Operators
and
Matrices
Volume 10, Number 1 (2016), 163–188
doi:10.7153/oam-10-11

# AUTOMORPHISMS OF SOME STRUCTURAL INFINITE MATRIX RINGS

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(*Communicated by H. Radjavi*)

*Abstract.* We define an analog of a structural matrix ring in the ring of column-finite infinite matrices. We describe the form of its automorphisms.

## 1. Introduction and statement of results

Let $\mathscr{M}_{Cf}(F)$ denote the set of all $\mathbb{N}\times\mathbb{N}$ matrices over a field $F$ such that every column of each matrix has only a finite number of nonzero entries. These matrices are called column-finite and the mentioned set forms an algebra. One of its subalgebras is $\mathscr{T}_\infty(F)$ which consists of all infinite upper triangular matrices.

Let $\lesssim$ be a preorder on $\mathbb{N}$, i.e. a reflexive and transitive relation. Denoting the entry in position $(n,m)$ of a matrix $x$ in $\mathscr{M}_{Cf}(F)$ by $x_{nm}$, we define $\mathscr{M}_{Cf}(\lesssim,F)$ and $\mathscr{T}_\infty(\lesssim,F)$ as follows:

$$\mathscr{M}_{Cf}(\lesssim,F) := \left\{x\in\mathscr{M}_{Cf}(F):\ \text{if } (n,m)\notin\lesssim,\text{ then } x_{nm}=0\right\},$$

$$\mathscr{T}_\infty(\lesssim,F) := \{x\in\mathscr{T}_\infty(F):\ \text{if } (n,m)\notin\lesssim,\text{ then } x_{nm}=0\}.$$

One can check that these are rings. We will call them structural infinite matrix rings. Clearly $\mathscr{T}_\infty(F)=\mathscr{M}_{Cf}(\lesssim,F)$, where $(n,m)\in\lesssim$ if and only if $n\leqslant m$.

Obviously, since $\lesssim$ is reflexive, $\mathscr{M}_{Cf}(\lesssim,F)$ always contains $\mathscr{D}_\infty(F)$ – the ring of all infinite diagonal matrices.

Note that the sets $\mathscr{M}_{Cf}(\lesssim,F)$ and $\mathscr{T}_\infty(\lesssim,F)$ are defined in the same manner as

$$\mathscr{M}_k(\lesssim,F)=\{x\in\mathscr{M}_k(F):\ \text{if } (n,m)\notin\lesssim,\text{ then } x_{nm}=0\},$$

where $\mathscr{M}_k(F)$ is the ring of all $k\times k$ matrices over $F$. $\mathscr{M}_k(\lesssim,F)$ is called a structural matrix ring and first appeared in [15]. Automorphisms of such rings were investigated in quite a few papers, like [1, 3, 4].

In this article we will investigate automorphisms of some structural infinite matrix rings. Before we formulate our results we introduce two relations.

If $\lesssim$ is a given preorder, then by $\lesssim_{sym}$ we will understand the relation

$$(n,m)\in\lesssim_{sym}\quad\Leftrightarrow\quad (n,m),(m,n)\in\lesssim.$$

*Mathematics subject classification* (2010): 08A35, 15A30, 16S50.
*Keywords and phrases*: Automorphisms of matrices, structural matrix rings, infinite matrices.

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One can see that $\lesssim_{sym}$ is an equivalence relation. (See, for example, [15, p. 402] and [16, p. 423], where the relation $\sim_B$ is used for a reflexive and transitive Boolean matrix $B$.) Therefore we can divide $\mathbb{N}$ into the classes $C_n$ such that $\cup_n C_n = \mathbb{N}$ and $n,m \in C_k$ if and only if $(n,m) \in \lesssim_{sym}$. We may even assume that $\mathscr{M}_{Cf}(\lesssim, F)$ is block upper triangular (see, for example, [5, p. 1386] and [14, p. 5604]), and that if $n < m$, then the smallest element in $C_n$ is less than the smallest element in $C_m$. In fact, these assumptions imply that if $n < m$, then every element of $C_n$ is smaller than every element of $C_m$.

Notice that for any $\mathscr{T}_\infty(\lesssim, F)$, the relation $\lesssim_{sym}$ has only one-element classes and does not tell us much about our ring. Therefore we will also define another relation on $\mathbb{N}$ by the following rule.

Let $n_1, n_2, \ldots, n_k$ be distinct natural numbers. If

$$\forall 1 \leqslant i,j \leqslant k \quad (n_i,n_j) \in \lesssim \ \vee\ (n_j,n_i) \in \lesssim, \tag{1}$$

and

$$\forall 1 \leqslant i \leqslant k\ \forall m \neq n_1,\ldots,n_k \quad (n_i,m) \notin \lesssim \ \wedge\ (m,n_i) \notin \lesssim, \tag{2}$$

then we put $B_n = \{n_1, n_2, \ldots, n_k\}$, where $n$ is the ordinal number of the class. Again, if $n < m$, then the smallest element in $B_n$ is less than the smallest element in $B_m$. If conditions (1) and (2) hold for some infinite set $\{n_1, n_2, \ldots\}$, then we also denote it by $B_n$. (Note that the above relation is similar to $\equiv_\rho$ from [17, p. 3679].)

For example, if

$$\lesssim = \{(1,1),(2,2),(2,3),(3,3),(4,4)\} \cup \{(n,m) :\ 5 \leqslant n \leqslant m\},$$

then we can identify it with a symbolic matrix

$$\begin{pmatrix}
* & & & & & & & \\
 & * & * & & & & & \\
 & & * & & & & & \\
 & & & * & & & & \\
 & & & & * & * & * & \cdots \\
 & & & & & * & * & \\
 & & & & & & * & \\
 & & & & & & & \ddots
\end{pmatrix}.$$

For this $\lesssim$ we have $B_1 = \{1\}$, $B_2 = \{2,3\}$, $B_3 = \{4\}$, $B_4 = \{n \in \mathbb{N} :\ n \geqslant 5\}$ (and we do not have any other classes). We can also see that $C_n = \{n\}$ for all $n \in \mathbb{N}$.

If we consider another example, more precisely

$$\begin{pmatrix}
* & * & * & & & & \\
 & * & * & & & & \\
 & * & * & & & & \\
 & & & * & * & * & \cdots \\
 & & & & * & * & \cdots \\
 & & & & & * & \\
 & & & & & & \ddots
\end{pmatrix},$$

then we have $C_1 = \{1\}$, $C_2 = \{2,3\}$, $C_n = \{n+1\}$ for $n \geqslant 3$, and $B_1 = \{1,2,3\}$, $B_2 = \{4,5,6,\dots\}$.

With every $B_n$ and $C_m$ we can identify the subrings which will be denoted by $S(B_n)$ and $S(C_m)$ respectively, and are defined as follows:

$$S(B_n) := \left\{x \in \mathscr{M}_{Cf}(\lesssim, F) : x_{pr} = 0 \text{ if } p \notin B_n \text{ or } r \notin B_n\right\},$$

$$S(C_m) := \left\{x \in \mathscr{M}_{Cf}(\lesssim, F) : x_{pr} = 0 \text{ if } p \notin C_m \text{ or } r \notin C_m\right\}.$$

It can be noticed that $\mathscr{M}_{Cf}(\lesssim, F)$ is some sort of direct sum of $S(B_n)$'s. (This is analogous to the sum defined in [17, p. 3679] for the finite case.)

For the sake of convenience we will say that $\mathscr{M}_{Cf}(\lesssim, F)$ is a generalized direct sum of $S(B_n)$ and write $\mathscr{M}_{Cf}(\lesssim, F) = \overline{\oplus_{n\in N}} S(B_n)$. Note that from the definition of the classes $B_n$ it follows that $S(B_n)S(B_m) = \{0\}$ for $n \neq m$.

In our investigation some standard maps will appear. We introduce them here.

- If $g \in \mathscr{M}_{Cf}(\lesssim, F)$ is invertible, then we can define the map $\mathscr{I}nn_g$ by the rule $\mathscr{I}nn_g(x) = g^{-1}xg$. It is simply an inner automorphism of the ring $\mathscr{M}_{Cf}(\lesssim, F)$.
- For any automorphism $\sigma$ of the field $F$ we can define an automorphism of $\mathscr{M}_{Cf}(\lesssim, F)$ as follows:
$$(\overline{\sigma}(x))_{nm} = \sigma(x_{nm}).$$
The map $\overline{\sigma}$ is called an induced automorphism.

  Note that in [2] it was proved that every automorphism of $\mathscr{T}_n(R)$ – the ring of all $n \times n$ upper triangular matrices over a ring $R$ is a composition of $\mathscr{I}nn_g$ (for some invertible $g \in \mathscr{T}_n(R)$) and $\overline{\sigma}$ (for some automorphism $\sigma$ of the ring $R$).

  Now we would like to generalize $\overline{\sigma}$ somewhat. Suppose that $\mathscr{M}_{Cf}(\lesssim, F)$ is a generalized direct sum of some subrings $S(B_n) = \mathscr{M}_{Cf}(\lesssim_n, F)$. Clearly $\lesssim_n \cap \lesssim_m = \emptyset$ for $n \neq m$. Then having a family of automorphisms $(\sigma_n)_{n\in N}$ of $F$ we can define the map $\overline{(\sigma_n)_{n\in N}}$ as follows:
$$(\overline{(\sigma_n)_{n\in N}}(x))_{ij} = \begin{cases} \sigma_n(x_{ij}) & \text{if } (i,j) \in \lesssim_n \\ 0 & \text{otherwise.} \end{cases}$$
We will call it a generalized induced automorphism.

  For instance for $\mathbb{C}$ we have two automorphisms: $\sigma_1$ – the identity, and $\sigma_2$ – the complex conjugation. If
$$\lesssim = \{(1,1),(1,2),(2,1),(2,2)\} \cup \{(n,m) : 3 \leqslant n \leqslant m\},$$
then we can define $\overline{\sigma_2\sigma_1}$ as follows:
$$\overline{\sigma_2\sigma_1}\left(\begin{pmatrix} x_{11} & x_{12} & 0 & 0 & 0 & \cdots \\ x_{21} & x_{22} & 0 & 0 & 0 & \\ & & x_{33} & x_{34} & x_{35} & \\ & & & x_{44} & x_{45} & \\ & & & & x_{55} & \\ & & & & & \ddots \end{pmatrix}\right) = \begin{pmatrix} \overline{x_{11}} & \overline{x_{12}} & 0 & 0 & 0 & \cdots \\ \overline{x_{21}} & \overline{x_{22}} & 0 & 0 & 0 & \\ & & x_{33} & x_{34} & x_{35} & \\ & & & x_{44} & x_{45} & \\ & & & & x_{55} & \\ & & & & & \ddots \end{pmatrix}.$$

- If $\pi$ is a permutation of $\mathbb{N}$, then by $\hat{\pi}$ we will understand a map such that

$$(\hat{\pi}(x))_{\pi(n)\pi(m)} = x_{nm}.$$

- For the classes $\{B_n\}_{n\in N}$ and permutations $\pi$ of $N$ we define the maps $\mathscr{B}_\pi$ as follows: suppose that for some pairs of classes $B_n = \{n_1, n_2, \ldots\}$, $B_m = \{m_1, m_2, \ldots\}$ with $n_1 < n_2 < \ldots$ and $m_1 < m_2 < \ldots$, there exists a permutation $\pi$ such that $\pi(n_i) = m_i$; in this case $\mathscr{B}_\pi$ is defined by the rule:

$$(\mathscr{B}_\pi(x))_{m_i m_j} = x_{n_i n_j}.$$

For instance, if $\mathscr{M}_{Cf}(\lesssim, F)$ is given by

$\lesssim = \{(1,1),(1,2),(2,1),(2,2),(3,3),(3,4),(4,3),(4,4)\} \cup \{(n,m) :\ 5 \leqslant n \leqslant m\}$

and $\pi = (1\,2)$, then

$$\mathscr{B}_\pi\left(\begin{pmatrix} x_{11} & x_{12} & & & & & \\ x_{21} & x_{22} & & & & & \\ & & x_{33} & x_{34} & & & \\ & & x_{43} & x_{44} & & & \\ & & & & x_{55} & x_{56} & \cdots \\ & & & & & x_{66} & \\ & & & & & & \cdots \end{pmatrix}\right) = \begin{pmatrix} x_{33} & x_{34} & & & & & \\ x_{43} & x_{44} & & & & & \\ & & x_{11} & x_{12} & & & \\ & & x_{21} & x_{22} & & & \\ & & & & x_{55} & x_{56} & \cdots \\ & & & & & x_{66} & \\ & & & & & & \cdots \end{pmatrix}.$$

- We present here one more type of standard map that is defined only on triangular matrices.

  First we focus on $k \times k$ such matrices. Define $\mathscr{J}$ as the map on the ring $\mathscr{T}_k(F)$ such that

$$(\mathscr{J}(x))_{nm} = x_{k+1-m,k+1-n}.$$

  It is a standard isomorphism of $\mathscr{T}_k(F)$.

  Let now $\mathscr{S}_1 = S(B_n)$, $\mathscr{S}_2 = S(B_m)$ be isomorphic to the same subring $\mathscr{S}$ of either $\mathscr{T}_\infty(F)$ or $\mathscr{T}_k(F)$ for some $k \in \mathbb{N}$. If $\phi$ is an isomorphism from $\mathscr{S}_1$ onto $\mathscr{S}$ and $\psi$ is an isomorphism from $\mathscr{S}$ onto $\mathscr{S}_2$, then $\psi \cdot \mathscr{J} \cdot \phi$ is an isomorphism of $\mathscr{S}_1$ and $\mathscr{S}_2$. We will denote this map by $\mathscr{J}$ as well. If, like in the class of maps $\overline{(\sigma_n)_{n\in N}}$, our ring is a generalized direct sum of some $S(B_n)$'s and we would like to apply $\mathscr{J}$ to some of them, then we will denote such map by $\overline{(\chi_n)_{n\in N}}$, where $\chi_n$ is applied to $S(B_n)$ and can be equal to either $\mathscr{J}$ or the identity map.

Now we present our first result.

THEOREM 1. *Let $F$ be a field and let $\lesssim$ be a preorder on $\mathbb{N}$. The map $\phi$ is an automorphism of $\mathscr{T}_\infty(\lesssim, F) = \overline{\oplus_{n\in N}} S(B_n)$ if and only if*

$$\phi = \mathscr{I}nn_t \cdot \overline{(\chi_n)_{n\in N}} \cdot \overline{(\sigma_n)_{n\in N}} \cdot \mathscr{B}_\pi,$$

*where $t$ is an invertible upper triangular matrix in $\mathscr{T}_\infty(\lesssim, F)$, $\pi \in S(N)$ is such that $S(B_n) \sim S(B_{\pi(n)})$ for all $n \in N$, the map $\chi_n$ is either $\mathscr{J}$ or the identity map for all $n \in N$, and $(\sigma_n)_{n\in N}$ is a family of automorphisms of $F$.*

Our next results concern some subrings of $\mathscr{M}_{Cf}(F)$ that may contain $\mathscr{T}_\infty(F)$. The first one consists of all matrices of the form

$$\left(\begin{array}{c|c} g_1 & g_2 \\ \hline 0 & g_3 \end{array}\right) \qquad \text{with } g_1 \in \mathscr{M}_k(F),\ g_3 \in \mathscr{T}_\infty(F), \tag{3}$$

where $k$ can be arbitrary and $\mathscr{M}_k(F)$ denotes the ring of all $k \times k$ matrices over $F$. The group of units of this ring was introduced in [18] and is called the Vershik-Kerov group. Therefore, we will denote the ring of matrices of form (3) by $\mathscr{M}_{VK}(F)$.

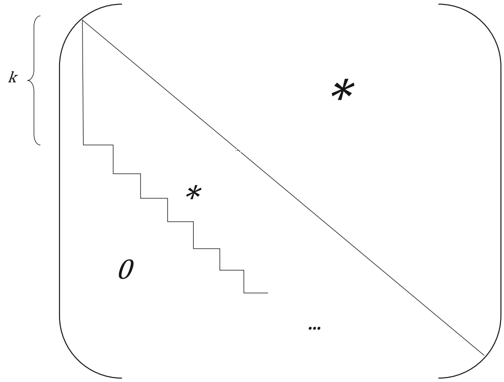
We define one more subring of $\mathscr{M}_{Cf}(F)$ containing $\mathscr{T}_\infty(F)$.

Consider the matrices $x$ for which the number

$$\sup_{x_{nm} \neq 0} (n-m) \tag{4}$$

is finite and positive. These are the matrices of the shape depicted in Figure 1. It can be checked that the set of all such matrices forms a ring. As it consists of all elements for which the maximal (over the columns) number of nonzero coefficients under the main diagonal is bounded, we will denote it by $\mathscr{M}_{\downarrow bound}(F)$.

Figure 1: *Picture to the definition of $\mathscr{M}_{\downarrow bound}(F)$. The supremum from Eq. (4) is here equal to $k$.*




For these rings we have the following theorem:

THEOREM 2. *Assume that $F$ is a field of characteristic different from 2 and $\lesssim$ is a preorder on $\mathbb{N}$ such that $\mathscr{M}_{Cf}(\lesssim, F) = \overline{\oplus_{n\in N} S(B_n)} \subseteq \mathscr{M}$, where either $\mathscr{M} = \mathscr{M}_{VK}(F)$ or $\mathscr{M} = \mathscr{M}_{\downarrow bound}(F)$. If the map $\phi$ is an automorphism of $\mathscr{M}_{Cf}(\lesssim, F)$, then*

$$\phi = \mathscr{I}nn_g \cdot \overline{(\sigma_n)_{n\in N}} \cdot \hat{\pi},$$

*for some invertible $g \in \mathscr{M}_{Cf}(\lesssim, F)$, some family $(\sigma_n)_{n\in N}$ of automorphisms of $F$, and $\pi \in S(\mathbb{N})$.*

## 2. Preliminaries

We start with presenting the notation and some simple results.

### 2.1. Notation

By $e_{nm}$ we mean the matrix with $1$ in the position $(n,m)$ and zeroes elsewhere.

The symbols $e_\infty$, $e_k$ are used for identity matrices, infinite, and $k\times k$, respectively. When some arguments can be applied to infinite as well as to finite dimensional matrices, we will write $e$ instead of $e_\infty$ and $e_k$.

By $x_{c(n)}$ we understand the $n$-th column of the matrix $x$. We write $x^T$ for the transpose of $x$. If $x$ is any square matrix and $g$ any invertible matrix of the same size, then we will write $x^g$ for the conjugation $g^{-1}xg$.

For any $\mathscr{M}_{Cf}(\lesssim,F)$ or $\mathscr{T}_\infty(\lesssim,F)$ the invertible elements of these rings form multiplicative groups which will be denoted by $\mathrm{GL}_{Cf}(\lesssim,F)$ and $\mathrm{T}_\infty(\lesssim,F)$ respectively.

We also introduce some notation for subrings of $\mathscr{T}_\infty(F)$. We put

$$\mathscr{D}_\infty(F)=\{x\in\mathscr{T}_\infty(F):\ x_{nm}=0\ \text{ for } n\neq m\},$$

$$\mathscr{N}T_\infty(F)=\{x\in\mathscr{T}_\infty(F):\ x_{nn}=0\ \text{ for all } n\in\mathbb{N}\}.$$

By $S(\mathbb{N})$ we will understand the set (that indeed forms a group) of all permutations of $\mathbb{N}$, and by $S_n$ - the group of all permutations of $\{1,2,\dots,n\}$. We will use the symbol supp for the support of a permutation.

If $A$ and $B$ are isomorphic rings, then we will write $A\sim B$.

The characteristic of a field $F$ will be denoted by $\mathrm{char}(F)$ and the group of its automorphisms by $\mathscr{A}ut(F)$.

### 2.2. Some general remarks

First we present some remarks that hold for all considered rings.

REMARK 1. For any field $F$, any preorder $\lesssim$ and any automorphism $\phi$ of $\mathscr{M}_{Cf}(\lesssim,F)$ we have, for every $n$:

1. $\phi(S(B_n))=S(B_{n'})$, where $S(B_n)$ and $S(B_{n'})$ are isomorphic to the same subring of either $\mathscr{T}_\infty(F)$ or $\mathscr{T}_k(F)$ for some $k\in\mathbb{N}$;
2. $\phi(S(C_n))=S(C_{n'})$, where $S(C_n)$ and $S(C_{n'})$ are isomorphic to the same subring of either $\mathscr{T}_\infty(F)$ or $\mathscr{T}_k(F)$ for some $k\in\mathbb{N}$.

LEMMA 1. *For any ring $\mathscr{M}_{Cf}(\lesssim,F)$ all the classes $C_n$ are finite.*

*Proof.* Suppose that the claim does not hold. Then for some $n$ we have $|C_n|=\infty$, say $C_n=\{n_k:\ k\in\mathbb{N}\}$. As $S(C_n)\subseteq\mathscr{M}_{Cf}(\lesssim,F)$, we should then have $\sum_{k\in\mathbb{N}}e_{n_kn_1}\in\mathscr{M}_{Cf}(\lesssim,F)$. Yet, the matrix $\sum_{k\in\mathbb{N}}e_{n_kn_1}$ is not column-finite – a contradiction. Hence $|C_n|<\infty$. $\square$

Later we will need results about the automorphisms of finite dimensional structural matrix rings. These were, in particular, described by S.P. Coelho. Here we cite her theorem.

THEOREM 3. (Thm. C, [3]) *Let $S$ be a structural matrix algebra. Then*

$$\mathscr{A}ut S = (\mathscr{C} \rtimes \mathscr{G}) \rtimes \mathscr{P}.$$

According to our notation, this means that every automorphism of $S$ has the form

$$\mathscr{I}nn_g \cdot \overline{\sigma} \cdot \hat{\pi}, \tag{5}$$

where $g \in S$, $\sigma$ is an automorphism of $F$, and $\pi$ is a permutation of $\{1,2,\dots,k\}$ (we assume here that the matrices in $S$ are $k \times k$).

PROPOSITION 1. (Prop. 4.1, [3]; see also [16]) *For any field $F$ and any preorder $\lesssim$, the set*

$$\left\{x \in \mathscr{M}_{Cf}(\lesssim, F) : \; x_{nm} = 0 \textit{ for } (n,m) \in \lesssim_{sym}\right\}$$

*is the Jacobson radical of $\mathscr{M}_{Cf}(\lesssim, F)$.*

The proof of this lemma is the same as the proof of Proposition 4.1 from [3]. As quite a few arguments are used there (in particular the proof uses Lemma 3.2 given in the same paper) and the proof does not use the finite dimension of the considered ring, we do not repeat it.

To get more information about radicals of structural matrix rings see [16, 12].

REMARK 2. It is well-known that if $R$ is an arbitrary ring with unity 1 and $\phi$ is an epimorphism of $R$, then $\phi(1) = 1$ and $\phi(x)$ is invertible if $x \in R$ is invertible.

Thus, in particular, if $F$ is a field, $\lesssim$ a preorder, and $\phi$ an epimorphism of the ring $\mathscr{M}_{Cf}(\lesssim, F)$, then

1. $\phi(e_\infty) = e_\infty$,;
2. if $x \in \mathscr{M}_{Cf}(\lesssim, F)$ is invertible, then so is $\phi(x)$.

In our proofs we are going to use some facts about idempotents. We start with some facts about their diagonalization.

LEMMA 2. ([13], Lemma 2.3) *Let $F$ be any field. If $x \in \mathscr{T}_\infty(F)$ is an idempotent, then there exists an invertible matrix $t \in \mathscr{T}_\infty(F)$ such that $x^t$ is a diagonal matrix.*

From the construction of $t$ given in the proof of the above lemma, we obtain:

COROLLARY 1. *If for $x \in \mathscr{T}_\infty(F)$ from Lemma 2 we have $x \in \mathscr{T}_\infty(\lesssim, F)$ for some preorder $\lesssim$, then $t \in \mathscr{T}_\infty(\lesssim, F)$.*

*Proof.* From the proof of Lemma 2 we get that the consecutive columns of $t$ can be found as follows. The first of them is $(1,0,0,0,\dots)^T$, so as $\mathscr{D}_\infty(F)\subseteq\mathscr{T}_\infty(\lesssim,F)$, we can informally say that the first column of $t$ 'is' in $\mathscr{T}_\infty(\lesssim,F)$ . If the first $n$ found columns form a matrix $t_n$ such that

$$\left(\begin{array}{c|c} t_n & 0 \\ \hline 0 & e_\infty \end{array}\right)$$

is in $\mathscr{T}_\infty(\lesssim,F)$, then the $(n+1)$-th column is equal to

$$\begin{pmatrix} (t_n x_{c(n+1)})_1 \\ \vdots \\ (t_n x_{c(n+1)})_n \\ z \end{pmatrix} \qquad \text{for some } z\in\{1,-1\}.$$

As the matrices

$$\left(\begin{array}{c|c} t_n & 0 \\ \hline 0 & e_\infty \end{array}\right), \qquad \left(\begin{array}{c|c|c} 0 \;\cdots & (x_{c(n+1)})_1 & 0\;\cdots \\ \vdots & \vdots & \\ 0\;\cdots & (x_{c(n+1)})_n & 0\;\cdots \\ \hline 0\;\cdots & 0 & 0\;\cdots \\ \hline \vdots & & \vdots \end{array}\right) \qquad \text{and } \pm e_{n+1,n+1}$$

are in $\mathscr{T}_\infty(\lesssim,F)$, also the $(n+1)$-th column 'is' in $\mathscr{T}_\infty(\lesssim,F)$. This means that for each $n$, $m$ we have $(n,m)\notin\lesssim$, then $(t_k)_{nm}=0$ for all $k\in\mathbb{N}$ and consequently $t_{nm}=0$. Thus $t\in\mathscr{T}_\infty(\lesssim,F)$. $\square$

Now we wish to generalize Lemma 2 a bit.

LEMMA 3. *Let $F$ be a field. If $x\in\mathscr{M}_{VK}(F)$ is an idempotent, then there exists an invertible matrix $g\in\mathscr{M}_{VK}(F)$ such that $x^g$ is a diagonal matrix.*

*Proof.* Since $x\in\mathscr{M}_{VK}(F)$, we can write that

$$x=\left(\begin{array}{c|c} x_1 & x_2 \\ \hline 0 & x_3 \end{array}\right) \qquad \text{with } x_1\in\mathscr{M}_k(F),\ x_3\in\mathscr{T}_\infty(F)$$

for some $k\in\mathbb{N}$. One knows that there exists some $g_1\in\mathscr{M}_k(F)$ such that $x_1^{g_1}$ is a diagonal matrix $d_1$, i.e.

$$y:=x^{\left(\begin{array}{c|c} t_1 & 0 \\ \hline 0 & e_\infty \end{array}\right)}=\left(\begin{array}{c|c} d_1 & x_2' \\ \hline 0 & x_3 \end{array}\right).$$

Clearly, $y$ is upper triangular. Hence, we can apply Lemma 2 to it and for some $t$ we have $x^{g_1t}=y^t\in\mathscr{D}_\infty(F)$. $\square$

LEMMA 4. *Let $F$ be a field. If $x \in \mathscr{M}_{\downarrow\, bound}(F)$ is an idempotent, then there exists an invertible matrix $g \in \mathscr{M}_{\downarrow\, bound}(F)$ such that $x^g$ is a diagonal matrix.*

*Proof.* As $x \in \mathscr{M}_{\downarrow\, bound}(F)$, we can assume that $x$ is of the form as depicted in Fig. 1. Define ${}_1x_p$ as follows:

$$({}_1x_p)_{nm} = x_{(p-1)k+n,(p-1)k+m}$$

For example for $k = 2$ the matrices ${}_1x_p$ are blocks of $x$ as depicted below:

$$x = \begin{pmatrix} \boxed{{}_1x_1} & * & * \cdots \\ \boxed{*}\,\boxed{{}_1x_2} & & * \\ & \boxed{*}\,\boxed{{}_1x_3} & \\ & & \boxed{*}\ \ddots \\ & & \ddots \end{pmatrix}.$$

For every ${}_1x_p$ there exists an extension field ${}_1F_p$ of $F$ and ${}_1g_p \in \mathscr{M}_k({}_1F_p)$ such that $({}_1x_p)^{{}_1g_p}$ is a Jordan form of ${}_1x_p$ (for more details see [9] or some other classical textbook). We define $g_1$ by the rule

$$(g_1)_{nm} = \begin{cases} ({}_1g_p)_{n'm'} & \text{if } n = n' + (p-1)k,\ m = m' + (p-1)k \\ & \text{for some } p \in \mathbb{N},\ 1 \leqslant n', m' \leqslant k \\ 0 & \text{otherwise.} \end{cases}$$

Consider now $x_1 := x^{g_1}$. Notice that in $x_1$ the blocks that are on the same places as ${}_1x_1$, ${}_1x_2$, ${}_1x_3$, $\ldots$ used to be in $x$ are now upper triangular, i.e. we have obtained some $0$-s under the main diagonal.

Define now ${}_2x_p$ as follows:

$$({}_2x_p)_{nm} = (x_1)_{(p-1)k+n+1,(p-1)k+m+1}$$

For example for $k = 2$ we have

$$x_1 = \begin{pmatrix} \boxed{*} & * & * \cdots \\ \boxed{0}\,\boxed{{}_2x_1} & & * \\ & \boxed{0}\,\boxed{{}_2x_2} & \\ & & \boxed{0}\ \ddots \\ & & \ddots \end{pmatrix}.$$

Again for every ${}_2x_p$ there exists an extension ${}_2F_p$ and of $F$ and ${}_2g_p \in \mathscr{M}_k({}_2F_p)$ such that ${}_2x_p^{{}_2g_p}$ is a Jordan form of ${}_2x_p$. We define $g_2$ by

$$(g_2)_{nm} = \begin{cases} ({}_2g_p)_{n'm'} & \text{if } n = n' + 1 + (p-1)k,\ m = m' + 1 + (p-1)k \\ & \text{for some } p \in \mathbb{N},\ 1 \leqslant n', m' \leqslant k \\ 1 & \text{if } n = m = 1 \\ 0 & \text{otherwise.} \end{cases}$$

In $x_2^{g_2}$ we have obtained some 'new' zero coefficients under the main diagonal. Moreover, as we multiply the block matrices, the coefficients we have obtained by the conjugation $x^{g_1}$ are still equal to $0$.

Analogously, we define $x_3$, $g_3$, …, $x_k$, $g_k$ and finally obtain that $x_k^{g_k} = x^{g_1 g_2 \cdots g_k}$ is an upper triangular idempotent. Now we can apply Lemma 2 to this matrix – for some $t \in \mathscr{T}_\infty(F)$ we have $x^{g_1 g_2 \cdots g_k t} \in \mathscr{D}_\infty(F)$. □

Now we can obtain some information about values of automorphisms of structural infinite matrix rings.

LEMMA 5. *Suppose that $F$ is a field and $\lesssim$ a preorder. If $x$ is a rank one idempotent from either $\mathscr{M}_{\downarrow bound}(\lesssim, F)$ or $\mathscr{M}_{VK}(\lesssim, F)$, then there exists a matrix $g$ in $\mathscr{M}_{\downarrow bound}(\lesssim, F)$ or $\mathscr{M}_{VK}(\lesssim, F)$ respectively such that $x^g = e_{kk}$ for some $k \in \mathbb{N}$.*

*Proof.* In the proof we will assume that $x \in \mathscr{M}_{\downarrow bound}(F) \subset \mathscr{M}_{Cf}(F)$. The case when $x \in \mathscr{M}_{VK}(F)$ is exactly the same.

Since $x$ has rank one and is in $\mathscr{M}_{\downarrow bound}(\lesssim, F)$, it must be of the form

$$\left(\begin{array}{c|c} x_1 & x_2 \\ \hline 0 & 0 \end{array}\right) \qquad \text{with } x_1 \in \mathscr{M}_k(F)$$

for some $k \in \mathbb{N}$. As $x$ is idempotent we have $x_1 x_2 = x_2$. One can check that

$$\left(\begin{array}{c|c} x_1 & x_2 \\ \hline 0 & 0 \end{array}\right)^{\left(\begin{array}{c|c} e_k & x_2 \\ \hline 0 & e_\infty \end{array}\right)} = \left(\begin{array}{c|c} x_1 & 0 \\ \hline 0 & 0 \end{array}\right).$$

As

$$\left(\begin{array}{c|c} e_k & x_2 \\ \hline 0 & e_\infty \end{array}\right) \in \mathscr{M}_{\downarrow bound}(\lesssim, F),$$

it suffices to focus on $x_1$. Clearly $x_1 \in \mathscr{M}_k(\lesssim', F)$ where $\lesssim'$ is a preorder on $\{1, 2, \ldots, k\}$ such that $(i, j) \in \lesssim'$ if and only if $(i, j) \in \lesssim$ and $1 \leqslant i, j \leqslant k$.

From $\operatorname{rank}(x_1) = 1$ it follows that

$$x_1 = \begin{pmatrix} \alpha_1\beta_1 & \alpha_1\beta_2 & \cdots & \alpha_1\beta_k \\ \alpha_2\beta_1 & \alpha_2\beta_2 & \cdots & \alpha_2\beta_k \\ \vdots & & & \vdots \\ \alpha_k\beta_1 & \alpha_k\beta_2 & \cdots & \alpha_k\beta_k \end{pmatrix}$$

for some $\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k \in F$.

There exist $1 \leqslant p, r \leqslant k$ such that $\alpha_r, \beta_p \neq 0$, otherwise $x_1$ would be the zero matrix, so the rank would not be equal to 1. Let $\pi$ be thepermutation $(1\ r)$ if $\alpha_1 = 0$ and the identity in the case when $\alpha_1 \neq 0$. Let $p_\pi$ be the permutation matrix determined by $\pi$. We have

$$y_1 := x_1^{p_\pi^{-1}} = \begin{pmatrix} \alpha_1'\beta_1' & \alpha_1'\beta_2' & \cdots & \alpha_1'\beta_k' \\ \alpha_2'\beta_1' & \alpha_2'\beta_2' & \cdots & \alpha_2'\beta_k' \\ \vdots & & & \vdots \\ \alpha_k'\beta_1' & \alpha_k'\beta_2' & \cdots & \alpha_k'\beta_k' \end{pmatrix}$$

for some $\alpha_1', \ldots, \alpha_k', \beta_1', \ldots, \beta_k' \in F$, $\alpha_1', \beta_1' \neq 0$.

The matrix $y_1$ is in $\mathscr{M}_k(\lesssim'', F)$ for some preorder $\lesssim''$. Now it suffices to prove that there exists $h_1 \in \mathscr{M}_k(\lesssim'', F)$ such that $y_1^{h_1} = e_{ii}$ for some $i \in \mathbb{N}$.

Assume that $\alpha_{j_1}', \ldots, \alpha_{j_s}' \neq 0$ (with $j_1 < j_2 < \ldots < j_s$) and the other $\alpha'$-s are equal to zero.

We put $h_1' = e_k - \sum_{i=2}^{s} \alpha_{j_i}'(\alpha_1')^{-1} e_{j_i 1}$. As $\alpha_{j_2}', \ldots, \alpha_{j_s}' \neq 0$, it follows that $h_1' \in \mathscr{M}_k(\lesssim'', F)$. Then

$$z_1 := y_1^{h_1'} = \begin{pmatrix} \gamma_1 & \gamma_2 & \cdots & \gamma_n \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \qquad \text{for some } \gamma_1, \gamma_2, \ldots, \gamma_n \in F.$$

Moreover, as $z_1$ is idempotent, $\gamma_1 = 1$.

Now we put $h_1'' = e_k + \sum_{i=2}^{k} \gamma_i e_{1i}$ to get $z_1^{h_1''} = e_{11}$. Obviously $h_1'' \in \mathscr{M}_k(\lesssim'', F)$.

From $h_1'h_1'' \in \mathscr{M}_k(\lesssim'', F)$, we conclude that $(h_1'h_1'')^{p_\pi} \in \mathscr{M}_k(\lesssim', F)$, and consequently

$$h = \left(\begin{array}{c|c} (h_1'h_1'')^{p_\pi} & 0 \\ \hline 0 & e_\infty \end{array}\right) \in \mathscr{M}_{\downarrow bound}(\lesssim, F).$$

Thus $x^{gh} = e_{\pi^{-1}(1)\pi^{-1}(1)}$ for some $g, h \in \mathscr{M}_{\downarrow bound}(\lesssim, F)$. $\square$

The facts proven above are useful when we consider some properties of homomorphisms of different structural infinite matrix rings.

LEMMA 6. *Let $F$ be a field and $\lesssim$ a preorder such that $\mathscr{M}_{Cf}(\lesssim, F)$ is contained in $\mathscr{M}_{\downarrow bound}(\lesssim, F)$ or $\mathscr{M}_{VK}(\lesssim, F)$. If $\phi$ is a homomorphism of $\mathscr{M}_{Cf}(\lesssim, F)$, then there exists $g \in \mathscr{M}_{Cf}(F)$ such that for all $n \in \mathbb{N}$ we have $(\phi(e_{nn}))^g = \sum_{i \in I_n} e_{ii}$ for some disjoint sets $I_n \subset \mathbb{N}$.*

*Proof.* From Lemmas 3, 4 we know that for every $n \in \mathbb{N}$ there exists $h_n \in \mathscr{M}_{Cf}(F)$ such that $(\phi(e_{nn}))^{h_n} = \sum_{i \in I_n} e_{ii}$ for some sets $I_n \subseteq \mathbb{N}$. Obviously, we focus on $n$'s satisfying $I_n \neq \emptyset$. Consider the least element in the union $\cup_{n \in \mathbb{N}} I_n$. It is in one of the sets $I_n$, say in $I_{n_1}$. Let us put $g_1 = h_{n_1}$. We have $(\phi(e_{n_1 n_1}))^{h_{n_1}} = \sum_{i \in I_{n_1}} e_{ii}$. Notice that from $e_{n_1 n_1} e_{nn} = e_{nn} e_{n_1 n_1} = 0$ for all $n \neq n_1$, we get

$$(\phi(e_{nn}))_{i i_{n_1}} = (\phi(e_{nn}))_{i_{n_1} i} = 0 \qquad \text{for all } i_{n_1} \in I_{n_1},\ i \neq i_{n_1}. \tag{6}$$

Consider now the minimal element in the set

$$\bigcup_{\substack{n\in\mathbb{N}\\ n\neq n_1}} I_n.$$

Say it is in $I_{n_2}$. For $h_{n_2}$ we have $((\phi(e_{n_2n_2}))^{h_{n_1}})^{h_{n_2}} = \sum_{i\in I_{n_2}} e_{ii}$. Moreover, by (6) the matrix $h_{n_2}$ is such that $((\phi(e_{n_1n_1}))^{h_{n_1}})^{h_{n_2}}$ is still equal to $\sum_{i\in I_{n_1}} e_{ii}$. Define $g_2$ as $h_{n_1}h_{n_2}$. Observe that as $i_{n_1}$ – the minimal element in $I_{n_1}$ is less than $i_{n_2}$ – the minimal element in $I_{n_2}$, the entries in the $(i_{n_2}-1)\times(i_{n_2}-1)$ left upper block of $g_1$ are the same as the entries in the $(i_{n_2}-1)\times(i_{n_2}-1)$ left upper block of $g_2$.

In the same way we construct the infinite sequence $g_1$, $g_2$, $g_3$, .... The matrix $g$ from the claim is defined by the condition $g_{nm} = (g_k)_{nm}$, where $k$ is any number such that $\min_i((\phi(e_{kk})_{ii})^{h_k} \neq 0) \geqslant \max(n,m)$. □

At the end of this section we observe that the matrices $\phi(e_{nn})$ have a great meaning for all the image of $\phi$.

LEMMA 7. *Suppose $F$ is a field, $S$ - any subring of $\mathscr{M}_{Cf}(F)$. If $\phi$ is an epimorphism of $S$ such that*

- *for every $k\in\mathbb{N}$ either there exists $n\in\mathbb{N}$ such that $\phi(e_{kk}) = e_{nn}$ or $\phi(e_{kk}) = 0$,*
- *for every $n\in\mathbb{N}$ there exists $k\in\mathbb{N}$ such that $\phi(e_{kk}) = e_{nn}$,*

*then $\phi$ is determined by the values that it takes at the matrices that have only a finite number of nonzero entries.*

*Proof.* To determine $\phi(x)$ we need to know $(\phi(x))_{nm}$ for all $n,m\in\mathbb{N}$. By the assumption we have $\phi(e_{kk}) = e_{nn}$ and $\phi(e_{ll}) = e_{mm}$ for some $k$, $l$. Then

$$\begin{aligned}(\phi(x))_{nm} &= (e_{nn}\phi(x)e_{mm})_{nm} = [\phi(e_{kk})\phi(x)\phi(e_{ll})]_{nm}\\ &= (\phi(e_{kk}xe_{ll}))_{nm} = (\phi(x_{kl}e_{kl}))_{nm}\end{aligned}$$

and the claim follows. □

## 3. Upper triangular matrices

Before we begin, let us note that the automorphisms of algebras of triangular (or somehow connected to triangular) matrices are of interest to many researchers; for instance they were investigated in [2, 11, 10].

### 3.1. Proof of Theorem 1

We start this section with a proposition that is a corollary from Proposition 1 cited in the preliminary section.

PROPOSITION 2. *Suppose that $F$ is a field and $\lesssim$ - a preorder. The set $\mathcal{N}T_\infty(F)\cap \mathcal{T}_\infty(\lesssim,F)$ is the Jacobson radical of $\mathcal{T}_\infty(\lesssim,F)$.*

From the preceding proposition and properties of homomorphisms we get now

COROLLARY 2. *Let $F$ be a field and $\lesssim$ a preorder. If $\phi$ is a homomorphism of $\mathcal{T}_\infty(\lesssim,F)$, then*

$$\phi(\mathcal{T}_\infty(\lesssim,F)\cap \mathcal{N}T_\infty(F)) \subseteq \mathcal{T}_\infty(\lesssim,F)\cap \mathcal{N}T_\infty(F).$$

Now we will get back to our maps.

LEMMA 8. *Let $F$ be a field, $\lesssim$ a preorder, and $\phi$ an epimorphism of $\mathcal{T}_\infty(\lesssim,F)$ such that for every $n$ the matrix $\phi(e_{nn})$ is diagonal. Then for every $k\in\mathbb{N}$ there exists $n\in\mathbb{N}$ such that $\phi(e_{nn})=e_{kk}$.*

*Proof.* Suppose first that, for some $n\in\mathbb{N}$, $\phi(e_{nn})=\sum_{i\in I_n} e_{ii}$ with $|I_n|>1$. Denote by $i_n$ the least element in $I_n$. As $\phi$ is onto, there exists $d\in\mathcal{T}_\infty(\lesssim,F)$ such that $\phi(d)=e_{i_ni_n}$. By Corollary 2 we may assume that $d$ is diagonal. Notice now that

$$e_{i_ni_n}=e_{i_ni_n}\sum_{i\in I_n} e_{ii}=\phi(d)\phi(e_{nn})=\phi(de_{nn}).$$

Therefore we may assume that $d=\alpha e_{nn}$ for some $\alpha\in F^*$. Clearly, $\alpha\neq 1$. Moreover we have $\phi(\alpha^2 e_{nn})=(\phi(\alpha e_{nn}))^2=e_{i_ni_n}$, so $\phi(\alpha^2 e_{nn})-\phi(\alpha e_{nn})=0$. Consider now the matrix $a:=e_\infty+(\alpha^2-\alpha-1)e_{nn}$. As $\alpha\neq 0,1$, we have that $a$ is invertible, so, by Remark 2, $\phi(a)$ is also invertible. However,

$$\begin{aligned}(\phi(a))_{i_ni_n} &= (e_{i_ni_n}\phi(a))_{i_ni_n}=(\phi(\alpha e_{nn})\phi(a))_{i_ni_n}=(\phi(\alpha e_{nn}a))_{i_ni_n}\\ &= (\phi((\alpha^2-\alpha)e_{nn}))_{i_ni_n}=0,\end{aligned}$$

contradicting the invertibility of $\phi(a)$.

Therefore, for every $n$, either $\phi(e_{nn})=e_{kk}$ for some $k\in\mathbb{N}$ or $\phi(e_{nn})=0$.
Moreover, as $\phi$ is onto, for every $k$ there must exist $n$ such that $\phi(e_{nn})=e_{kk}$. □

From the above lemma we can easily obtain

LEMMA 9. *For any field $F$, any preorder $\lesssim$, any epimorphism $\phi$ of $\mathcal{T}_\infty(\lesssim,F)$ such that for every $k$ the matrix $\phi(e_{kk})$ is equal to either $0$ or $e_{nn}$ for some $n\in\mathbb{N}$, and $x\in\mathcal{T}_\infty(\lesssim,F)$ we have*

$$(\phi(x))_{nn}=\phi(x_{kk}e_{kk})_{nn},$$

*where $k$ is a number such that $\phi(e_{kk})=e_{nn}$.*

*Proof.* By Lemma 8 we have

$$(\phi(x))_{nn}=(e_{nn}\phi(x)e_{nn})_{nn}=(\phi(e_{kk})\phi(x)\phi(e_{kk}))_{nn}=(\phi(x_{kk}e_{kk})_{nn}. \tag{7}$$

□

From the two latter lemmas we can derive some more consequences.

LEMMA 10. *Let $F$ be a field, $\lesssim$ a preorder and $\phi$ a homomorphism of $\mathscr{T}_\infty(\lesssim,F)$ such that, for every $k\in\mathbb{N}$, $\phi(e_{kk})$ is equal to either $0$ or $e_{nn}$ for some $n$, and for every $n\in\mathbb{N}$ there exists exactly one $k$ such that $\phi(e_{kk})=e_{nn}$. For any $n<m$, $\alpha\in F$, we have one of the following cases:*

1. *$\phi(\alpha e_{nm})=\alpha' e_{n'm'}$ in the case when $(n,m)\in\lesssim$, $\phi(e_{nn})=e_{n'n'}$, $\phi(e_{mm})=e_{m'm'}$, and $n'<m'$;*
2. *$\phi(\alpha e_{nm})=\alpha' e_{m'n'}$ in the case when $(n,m)\in\lesssim$, $\phi(e_{nn})=e_{n'n'}$, $\phi(e_{mm})=e_{m'm'}$, and $m'<n'$;*
3. *$\phi(\alpha e_{nm})=0$ in the case when either $\phi(e_{nn})=0$ or $\phi(e_{mm})=0$, or $(n,m)\notin\lesssim$.*

(Note that the coefficients $\alpha'$ in points (1), (2) of the claim may be equal to $0$.)

*Proof.* This follows easily from the fact that $e_{nn}+\alpha e_{nm}$ and $e_{mm}+\alpha e_{nm}$ are idempotents. Clearly, if $(n,m)\notin\lesssim$, then the latter two matrices are not in $\mathscr{T}_\infty(\lesssim,F)$, so there is not point in discussing their images. Let us then assume that $(n,m)\in\lesssim$. Once again we repeat that, by Corollary 2,

$$\phi(\alpha e_{nm})\in\mathscr{N}T_\infty(F). \tag{8}$$

If $\phi(e_{nn})=0$, then $\phi(e_{nn})+\phi(\alpha e_{nm})$ is idempotent only in the case when $\phi(\alpha e_{nm})=0$. The same holds when $\phi(e_{mm})=0$.

Consider now the case when $\phi(e_{nn})=e_{n'n'}$, $\phi(e_{mm})=e_{m'm'}$. We have

$$e_{n'n'}+e_{n'n'}\phi(\alpha e_{nm})+\phi(\alpha e_{nm})e_{n'n'}+(\phi(\alpha e_{nm}))^2=e_{n'n'}+\phi(\alpha e_{nm}), \tag{9a}$$

$$e_{m'm'}+e_{m'm'}\phi(\alpha e_{nm})+\phi(\alpha e_{nm})e_{m'm'}+(\phi(\alpha e_{nm}))^2=e_{m'm'}+\phi(\alpha e_{nm}), \tag{9b}$$

which force

$$e_{n'n'}\phi(\alpha e_{nm})+\phi(\alpha e_{nm})e_{n'n'}=e_{m'm'}\phi(\alpha e_{nm})+\phi(\alpha e_{nm})e_{m'm'}.$$

From the above equality we get $\phi(\alpha e_{nm})=\alpha_1 e_{n'm'}+\alpha_2 e_{m'n'}$. Since $\phi(\alpha e_{nm})\in\mathscr{T}_\infty(F)$, $\alpha_1=0$ for $n'>m'$ and $\alpha_2=0$ if $n'<m'$. $\square$

Now we can prove our first main result.

*Proof of Theorem 1.* From Lemmas 6 and 8 we know that there exists $t\in\mathscr{T}_\infty(\lesssim,F)$ such that for every $n\in\mathbb{N}$ either $(\phi(e_{nn}))^t=0$ or $(\phi(e_{nn}))^t=e_{k_nk_n}$. Moreover, as $\phi$ is injective, for every $n$ the second possibility holds.

Consider $\psi=\mathscr{I}nn_t\cdot\phi$ instead of $\phi$. Let us focus on the classes $\{B_n\}_{n\in N}$. It is easily seen that if $\psi(e_{kk})=e_{k'k'}$, $\psi(e_{ll})=e_{l'l'}$ and the numbers $k$, $l$ are in the same class, then $k'$, $l'$ also should be in the same class, and conversely. Therefore we should have $\psi(B_n)=B_m$ for $B_n\sim B_m$. Hence there exists $\pi\in S(\mathbb{N})$ such that for every $n$ we have $\psi(B_n)=B_{\pi(n)}$. As we have stated before, $\pi$ satisfies the condition $B_n\sim B_{\pi(n)}$ for every $n$.

Hence, we need to consider automorphisms from $S(B_n)$ to $S(B_m)$, where $S(B_n)$, $S(B_m)$ are both isomorphic to the same subring of either $\mathscr{T}_k(F)$ (for a fixed $k$) or $\mathscr{T}_\infty(F)$, that contains $\mathscr{D}_k(F)$ or $\mathscr{D}_\infty(F)$, respectively.

Let $B_n = \{i_1, i_2, \ldots, i_k\}$, $B_m = \{j_1, j_2, \ldots, j_k\}$. For $k$ distinct $j_{p_r}$'s ($1 \leqslant r \leqslant k$) we have

$$\phi(e_{i_1i_1}) = e_{j_{p_1}j_{p_1}}, \quad \phi(e_{i_2i_2}) = e_{j_{p_2}j_{p_2}}, \quad \ldots, \quad \phi(e_{i_ki_k}) = e_{j_{p_k}j_{p_k}}$$

and

$$\phi(\alpha e_{i_si_s}) = \alpha_1 e_{j_{p_s}j_{p_r}} + \alpha_2 e_{j_{p_r}j_{p_s}} \qquad \text{with } \alpha_1\alpha_2 = 0,\ \alpha_1 + \alpha_2 \neq 0. \tag{10}$$

By (10) and the fact that $e_{mm} + \alpha e_{mk} + \beta e_{nm} + \alpha\beta e_{nk}$ is idempotent for any $n < m < k$ we have either $j_{p_1} < j_{p_2} < \cdots < j_{p_k}$ or $j_{p_1} > j_{p_2} > \cdots > j_{p_k}$, so either $j_{p_1} = j_1$, $j_{p_2} = j_2$, $\ldots$, $j_{p_k} = j_k$ or $j_{p_1} = j_k$, $j_{p_2} = j_{k-1}$, $\ldots$, $j_{p_k} = j_1$.

If the second posibility holds, let us apply to the images of our blocks the map $\mathscr{J}$.

The infinite dimensional case is almost the same, but clearly in that case we can only have $j_{p_1} = j_1$, $j_{p_2} = j_2$, $j_{p_3} = j_3$, $\ldots$

Now it suffices to consider automorphisms $\psi'$ such that $\psi' : \mathscr{T}_k(F) \to \mathscr{T}_k(F)$ or $\psi' : \mathscr{T}_\infty(F) \to \mathscr{T}_\infty(F)$ and $\psi'(\alpha e_{ij}) = \alpha' e_{ij}$.

We can write that $\psi'(\alpha e_{ij}) = f_{ij}(\alpha)e_{ij}$ for some $f_{ij} : F \to F$. From

$$\psi'(\alpha e_{ij}) = \psi'(\alpha e_{ii} \cdot e_{ij}) = \psi'(e_{ij} \cdot \alpha e_{jj})$$

we get

$$f_{ij}(\alpha) = f_{ii}(\alpha) f_{ij}(1) = f_{jj}(\alpha) f_{ij}(1). \tag{11}$$

If $f_{ij}(1) = 0$, then we have $f_{ij}(\alpha) = 0$ for $\alpha \in F$. Notice that, as $i$, $j$ are in the same class for each $i$ there exists $j$ such that $f_{ij} \neq 0$, so by (11) $f_{ii} = f_{jj}$. Let us write $f_1$ for all $f_{ii}$.

Moreover, if $f_{ij}(1) \neq 0$, then $f_{ij}(\alpha) \neq 0$ for $\alpha \neq 0$. We have then $f_{ij}(\alpha) = f_1(\alpha) f_{ij}(1)$.

We will show now that in our ring there exists an upper triangular matrix $t$ satisfying the following conditions:

- $t_{ii} = 1$ for all $i$,
- for every $i$ and $j$, if $f_{ij}(1) \neq 0$, then $(f_{ij}(1)e_{ij})^t = e_{ij}$.

We construct this $t$ using induction on columns.

First we set $t_1 = t_1' = e$.

Now we look for $t_2$ of the form $e + t_{12}e_{12}$ such that

$$(f_{12}(1)e_{12})^{t_2} = \begin{cases} e_{12} & \text{if } f_{12}(1) \neq 0 \\ 0 & \text{if } f_{12}(1) = 0. \end{cases}$$

From the calculations it follows that the coefficient $t_{12}$ must satisfy the condition

$$f_{12}(1) = \begin{cases} 1 + t_{12} & \text{if } f_{12}(1) \neq 0 \\ t_{12} & \text{if } f_{12}(1) = 0. \end{cases}$$

Hence

$$t_{12} = \begin{cases} f_{12}(1) - 1 & \text{if } f_{12}(1) \neq 0 \\ 0 & \text{if } f_{12}(1) = 0, \end{cases}$$

and $t_2$ is now determined. We put $t_2' = t_1' t_2$. It can be seen that $t_2'$ is in $\mathscr{T}_k(\lesssim, F)$ or in $\mathscr{T}_\infty(\lesssim, F)$ respectively.

Next we consider $(S(B_{\pi(n)}))^{t_2'}$. Obviously, the functions $f_{ij}$ might have changed, so we denote them now by $f_{ij}^{(2)}$. Clearly, if $f_{12}(1) \neq 0$, then $f_{12}^{(2)}(1) = 1$.

Suppose that we have constructed the sequences $t_1, t_2, \ldots, t_l$ and $t_1', t_2', \ldots, t_l'$ such that $f_{ij}^{(l)}(1)$ for $1 \leqslant i, j \leqslant l$ is equal to either $1$ or $0$. Now we wish to find $t_{l+1}$ such that $t_{l+1} = e_\infty + \sum_{i=1}^{l} t_{i,l+1} e_{i,l+1}$ and satisfying the condition that $f_{ij}^{(l+1)}(1)$ will be equal to either $1$ or $0$. From the condition

$$(f_{i,l+1}^{(l)}(1) e_{i,l+1})^{t_{l+1}} = \begin{cases} e_{i,l+1} & \text{if } f_{i,l+1}^{(l)}(1) \neq 0 \\ 0 & \text{if } f_{i,l+1}^{(l)}(1) = 0, \end{cases}$$

we get

$$t_{i,l+1} = \begin{cases} f_{i,l+1}^{(l)} - 1 & \text{if } f_{i,l+1}^{(l)}(1) \neq 0 \\ 0 & \text{if } f_{i,l+1}^{(l)}(1) = 0. \end{cases}$$

Now we put $t_{l+1}' = t_l' t_{l+1}$. Again, we can see that $t_{l+1}' \in \mathscr{T}_\infty(\lesssim, F)$.

One can check that the first $l$ columns of $t_{l+1}'$ are the same as the first $l$ columns of $t_l'$. Thus, it can be noticed that the desired $t$ fulfills the condition $t_{ij} = (t_j')_{ij}$.

Observe that this $t$ was found to ensure $f_{ij}(1) = 1$ only for the functions from the subring $S(B_{\pi(n)})$. Denote it then by $t_{\pi(n)}$. As $S(B_n)S(B_m) = \{0\}$, from the construction of $t_{\pi(n)}$ it follows that all the $t_{\pi(n)}$'s commute and

$$t_{\pi(n)} t_{\pi(m)} = t_{\pi(n)} + t_{\pi(m)} - e_\infty \qquad \text{for any } n \neq m. \tag{12}$$

Now we have $(\psi(\alpha e_{ij}))^t = f_1(\alpha) e_{ij}$. From $\psi((\alpha+\beta)e_{ii}) = \psi(\alpha e_{ii}) + \psi(\beta e_{ii})$, $\psi((\alpha \cdot \beta)e_{ii}) = \psi(\alpha e_{ii}) \cdot \psi(\beta e_{ii})$, surjectivity of $\psi$, and fact that $\psi$ preserves invertible matrices, we obtain that $f_1$ is an automorphism of $F$.

Hence, we can write that

$$\phi = \mathscr{I}nn_t \cdot \overline{(\chi_n \cdot \mathscr{I}nn_{t_{\pi(n)}} \cdot \sigma_n)_{n \in N}} \cdot \mathscr{B}_\pi.$$

We can replace $\chi_n \cdot \mathscr{I}nn_{t_{\pi(n)}}$ with $\mathscr{I}nn_{t'_{\pi(n)}} \cdot \chi_n$, and by (12) we can also replace $\overline{(\mathscr{I}nn_{t'_{\pi(n)}})_n}$ with $\mathscr{I}nn_{t'}$, where

$$t'_{ij} = \begin{cases} (t'_{\pi(n)})_{ij} & \text{if } i, j \in B_n \\ 0 & \text{otherwise.} \end{cases}$$

Thus we have

$$\phi = \mathscr{I}nn_t \cdot \mathscr{I}nn_{t'} \cdot \overline{(\chi_n \cdot \sigma_n)_{n \in N}} \cdot \mathscr{B}_\pi = \mathscr{I}nn_{t''} \cdot \overline{(\chi_n)_{n \in N}} \cdot \overline{(\sigma_n)_{n \in N}} \cdot \mathscr{B}_\pi$$

for the matrices of the form $\alpha e_{nm}$. The result for the matrices with a finite number of nonzero entries follows from additivity. By Lemma 7 this completes the proof. $\square$

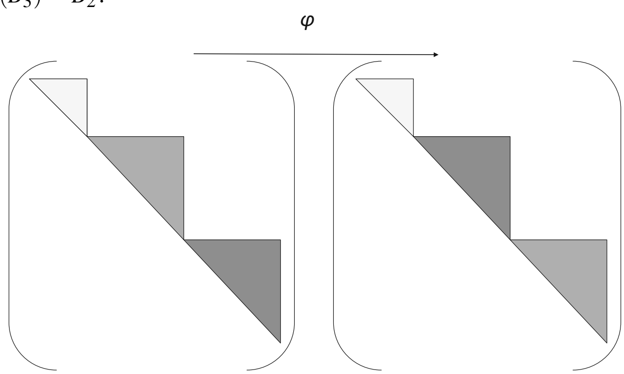
### 3.2. Consequences and examples

In this paragraph we will give some more comments about homomorphisms of the rings $\mathscr{T}_\infty(\lesssim, F)$.

First we would like to notice that using the argumentation from the proof of Theorem 1 it is easy to formulate

COROLLARY 3. *Let $F$ be a field and $\lesssim$ a preorder. If the map $\phi$ is an epimorphism of $\mathscr{T}_\infty(\lesssim, F) = \overline{\oplus_{n\in N}} S(B_n)$, then $\phi$ is of the form $\phi = \mathscr{I}nn_t \cdot \psi$ with $\psi$ satisfying the condition that for every $n \in N$ we have $\psi(S(B_n)) \subseteq \cup_m S(B_m)$, where $S(B_n)$ is isomorphic to a ring $\mathscr{T}_{n'}(\lesssim, F)$ and the $S(B_m)$'s are isomorphic to rings $\mathscr{T}_{n_m}(\lesssim, F)$, and $\sum_m n_m \leqslant n'$.*

Figure 2: *Picture depicting the images of $B_i$ of $\phi$. Here the yellow block is $B_1$, the green is $B_2$ and the blue one is $B_3$ (next blocks are not shown in the picture), and we have $\phi(B_1) = B_1$, $\phi(B_2) = B_3$, $\phi(B_3) = B_2$.*




An example of how such a map can act is given in Figure 3. Another example is given below.

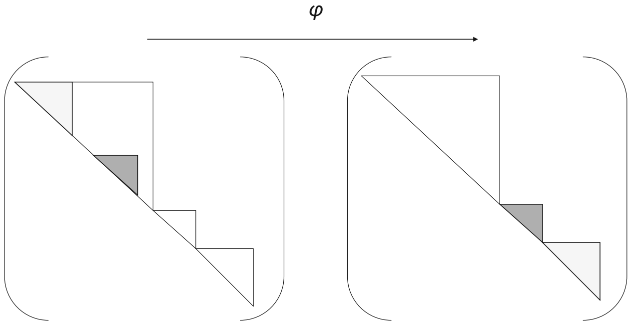
EXAMPLE 1. Let

$$\lesssim = \{(1,1)\} \cup \{(2,2),(2,3),(3,3)\} \cup \{(4,4),(4,5),(4,6),(5,5),(5,6),(6,6)\} \cup \ldots,$$

i.e. we identify $\mathscr{T}_\infty(\lesssim, F)$ with the symbolic matrix

$$\begin{pmatrix} * & & & & & & \\ & * & * & & & & \\ & & * & & & & \\ & & & * & * & * & \\ & & & & * & * & \\ & & & & & * & \\ & & & & & & \ddots \end{pmatrix},$$

Figure 3: *Picture depicting some exemplary epimorphism of* $\mathcal{T}_\infty(\lesssim, F)$. *Here* $\phi(B_1)$ *is equal to* $B_2 \cup B_3$ *(the images of other classes are not shown in the picture).*




where each block (and each class $B_n$) is $1$ bigger than the preceding block.

An example of an epimorphism of such ring can be $\phi$ given as below:

$$\phi\left(\begin{pmatrix}
x_{11} &&&&&&&&&& \\
& x_{22} & x_{23} &&&&&&&& \\
&& x_{33} &&&&&&&& \\
&&& x_{44} & x_{45} & x_{46} &&&&& \\
&&&& x_{55} & x_{56} &&&&& \\
&&&&& x_{66} &&&&& \\
&&&&&& x_{77} & x_{78} & x_{79} & x_{7,10} & \\
&&&&&&& x_{88} & x_{89} & x_{8,10} & \\
&&&&&&&& x_{99} & x_{9,10} & \\
&&&&&&&&& x_{10,10} & \\
&&&&&&&&&& \ddots
\end{pmatrix}\right)$$
$$= \begin{pmatrix}
x_{66} &&&&&&&&&& \\
& x_{55} & x_{45} &&&&&&&& \\
&& x_{44} &&&&&&&& \\
&&& x_{77} & x_{78} & x_{79} &&&&& \\
&&&& x_{88} & x_{89} &&&&& \\
&&&&& x_{99} &&&&& \\
&&&&&& x_{11,11} & x_{11,12} & x_{11,13} & x_{11,14} & \\
&&&&&&& x_{12,12} & x_{12,13} & x_{12,14} & \\
&&&&&&&& x_{13,13} & x_{13,14} & \\
&&&&&&&&& x_{14,14} & \\
&&&&&&&&&& \ddots
\end{pmatrix}.$$

Another corollary from our results proven in this section concerns the group of automorphisms of $\mathcal{T}_\infty(\lesssim, F)$. We have

THEOREM 4. *Let $F$ be a field, $\lesssim$ - a preorder. The group of automorphisms of $\mathscr{T}_\infty(\lesssim,F)=\overline{\oplus_{n\in N}S(B_n)}$ is isomorphic to a subgroup of*

$$\mathrm{T}_\infty(\lesssim,F)\wr((\mathbb{Z}_2^N\times(\mathscr{A}ut(F))^N)\wr\mathscr{P}erm),$$

*where $\mathscr{P}erm$ is a subgroup of $S(N)$ satisfying the condition: if $\pi\in\mathscr{P}erm$, then $B_n\sim B_{\pi(n)}$.*

*Proof.* First observe that

- the group of all inner automorphisms of $\mathscr{T}_\infty(\lesssim,F)$ is isomorphic to the group $\mathrm{T}_\infty(\lesssim,F)$,
- the group of all automorphisms of $\mathscr{T}_\infty(\lesssim,F)$ of the form $\overline{(\chi_n)_{n\in N}}$ is isomorphic to a subgroup of $\mathbb{Z}_2^N$ (note that this group does not have to be equal to $\mathbb{Z}_2^N$),
- the group of all automorphisms of $\mathscr{T}_\infty(\lesssim,F)$ of the form $\overline{(\sigma_n)_{n\in N}}$ is isomorphic to the group $\mathscr{A}ut(F)^N$,
- the group of all automorphisms of $\mathscr{T}_\infty(\lesssim,F)$ of the form $\mathscr{B}_\pi$ is isomorphic to some subgroup $\mathscr{P}erm$ of $S(\mathbb{N})$ (again, usually it is not equal to $S(\mathbb{N})$).

Moreover, it can be noticed that the groups $G_1$, $G_2$ that are isomorphic to $\mathbb{Z}_2^N$ and $(\mathscr{A}ut(F))^N$ satisfy the conditions that $G_1\cap G_2$ consists only of the identity map and we have $G_1G_2=G_2G_1$.

Now notice that

$$\begin{aligned}(\overline{(\chi_n)_n}\cdot\overline{(\sigma_n)_n}\cdot\mathscr{B}_\pi)\cdot(\overline{(\chi_n')_n}\cdot\overline{(\sigma_n')_n}\cdot\mathscr{B}_{\pi'})&=\overline{(\chi_n\cdot\sigma_n)_n}\cdot\mathscr{B}_\pi\cdot\overline{(\chi_n'\cdot\sigma_n')_n}\cdot\mathscr{B}_{\pi'}\\&=\overline{(\chi_n\cdot\sigma_n\cdot\chi'_{\pi(n)}\cdot\sigma'_{\pi(n)})_n}\cdot\mathscr{B}_\pi\cdot\mathscr{B}_{\pi'}\\&=\overline{(\chi_n\cdot\chi'_{\pi(n)})_n}\cdot\overline{(\sigma_n\cdot\sigma'_{\pi(n)})_n}\cdot\mathscr{B}_{\pi\pi'}.\end{aligned}$$

Hence we have obtained that the group of these maps $\psi$ is isomorphic to a subgroup of $(\mathbb{Z}_2^N\times(\mathscr{A}ut(F))^N)\wr\mathscr{P}erm$.

Analogously

$$(\mathscr{I}nn_t\cdot\psi)\cdot(\mathscr{I}nn_{t'}\cdot\psi')=\mathscr{I}nn_t\cdot\mathscr{I}nn_{\psi(t')}\cdot\psi\cdot\psi'=\mathscr{I}nn_{t\psi(t')}\cdot(\psi\cdot\psi'),$$

so our group is isomorphic to a subgroup of

$$\mathrm{T}_\infty(\lesssim,F)\wr((\mathbb{Z}_2^N\times(\mathscr{A}ut(F))^N)\wr\mathscr{P}erm).$$

Thus, the claim follows. $\square$

EXAMPLE 2. Let $\lesssim=\{(n,n):\ n\in\mathbb{N}\}\cup\{(2,3),(4,5)\}$. In this case the matrices in $\mathscr{T}_\infty(\lesssim, F)$ are of the ‘shape’:

$$\begin{pmatrix} * & & & & & & & \\ & * & * & & & & & \\ & & * & & & & & \\ & & & * & * & & & \\ & & & & * & & & \\ & & & & & * & & \\ & & & & & & * & \\ & & & & & & & \ddots \end{pmatrix}.$$

We have $B_1 = \{1\}$, $B_2 = \{2,3\}$, $B_3 = \{4,5\}$, $B_n = \{n+2\}$ for $n \geqslant 4$.

One can see that we can permute $B_2$ only with $B_3$ and itself, and the other classes with each other. Hence

$$\mathscr{P}erm = \{\pi_1\pi_2 :\ \pi_1 \in \{\mathrm{id},\ (23)\},\ 2,3 \notin \mathrm{supp}(\pi_2)\}. \tag{13}$$

Now notice that regardless of whether $\phi$ maps $B_2$ to $B_2$ or $B_3$, we always can (but we do not have to) apply $\mathscr{J}$ to $\phi(B_2)$, $\phi(B_3)$. Clearly, in each case, the automorphisms of $F$ applied to our classes are arbitrary.

According to that,

$$\mathscr{A}ut(\mathscr{T}_\infty(\lesssim, F)) \sim (\mathrm{T}_\infty(\lesssim, F) \wr ((\mathbb{Z}_2^2 \times (\mathscr{A}ut(F))^{\mathbb{N}}) \wr \mathscr{P}erm)),$$

where $\mathscr{P}erm$ is given by formula (13).

EXAMPLE 3. Now we choose $\lesssim$ as follows.

$$\lesssim = \{(n,n) :\ n \in \mathbb{N}\} \cup \{(1,2),(1,3),(4,6),(5,6)\} \cup \{(n,m) :\ 7 \leqslant n < m\}$$

In this case we have $B_1 = \{1,2,3\}$, $B_2 = \{4,5,6\}$, $B_3 = \{n :\ n \geqslant 7\}$.

Obviously $\phi(B_3) = B_3$ and $\phi(B_1)$ is either $B_1$ or $B_2$. If $\phi(B_1) = B_1$, the map $\mathscr{J}$ is not applied, whereas if $\phi(B_1) = B_2$ the map $\mathscr{J}$ has to be applied. Therefore, the subgroup of the maps $\psi$ is isomorphic to

$$\{((0,0,0),(\mathscr{A}ut(F))^3),((1,1,0),(\mathscr{A}ut(F))^3)\} \sim (\mathbb{Z}_2 \times (\mathscr{A}ut(F))^3).$$

Hence

$$\mathrm{T}_\infty(\lesssim, F) \wr (\mathbb{Z}_2 \times (\mathscr{A}ut(F))^3).$$

We conclude this section with one more comment.

In the proof of Theorem 1 we have shown that if $\psi(\alpha e_{ij}) = \alpha' e_{ij}$, then there exists a matrix $t$ such that $(\psi(e_{ij}))^t$ is equal to either $e_{ij}$ or $0$. It should be mentioned that in the case when the ring can be written as a generalized direct sum of subrings that are isomorphic to (the whole) $\mathscr{T}_k(F)$ (for possibly various $k \in \mathbb{N}$) or $\mathscr{T}_\infty(F)$, then we can also choose $t$ to be diagonal.

## 4. Proof of Theorem 2

### 4.1. Proof of the theorem

Also in this case we start with some lemmas.

LEMMA 11. *Let $F$ be a field of characteristic different from 2 and let $\lesssim$ be a preorder such that $\mathscr{M}_{Cf}(\lesssim,F)$ is contained in $\mathscr{M}_{VK}(F)$ or $\mathscr{M}_{\downarrow bound}(F)$. If $\phi$ is an automorphism of $\mathscr{M}_{Cf}(\lesssim,F)$ such that for all $n\in\mathbb{N}$ the matrices $\phi(e_{nn})$ are diagonal, then there exists $\pi\in S(\mathbb{N})$ such that $\phi(e_{nn})=e_{\pi(n)\pi(n)}$.*

*Proof.* From Lemma 6 we know that $\phi(e_{nn})=\sum_{i\in I_n}e_{ii}$ for some pairwise disjoint sets $I_n$. Moreover, as $\phi$ is injective, we have $I_n\neq\emptyset$. Hence, we need to prove that $|I_n|=1$ and $\cup_{n\in\mathbb{N}}I_n=\mathbb{N}$.

Suppose first that for some $n$ we have $|I_n|>1$. Let $i_n$ be the least element in $I_n$. As $\phi$ is onto, there exists $x\in\mathscr{M}_{Cf}(\lesssim,F)$ such that $\phi(x)=e_{i_ni_n}$. Clearly, $x\neq e_{nn}$. One can see that $\phi(x)$ and $\phi(e_{nn})-\phi(x)$ are idempotents, so as $\phi$ is an automorphism, their preimages are idempotents as well. Hence, we have

$$x^2=x \tag{14}$$

$$e_{nn}-e_{nn}x-xe_{nn}+x^2=e_{nn}-x. \tag{15}$$

Substituting (14) into (15) we get $2x=xe_{nn}+e_{nn}x$. The latter yields $x=\alpha e_{nn}$ for some $\alpha\in F\setminus\{0,1\}$. Moreover we have $\phi(\alpha^2e_{nn})=(\phi(\alpha e_{nn}))^2=e_{i_ni_n}$, so $\phi(\alpha^2e_{nn})-\phi(\alpha e_{nn})=0$. Let $a:=e_\infty+(\alpha^2-\alpha-1)e_{nn}$. Since $\alpha\neq 0,1$, $a$ is invertible. Hence, by Remark 2, $\phi(a)$ should be invertible as well. However

$$(\phi(a))_{i_ni_n}=e_{i_ni_n}\phi(a)=\phi(\alpha e_{nn})\phi(a)=\phi(\alpha e_{nn}a)=\phi((\alpha^2-\alpha)e_{nn})=0,$$

so $\phi(a)$ is not invertible – a contradiction.

Therefore for every $n$, either $\phi(e_{nn})=e_{kk}$ for some $k\in\mathbb{N}$, or $\phi(e_{nn})=0$.

Suppose now that $\cup_{n\in\mathbb{N}}I_n\neq\mathbb{N}$. Let $k\notin\cup_{n\in\mathbb{N}}I_n$. There exists $x\in\mathscr{M}_{Cf}(\lesssim,F)$, $x\neq e_{nn}$ for all $n\in\mathbb{N}$, such that $\phi(x)=e_{kk}$. Hence, for all $n\in\mathbb{N}$ the matrices $\phi(e_{nn})+\phi(x)$ and $\phi(x)$ are idempotents. Again, as $\phi$ is a bijection, we must have

$$\begin{aligned}x^2&=x\\ e_{nn}+e_{nn}x+xe_{nn}+x^2&=e_{nn}+x \qquad \text{for all } n\in\mathbb{N},\end{aligned}$$

which forces $e_{nn}x+xe_{nn}=0$ for all $n\in\mathbb{N}$. Hence, $x$ should be diagonal. Moroeover, as $\operatorname{char}(F)\neq 2$, we must have $x=0$, which is a contradiction.

Summing up, $\phi(e_{nn})=e_{\pi(n)\pi(n)}$ for some $\pi\in S(\mathbb{N})$. $\square$

LEMMA 12. *Suppose that $F$ is a field such that $\operatorname{char}(F)\neq 2$, $\lesssim$ is a preorder, and $\mathscr{M}_{Cf}(\lesssim,F)$ is contained in $\mathscr{M}_{\downarrow bound}(F)$ or $\mathscr{M}_{VK}(F)$. If $\phi$ is an automorphism of $\mathscr{M}_{Cf}(\lesssim,F)$, then there exist $g\in\mathscr{M}_{Cf}(\lesssim,F)$ and $\pi\in S(\mathbb{N})$ such that for all $n\in\mathbb{N}$ we have $(\phi(e_{nn}))^g=e_{\pi(n)\pi(n)}$.*

*Proof.* From Lemmas 3 and 4 we know that there exists $g \in \mathcal{M}_{Cf}(F)$ such that $(\phi(e_{nn}))^g$ are diagonal. From Lemma 11 we have learned that in this case we must have $(\phi(e_{nn}))^g = e_{\pi(n)\pi(n)}$ for some $\pi \in S(\mathbb{N})$. Now notice that, by Lemma 5, $g$ is in $\mathcal{M}_{Cf}(\lesssim, F)$. This completes the proof. $\square$

LEMMA 13. *If $F$ is a field, $\lesssim$ a preorder, and $\phi$ an automorphism of the ring $\mathcal{M}_{Cf}(\lesssim, F)$ such that for some $\phi \in S(\mathbb{N})$ we have $\phi(e_{nn}) = e_{\pi(n)\pi(n)}$ for every $n \in \mathbb{N}$, then*

1. $(\phi(x))_{\pi(n)\pi(n)} = (\phi(x_{nn}e_{nn}))_{\pi(n)\pi(n)}$ *for all* $n \in \mathbb{N}$;
2. *for any* $(n,m) \in \lesssim$ *we have* $\phi(\alpha e_{nm}) = \alpha_1 e_{\pi(n)\pi(m)} + \alpha_2 e_{\pi(m)\pi(n)}$ *with* $\alpha_1\alpha_2 = 0$, $\alpha_1 + \alpha_2 \neq 0$.

*Proof.* The first point follows from (7). The second point is a consequence of equations (9a), (9b), the fact that $e_{\pi(n)\pi(n)} + \alpha_1 e_{\pi(n)\pi(m)} + \alpha_2 e_{\pi(m)\pi(n)}$ can be idempotent only if $\alpha_1\alpha_2 = 0$, and bijectivity of $\phi$. $\square$

Now we prove our second and third main result.

*Proof of Theorem 2.* According to Lemma 12, for some matrix $g \in \mathcal{M}_{Cf}(\lesssim, F)$ we have $\phi = \mathcal{I}nn_g \cdot \psi$, where $\psi(e_{nn}) = e_{\pi(n)\pi(n)}$ for all $n \in \mathbb{N}$ and $\pi \in S(\mathbb{N})$. Consider then $\psi$.

By Remark 1, $\psi(S(B_n)) = S(B_{n'})$ for $S(B_n) \sim S(B_{n'})$. Consider then the isomorphic pairs $S(B_n)$, $S(B_{n'})$.

Let $S(B_n) \supset S(C_{i_1}), S(C_{i_2}), \ldots$, and $S(B_{n'}) \supset S(C_{i'_1}), S(C_{i'_2}), \ldots$ . Again, by Remark 1, for every $p$ there exists $r$ such that $\psi(S(C_{i_p})) = C_{i'_r}$. From Lemma 1 we know that $S(C_{i_p})$, $S(C_{i'_r})$ are isomorphic to the same subring of $\mathcal{M}_k(F)$ for some finite $k$, i.e.

$$S(C_{i_p}) \sim \mathcal{M}_k(\lesssim', F), \quad S(C_{i'_r}) \sim \mathcal{M}_k(\lesssim'', F), \quad \mathcal{M}_k(\lesssim', F) \sim \mathcal{M}_k(\lesssim'', F).$$

From the last relation it follows that there exists a permutation $\pi'$ of $\{1, 2, \ldots, k\}$ such that $\pi'(\lesssim') = \lesssim''$. From this and Theorem 3 we obtain then that $S(C_{i'_r}) = \mathcal{I}nn_g \cdot \overline{\sigma} \cdot \hat{\pi}'$. As $g$, $\sigma$ and $\pi'$ are determined for $C_{i_p}$, we can denote them by $\mathcal{I}nn_{g_{i_p}}$, $\sigma_{i_p}$, $\pi'_{i_p}$.

Let $(i,j) \in \lesssim$, $i \in C_{i_{p_1}}$, $j \in C_{i_{p_2}}$ with $p_2 \neq p_1$. Then

$$\begin{aligned}\sigma_{i_{p_1}}(\alpha)\psi(e_{ij}) &= \psi(\alpha e_{ii})\psi(e_{ij}) = \psi(\alpha e_{ii} \cdot e_{ij}) = \psi(\alpha e_{ij}) \\ &= \psi(e_{ij} \cdot \alpha e_{jj}) = \psi(e_{ij})\psi(\alpha e_{jj}) = \sigma_{i_{p_2}}\psi(e_{ij}).\end{aligned}$$

Since $\psi(e_{ij}) \neq 0$, the above equation forces $\sigma_{i_{p_1}}(\alpha) = \sigma_{i_{p_2}}(\alpha)$ for all $\alpha \in F$. As all the classes $C_{i_p}$ are contained in one class $B_n$, we have $\sigma_{i_{p_1}} = \sigma_{i_{p_2}}$ for any $i_{p_1}$, $i_{p_2}$.

Hence, we can write that

$$\begin{aligned}\phi &= \mathcal{I}nn_g \cdot \overline{(\mathcal{I}nn_{g_n} \cdot \sigma_n \cdot \hat{\pi}_n)_{n \in N}} \cdot \mathcal{B}_\pi \\ &= \mathcal{I}nn_g \cdot \mathcal{I}nn_{g'} \cdot \overline{(\sigma_n)}_{n \in N} \cdot \hat{\pi}' \cdot \mathcal{B}_\pi = \mathcal{I}nn_{g''} \cdot \overline{(\sigma_n)}_{n \in N} \cdot \hat{\pi}''. \quad \square\end{aligned}$$

### 4.2. The automorphism group and some examples

Just as in the previous section from Theorem 2 we get

THEOREM 5. *Suppose that $F$ is a field, $\lesssim$ is a preorder and $\mathscr{M}_{Cf}(\lesssim, F) = \overline{\oplus_{n\in N} S(B_n)}$ is contained in $\mathscr{M}_{VK}(F)$ or $\mathscr{M}_{\downarrow bound}(F)$. The group of automorphisms of $\mathscr{M}_{Cf}(\lesssim, F)$ is isomorphic to a subgroup of*

$$\mathrm{GL}_{Cf}(\lesssim, F) \wr [(\mathscr{A}ut(F))^N \wr S(\mathbb{N})].$$

*Proof.* As in the proof of Theorem 4 we notice that

- the group of inner automorphisms of $\mathscr{M}_{Cf}(\lesssim, F)$ is isomorphic to $\mathrm{GL}_{Cf}(\lesssim, F)$,
- the group of all automorphisms of the form $\overline{(\sigma_n)_{n\in N}}$ is isomorphic to $(\mathscr{A}ut(F))^N$,
- the group of all $\hat{\pi}$ is isomorphic to a subgroup of $S(\mathbb{N})$,

and we have

$$(\overline{(\sigma_n)}_{n\in N} \cdot \hat{\pi}) \cdot (\overline{(\sigma_n')}_{n\in N} \cdot \hat{\pi}') = \overline{(\sigma_n)_{n\in N}} \cdot \overline{(\sigma_{\pi(n)}')_{n\in N}} \cdot \hat{\pi} \cdot \hat{\pi}' = \overline{(\sigma_n \cdot \sigma_{\pi(n)}')_{n\in N}} \cdot (\hat{\pi} \cdot \hat{\pi}'),$$

and

$$(\mathscr{I}nn_g \cdot \psi) \cdot (\mathscr{I}nn_{g'} \cdot \psi') = \mathscr{I}nn_g \cdot \mathscr{I}nn_{\psi(g')} \cdot \psi \cdot \psi' = \mathscr{I}nn_{g\psi(g')} \cdot (\psi \cdot \psi'),$$

so the result follows. $\square$

Let us present some automorphism groups for a few rings.

EXAMPLE 4. Let $\lesssim = \{(n,n) : n \in \mathbb{N}\} \cup \{(1,2),(4,3)\}$, so we identify the ring $\mathscr{M}_{Cf}(\lesssim, F)$ with

$$\begin{pmatrix} * & * & & & & & \\ & * & & & & & \\ & & * & & & & \\ & & * & * & & & \\ & & & & * & & \\ & & & & & * & \\ & & & & & & \ddots \end{pmatrix}.$$

We have $B_1 = \{1,2\}$, $B_2 = \{3,4\}$, $B_n = \{n+2\}$ for $n \geqslant 3$, so automorphisms can permute $S(B_1)$ and $S(B_2)$ with each other, but with no other subring, and permute $B_n$ for $n \geqslant 3$ with each other. Hence, we have the following group of permutations:

$$\mathscr{P}erm = \{\pi_1\pi_2 : \pi_1 \in \{\mathrm{id}, (1\,4)(2\,3)\},\ 1,2,3,4 \notin \mathrm{supp}(\pi_2)\}$$

and

$$\mathscr{A}ut(\mathscr{M}_{Cf}(\lesssim, F)) \sim G \leqslant \mathrm{GL}_{Cf}(\lesssim, F) \wr [(\mathscr{A}ut(F))^{\mathbb{N}} \wr \mathscr{P}erm].$$

It can be noticed that $G$ is isomorphic to

$$[(\mathrm{T}_2(F))^2 \wr [(\mathscr{A}ut(F))^2 \wr S_2]] \times [(\mathscr{A}ut(F))^{\mathbb{N}} \wr S(\mathbb{N})].$$

EXAMPLE 5. Consider the ring $\mathscr{M}_{Cf}(\lesssim, F)$, where

$$\lesssim = \{(1,1),(1,2),(2,1),(2,2),(3,3),(3,4),(4,3),(4,4)\} \cup \{(n,m):\ 5 \leqslant n \leqslant m\},$$

i.e. $\mathscr{M}_{Cf}(\lesssim, F)$ is of the following 'shape':

$$\begin{pmatrix} * & * & & & & & & \\ * & * & & & & & & \\ & & * & * & & & & \\ & & * & * & & & & \\ & & & & * & * & * & \cdots \\ & & & & & * & * & \\ & & & & & & * & \\ & & & & & & & \ddots \end{pmatrix}.$$

The group of its automorphisms is isomorphic to the direct product of automorphisms of the matrices of the form

$$\begin{pmatrix} * & * & & \\ * & * & & \\ & & * & * \\ & & * & * \end{pmatrix}$$

and the group of automorphisms of $\mathscr{T}_\infty(F)$.

For the second ring the group of automorphisms is $\mathrm{T}_\infty(F) \wr \mathscr{A}ut(F)$, whereas for the first

$$(\mathrm{GL}_2(F))^2 \wr [(\mathscr{A}ut(F))^2 \wr \mathscr{P}erm],$$

where

$$\mathscr{P}erm = \{\pi \in S_4:\ \text{ either } \pi(1), \pi(2) \in \{1,2\} \text{ or } \pi(1), \pi(2) \in \{3,4\}\}.$$

## 5. Some additional comments

We end the paper with a few more remarks.

1. Additional to Propositions 2, 3 from [8] (see also [7]) the two propositions below can be proven.

PROPOSITION 3. *If $\lesssim$ is a preorder on $\mathbb{N}$ and $R$ is an associative ring, then the following conditions are equivalent.*

1. *$\lesssim$ is a linear order.*

2. *$\mathscr{M}_{Cf}(\lesssim, R)$ is a permutation conjugate of $\mathscr{T}_\infty(F)$.*

PROPOSITION 4. *If $\lesssim$ is a preorder on $\mathbb{N}$ and $R$ is associative ring, then the following conditions are equivalent.*

1. *$\lesssim$ is an order.*

2. *$\mathscr{M}_{Cf}(\lesssim,R)$ is the intersection of some permutation conjugates of $\mathscr{T}_\infty(R)$.*

The proofs are adapted from [7].

*Proof of Proposition 3.* Suppose that (1) holds and consider $(\mathbb{N},\leqslant)$, where $\leqslant$ is a natural order on $\mathbb{N}$. Then $(\mathbb{N},\lesssim)$ and $(\mathbb{N},\leqslant)$ are isomorphic. Let $\pi$ be an isomorphism between them. Then for $p \in \mathscr{M}_{Cf}(R)$ defined by the rule $p_{nm} = \delta(m,\pi(n))$ we have $p\mathscr{M}_{Cf}(\lesssim,F)p^{-1} = \mathscr{T}_\infty(F)$.

On the other hand, if for some permutation matrix $p \in \mathscr{M}_{Cf}(R)$ we have the equality $p\mathscr{M}_{Cf}(\lesssim,R)p^{-1} = \mathscr{T}_\infty(R)$, then $\mathscr{M}_{Cf}(\lesssim,R)$ and $\mathscr{T}_\infty(R)$ are isomorphic, and so are $(\mathbb{N},\lesssim)$, $(\mathbb{N},\leqslant)$. Thus, $\lesssim$ must be a linear order. $\square$

*Proof of Proposition 4.* It is known (see e.g. [6, p. 41]) that every order is an intersection of some linear orders. Hence, $\mathscr{M}_{Cf}(\lesssim,R)$ is an intersection of some $\mathscr{M}_{Cf}(\leqslant,R)$, where $\leqslant$ are linear orders. Consequently, by Proposition 3, it is an intersection of some conjugates of $\mathscr{T}_\infty(R)$. $\square$

2. The proofs presented in Sections 3, 4 are based on the form of the elements $\phi(e_{nn})$ and fact that these matrices can be diagonalized. Hence, it is natural to ask under which conditions an infinite matrix is diagonalizable. Some answers to this question are given in [19].

*Acknowledgement.* The second author was supported by the National Research Foundation (of South Africa) under grant no. UID 72375. Any opinion, findings and conclusions or recommendations expressed in this material are those of the authors and therefore the National Research Foundation does not accept any liability in regard thereto.

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(Received February 28, 2015)

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