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PERIPHERAL LOCAL SPECTRUM PRESERVERS AND MAPS INCREASING THE LOCAL SPECTRAL RADIUS

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Abstract. We address two long standing problems in the context of local spectral radius preservers. First, we completely describe the form of maps preserving the peripheral local spectrum of product or triple product of operators. Second, we establish the automatic continuity of linear maps increasing the local spectral radius of operators at a fixed nonzero vector.

1. Introduction

Throughout this paper, X and Y denote infinite-dimensional complex Banach spaces, and $\mathscr{B}(X,Y)$ denotes the space of all bounded linear maps from X into Y. When $X = Y$, we simply write $\mathscr{B}(X)$ instead of $\mathscr{B}(X,X)$. The *local resolvent* set, $\rho_T(x)$, of an operator $T \in \mathscr{B}(X)$ at a point $x \in X$ is the union of all open subsets U of the complex plane $\mathbb{C}$ for which there is an analytic function $\phi : U \to X$ such that $(T-\lambda)\phi(\lambda) = x,\ (\lambda \in U)$. Clearly $\rho_T(x)$ contains the resolvent set $\rho(T)$ of T, but this containment could be proper. The *local spectrum* of T at x is defined by

$$\sigma_T(x) := \mathbb{C}\backslash\rho_T(x),$$

and thus it is a closed subset of $\sigma(T)$, the spectrum of T. The *local spectral radius* of T at x is defined by

$$\mathrm{r}_T(x) := \limsup_{n\to+\infty} \|T^n x\|^{\frac{1}{n}},$$

and coincides with maximum modulus of $\sigma_T(x)$ provided that T has the *single-valued extension property* (SVEP). Recall that $T \in \mathscr{B}(X)$ is said to have SVEP provided that for every open subset U of $\mathbb{C}$, the equation $(T-\lambda)\phi(\lambda) = 0,\ (\lambda \in U)$, has no non-trivial analytic solution ϕ. Every operator $T \in \mathscr{B}(X)$ for which the interior of its point spectrum, $\sigma_p(T)$, is empty enjoys this property. The set

$$\gamma_T(x) := \{\lambda \in \sigma_T(x) : |\lambda| = \mathrm{r}_T(x)\}$$

is called the *peripheral local spectrum* of T at x. Note that $\gamma_T(x) = \emptyset$ provided that $\max\{|\lambda| : \lambda \in \sigma_T(x)\} < \mathrm{r}_T(x)$. The remarkable books by P. Aiena [1] and by K.B.

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Laursen, M.M. Neumann [27] provide an excellent exposition as well as a rich bibliography of the local spectral theory.

The problem of linear preservers of local spectra of matrices and operators was initiated by A. Bourhim and T. Ransford in [10], and then it was continued by several authors; see for instance [4, 8, 11, 14, 15, 16, 17, 19, 26] and the references therein. However, we should add that the literature on this subject is very extensive. In [11], J. Bračič and V. Müller characterized surjective linear and continuous mappings on $\mathscr{B}(X)$ preserving the local spectrum (local spectral radius) at a fixed nonzero vector e of X, and thus extending the main results of [9, 21] to infinite-dimensional Banach spaces. In [17, Theorem 1.2], C. Costara showed that every linear surjective map on $\mathscr{B}(X)$ decreasing the local spectral radius at a nonzero vector of X is automatically continuous. In [14], C. Costara characterized linear maps on the algebra $M_n(\mathbb{C})$ of all $n\times n$ complex matrices preserving the local spectrum or local spectral radius at non-fixed vectors. He, in particular, showed that if φ is a linear map on $M_n(\mathbb{C})$ then for every $T\in M_n(\mathbb{C})$ there exists a nonzero vector $x_T\in\mathbb{C}^n$ such that $\sigma_{\varphi(T)}(x_T)\cap\sigma_T(x_T)\neq\emptyset$ if and only if φ is an automorphism or an anti-automorphism on $M_n(\mathbb{C})$.

Besides linear local spectra preservers, nonlinear maps preserving different local spectra were considered by a number of authors. In [16], C. Costara described surjective linear maps on $\mathscr{B}(X)$ which preserve operators of local spectral radius zero at points of X. He showed, in particular, that if φ is a surjective linear map on $\mathscr{B}(X)$ such that for every $x\in X$ and $T\in\mathscr{B}(X)$, we have $\mathrm{r}_T(x)=0$ if and only if $\mathrm{r}_{\varphi(T)}(x)=0$, then there exists a nonzero scalar c such that $\varphi(T)=cT$ for all $T\in\mathscr{B}(X)$. This result has been extended in [7] to the nonlinear setting where it shown that if φ is a surjective (not necessarily linear) map on $\mathscr{B}(X)$ satisfying $\mathrm{r}_{T-S}(x)=0$ if and only if $\mathrm{r}_{\varphi(T)-\varphi(S)}(x)=0$, for every $x\in X$ and $S,\ T\in\mathscr{B}(X)$, then there is a nonzero scalar $c\in\mathbb{C}$ and an operator $A\in\mathscr{B}(X)$ such that $\varphi(T)=cT+A$ for all $T\in\mathscr{B}(X)$. In [15], C. Costara described surjective maps φ on $M_n(\mathbb{C})$ preserving the local spectral radius distance and showed that if x_0 is a nonzero vector of $\mathbb{C}^n$, then a surjective map φ on $M_n(\mathbb{C})$ satisfies $\varphi(0)=0$ and

$$\mathrm{r}_{\varphi(T)-\varphi(S)}(x_0)=\mathrm{r}_{T-S}(x_0),\ (T,\ S\in M_n(\mathbb{C})) \tag{1.1}$$

if and only if there exists an invertible matrix $A\in M_n(\mathbb{C})$ and unimodular scalar $\alpha\in\mathbb{C}$ such that either $Ax_0=x_0$ and $\varphi(T)=ATA^{-1}$ for all $T\in M_n(\mathbb{C})$ or $A\overline{x_0}=x_0$ and $\varphi(T)=A\overline{T}A^{-1}$ for all $T\in M_n(\mathbb{C})$, where $\overline{x_0}$ is the complex conjugation of x_0. In [5, 6], Bourhim and Mashreghi determined the structure of all surjective maps on $\mathscr{B}(X)$ preserving the local spectrum at a nonzero fixed vector of product and triple product of operators.

In this paper, we settle two important problems in this field. First, we characterize surjective maps on $\mathscr{B}(X)$ 'preserving the peripheral local spectrum' at a nonzero fixed vector of product and triple product of operators. Second, we show that any linear surjective map on $\mathscr{B}(X)$ 'increasing the local spectral radius' at a nonzero vector of X is bijective and continuous. The main tools in the proofs of our results are the characterization of the linearly independence of two operators and the characterization of rank one operators in term of the peripheral local spectrum at a nonzero fixed vector of product and triple product of operators. These marginal results by themselves are interesting.

2. Statement of the main results

In this section, we gather the statement of our main results. However, to prove each theorem some further tools are needed which are developed in subsequent sections. Each case is discussed below.

In Section 4, we first establish a local spectral identity principle that characterizes the linear dependence of two operators in term of the peripheral local spectrum of product of operators. Second, we provide a local spectral characterization of rank one operators in term of the peripheral local spectrum of product of operators. These are the essential ingredients in establishing the following result that describes the structure of all surjective maps on $\mathscr{B}(X)$ preserving the peripheral local spectrum at a nonzero fixed vector of product of operators.

THEOREM 2.1. *Let $x_0 \in X$ and $y_0 \in Y$ be two nonzero vectors. A map φ from $\mathscr{B}(X)$ onto $\mathscr{B}(Y)$ satisfies*

$$\gamma_{\varphi(T)\varphi(S)}(y_0) = \gamma_{TS}(x_0), \qquad (T,\, S \in \mathscr{B}(X)), \tag{2.2}$$

if and only if there exists a bijective bounded linear mapping A from X into Y such that $Ax_0 = y_0$, and either $\varphi(T) = -ATA^{-1}$ for all $T \in \mathscr{B}(X)$ or $\varphi(T) = ATA^{-1}$ for all $T \in \mathscr{B}(X)$.

In Section 5, we characterize maps preserving the peripheral local spectrum at a fixed vector of triple product of operators. More explicitly, we prove the following result. Its proof uses as well a local spectral identity principle that characterizes the linear dependence of two operators a local spectral characterization of rank one operators in term of the peripheral local spectrum of triple product of operators.

THEOREM 2.2. *Let $x_0 \in X$ and $y_0 \in Y$ be two nonzero vectors. A map φ from $\mathscr{B}(X)$ onto $\mathscr{B}(Y)$ satisfies*

$$\gamma_{\varphi(T)\varphi(S)\varphi(T)}(y_0) = \gamma_{TST}(x_0), \qquad (T,\, S \in \mathscr{B}(X)), \tag{2.3}$$

if and only if there exists a bijective mapping $A \in \mathscr{B}(X,Y)$ such that $Ax_0 = y_0$ and $\varphi(T) = \lambda ATA^{-1}$ for all $T \in \mathscr{B}(X)$, where λ is a third root of unity, i.e., $\lambda^3 = 1$.

In Section 6, we turn our attention to linear maps increasing the local spectral radius at a nonzero fixed vector of X. We show that any linear surjective map φ from $\mathscr{B}(X)$ into $\mathscr{B}(Y)$ increasing the local spectral radius at a nonzero vector of X is bijective, continuous and spectrally bounded from below; i.e., there is a constant m such that $\mathrm{r}(T) \leqslant m\mathrm{r}(\varphi(T))$ for all $T \in \mathscr{B}(X)$. If the reverse inequality is satisfied then φ is called spectrally bounded. Here, $\mathrm{r}(T)$ denotes the classical spectral radius of any operator $T \in \mathscr{B}(X)$.

THEOREM 2.3. *Let $x_0 \in X$ be a fixed nonzero element, and let φ be a surjective linear map from $\mathscr{B}(X)$ into $\mathscr{B}(Y)$. If there is a constant $M > 0$ such that*

$$\mathrm{r}_T(x_0) \leqslant M\mathrm{r}(\varphi(T)) \tag{2.4}$$

for all $T \in \mathscr{B}(X)$, then φ is a continuous bijective map spectrally bounded from below.

A few comments must be added to the statement of the result. In [29], Šemrl described spectrally bounded maps on $\mathscr{L}(\mathscr{H})$ when $\mathscr{H}$ is an infinite-dimensional complex Hilbert space and provided an example showing that there are infinite-dimensional Banach spaces X and spectrally bounded maps on $\mathscr{B}(X)$ that are of standard forms. In general, the complete description of all surjective linear maps from $\mathscr{B}(X)$ into $\mathscr{B}(Y)$ that are spectrally bounded or spectrally bounded from below is still unknown and remains an open problem. Of course, if such a description is obtained, then one would be able to characterize surjective linear maps on $\mathscr{B}(X)$ increasing or decreasing the local spectral radius of operators at a fixed nonzero vector. In [20], Fošner and Šemrl obtained a characterization of the surjective linear maps on $\mathscr{B}(X)$ that are both spectrally bounded and spectrally bounded from below. The obtained forms are somehow different from the ones appeared in [12] where Brešar and Šemrl showed that a surjective linear map on $\mathscr{B}(X)$ preserves the spectral radius if and only if it is either an automorphism or anti-automorphism multiplied by a scalar of modulus one.

3. Preliminaries and auxiliary results

In this section, we fix some notions and collect some useful lemmas needed for the proof of our main results. We also establish some results which are interesting in their own right. The first lemma summarizes some basic properties of the local spectrum which will be used frequently.

LEMMA 3.1. *For an operator $T \in \mathscr{B}(X)$, vectors $x,\ y \in X$ and a nonzero scalar $\alpha \in \mathbb{C}$, the following statements hold.*

1. *If T has SVEP, then $\sigma_T(x) \neq \emptyset$ provided that $x \neq 0$.*
2. *$\sigma_T(\alpha x) = \sigma_T(x)$ if $\alpha \neq 0$, and $\sigma_{\alpha T}(x) = \alpha\sigma_T(x)$.*
3. *$\sigma_T(x+y) \subset \sigma_T(x) \cup \sigma_T(y)$. The equality holds if $\sigma_T(x) \cap \sigma_T(y) = \emptyset$.*
4. *If T has SVEP, $x \neq 0$ and $Tx = \lambda x$ for some $\lambda \in \mathbb{C}$, then $\sigma_T(x) = \{\lambda\}$.*
5. *If T has SVEP and $Tx = \alpha y$, then $\sigma_T(y) \subset \sigma_T(x) \subset \sigma_T(y) \cup \{0\}$.*
6. *If $R \in \mathscr{B}(X)$ commutes with T, then $\sigma_T(Rx) \subset \sigma_T(x)$.*
7. *$\sigma_{T^n}(x) = \{\sigma_T(x)\}^n$ for all $x \in X$ and $n \geqslant 1$.*

Proof. See for instance [1, 27]. □

For any operator $T \in \mathscr{B}(X)$, let T^* be its adjoint on the dual space X^* of X. For every nonzero $x \in X$ and $f \in X^*$, let $x \otimes f$ denote the rank one operator defined on X by

$$(x \otimes f)(y) := f(y)x.$$

Note that every rank one operator on X can be written in this way and every finite rank operator is a finite sum of rank one operators.

The second lemma is a useful observation needed to establish the linearity of surjective maps $\varphi : \mathscr{B}(X) \to \mathscr{B}(Y)$ satisfying either (2.2) or (2.3).

LEMMA 3.2. *If x_0 is a nonzero vector in X and R is a rank one operator in $\mathscr{B}(X)$, then following assertions hold.*

1. $\gamma_{(T+S)R}(x_0) = \gamma_{TR}(x_0) + \gamma_{SR}(x_0)$ *for all* $T,\ S \in \mathscr{B}(X)$.

2. $\gamma_{R(T+S)R}(x_0) = \gamma_{RTR}(x_0) + \gamma_{RSR}(x_0)$ *for all operator* $T,\ S \in \mathscr{B}(X)$.

Proof. The proof relies on the following fact that

$$\gamma_{x\otimes f}(x_0) = \begin{cases} \{0\} & \text{if } f(x_0) = 0 \\ \\ \{f(x)\} & \text{if } f(x_0) \neq 0 \end{cases} \tag{3.5}$$

for all $x \in X$ and $f \in X^*$. $\square$

We close this section with two more lemmas needed in the sequel.

LEMMA 3.3. *Let $x_0 \in X$ and $y_0 \in Y$ be nonzero vectors, and let $A : X \to Y$ and $B : X^* \to Y^*$ be bijective linear transformations. The following statements are equivalent.*

1. *For every $x \in X$ and $f \in X^*$, we have* $\mathrm{r}_{x\otimes f}(x_0) = \mathrm{r}_{Ax\otimes Bf}(y_0)$.

2. *A is continuous, $B = \alpha A^{*-1}$ and $Ax_0 = \beta y_0$ for some nonzero scalars $\alpha,\ \beta \in \mathbb{C}$ with $|\alpha| = 1$.*

Proof. If A is continuous, $B = \alpha A^{-1}$ and $Ax_0 = \beta y_0$ for some nonzero scalars $\alpha,\ \beta \in \mathbb{C}$ with $|\alpha| = 1$, then

$$\mathrm{r}_{Ax\otimes Bf}(y_0) = \mathrm{r}_{\alpha A(x\otimes f)A^{-1}}(y_0) = |\alpha|\mathrm{r}_{x\otimes f}(A^{-1}y_0) = \mathrm{r}_{x\otimes f}(\beta^{-1}x_0) = \mathrm{r}_{x\otimes f}(x_0)$$

for all $x \in X$ and $f \in X^*$. This establishes the implication $(2) \Rightarrow (1)$.

Conversely, assume that $\mathrm{r}_{x\otimes f}(x_0) = \mathrm{r}_{Ax\otimes Bf}(y_0)$ for all $x \in X$ and $f \in X^*$. Let $x \in X$ and $f \in X^*$. First, note that

$$\mathrm{r}_{x\otimes f}(x_0) = \begin{cases} 0 & \text{if } f(x_0) = 0, \\ \\ |f(x)| & \text{if } f(x_0) \neq 0. \end{cases} \tag{3.6}$$

Second, let us show that

$$|f(x)| = |Bf(Ax)|. \tag{3.7}$$

Assume first that $f(x_0) \neq 0$, and note that, since $\mathrm{r}_{Ax_0\otimes Bf}(y_0) = \mathrm{r}_{x_0\otimes f}(x_0) = |f(x_0)|$, we have $Bf(y_0) \neq 0$. Thus (3.6) shows that

$$|f(x)| = \mathrm{r}_{x\otimes f}(x_0) = \mathrm{r}_{Ax\otimes Bf}(y_0) = |Bf(Ax)|.$$

This ensures that (3.7) holds in this case. If, however, $f(x_0) = 0$, take a linear functional $g \in X^*$ such that $g(x_0) \neq 0$ and note that it follows from what has been shown previously that

$$|(f+\lambda g)(x)| = |B(f+\lambda g)(Ax)| = |Bf(Ax) + \lambda Bg(Ax)|$$

for all nonzero scalars $\lambda \in \mathbb{C}$. Letting λ goes to 0, we get $|f(x)| = |Bf(Ax)|$ which establishes (3.7) in this case too; as desired.

Now, let us show that A is continuous and $B = A^{-1}$. Let $(x_n)_n$ be a sequence in X converging to 0, and let $y \in Y$ such that $\lim_{n\to\infty} Ax_n = y$. For every $f \in X^*$, we have

$$|Bf(y)| = \lim_{n\to\infty} |Bf(Ax_n)| = \lim_{n\to\infty} |f(x_n)| = 0,$$

and thus, B is bijective and $f \in X^*$ is an arbitrary linear functional, we see that $y = 0$. The closed graph theorem tells us that A is continuous.

Moreover, if $f \in X^*$ is a fixed linear functional, then for every $x \in X$, we have

$$|f(x)| = |Bf(Ax)| = |A^*Bf(x)|,$$

and thus $A^*B = \alpha \mathbf{1}_X$ for some nonzero scalar $\alpha \in \mathbb{C}$. Now, we show that x_0 and $A^{-1}y_0$ are linearly independent. If not, there is a linear functional f in X^* such that $f(x_0) = 1$ and $f(A^{-1}y_0) = 0$ and

$$\begin{aligned} 1 &= \mathrm{r}_{x_0\otimes f}(x_0) = \mathrm{r}_{Ax_0\otimes Bf}(y_0) = \mathrm{r}_{\alpha Ax_0\otimes A^{*-1}f}(y_0) \\ &= \mathrm{r}_{\alpha A(x_0\otimes f)A^{-1}}(y_0) = \mathrm{r}_{\alpha x_0\otimes f}(A^{-1}y_0) = 0. \end{aligned}$$

This contradiction shows that there is a nonzero scalar $\beta \in \mathbb{C}$ such that $Ax_0 = \beta y_0$. To finish the proof, we show that α must has modulus one. Pick up a linear functional f in X^* such that $f(x_0) = 1$, and note that

$$\begin{aligned} 1 &= \mathrm{r}_{x_0\otimes f}(x_0) = \mathrm{r}_{Ax_0\otimes Bf}(y_0) = \mathrm{r}_{\alpha A(x_0\otimes f)A^{-1}}(y_0) \\ &= \mathrm{r}_{\alpha x_0\otimes f}(A^{-1}y_0) = \mathrm{r}_{\alpha x_0\otimes f}(x_0) = |\alpha|. \end{aligned}$$

The proof is therefore complete. $\square$

REMARK 3.4. If the local spectral radius is replaced by the peripheral local spectrum in the first statement of the above lemma, then it is easy to see that α must be 1 in the second statement.

LEMMA 3.5. *Let $x_0 \in X$ and $y_0 \in Y$ be nonzero vectors, and let $C : X^* \to Y$ and $D : X \to Y^*$ be bijective linear mappings. Then there are $x \in X$ and $f \in X^*$ such that $\mathrm{r}_{x\otimes f}(x_0) \neq \mathrm{r}_{Cf\otimes Dx}(y_0)$.*

Proof. Choose a nonzero linear functional $g \in Y^*$ such that $g(y_0) = 0$ and set $x = D^{-1}g$. Because x and x_0 are nonzero vectors in X, one can find a linear functional $f \in X^*$ such that $f(x_0) \neq 0$ and $f(x) \neq 0$. Then $0 \neq |f(x)| = \mathrm{r}_{x\otimes f}(x_0)$ and $\mathrm{r}_{Cf\otimes Dx}(y_0) = \mathrm{r}_{Cf\otimes g}(y_0) = 0$; as desired. $\square$

4. Proof of theorem 2.1

To prove Theorem 2.1, we need two auxiliary, but important, result: first, a local spectral identity principle that characterizes the linear dependence of two operators in term of the peripheral local spectrum; second, a spectral characterization of rank one operators in term of the peripheral local spectrum.

THEOREM 4.1. *Let $x_0 \in X$ be a nonzero vector, and let $A,\ B \in \mathscr{B}(X)$. The following statements are equivalent.*

1. $A = \alpha B$ *for some nonzero scalar* $\alpha \in \mathbb{C}$.

2. *For every* $T \in \mathscr{B}(X)$, *we have* $\mathrm{r}_{AT}(x_0) = 0$ *if and only if* $\mathrm{r}_{BT}(x_0) = 0$.

3. *For every rank one operator* $T \in \mathscr{B}(X)$, *we have* $\mathrm{r}_{AT}(x_0) = 0$ *if and only if* $\mathrm{r}_{BT}(x_0) = 0$.

Proof. The implications $(1) \Rightarrow (2)$ and $(2) \Rightarrow (3)$ are trivial. We only need to establish $(3) \Rightarrow (1)$.

Assume that (3) holds, i.e., $\mathrm{r}_{AT}(x_0) = 0$ if and only if $\mathrm{r}_{BT}(x_0) = 0$ for all rank one operators $T \in \mathscr{B}(X)$. Let us first show that Ax_0 and Bx_0 are linearly dependent. Suppose to the contrary that Ax_0 and Bx_0 are linearly independent. This implies that x_0, Ax_0 and Bx_0 are linearly dependent. Since, if not, there is a linear functional $f_0 \in X^*$ such that $f_0(x_0) = f_0(Ax_0) = 1$ and $f_0(Bx_0) = 0$. For $T_0 := x_0 \otimes f_0$, we have $(AT_0)^n x_0 = Ax_0$ for all $n \geqslant 1$ and $\mathrm{r}_{AT_0}(x_0) = 1$. We also have $(BT_0)^n x_0 = 0$ for all $n \geqslant 2$ and thus $\mathrm{r}_{BT_0}(x_0) = 0$. This is a contradiction. Thus, there are constants α and β in $\mathbb{C}$ such that $x_0 = \alpha Ax_0 + \beta Bx_0$. Note that either $\alpha \neq 0$ or $\beta \neq 0$ and thus we may and shall assume that $\alpha \neq 0$. Let $f_1 \in X^*$ be a linear functional such that $f_1(Ax_0) = 1$ and $f_1(Bx_0) = 0$. For $T_1 := x_0 \otimes f_1$, we have $(AT_1)^n x_0 = \alpha Ax_0$ for all $n \geqslant 1$ and $\mathrm{r}_{AT_1}(x_0) = 1$. We also have $(BT_1)^n x_0 = 0$ for all $n \geqslant 2$ and $\mathrm{r}_{BT_1}(x_0) = 0$. This contradiction shows that $Ax_0 = \alpha_{x_0} Bx_0$ for some nonzero scalar α_{x_0}.

Now, let x be an arbitrary vector in X, and let $S \in \mathscr{B}(X)$ be an operator such that $Sx_0 = x$. Replacing T by ST in the third statement, we note that $\mathrm{r}_{AST}(x_0) = 0$ if and only if $\mathrm{r}_{BST}(x_0) = 0$ for all rank one operators $T \in \mathscr{B}(X)$. By what has been shown above, there is α_x that $Ax = ASx_0 = \alpha_x BSx_0 = \alpha_x Bx$ for some nonzero scalar α_x. Thus, there is a nonzero scalar $\alpha \in \mathbb{C}$ such that $A = \alpha B$. $\square$

As a consequence, we obtain the following corollary which characterizes, in term of the peripheral local spectrum, when two operators are the same.

COROLLARY 4.2. *Let* $x_0 \in X$ *be a nonzero vector, and let* A, $B \in \mathscr{B}(X)$. *The following statements are equivalent.*

1. $A = B$.

2. $\gamma_{AT}(x_0) = \gamma_{BT}(x_0)$ *for all operators* $T \in \mathscr{B}(X)$.

3. $\gamma_{AT}(x_0) = \gamma_{BT}(x_0)$ *for all rank one operators* $T \in \mathscr{B}(X)$.

Proof. We only need to show that $(3) \Rightarrow (1)$. So, assume that $\gamma_{AT}(x_0) = \gamma_{BT}(x_0)$ for all rank one operators $T \in \mathscr{B}(X)$, and note that there is a nonzero $\alpha \in \mathbb{C}$ such that $A = \alpha B$, by Theorem 4.1. To show that such α must be one, assume first that there is $x \in X$ such that Bx and x_0 are linearly independent, and let $f \in X^*$ be a linear functional such that $f(Bx) = f(x_0) = 1$. Set $T := x \otimes f$, and note that $BT = Bx \otimes f$ and thus $\gamma_{BT}(x_0) = \gamma_{BT}(x_0 - Bx) \cup \gamma_{BT}(Bx) = \{0,\ 1\}$. It follows that $\{1\} = \gamma_{BT}(x_0) = \gamma_{AT}(x_0) = \gamma_{\alpha BT}(x_0) = \{\alpha\}$, and $\alpha = 1$.

Now, if Bx and x_0 are linearly dependent for all $x \in X$, then either $B = 0$ and there is nothing to prove, or $B = x_0 \otimes f$ for some $f \in X^*$. If the last case occurs, pick up $x \notin \ker(f)$ such that $f(x) = 1$ and a linear functional $g \in X^*$ such that $g(x_0) = 1$. Let $T := x \otimes g$, and note that $Bx = x_0$ and that $BTx_0 = Bx = x_0$. It follows that $\{1\} = \gamma_{BT}(x_0) = \gamma_{AT}(x_0) = \gamma_{\alpha BT}(x_0) = \{\alpha\}$. $\square$

The following theorem, which may be of independent interest, gives a spectral characterization of rank one operators in term of the peripheral local spectrum.

THEOREM 4.3. *For a nonzero vector x_0 of X and a nonzero operator $R \in \mathscr{B}(X)$, the following statements are equivalent.*

1. *R has rank one.*
2. *$\gamma_{RT}(x_0)$ is a singleton for all $T \in \mathscr{B}(X)$.*
3. *$\gamma_{RT}(x_0)$ is a singleton for all operators $T \in \mathscr{B}(X)$ of rank at most two.*

Proof. Obviously, if R has rank one and $T \in \mathscr{B}(X)$ is an arbitrary operator, then RT has rank one too and thus $\gamma_{RT}(x_0)$ is a singleton. This shows that the implication $(1) \Rightarrow (2)$ always holds. So, we only need to establish the implication $(3) \Rightarrow (1)$.

Assume that $\gamma_{RT}(x_0)$ is a singleton for all operators $T \in \mathscr{B}(X)$ of rank at most two. Suppose, by the way of contradiction, that R has rank at least two, and let us first show that x_0 is not in the range of R. If x_0 is not in the range of R, then there are $u,\ v \in X$ such that $x_0 = Ru$ and $x_1 = Rv$ are linearly independent. Let f_0 and f_1 be two linear functionals in X^* such that $f_i(x_j) = \delta_{ij}$, where δ_{ij} is the delta Kronecker symbol. For $T_0 := (2v - u) \otimes f_0 + v \otimes f_1$, we have $RTx_1 = Rv = x_1$ and

$$RT(x_0 - x_1) = R(2v - u) - x_1 = -(x_0 - x_1).$$

Thus,

$$\sigma_{RT}(x_0) = \sigma_{RT}(x_1) \cup \sigma_{RT}(x_0 - x_1) = \{-1, 1\},$$

and $\gamma_{RT}(x_0) = \{-1, 1\}$ contains two different elements. This contradiction shows that x_0 is not in the range of R.

Now, pick two elements $u = Rx$ and $v = Ry$ from the range of R so that x_0, u and v are linearly independent. Choose two linear functionals f and g in X^* such that

$$f(x_0) = f(u) = 1,\ f(v) = 0,$$

and

$$g(x_0) = -g(v) = 1,\ g(u) = 0.$$

For $R_1 := x \otimes f + y \otimes g$, we have

$$RR_1u = u,\ RR_1x_0 = u + v \text{ and } RR_1(RR_1x_0 - u) = -v = -(RR_1x_0 - u).$$

Thus, $\sigma_{RR_1}(u) = \{1\}$ and $\sigma_{RR_1}(RR_1x_0 - u) = \{-1\}$, and therefore

$$\sigma_{RR_1}(RR_1x_0) = \sigma_{RR_1}(RR_1x_0) \cup \sigma_{RR_1}(RR_1x_0) = \{-1,\ 1\}.$$

From this, we see that

$$\{-1,\ 1\} = \sigma_{RR_1}(RR_1x_0) \subset \sigma_{RR_1}(x_0) \subset \sigma_{RR_1}(RR_1x_0) \cup \{0\} = \{-1,\ 0,\ 1\},$$

and $\gamma_{RR_1}(x_0) = \{-1,\ 1\}$ contains two different elements. This contradiction shows that R has rank one, and establishes the implication $(3) \Rightarrow (1)$. The proof is therefore complete. $\square$

We collected all necessary ingredients and we therefore are ready to prove our first main result, i.e., Theorem 2.1.

Proof of Theorem 2.1. We only need to establish the 'only if' part whose proof is long and delicate. Hence, we break it into several steps. Assume that φ satisfies (2.2).

Step 1: We show that φ is injective and $\varphi(0) = 0$. If $\varphi(A) = \varphi(B)$ for some operators A and B in $\mathscr{B}(X)$, we get

$$\gamma_{AT}(x_0) = \gamma_{\varphi(A)\varphi(T)}(y_0) = \gamma_{\varphi(B)\varphi(T)}(y_0) = \gamma_{BT}(x_0)$$

for all $T \in \mathscr{B}(X)$. By Corollary 4.2, we have $A = B$ and thus φ is injective. But since φ is assumed to be surjective, the map φ is, in fact, bijective. For the second part of the claim, note that for every $T \in \mathscr{B}(X)$, we have $\{0\} = \gamma_{0\times T}(x_0) = \gamma_{\varphi(0)\varphi(T)}(y_0)$. Again by Corollary 4.2 and the bijectivity of φ, we see that $\varphi(0) = 0$.

Step 2: We show that either

$$\gamma_{\varphi(T)}(y_0) = \gamma_T(x_0) \tag{4.8}$$

or

$$\gamma_{\varphi(T)}(y_0) = -\gamma_T(x_0) \tag{4.9}$$

for all $T \in \mathscr{B}(X)$. To do so, we first prove that $\varphi(\mathbf{1}) = \mathbf{1}$ or $\varphi(\mathbf{1}) = -\mathbf{1}$. Indeed, for every $T \in \mathscr{B}(X)$, we have

$$\{0\} = \gamma_T(x_0) = \gamma_{T^2}(x_0) \iff \{0\} = \gamma_{\varphi(T)^2}(y_0) = \gamma_{\varphi(T)}(y_0),$$

and thus

$$\gamma_{\varphi(T)}(y_0) = \{0\} \iff \gamma_T(x_0) = \{0\} = \gamma_{\mathbf{1}T}(x_0) \iff \gamma_{\varphi(\mathbf{1})\varphi(T)}(y_0) = \{0\}.$$

By the subjectivity of φ and Lemma 4.1, we see that $\varphi(\mathbf{1}) = \alpha\mathbf{1}$ for some nonzero scalar $\alpha \in \mathbb{C}$. Such α must be 1 or -1 since

$$\{1\} = \gamma_{\mathbf{1}^2}(x_0) = \gamma_{\varphi(\mathbf{1})^2}(y_0) = \gamma_{\alpha^2\mathbf{1}}(y_0) = \{\alpha^2\}.$$

If $\varphi(\mathbf{1}) = \mathbf{1}$, then for every $T \in \mathscr{B}(X)$, we have

$$\gamma_T(x_0) = \gamma_{\mathbf{1}\times T}(x_0) = \gamma_{\varphi(\mathbf{1})\varphi(T)}(y_0) = \gamma_{\mathbf{1}\times\varphi(T)}(y_0) = \gamma_{\varphi(T)}(y_0),$$

and (4.8) is established. If $\varphi(\mathbf{1}) = -\mathbf{1}$, then

$$\gamma_T(x_0) = \gamma_{\mathbf{1}\times T}(x_0) = \gamma_{\varphi(\mathbf{1})\varphi(T)}(y_0) = \gamma_{-\mathbf{1}\times\varphi(T)}(y_0) = -\gamma_{\varphi(T)}(y_0)$$

for all $T \in \mathscr{B}(X)$. This establishes (4.9).

Step 3: The next goal is to show that φ is a linear map preserving rank one operators in both directions. Let $R \in \mathscr{B}(X)$ be a rank one operator, and note that, since $\varphi(0) = 0$ and φ is bijective, $\varphi(R) \neq 0$. Moreover, for every operator $S = \varphi(T) \in \mathscr{B}(Y)$, we have $\gamma_{RT}(x_0) = \gamma_{\varphi(R)\varphi(T)}(y_0) = \gamma_{\varphi(R)S}(y_0)$ contains at most one element. By Theorem 4.3, we see that $\varphi(R)$ has rank one. The converse holds since φ is bijective and φ^{-1} satisfies (2.2) too, and thus φ preserves rank one operators in both directions.

Step 4: To establish the linearity of φ, let us first show that φ is homogenous. For every $\lambda \in \mathbb{C}$ and $S,\ T \in \mathscr{B}(X)$, we have

$$\gamma_{\lambda\varphi(S)\varphi(T)}(y_0) = \lambda\gamma_{\varphi(S)\varphi(T)}(y_0) = \lambda\gamma_{ST}(x_0) = \gamma_{\lambda ST}(x_0) = \gamma_{\varphi(\lambda S)\varphi(T)}(y_0).$$

Since φ is bijective, Corollary 4.2 shows that $\varphi(\lambda S) = \lambda\varphi(S)$ for all $S \in \mathscr{B}(X)$ and $\lambda \in \mathbb{C}$; as desired. Now, to show that φ is additive keep in mind that φ preserves rank one operators in both directions. Let $R \in \mathscr{B}(X)$ be a rank one operator and $T,\ S \in \mathscr{B}(X)$, and note that, by Lemma 3.2, we have

$$\begin{aligned}\gamma_{\varphi(T+S)\varphi(R)}(y_0) &= \gamma_{(T+S)R}(x_0) = \gamma_{TR}(x_0) + \gamma_{SR}(x_0)\\ &= \gamma_{\varphi(T)\varphi(R)}(y_0) + \gamma_{\varphi(S)\varphi(R)}(y_0)\\ &= \gamma_{\big(\varphi(T)+\varphi(S)\big)\varphi(R)}(y_0).\end{aligned}$$

By the arbitrariness of the rank one operator R, the bijectivity of φ and Lemma 4.1, we deduce that

$$\varphi(T+S) = \varphi(T) + \varphi(S)$$

for all $S,\ T \in \mathscr{B}(X)$, and φ is linear.

Step 5: We show that φ takes the desired form. Since φ is a bijective linear map preserving the rank one operators in both directions, either there are bijective linear mappings $A : X \to Y$ and $B : X^* \to Y^*$ such that

$$\varphi(x \otimes f) = Ax \otimes Bf,\ (x \in X,\ f \in X^*), \tag{4.10}$$

or there are bijective linear mappings $C : X^* \to Y$ and $D : X \to Y^*$ such that

$$\varphi(x \otimes f) = Cf \otimes Dx,\ (x \in X,\ f \in X^*); \tag{4.11}$$

see for instance [31, Theorem 3.3]. By Lemma 3.5, the second form can not occur, and thus φ only takes the form (4.10). Since φ satisfies either (4.8) or (4.9), Lemma 3.3 shows that A is continuous, $B = \alpha A^{*-1}$ and $Ax_0 = \beta y_0$ for some nonzero scalars $\alpha,\ \beta \in \mathbb{C}$ with $\alpha = \pm 1$. After replacing A by $\beta^{-1}A$, we may and shall assume that $Ax_0 = y_0$ and keep in mind that $\varphi(x \otimes f) = \pm A(x \otimes f)A^{-1}$ for all $x \in X$ and $f \in X^*$. Now, for every rank one operator $R \in \mathscr{B}(X)$ and every $T \in \mathscr{B}(X)$, we have

$$\gamma_{\pm ATA^{-1}\varphi(R)}(y_0) = \gamma_{ATA^{-1}ARA^{-1}}(y_0) = \gamma_{ATRA^{-1}}(y_0) = \gamma_{TR}(x_0) = \gamma_{\varphi(T)\varphi(R)}(y_0).$$

By Corollary 4.2, we see that $\varphi(T) = \pm ATA^{-1}$ for all $T \in \mathscr{B}(X)$. The proof is now complete. $\square$

5. Proof of theorem 2.2

For the proof of Theorem 2.2 we need two new ingredients that we establish below. The first result is a *local spectral identity principle* that characterizes in term of local spectral radius of triple product of operators when two given operators are linearly dependent. The second result characterizes rank one operators in term of the peripheral local spectrum of triple product of operators.

THEOREM 5.1. *Let $x_0 \in X$ be a nonzero vector, and let $A,\ B \in \mathscr{B}(X)$. The following statements are equivalent.*

1. *$A = \alpha B$ for some nonzero scalar $\alpha \in \mathbb{C}$.*

2. *$\mathrm{r}_{TAT}(x_0) = \mathrm{r}_{TBT}(x_0)$ for all operators $T \in \mathscr{B}(X)$.*

3. $\mathrm{r}_{TAT}(x_0) = \mathrm{r}_{TBT}(x_0)$ *for all rank one operators* $T \in \mathscr{B}(X)$.

Proof. We only need to show that the implication $(3) \Rightarrow (1)$ holds.

Assume that $\mathrm{r}_{TAT}(x_0) = \mathrm{r}_{TBT}(x_0)$ for all rank one operators $T \in \mathscr{B}(X)$, and let x be a nonzero vector from X and $f \in X^*$ a linear functional for which $f(x_0) \neq 0$ and $f(x) \neq 0$. For $T := x \otimes f$, we have

$$|f(x)f(Ax)| = \mathrm{r}_{TAT}(x_0) = \mathrm{r}_{TBT}(x_0) = |f(x)f(Bx)|,$$

and thus

$$|f(Ax)| = |f(Bx)| \tag{5.12}$$

for all linear functionals $f \in X^*$ satisfying $f(x_0) \neq 0$ and $f(x) \neq 0$. Now, let $f \in X^*$ be a linear functional in X^* for which $f(x_0) \neq 0$ and $f(x) = 0$, and note that x and x_0 must be linearly independent. Pick up a linear functional $g \in X^*$ such that $g(x_0) = 0$ and $g(x) = 1$, and note that $(f + \lambda g)(x_0) = f(x_0) \neq 0$ and $(f + \lambda g)(x) = \lambda \neq 0$ for all nonzero scalars $\lambda \in \mathbb{C}$. Thus (5.12) applied to $(f + \lambda g)$ shows that

$$|(f + \lambda g)(Ax)| = |(f + \lambda g)(Bx)|$$

for all nonzero scalars $\lambda \in \mathbb{C}$. Letting λ goes to 0, we get $|f(Ax)| = |f(Bx)|$ for all $f \in X^*$ for which $f(x_0) \neq 0$ and $f(x) = 0$. This and (5.12) show that

$$|f(Ax)| = |f(Bx)| \tag{5.13}$$

for all linear functionals $f \in X^*$ satisfying $f(x_0) \neq 0$. Finally, let $f \in X^*$ be a linear functional in X^* for which $f(x_0) = 0$ and pick up a linear functional $h \in X^*$ such that $h(x_0) = 1$. Since $(f + \lambda h)(x_0) = \lambda \neq 0$ for all nonzero scalars $\lambda \in \mathbb{C}$, the identity (5.13) applied to $(f + \lambda g)$ shows that

$$|(f + \lambda h)(Ax)| = |(f + \lambda h)(Bx)|$$

for all nonzero scalars $\lambda \in \mathbb{C}$. Letting λ goes to 0, we get $|f(Ax)| = |f(Bx)|$ for all $f \in X^*$, and Ax and Bx are linearly dependent for all $x \in X$. Hence, $A = \alpha B$ for some nonzero scalar $\alpha \in \mathbb{C}$; as desired. □

In the following corollary, we exploit the above result to characterize, in terms of local spectral radius of triple product of operators, when two given operators coincide.

COROLLARY 5.2. *Let* $x_0 \in X$ *be a nonzero vector, and let* A, $B \in \mathscr{B}(X)$. *The following statements are equivalent.*

1. $A = B$.

2. $\gamma_{TAT}(x_0) = \gamma_{TBT}(x_0)$ *for all operators* $T \in \mathscr{B}(X)$.

3. $\gamma_{TAT}(x_0) = \gamma_{TBT}(x_0)$ *for all rank one operators* $T \in \mathscr{B}(X)$.

Proof. We only need to show that $(3) \Rightarrow (1)$. So, assume that $\gamma_{TAT}(x_0) = \gamma_{TBT}(x_0)$ for all rank one operators $T \in \mathscr{B}(X)$, and note that there is a nonzero scalar $\alpha \in \mathbb{C}$ such that $A = \alpha B$, by Theorem 5.1. If $A = 0$, then there is nothing to prove since in this case

$B = 0$ too. So, assume $Ax \neq 0$ for some vector $x \in X$, and let g be a linear functional in X^* such that $g(x_0) \neq 0$, $g(x) \neq 0$ and $g(Ax) \neq 0$. For $T := x \otimes g$, we have

$$\{g(Ax)g(x)\} = \gamma_{TAT}(x_0) = \gamma_{TBT}(x_0) = \{g(Bx)g(x)\} = \{\alpha g(Ax)g(x)\},$$

and $\alpha = 1$. Thus $A = B$, and the proof is complete. $\square$

The following result characterizes rank one operators in terms of the peripheral local spectrum of triple product of operators.

THEOREM 5.3. *Let x_0 be a nonzero vector of X. For a nonzero operator $R \in \mathscr{B}(X)$, the following are equivalent.*

1. *R has rank one.*

2. *$\gamma_{TRT}(x_0)$ is a singleton for all $T \in \mathscr{B}(X)$.*

Proof. Obviously, if R has rank one and $T \in \mathscr{B}(X)$ is an arbitrary operator, then TRT has rank one too and thus $\gamma_{TRT}(x_0)$ is a singleton. This shows that the implication $(1) \Rightarrow (2)$ always holds.

Conversely, assume that R has rank at least two, and let us show that there exists $T \in \mathscr{B}(X)$ such that $\gamma_{TRT}(x_0)$ contains at least two elements. We shall discuss two cases.

Case 1. If there exist two vectors x_1, $x_2 \in X$ such that x_0, Rx_1 and Rx_2 are linearly independent, then there also exists $x \in X$ such that x, x_0, Rx_1 and Rx_2 are linearly independent. Hence, there exists an operator $T \in \mathscr{B}(X)$ of a finite rank such that $Tx_0 = x_1$, $Tx = x_2$, $TRx_1 = x_0 - 2x$ and $TRx_2 = -x$. Then we have

$$TRTx = TRx_2 = -x \text{ and } TRTx_0 = TRx_1 = x_0 - 2x,$$

and consequently, $TRT(x_0 - x) = x_0 - x$. Thus,

$$\sigma_{TRT}(x_0) = \sigma_{TRT}(x) \cup \sigma_{TRT}(x_0 - x) = \{-1, 1\},$$

and $\gamma_{TRT}(x_0) = \{-1, 1\}$ contains two different scalars.

Case 2. If x_0, Rx_1 and Rx_2 are linearly dependent for all x_1, $x_2 \in X$, then R has rank 2 and its image contains x_0. So, $R := x_1 \otimes f_1 + x_2 \otimes f_2$ and $x_0 = \alpha_1 x_1 + \alpha_2 x_2$ for some linearly independent vectors x_1, $x_2 \in X$, linearly independent linear functionals f_1, $f_2 \in X^*$ and α_1, $\alpha_2 \in \mathbb{C}$. If both α_1 and α_2 are nonzero scalars, then take z_1 and z_2 in X linearly independent of x_1 and x_2 such that $f_1(z_1) = \alpha_1^{-1}$, $f_1(z_2) = 0$, $f_2(z_1) = 0$ and $f_2(z_2) = -\alpha_2^{-1}$. Now, let $x := x_0 - z_1 - z_2 \neq 0$ and define $Tx_i = z_i$ and $Tz_i = \alpha_i z_i$. Note that $Tx = 0$, $TRTz_1 = z_1$ and $TRTz_2 = -z_2$. It follows that

$$\sigma_{TRT}(x_0) = \sigma_{TRT}(x + z_1 + z_2) = \sigma_{TRT}(x) \cup \sigma_{TRT}(z_1) \cup \sigma_{TRT}(z_2) = \{-1, 0, 1\},$$

and $\gamma_{TRT}(x_0) = \{-1, 1\}$ contains two different scalars.

If $\alpha_2 = 0$, then $x_0 = \alpha_1(x_1 - x_2) + \alpha_1 x_2$ and $R = (x_1 - x_2) \otimes f_1 + x_2 \otimes (f_1 + f_2)$. By what has shown above, there is $T \in \mathscr{B}(X)$ such that $\gamma_{TRT}(x_0) = \{-1, 1\}$ contains two different scalars.

The case when $\alpha_1 = 0$ is similar, and thus the implication $(2) \Rightarrow (1)$ is established. $\square$

We collected all required ingredients and we proceed to prove our second main result, i.e. Theorem 2.2.

Proof of Theorem 2.2. Checking the 'if' part is straightforward, and we therefore just deal with the 'only if' part. So assume that φ satisfies (2.3), and we proceed to show that φ takes the desired form.

Step 1: We first show that φ is injective and $\varphi(0)=0$. Assume that $\varphi(A)=\varphi(B)$ for some $A,\,B\in\mathscr{B}(X)$, and note that

$$\gamma_{TAT}(x_0)=\gamma_{\varphi(T)\varphi(A)\varphi(T)}(y_0)=\gamma_{\varphi(T)\varphi(B)\varphi(T)}(y_0)=\gamma_{TBT}(x_0)$$

for all $T\in\mathscr{B}(X)$. By Theorem 5.1, we see that $A=B$ and φ is injective and thus it is, in fact, bijective. In a similar way, we show that $\varphi(0)=0$. For every $T\in\mathscr{B}(X)$, we have

$$\{0\}=\gamma_{T\times 0\times T}(x_0)=\gamma_{\varphi(T)\varphi(0)\varphi(T)}(y_0).$$

Again, by Theorem 5.1 and the bijectivity of φ, we see that $\varphi(0)=0$.

Step 2: We show that φ is a linear map preserving rank one operators in both directions. Let $R\in\mathscr{B}(X)$ be a rank one operator and note that $\gamma_{TRT}(x_0)$ contains exactly one element and so does $\gamma_{\varphi(T)\varphi(R)\varphi(T)}(x_0)$. By Theorem 5.3 and the bijectivity of φ, we see that $\varphi(R)$ is a rank one operator. Since φ is bijective and φ^{-1} satisfies (2.3), we see that if $\varphi(R)$ is a rank one operator, then so is R.

Step 3: Let us verify that φ is linear. Let us first show that φ is homogenous. For every $\lambda\in\mathbb{C}$ and $A,\,T\in\mathscr{B}(X)$, we have

$$\begin{aligned}\gamma_{\varphi(T)(\lambda\varphi(A))\varphi(T)}(y_0)&=\lambda\gamma_{\varphi(T)\varphi(A)\varphi(T)}(y_0)\\&=\lambda\gamma_{TAT}(x_0)\\&=\gamma_{T(\lambda A)T}(x_0)\\&=\gamma_{\varphi(T)\varphi(\lambda A)\varphi(T)}(y_0).\end{aligned}$$

Since φ is surjective, Theorem 5.1 shows that $\varphi(\lambda A)=\lambda\varphi(A)$ for all $\lambda\in\mathbb{C}$ and $A\in\mathscr{B}(X)$; as desired.

To show that φ is additive, let $R\in\mathscr{B}(X)$ be a rank one operator and let $T,\,S\in\mathscr{B}(X)$. Then, by Lemma 3.2, we have

$$\begin{aligned}\gamma_{\varphi(R)\varphi(T+S)\varphi(R)}(y_0)&=\gamma_{R(T+S)R}(x_0)\\&=\gamma_{RTR}(x_0)+\gamma_{RSR}(x_0)\\&=\gamma_{\varphi(R)\varphi(T)\varphi(R)}(y_0)+\gamma_{\varphi(R)\varphi(S)\varphi(R)}(y_0)\\&=\gamma_{\varphi(R)\left(\varphi(T)+\varphi(S)\right)\varphi(R)}(y_0).\end{aligned}$$

By the arbitrariness of rank one operator R, the bijectivity of φ and Theorem 5.1, we deduce that

$$\varphi(T+S)=\varphi(T)+\varphi(S)$$

for all $T,\,S\in\mathscr{B}(X)$, and φ is linear.

Step 4: We show that $\varphi(\mathbf{1})=\lambda\mathbf{1}$ for some scalar $\lambda\in\mathbb{C}$ for which $\lambda^3=1$. Note that, since $\sigma_{T^n}(x_0)=\{\sigma_T(x_0)\}^n$ and $\mathrm{r}_{T^n}(x_0)=\{\mathrm{r}_T(x_0)\}^n$ for all $T\in\mathscr{B}(X)$ and $n\geqslant 1$, we see that $\gamma_{T^n}(x_0)=\{\gamma_T(x_0)\}^n$ for all $T\in\mathscr{B}(X)$ and $n\geqslant 1$. Therefore,

$$\{\gamma_T(x_0)\}^3=\gamma_{T^3}(x_0)=\gamma_{\varphi(T)^3}(y_0)=\left\{\gamma_{\varphi(T)}(y_0)\right\}^3 \tag{5.14}$$

for all $T\mathscr{B}(X)$. Let $R \in \mathscr{B}(X)$ be an operator such that $\varphi(R) = \mathbf{1}$, and let us first verify that Rx and x are linearly dependent for all $x \in X$. Fix a nonzero vector $x \in X$, and let us first show that either x and Rx are linearly dependent or x_0 and Rx are linearly dependent. Assume to the contrary that Rx is linearly independent with both x and x_0, and pick up a linear functional $f_1 \in X^*$ such that $f_1(Rx) = 0$, $f_1(x) \neq 0$ and $f_1(x_0) \neq 0$. For $T_1 := x \otimes f_1$, we have $T_1RT_1x_0 = 0$ and

$$\{0\} = \gamma_{T_1RT_1}(x_0) = \gamma_{\varphi(T_1)\varphi(R)\varphi(T_1)}(y_0) = \gamma_{\varphi(T_1)^2}(y_0).$$

This shows that $\gamma_{\varphi(T_1)}(y_0) = \{0\}$ and thus, by (5.14), we have

$$\{0\} = \{\gamma_{\varphi(T_1)}(y_0)\}^3 = \{\gamma_{T_1}(x_0)\}^3.$$

This shows that $\gamma_{T_1}(x_0) = \{0\}$ and thus $\mathrm{r}_{T_1}(x_0) = 0$. Since $T_1x_0 = f_1(x_0)x$ and $f_1(x_0) \neq 0$, we have

$$|f_1(x)| = \mathrm{r}_{T_1}(x) = \mathrm{r}_{T_1}(T_1x_0) = \mathrm{r}_{T_1}(x_0) = 0.$$

This contradicts the fact that $f_1(x) \neq 0$ and shows that either x and Rx are linearly dependent or x_0 and Rx are linearly dependent. In particular, when $x = x_0$ we note that Rx_0 and x_0 are linearly independent.

Next, we may and shall assume that x and x_0 are linearly independent and show that x_0 and Rx are linearly independent so that x and Rx are linearly dependent. Assume for the sake of contradiction that $Rx = \beta_0 x_0$ for some scalar $\beta_0 \in \mathbb{C}$. If $\beta_0 = 0$, then take a linear functional $f_2 \in X^*$ such that $f_2(x) = f_2(x_0) = 1$ and let $T_2 := x \otimes f_2$. Just as above, one gets that

$$\{0\} = \{\gamma_{\varphi(T_2)}(y_0)\}^3 = \{\gamma_{T_2}(x_0)\}^3,$$

and

$$1 = \mathrm{r}_{T_2}(x) = \mathrm{r}_{T_2}(T_2x_0) = \mathrm{r}_{T_2}(x_0) = 0.$$

This contradiction shows that β_0 must be different from 0. Now, pick up a linear functional $f_3 \in X^*$ such that $f_3(x_0) = 1$ and $f_3(x) = 2\beta_0$. For $T_3 := x \otimes f_3$, we have $T_3RT_3x_0 = \beta_0 x$ and $T_3RT_3x = 2\beta_0^2 x$. Thus, by Lemma 3.1, we have

$$\{2\beta_0^2\} = \sigma_{T_3RT_3}(x) \subset \sigma_{T_3RT_3}(x_0) \subset \sigma_{T_3RT_3}(x) \cup \{0\} = \{0,\, 2\beta_0^2\}.$$

Hence, $\{2\beta_0^2\} = \gamma_{T_3RT_3}(x_0) = \gamma_{\varphi(T_3)^2}(y_0)$. Since $\varphi(T_3)$ has rank one, $\gamma_{\varphi(T_3)}(y_0)$ contains one element and thus either $\gamma_{\varphi(T_3)}(y_0) = \{\sqrt{2}\beta_0\}$ or $\gamma_{\varphi(T_3)}(y_0) = \{-\sqrt{2}\beta_0\}$. But, as $T_3x_0 = x$ and $T_3x = 2\beta_0 x$, we have $\{2\beta_0\} = \sigma_{T_3}(x) \subset \sigma_{T_3}(x_0) \subset \sigma_{T_3}(x) \cup \{0\} = \{0,\, 2\beta_0\}$ and $\gamma_{T_3}(x_0) = \{2\beta_0\}$. By (5.14), we see that either $2\beta_0 = \sqrt{2}\beta_0$ or $2\beta_0 = -\sqrt{2}\beta_0$ which is impossible. This contradiction shows that x_0 and Rx are linearly independent and thus x and Rx are linearly dependent. The arbitrariness of x and the linearity of φ show that $\varphi(\mathbf{1}) = \lambda\mathbf{1}$ for some scalar $\lambda \in \mathbb{C}$ and by (5.14), we see that $\lambda^3 = 1$. As $\lambda^{-1}\varphi$ satisfies (2.3) as well, we may and shall assume that $\varphi(\mathbf{1}) = \mathbf{1}$. Then

$$\gamma_T(x_0) = \gamma_{\mathbf{1}T\mathbf{1}}(x_0) = \gamma_{\varphi(\mathbf{1})\varphi(T)\varphi(\mathbf{1})}(y_0) = \gamma_{\varphi(T)}(y_0) \tag{5.15}$$

for all $T \in \mathscr{B}(X)$.

Step 5: It remains to check that φ takes the desired form. Since φ is a bijective linear map preserving rank one operators in both directions, either there are bijective linear mappings $A : X \to Y$ and $B : X^* \to Y^*$ such that

$$\varphi(x \otimes f) = Ax \otimes Bf,\ (x \in X,\ f \in X^*), \tag{5.16}$$

or there are bijective linear mappings $C : X^* \to Y$ and $D : X \to Y^*$ such that

$$\varphi(x \otimes f) = Cf \otimes Dx, \ (x \in X, \ f \in X^*); \tag{5.17}$$

see for instance [31, Theorem 3.3]. By Lemma 3.5 and Lemma 3.3, we see that φ only takes the first form such that A is continuous, $B = A^{*-1}$ and $Ax_0 = \beta y_0$ for some nonzero scalar $\beta \in \mathbb{C}$. After replacing A by $\beta^{-1}A$, we may and shall assume that $Ax_0 = y_0$. To finish the proof, note that for every rank one operator $R \in \mathscr{B}(X)$ and every $T \in \mathscr{B}(X)$, we have

$$\begin{aligned}
\gamma_{\varphi(R)ATA^{-1}\varphi(R)}(y_0) &= \gamma_{ARA^{-1}ATA^{-1}ARA^{-1}}(y_0) \\
&= \gamma_{ARTRA^{-1}}(y_0) \\
&= \gamma_{RTR}(x_0) \\
&= \gamma_{\varphi(R)\varphi(T)\varphi(R)}(y_0).
\end{aligned}$$

Hence, by Theorem 5.1, we see that $\varphi(T) = ATA^{-1}$ for all $T \in \mathscr{B}(X)$. The proof is now complete. □

6. Proof of theorem 2.3

As in the previous two cases, before giving the proof on the main result we need some auxiliary lemmas. The first lemma is quoted from [11], and will be used in the proof of Theorem 2.3. Recall that the surjectivity spectrum of an operator $T \in \mathscr{B}(X)$ is defined by

$$\sigma_{su}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not onto}\}.$$

It is a nonempty compact subset of $\sigma(T)$.

LEMMA 6.1. *Let $T \in \mathscr{B}(X)$, and let $x_0 \in X$ be a nonzero vector. Then, for each $\lambda \in \sigma_{su}(T)$ and every $\varepsilon > 0$, there is $S \in \mathscr{B}(X)$ such that $\|T - S\| < \varepsilon$ and $\lambda \in \sigma_S(x_0)$.*

Proof. See [11]. □

In [17, Lemma 2.1], Costara showed that if $A \in \mathscr{B}(X)$ and x_0 is a nonzero vector of X such that $\mathrm{r}_{T-A}(x_0) \leqslant \mathrm{r}(T)$ for all $T \in \mathscr{B}(X)$, then $A = 0$. A generalization of such lemma was obtained in [7], which is similar to a local version of Zemánek's spectral characterization of the radical [2, Theorem 5.3.1].

LEMMA 6.2. *For an operator $A \in \mathscr{B}(X)$ and a nonzero fixed vector $x_0 \in X$, the following statements are equivalent.*

(i) $A = 0$.

(ii) $\mathrm{r}_{T+A}(x_0) \leqslant \mathrm{r}_T(x_0)$ *for all operators* $T \in \mathscr{B}(X)$.

(iii) $\mathrm{r}_{T+A}(x_0) \leqslant \mathrm{r}(T)$ *for all operators* $T \in \mathscr{B}(X)$.

(iv) $\mathrm{r}(T+A) \leqslant \mathrm{r}(T)$ *for all operators* $T \in \mathscr{B}(X)$.

(v) $\mathrm{r}(T+A)=0$ *for all nilpotent operators* $T\in\mathscr{B}(X)$ *of at most rank one.*

Proof. See [7, 17]. □

We are now in a position to prove our third main result, i.e., Theorem 2.3.

Proof of Theorem 2.3. Assume that there is a constant $M>0$ such that $\mathrm{r}_T(x_0)\leqslant M\mathrm{r}(\varphi(T))$ for all $T\in\mathscr{B}(X)$, and let us prove that φ is injective. Suppose that $\varphi(T_0)=0$ for some $T_0\in\mathscr{B}(X)$, and let us show that $T_0=0$. Let $T\in\mathscr{B}(X)$, and consider the following function

$$f(\lambda):=\mathrm{r}_{\lambda T_0+T}(x_0),\ (\lambda\in\mathbb{C}).$$

Clearly, we have

$$f(\lambda)\leqslant f^*(\lambda)\leqslant M\mathrm{r}(\varphi(\lambda T_0+T))=M\mathrm{r}(\varphi(T))$$

for all $\lambda\in\mathbb{C}$. Here, $f^*(\lambda):=\limsup_{\mu\to\lambda}f(\mu)$, $(\lambda\in\mathbb{C})$, is the upper regularization of f. It is subharmonic (see for example [17]), and thus Liouville's Theorem implies that f^* is a constant function. It follows, in particular, that $f^*(1)=f^*(0)=\limsup_{\mu\to 0}\mathrm{r}_{\mu T_0+T}(x_0)$. This together with the upper semicontinuity of the spectral radius entail that

$$\mathrm{r}_{T_0+T}(x_0)=f(1)\leqslant f^*(1)=\limsup_{\mu\to 0}\mathrm{r}_{\mu T_0+T}(x_0)\leqslant\limsup_{\mu\to 0}\mathrm{r}(\mu T_0+T)\leqslant\mathrm{r}(T).$$

As T is an arbitrary operator, Lemma 6.2 implies that $T_0=0$ and φ is injective; as desired. Since φ is now a bijective map and its inverse φ^{-1} satisfies

$$\mathrm{r}_{\varphi^{-1}(S)}(x_0)\leqslant M\mathrm{r}(S)$$

for all $S\in\mathscr{B}(Y)$. By Theorem [17, Theorem 1.2] and the open mapping theorem, we deduce that both φ^{-1} and φ are continuous.

To show that φ is spectrally bounded from below, we argue as in [11] to show that $\mathrm{r}(T)\leqslant M\mathrm{r}(\varphi(T))$ for all $T\in\mathscr{B}(X)$. Pick up an operator and let $\lambda\in\sigma_{su}(T)$ such that $\mathrm{r}(T)=|\lambda|$. By Lemma 6.1, there is a sequence of operators $(T_n)_n\subset\mathscr{B}(X)$ converging to T such that $\lambda\in\sigma_{T_n}(x_0)$ for all n. Since φ is continuous and the spectral radius is upper semi-continuous, we have

$$M\mathrm{r}(\varphi(T))\geqslant M\limsup_{n\to\infty}\mathrm{r}(\varphi(T_n))\geqslant\limsup_{n\to\infty}\mathrm{r}_{T_n}(x_0)\geqslant|\lambda|=\mathrm{r}(T),$$

and φ is spectrally bounded from below. □

Finally, we turn our attention to some consequences of Theorem 2.3. The first one was established in [8] by Bourhim under the extra condition of the continuity.

COROLLARY 6.3. *Let $\mathscr{H}$ and $\mathscr{K}$ be two infinite-dimesional complex Hilbert spaces. Let $h_0\in\mathscr{H}$ and $k_0\in\mathscr{K}$ be fixed nonzero elements. For a linear map φ from $\mathscr{L}(\mathscr{H})$ onto $\mathscr{L}(\mathscr{K})$, the following are equivalent.*

(i) *There are two constants m, $M>0$ such that $m\mathrm{r}_T(h_0)\leqslant\mathrm{r}_{\varphi(T)}(k_0)\leqslant M\mathrm{r}_T(h_0)$ for all $T\in\mathscr{L}(\mathscr{H})$.*

(ii) *There is a constant $M>0$ such that $\mathrm{r}_{\varphi(T)}(k_0)\leqslant M\mathrm{r}_T(h_0)$ for all $T\in\mathscr{L}(\mathscr{H})$.*

(iii) *There is a constant* $m>0$ *such that* $m\mathrm{r}_T(h_0)\leqslant \mathrm{r}_{\varphi(T)}(k_0)$ *for all* $T\in\mathscr{L}(\mathscr{H})$.

(iv) *There is a bijection* $A\in\mathscr{L}(\mathscr{H},\mathscr{K})$ *and a nonzero scalar* α *such that* $Ah_0=k_0$ *and* $\varphi(T)=\alpha ATA^{-1}$ *for all* $T\in\mathscr{L}(\mathscr{H})$.

Proof. Combine [8, Corollary 5.5], [17, Theorem 1.2] and Theorem 2.3. □

Let $T\in\mathscr{B}(X)$ be a bounded linear operator and $x\in X$ a fixed vector. The *analytic residuum* of T, denoted by $\mathfrak{R}(T)$, is the largest open set $U\subseteq\mathbb{C}$ for which the equation $(T-\lambda)\varphi(\lambda)=0$, $(\lambda\in U)$, has no nontrivial analytic solution φ on U. Its closure is denoted by S_T, and the set $S_T\cup\sigma_T(x)$ is also called the *local spectrum* of T at x and is denoted by $\sigma_x(T)$ instead of $\sigma_T(x)$. Note that, unlike the standard local spectrum, $\sigma_x(T)$ is a nonempty closed set if $x\neq 0$, and that

$$\Gamma_T(x):=\max\{|\lambda|:\lambda\in\sigma_T(x)\}\leqslant \mathrm{r}_T(x)\leqslant\max\{|\lambda|:\lambda\in\sigma_x(T)\}\leqslant\mathrm{r}(T);\qquad(6.18)$$

see for instance [24, II. 14. Theorem 12]. If, however, T has the SVEP, then obviously the left three local spectral radii coincide.

The following result describes surjective linear maps φ on $\mathscr{B}(X)$ that preserve the above local spectrum at a nonzero fixed vector of X. Its proof is on the straightforward side, and will be included here for the sake of completeness.

COROLLARY 6.4. *Let* $\varphi:\mathscr{B}(X)\to\mathscr{B}(Y)$ *be a surjective linear map, and let* $x_0\in X$ *and* $y_0\in Y$ *be nonzero vectors. The following statements are equivalent.*

(i) $\sigma_{x_0}(T)=\sigma_{y_0}(\varphi(T))$ *for all* $T\in\mathscr{B}(X)$.

(ii) $\sigma_{x_0}(T)\subset\sigma_{y_0}(\varphi(T))$ *for all* $T\in\mathscr{B}(X)$.

(iii) $\sigma_{x_0}(T)\supset\sigma_{y_0}(\varphi(T))$ *for all* $T\in\mathscr{B}(X)$.

(iv) *There is a bijection* $A\in\mathscr{B}(X,Y)$ *such that* $Ax_0=y_0$ *and* $\varphi(T)=ATA^{-1}$ *for all* $T\in\mathscr{B}(X)$.

Proof. Clearly, $(iv)\Rightarrow(i)\Rightarrow(ii)$ and $(iv)\Rightarrow(i)\Rightarrow(iii)$. So, we only need to show that $(ii)\Rightarrow(iv)$ and $(iii)\Rightarrow(iv)$.

Assume that $\sigma_{x_0}(T)\supset\sigma_{y_0}(\varphi(T))$ for all $T\in\mathscr{B}(X)$, and note that it follows from (6.18) that

$$\mathrm{r}_{\varphi(T)}(y_0)\leqslant\max\{|\lambda|:\lambda\in\sigma_{y_0}(\varphi(T))\}\leqslant\max\{|\lambda|:\lambda\in\sigma_{x_0}(T)\}\leqslant\mathrm{r}(T)$$

for all $T\in\mathscr{B}(X)$. By [17, Theorem 1.2], φ is a continuous map. Now, let us prove that

$$\sigma(\varphi(T))\subset\sigma(T)\qquad(6.19)$$

for all $T\in\mathscr{B}(X)$. Since $S_{\varphi(T)}\subset\sigma_{y_0}(\varphi(T))\subset\sigma_{x_0}(T)\subset\sigma(T)$ and $\sigma(\varphi(T))=S_{\varphi(T)}\cup\sigma_{su}(\varphi(T))$ for all $T\in\mathscr{B}(X)$, we only need to show that $\sigma_{su}(\varphi(T))\subset\sigma(T)$ for all $T\in\mathscr{B}(X)$. Indeed, given an operator $T\in\mathscr{B}(X)$ and $\lambda\in\sigma_{su}(\varphi(T))$, by Lemma 6.1, there is a sequence $(S_n)_n$ of operators in $\mathscr{B}(Y)$ such that $\|S_n-\varphi(T)\|<n^{-1}$ and $\lambda\in\sigma_{S_n}(y_0)\subset\sigma_{y_0}(S_n)$. Applying the open mapping theorem, one can find a sequence

$(T_n)_n \subset \mathscr{B}(X)$ converging to T such that $\varphi(T_n) = S_n$ for all n, and thus $\lambda \in \sigma_{x_0}(T_n) \subset \sigma(T_n)$ for all n. Hence, $T_n - \lambda$ is not invertible for all n and so is $\lim_n(T_n - \lambda) = T - \lambda$ since the collection of all noninvertible operators in $\mathscr{B}(X)$ is closed. It follows that $\sigma_{su}(\varphi(T)) \subset \sigma(T)$ for all $T \in \mathscr{B}(X)$, and thus (6.19) is established. Now, to show that φ is injective it suffices, in view of [30, Theorem 3.4], to show that φ does not vanish at an operator of rank one. In fact, we shall show that $\varphi(x_0 \otimes f) \neq 0$ for all linear functionals $f \in X^*$ for which $f(x_0) = 1$. Assume for the sake of contradiction that there is a linear functional $f \in X^*$ such that $f(x_0) = 1$ and $\varphi(x_0 \otimes f) = 0$. By (6.19), we have $\sigma(\varphi(\mathbf{1})) \subset \sigma(\mathbf{1}) = \{1\}$, and thus $\varphi(\mathbf{1})$ has the SVEP and $\sigma(\varphi(\mathbf{1})) = \sigma_{\varphi(\mathbf{1})}(y_0) = \sigma_{y_0}(\varphi(\mathbf{1})) = \{1\}$. We also have $(\mathbf{1} + x_0 \otimes f)(x_0) = 2x_0$ and $\varphi(\mathbf{1} + x_0 \otimes f) = \varphi(\mathbf{1})$, and thus

$$\{1\} = \sigma_{y_0}(\varphi(\mathbf{1})) = \sigma_{y_0}(\varphi(\mathbf{1} + x_0 \otimes f)) \subset \sigma_{x_0}(\mathbf{1} + x_0 \otimes f) = \{2\}.$$

This contradiction shows that φ is injective and thus [30, Theorem 3.4] implies that either $\varphi(T) = ATA^{-1}$, $(T \in \mathscr{B}(X))$, for some isomorphism $A \in \mathscr{B}(X,Y)$, or $\varphi(T) = BT^*B^{-1}$, $(T \in \mathscr{B}(X))$, for some isomorphism $B \in \mathscr{B}(X^*,Y)$. Now, with the proof of Lemma 3.3 and Lemma 3.5, one can show that φ takes only the first form with A can be supposed to satisfy $Ax_0 = y_0$. This establishes the implication $(iii) \Rightarrow (iv)$.

Assume that $\sigma_{x_0}(T) \subset \sigma_{y_0}(\varphi(T))$ for all $T \in \mathscr{B}(X)$, and note that it follows from (6.18) that

$$\mathrm{r}_T(x_0) \leqslant \max\{|\lambda| : \lambda \in \sigma_{x_0}(T)\} \leqslant \max\{|\lambda| : \lambda \in \sigma_{y_0}(\varphi(T))\} \leqslant \mathrm{r}(\varphi(T))$$

for all $T \in \mathscr{B}(X)$. By Theorem 2.3, φ is a continuous bijective map and thus the implication $(iii) \Rightarrow (iv)$ applied to φ^{-1} shows that the implication $(ii) \Rightarrow (iv)$ always holds too. $\square$

7. Comments and open problems

In this section, we make some remarks and comments on nonlinear preservers of local spectral radius and discuss some related problems. First, we would like to point out that the restriction to infinite-dimensional Banach spaces in the statement of our main results is just for the sake of simplicity. Second, note that the local spectral radius of an operator $T \in \mathscr{B}(X)$ at a vector $x \in X$ could be defined by

$$\Gamma_T(x) := \max\{|z| : z \in \sigma_T(x)\}$$

with the convention that $\max \emptyset = -\infty$, and the peripheral local spectrum of T at x could be given by

$$\Lambda_T(x) := \{\lambda \in \sigma_T(x) : |\lambda| = \Gamma_T(x)\}.$$

Of course, $\Lambda_T(x) = \gamma_T(x)$ and $\mathrm{r}_T(x) = \Gamma_T(x)$ when T has the SVEP. Here, we would like to point out that our above results and their proofs remain valid when replacing $\gamma_T(x_0)$ by $\Lambda_T(x_0)$ and $\mathrm{r}_T(x_0)$ by $\Gamma_T(x_0)$.

In the sequel, let $x_0 \in X$ and $y_0 \in Y$ be two nonzero vectors. Our study can be viewed as a step towards the study of some further challenging problems on local spectral radius preservers. We mention two major open problems. First, which maps φ from $\mathscr{B}(X)$ onto $\mathscr{B}(Y)$ satisfy

$$\mathrm{r}_{\varphi(T)\varphi(S)}(y_0) = \mathrm{r}_{TS}(x_0), \qquad (T,\, S \in \mathscr{B}(X))? \tag{7.20}$$

Second, which maps φ from $\mathscr{B}(X)$ onto $\mathscr{B}(Y)$ satisfy

$$\mathrm{r}_{\varphi(T)\varphi(S)\varphi(T)}(y_0) = \mathrm{r}_{TST}(x_0), \qquad (T,\, S \in \mathscr{B}(X))? \tag{7.21}$$

Similar questions can be asked when replacing $\mathrm{r}_T(x_0)$ by $\Gamma_T(x_0)$.

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