2}_{n\text{-times}} \times \underbrace{l_1 \times \cdots \times l_1}_{k\text{-times}} \tag{$*$}$$

From this point the proof is similar to that of Proposition 1.

Let $m \in \mathbb{N}$. Taking in $(*)$ $x_1 = \cdots = x_{n+k} = e_m$ we get

$$\lambda_m = \sum_{i=1}^{\infty} \left\langle e_m, x_i^1\right\rangle \cdots \langle e_m, x_i^n\rangle\, \psi_i^{n+1}(e_m) \cdots \psi_i^{n+k}(e_m)\, \alpha_i .$$

Since $n \geqslant 2$ we deduce

$$\begin{aligned} |\lambda_m| &\leqslant \sum_{i=1}^{\infty} \left|\left\langle e_m, x_i^1\right\rangle\right| \cdots \left|\langle e_m, x_i^n\rangle\right| \left|\psi_i^{n+1}(e_m)\right| \cdots \left|\psi_i^{n+k}(e_m)\right| |\alpha_i| \\ &\leqslant \sum_{i=1}^{\infty} \left|\left\langle e_m, x_i^1\right\rangle\right| \left|\left\langle e_m, x_i^2\right\rangle\right| \left\|x_i^3\right\| \cdots \|x_i^n\| \left\|\psi_i^{n+1}\right\| \cdots \left\|\psi_i^{n+k}\right\| |\alpha_i| . \end{aligned}$$

(If $n=2$, above and in the sequel the terms $\left\|x_i^3\right\|\cdots\|x_i^n\|$ do not appear). By Cauchy-Schwarz's and Bessel's inequalities we deduce

$$\begin{aligned}
\sum_{m=1}^{\infty}|\lambda_m| &\leqslant \sum_{i=1}^{\infty}\left\|x_i^3\right\|\cdots\|x_i^n\|\left\|\psi_i^{n+1}\right\|\cdots\left\|\psi_i^{n+k}\right\||\alpha_i|\left(\sum_{m=1}^{\infty}\left|\left\langle e_m,x_i^1\right\rangle\right|\left|\left\langle e_m,x_i^2\right\rangle\right|\right)\\
&\leqslant \sum_{i=1}^{\infty}\left\|x_i^3\right\|\cdots\|x_i^n\|\left\|\psi_i^{n+1}\right\|\cdots\left\|\psi_i^{n+k}\right\|\\
&\quad\times|\alpha_i|\left(\sum_{m=1}^{\infty}\left|\left\langle e_m,x_i^1\right\rangle\right|^2\right)^{\frac{1}{2}}\left(\sum_{m=1}^{\infty}\left|\left\langle e_m,x_i^2\right\rangle\right|^2\right)^{\frac{1}{2}}\\
&= \sum_{i=1}^{\infty}\left\|x_i^1\right\|\left\|x_i^2\right\|\left\|x_i^3\right\|\cdots\|x_i^n\|\left\|\psi_i^{n+1}\right\|\cdots\left\|\psi_i^{n+k}\right\||\alpha_i|<\infty
\end{aligned}$$

thus $(\lambda_i)_{i\in\mathbb{N}}\in l_1$. If $(\lambda_i)_{i\in\mathbb{N}}\in l_1$, then obvious T is nuclear. □

The fact that the answer to the Question 2 is **No** follows from the following example.

PROPOSITION 5. *Let $n\geqslant 2$ be a natural number, k a natural number, $(\lambda_i)_{i\in\mathbb{N}}\in l_2$ and x_0, $y_0\in l_2$ with $\|x_0\|=\|y_0\|=1$ and the diagram*

$$\underbrace{l_2\times\cdots\times l_2}_{n\text{-times}}\times\underbrace{l_1\times\cdots\times l_1}_{k\text{-times}}\overset{\left(I_{l_2},\ldots,I_{l_2},J,\ldots,J\right)}{\longrightarrow}\underbrace{l_2\times\cdots\times l_2}_{n\text{-times}}\times\underbrace{l_2\times\cdots\times l_2}_{k\text{-times}}$$
$$\overset{B}{\to}l_2\overset{A}{\to}l_2$$

where $J:l_1\hookrightarrow l_2$ is the canonical inclusion,

$$B(x_1,\ldots,x_n,x_{n+1},\ldots,x_{n+k})=\sum_{i=1}^{\infty}\lambda_i\langle x_1,e_i\rangle\cdots\langle x_n,e_i\rangle\langle x_{n+1},e_i\rangle\cdots\langle x_{n+k},e_i\rangle x_0$$

and $A(x)=\langle x,x_0\rangle y_0$. Then: J is 2-summing, B is multiple 2-summing, A is 2-summing, but $A\circ B\circ\left(I_{l_2},\ldots,I_{l_2},J,\ldots,J\right)$ is nuclear if and only if $(\lambda_i)_{i\in\mathbb{N}}\in l_1$.

Proof. Since $(\lambda_i)_{i\in\mathbb{N}}\in l_2$ it follows that B is Hilbert-Schmidt and hence B is multiple 2-summing. Let us note that since $\|x_0\|=1$ we have

$$\begin{aligned}
&\left[A\circ B\circ\left(I_{l_2},\ldots,I_{l_2},J,\ldots,J\right)\right](x_1,\ldots,x_n,x_{n+1},\ldots,x_{n+k})\\
&=\sum_{i=1}^{\infty}\lambda_i\langle x_1,e_i\rangle\cdots\langle x_n,e_i\rangle\langle x_{n+1},e_i\rangle\cdots\langle x_{n+k},e_i\rangle A(x_0)\\
&=\sum_{i=1}^{\infty}\lambda_i\langle x_1,e_i\rangle\cdots\langle x_n,e_i\rangle\langle x_{n+1},e_i\rangle\cdots\langle x_{n+k},e_i\rangle y_0
\end{aligned}$$

Let us suppose that $A\circ B\circ\left(I_{l_2},\dots,I_{l_2},J,\dots,J\right)$ is nuclear. By definition and Riesz's theorem there exists $\left(x_i^j\right)_{i\in\mathbb{N}}\subset l_2$ $(1\leqslant j\leqslant n)$, $\left(\psi_i^j\right)_{i\in\mathbb{N}}\subset l_1^*$ $(n+1\leqslant j\leqslant n+k)$, $(y_i)_{i\in\mathbb{N}}\subset l_2$ such that $\sum\limits_{i=1}^{\infty}\left\|x_i^1\right\|\cdots\left\|x_i^n\right\|\left\|\psi_i^{n+1}\right\|\cdots\left\|\psi_i^{n+k}\right\|\|y_i\|<\infty$ and

$$\begin{aligned}&\left[A\circ B\circ\left(I_{l_2},\dots,I_{l_2},J,\dots,J\right)\right](x_1,\dots,x_n,x_{n+1},\dots,x_{n+k})\\&=\sum_{i=1}^{\infty}\left\langle x_1,x_i^1\right\rangle\cdots\left\langle x_n,x_i^n\right\rangle\psi_i^{n+1}(x_{n+1})\cdots\psi_i^{n+k}(x_{n+k})\,y_i\end{aligned}$$

$$\text{for }(x_1,\dots,x_n,x_{n+1},\dots,x_{n+k})\in\underbrace{l_2\times\cdots\times l_2}_{n\text{-times}}\times\underbrace{l_1\times\cdots\times l_1}_{k\text{-times}}$$

i.e.

$$\begin{aligned}&\sum_{i=1}^{\infty}\lambda_i\langle x_1,e_i\rangle\cdots\langle x_n,e_i\rangle\langle x_{n+1},e_i\rangle\cdots\langle x_{n+k},e_i\rangle\,y_0\\&=\sum_{i=1}^{\infty}\left\langle x_1,x_i^1\right\rangle\cdots\left\langle x_n,x_i^n\right\rangle\psi_i^{n+1}(x_{n+1})\cdots\psi_i^{n+k}(x_{n+k})\,y_i\end{aligned}$$

$$\text{for }(x_1,\dots,x_n,x_{n+1},\dots,x_{n+k})\in\underbrace{l_2\times\cdots\times l_2}_{n\text{-times}}\times\underbrace{l_1\times\cdots\times l_1}_{k\text{-times}}$$

and by the continuity of the scalar product in the first variable and $\|y_0\|=1$ we get

$$\begin{aligned}&\sum_{i=1}^{\infty}\lambda_i\langle x_1,e_i\rangle\cdots\langle x_n,e_i\rangle\langle x_{n+1},e_i\rangle\cdots\langle x_{n+k},e_i\rangle\\&=\sum_{i=1}^{\infty}\left\langle x_1,x_i^1\right\rangle\cdots\left\langle x_n,x_i^n\right\rangle\psi_i^{n+1}(x_{n+1})\cdots\psi_i^{n+k}(x_{n+k})\langle y_i,y_0\rangle\end{aligned}$$

$$\text{for }(x_1,\dots,x_n,x_{n+1},\dots,x_{n+k})\in\underbrace{l_2\times\cdots\times l_2}_{n\text{-times}}\times\underbrace{l_1\times\cdots\times l_1}_{k\text{-times}}.$$

Since

$$\begin{aligned}&\sum_{i=1}^{\infty}\left\|x_i^1\right\|\cdots\left\|x_i^n\right\|\left\|\psi_i^{n+1}\right\|\cdots\left\|\psi_i^{n+k}\right\||\langle y_i,y_0\rangle|\\&\leqslant\sum_{i=1}^{\infty}\left\|x_i^1\right\|\cdots\left\|x_i^n\right\|\left\|\psi_i^{n+1}\right\|\cdots\left\|\psi_i^{n+k}\right\|\|y_i\|<\infty\end{aligned}$$

we deduce that the operator $T:\underbrace{l_2\times\cdots\times l_2}_{n\text{-times}}\times\underbrace{l_1\times\cdots\times l_1}_{k\text{-times}}\to\mathbb{K}$ defined by

$$T(x_1,\dots,x_n,x_{n+1},\dots,x_{n+k})=\sum_{i=1}^{\infty}\lambda_i\langle x_1,e_i\rangle\cdots\langle x_n,e_i\rangle\langle x_{n+1},e_i\rangle\cdots\langle x_{n+k},e_i\rangle.$$

is nuclear. By Proposition 4 it follows that $(\lambda_i)_{i\in\mathbb{N}}\in l_1$. If $(\lambda_i)_{i\in\mathbb{N}}\in l_1$, then obvious $A\circ B\circ\left(I_{l_2},\dots,I_{l_2},J,\dots,J\right)$ is nuclear. $\square$

4. A second multilinear variant of Grothendieck's composition theorem

The following notion was suggested by the result which will be proved in Theorem 4 below.

DEFINITION 1. Let H_1, ..., H_k, H be Hilbert spaces. A bounded k-linear operator $T : H_1 \times \cdots \times H_k \to H$ is called *nuclear of Hilbert-Schmidt type* if there exists $(\psi_n)_{n\in\mathbb{N}} \subset HS(H_1,...,H_k;\mathbb{K})$, $(y_n)_{n\in\mathbb{N}} \subset H$ such that $\sum_{n=1}^{\infty} \|\psi_n\|_{HS(H_1,...,H_k;\mathbb{K})} \|y_n\| < \infty$ and $T(x_1,...,x_k) = \sum_{n=1}^{\infty} \psi_n(x_1,...,x_k) y_n$ for $(x_1,...,x_k) \in H_1 \times \cdots \times H_k$. *Such a representation will be called a nuclear Hilbert-Schmidt type representation of* T. In this case $\|T\|_{nuc-HS} = \inf \left\{ \sum_{n=1}^{\infty} \|\psi_n\|_{HS(H_1,...,H_k;\mathbb{K})} \|y_n\| \right\}$, where infimum is taken over all *nuclear Hilbert-Schmidt type representation of* T.

In the sequel we prove a second multilinear variant of Grothendieck's composition theorem, which compared to Proposition 3, seems to be the best possible general statement.

THEOREM 4. *Let* H_1, ..., H_n, H, K *be Hilbert spaces and the diagram* $H_1 \times \cdots \times H_n \xrightarrow{B} H \xrightarrow{A} K$. *If* B *is Hilbert-Schmidt and* A *is Hilbert-Schmidt, then* $A \circ B$ *is nuclear of Hilbert-Schmidt type and* $\|A \circ B\|_{nuc-HS} \leqslant \|A\|_{HS} \|B\|_{HS}$.

Proof. Let us denote by $\left(e_{i_j}^j\right)_{i_j \in I_j}$ an orthonormal basis in H_j $(1 \leqslant j \leqslant k)$ and by $(e_j)_{j\in J}$ an orthonormal basis in H. Let $(x_1,...,x_n) \in H_1 \times \cdots \times H_n$. Since B is n-linear and bounded

$$B(x_1,...,x_n) = \sum_{i_1 \in I_1} \left(\langle x_1, e_{i_1}^1 \rangle \cdots \left(\sum_{i_n \in I_n} \langle x_n, e_{i_n}^n \rangle B\left(e_{i_1}^1,...,e_{i_n}^n\right) \right) \cdots \right).$$

From Cauchy-Schwarz's inequality and the fact that B is Hilbert-Schmidt we have

$$\begin{aligned} &\sum_{(i_1,...,i_n)\in I_1\times\cdots\times I_n} \left|\langle x_1, e_{i_1}^1\rangle\right| \cdots \left|\langle x_n, e_{i_n}^n\rangle\right| \left\|B\left(e_{i_1}^1,...,e_{i_n}^n\right)\right\| \\ &\leqslant \left(\sum_{(i_1,...,i_n)\in I_1\times\cdots\times I_n} \left\|B\left(e_{i_1}^1,...,e_{i_n}^n\right)\right\|^2 \right)^{\frac{1}{2}} \\ &\quad \times \left(\sum_{(i_1,...,i_n)\in I_1\times\cdots\times I_n} \left|\langle x_1, e_{i_1}^1\rangle\right|^2 \cdots \left|\langle x_n, e_{i_n}^n\rangle\right|^2 \right)^{\frac{1}{2}} \\ &= \|B\|_{HS} \|x_1\| \cdots \|x_n\|. \end{aligned}$$

From here, by a Fubini's type result

$$\begin{aligned}&\sum_{i_1\in I_1}\left(\left\langle x_1,e^1_{i_1}\right\rangle\cdots\left(\sum_{i_n\in I_n}\left\langle x_n,e^n_{i_n}\right\rangle B\left(e^1_{i_1},\ldots,e^n_{i_n}\right)\right)\right)\\&=\sum_{(i_1,\ldots,i_n)\in I_1\times\cdots\times I_n}\left\langle x_1,e^1_{i_1}\right\rangle\cdots\left\langle x_n,e^n_{i_n}\right\rangle B\left(e^1_{i_1},\ldots,e^n_{i_n}\right)\end{aligned}$$

i.e.

$$B(x_1,\ldots,x_n)=\sum_{(i_1,\ldots,i_n)\in I_1\times\cdots\times I_n}\left\langle x_1,e^1_{i_1}\right\rangle\cdots\left\langle x_n,e^n_{i_n}\right\rangle B\left(e^1_{i_1},\ldots,e^n_{i_n}\right). \tag{1}$$

Also $B(x_1,\ldots,x_n)=\sum_{j\in J}\left\langle B(x_1,\ldots,x_n),e_j\right\rangle e_j$ from where

$$A(B(x_1,\ldots,x_n))=\sum_{j\in J}\left\langle B(x_1,\ldots,x_n),e_j\right\rangle A(e_j). \tag{2}$$

Let us define $\lambda_j: H_1\times\cdots\times H_n\to\mathbb{K}$ by $\lambda_j(x_1,\ldots,x_n)=\left\langle B(x_1,\ldots,x_n),e_j\right\rangle$. From the separate continuity of the scalar product and (1) we get

$$\begin{aligned}\lambda_j(x_1,\ldots,x_n)&=\left\langle B(x_1,\ldots,x_n),e_j\right\rangle\\&=\sum_{(i_1,\ldots,i_n)\in I_1\times\cdots\times I_n}\left\langle x_1,e^1_{i_1}\right\rangle\cdots\left\langle x_n,e^n_{i_n}\right\rangle\left\langle B\left(e^1_{i_1},\ldots,e^n_{i_n}\right),e_j\right\rangle.\end{aligned}$$

Then

$$\begin{aligned}\left\|\lambda_j\right\|^2_{HS(H_1,\ldots,H_n;\mathbb{K})}&=\sum_{(i_1,\ldots,i_n)\in I_1\times\cdots\times I_n}\left|\lambda_j\left(e^1_{i_1},\ldots,e^n_{i_n}\right)\right|^2\\&=\sum_{(i_1,\ldots,i_n)\in I_1\times\cdots\times I_n}\left|\left\langle B\left(e^1_{i_1},\ldots,e^n_{i_n}\right),e_j\right\rangle\right|^2\end{aligned}$$

and from here

$$\begin{aligned}\sum_{j\in J}\left\|\lambda_j\right\|^2_{HS(H_1,\ldots,H_n;\mathbb{K})}&=\sum_{(i_1,\ldots,i_n)\in I_1\times\cdots\times I_n}\left(\sum_{j\in J}\left|\left\langle B\left(e^1_{i_1},\ldots,e^n_{i_n}\right),e_j\right\rangle\right|^2\right)\\&=\sum_{(i_1,\ldots,i_n)\in I_1\times\cdots\times I_n}\left\|B\left(e^1_{i_1},\ldots,e^n_{i_n}\right)\right\|^2=\|B\|^2_{HS}.\end{aligned}$$

Again from Cauchy-Schwarz's inequality we have

$$\begin{aligned}\sum_{j\in J}\left\|\lambda_j\right\|_{HS(H_1,\ldots,H_n;\mathbb{K})}\left\|A(e_j)\right\|&\leqslant\left(\sum_{j\in J}\left\|\lambda_j\right\|^2_{HS(H_1,\ldots,H_2;\mathbb{K})}\right)^{\frac{1}{2}}\left(\sum_{j\in J}\left\|A(e_j)\right\|^2\right)^{\frac{1}{2}}\\&=\|B\|_{HS}\|A\|_{HS}.\end{aligned}$$

Let us denote $J_0=\left\{j\in J\mid\left\|\lambda_j\right\|_{HS(H_1,\ldots,H_n;\mathbb{K})}\left\|A(e_j)\right\|\neq 0\right\}$ and note that, as is well known, J_0 is countable. Then from (2) we deduce

$$A(B(x_1,\ldots,x_n))=\sum_{j\in J_0}\lambda_j(x_1,\ldots,x_n)A(e_j).$$

By Definition 1, $A \circ B$ is nuclear of Hilbert-Schmidt type and $\|A \circ B\|_{nuc-HS} \leqslant \|A\|_{HS} \|B\|_{HS}$. $\square$

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Dumitru Popa
Department of Mathematics
Ovidius University of Constanta
Bd. Mamaia 124, 900527 Constanta, Romania
e-mail: dpopa@univ-ovidius.ro