ACCUMULATION OF COMPLEX EIGENVALUES OF AN INDEFINITE STURM—LIOUVILLE OPERATOR WITH A SHIFTED COULOMB POTENTIAL

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Abstract. For a particular family of long-range potentials V, we prove that the eigenvalues of the indefinite Sturm–Liouville operator $A = \text{sign}(x)(-\Delta + V(x))$ accumulate to zero asymptotically along specific curves in the complex plane. Additionally, we relate the asymptotics of complex eigenvalues to the two-term asymptotics of the eigenvalues of associated self-adjoint operators.

1. Introduction

Given a real-valued potential V such that

$$V \in L^\infty(\mathbb{R}), \quad \lim_{x \to \pm\infty} V(x) = 0, \quad \limsup_{x \to \pm\infty} x^2 V(x) < -\frac{1}{4}, \tag{1}$$

consider a one-dimensional Schrödinger operator in $L^2(\mathbb{R})$

$$T := T_V := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x),$$
$$\text{Dom}(T) := \left\{ f \in L^2(\mathbb{R}) \mid f, f' \in AC(\mathbb{R}), \, Tf \in L^2(\mathbb{R}) \right\}. \tag{2}$$

It is well known that in this case the spectrum $\text{Spec}(T)$ is bounded from below, the essential spectrum $\text{Spec}_{\text{ess}}(T) = [0, \infty)$, and the negative spectrum $\text{Spec}(T) \cap (-\infty, 0)$ consists of eigenvalues accumulating to zero from below.

Let $J := \text{sign}(x)$ be the multiplication operator by ± 1 on $\mathbb{R}_\pm$. In what follows we consider the point spectrum of the operator

$$A := A_V := JT_V, \quad \text{Dom}(A) = \text{Dom}(T). \tag{3}$$

This operator is not self-adjoint (and not even symmetric) on $L^2(\mathbb{R})$, and its spectrum need not therefore be real. However, as $J^* = J^{-1} = J$, A can be treated as a self-adjoint operator in the Krein space $(L^2(\mathbb{R}), [\cdot, \cdot])$ with indefinite inner product

$$[f,g] := \langle Jf, g \rangle_{L^2(\mathbb{R})} = \int_{\mathbb{R}} f(x)\overline{g(x)} \operatorname{sign}(x)\mathrm{d}x$$

or equivalently as a J-self-adjoint operator [1]. Operators of type (3) have been studied both in the framework of operator pencils, cf. [9, 22], and of indefinite Sturm–Liouville problems [4, 6, 19, 21].

   In both settings the literature is extensive, starting mostly with Soviet contributions in the 1960s, including those by Krein, Langer, Gohberg, Pontryagin and Shkalikov. We refer to [22, 21] for reviews and bibliographies. In particular due to its many applications, for example in control theory, mathematical physics and mechanics, the field is still very active, with recent works on the theoretical, as well as numerical, aspects, (see e.g. [9, 10, 14, 27] and references therein).

   In the special case of indefinite Sturm–Liouville operators, it is well known that for positive potentials, $V \geqslant 0$, the spectrum of A_V is real and the operator A_V is similar to a self-adjoint operator [7, 8, 24, 20].

   The case $V \in L^1(\mathbb{R}, (1+|x|)\mathrm{d}x)$ has been considered in [16], where it is shown that A_V is self-adjoint iff $\operatorname{Spec}(A_V) = \mathbb{R}$. Finally, for (quasi-)periodic finite zone potentials, [18] explores some conditions under which A_V is similar to a self-adjoint or a normal operator. For a review of indefinite weighted Sturm–Liouville problems on a finite interval, see [12].

   Let us return to our original problem (3). Recently, there was a rapid growth of interest in the case of non-positive potentials, especially by Behrndt, Trunk, and collaborators [2, 4, 6, 5], clarifying the structure of their spectra and other properties as well as stating new conjectures on rather unusual spectral behaviour [3].

   The following known results are a particular case of [4, Theorem 1 and Theorem 2] and [5, Theorem 4.2].

   PROPOSITION 1. *For the operator A in* (3),

(a) $\operatorname{Spec}(A)$ *is symmetric with respect to* $\mathbb{R}$.

(b) $\operatorname{Spec}_{ess}(A) = \mathbb{R}$.

(c) $\operatorname{Spec}(A) \setminus \mathbb{R}$ *consists of eigenvalues of finite multiplicity.*

(d) *No point of* $\overline{\mathbb{R}} \setminus \{0\}$ *is an accumulation point of non-real eigenvalues of* A.

(e) *At least one of the following statements is true:*

   (i) *The non-real eigenvalues of A accumulate only to* 0;

   (ii) *There exist embedded eigenvalues of A in $\mathbb{R}_+$ that accumulate to* 0;

   (iii) *There exist embedded eigenvalues of A in $\mathbb{R}_-$ that accumulate to* 0;

   (iv) *The growth of $\lambda \mapsto (A - \mu)^{-1}$ near zero is not of finite order.*

(f) *If additionally V is even, $V(x) = V(-x)$, then $\mathrm{Spec}(A)$ is also symmetric with respect to $i\mathbb{R}$.*

(g) *The non-real spectrum of A is contained in the strip $|\mathrm{Im}\,\mu| < 2\|V\|_\infty$.*

Despite the amount of information on the structure of the spectrum of A, known proofs of Proposition 1 are not constructive and, in fact, we do not even know *a priori* which of the four statements (e)(i)–(iv) are true for a particular given potential. Some numerical experiments, cf. [4], have recently led to conjecture that statement (e)(i) in Proposition 1 may hold for many potentials satisfying (1), see Figure 1.
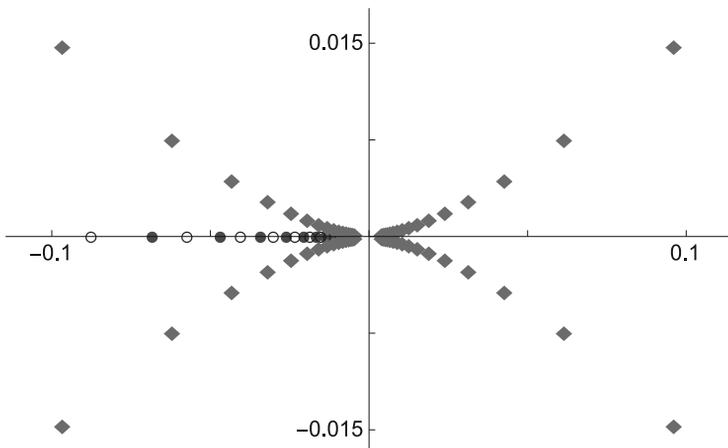

Figure 1: *A numerical example showing accumulation to 0 of complex eigenvalues (red diamonds) of the operator A_γ, $\gamma = 2.5$. The magenta and white circles on the negative real axis are the eigenvalues of T_γ corresponding to the eigenfunctions which are even or odd with respect to zero, cf. [4].*

In this paper we prove that for a particular family of potentials

$$V(x) = V_\gamma(x) = -\frac{\gamma}{1+|x|}, \quad \gamma > 0, \tag{4}$$

Proposition 1(e)(i) holds. Moreover we also prove (Theorem 6) that the complex eigenvalues of

$$T = T_\gamma := T_{V_\gamma}$$

accumulate to zero asymptotically along specific curves in the complex plane, and that the explicit asymptotics of complex eigenvalues of T_γ can be obtained from the asymptotics of eigenvalues of the self-adjoint operator

$$A = A_\gamma := A_{V_\gamma} \tag{5}$$

(or, more precisely, from the eigenvalues of its restriction on either even or odd (with respect to zero) subspace). We also extend these results to the more general non-

symmetric potentials

$$V_{\gamma_-,\gamma_+}(x) = \begin{cases} -\dfrac{\gamma_+}{1+|x|} & \text{if } x > 0 \\[2mm] -\dfrac{\gamma_-}{1+|x|} & \text{if } x < 0 \end{cases}, \quad \gamma_+, \gamma_- \in \mathbb{R}_+. \tag{6}$$

The rest of the paper is organised as follows. Section 2 contains the statements of our main results. The proofs, as well as some numerical examples, are in Sections 3–5; they are based on the explicit expressions for Jost solutions of the differential equation

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2}g(x) - \frac{\gamma}{1+x}g(x) = \mu g(x)$$

on $\mathbb{R}_+$ and a rather delicate asymptotic analysis involving Kummer functions. A brief exposition of some auxiliary results, mainly due to Temme [25], which we use in our proofs, is given in the Appendix.

2. Sharp asymptotics of the eigenvalues of the self-ajdiont operator

Let T_γ^D and T_γ^N denote the restrictions of the operator T_γ to $\mathbb{R}_+$ with Dirichlet and Neumann boundary condition at zero, resp. By the spectral theorem, for symmetric potentials $V_\gamma(x)$

$$\mathrm{Spec}(T_\gamma) = \mathrm{Spec}(T_\gamma^D) \cup \mathrm{Spec}(T_\gamma^N)$$

with account of multiplicities. Let $-\lambda_n^\#(\gamma)$ denote the eigenvalues of $T_\gamma^\#$, $\# = D$ or N, ordered increasingly. In what follows we often drop the explicit dependence on γ.

It is well-known that $-\lambda_n^\# < 0$ and $-\lambda_n^N < -\lambda_n^D < -\lambda_{n+1}^N$ for all $n \in \mathbb{N}$, and also that $-\lambda_n^\# \to 0-$ as $n \to \infty$.

Before stating our main results, we need some additional notation.

DEFINITION 2. Let $\mathscr{F}$ denote the class of piecewise smooth functions $F : \mathbb{R}_+ \to \mathbb{R}$ which have a discrete set of singularities (with no finite accumulation points). At each singularity both one-sided limits of F are $\pm\infty$ and differ by sign. Assume for simplicity that 0 is not a singularity of F, and that $F(0) = 0$. For $F \in \mathscr{F}$ we denote by $\Theta_F(x)$ the continuous branch of the multi-valued $\mathrm{Arctan}(F(x))$ such that $\Theta_F(0) = 0$.

REMARK 3. Away from the singularities of F, the function $\Theta_F(x)$ can be written in terms of the ordinary $\arctan(F(x))$ (which takes the values in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$) and the *total signed index* of F on $[0,x]$, which we denote by $Z_F(x)$, and which is defined as the total number of jumps from $+\infty$ to $-\infty$ on $[0,x]$ minus the total number of jumps in the opposite direction:

$$Z_F(x) := \left(\sum_{\{\tau \in (0,x] \,|\, \lim_{t \to \tau_-} F(t) = +\infty\}} - \sum_{\{\tau \in (0,x] \,|\, \lim_{t \to \tau_-} F(t) = -\infty\}} \right) 1. \tag{7}$$

Then

$$\Theta_F(x) = \arctan(F(x)) + \pi Z_F(x). \tag{8}$$

Obviously, $\Theta_{-F}(x) = -\Theta_F(x)$.

Our first result gives sharp two-term asymptotics of eigenvalues (accumulating to zero) of the self-adjoint operators $T_\gamma^{\#}$.

THEOREM 4. *As $n \to \infty$,*

$$\lambda_n^D(\gamma) = \frac{\gamma^2}{4n^2}\left(1 - \frac{2}{\pi n}\Theta_{R_1}(\gamma) + O\left(\frac{1}{n^2}\right)\right),$$

$$\lambda_n^N(\gamma) = \frac{\gamma^2}{4n^2}\left(1 - \frac{2}{\pi n}\Theta_{R_0}(\gamma) + O\left(\frac{1}{n^2}\right)\right),$$

where

$$R_k(\gamma) = \frac{J_k\left(2\sqrt{\gamma}\right)}{Y_k\left(2\sqrt{\gamma}\right)},$$

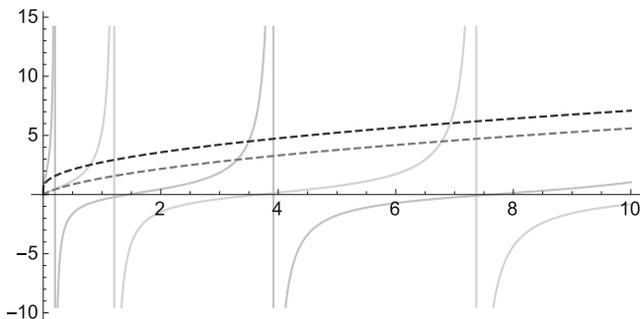*and J_k and Y_k denote the Bessel functions of the first and second kind, respectively.*

Figure 2: *$R_1(\gamma)$ and $R_0(\gamma)$ (resp. magenta and blue line) from Theorem 4 and the corresponding $\Theta_{R_1}(\gamma)$ and $\Theta_{R_0}(\gamma)$ (resp. magenta and blue dashed line).*
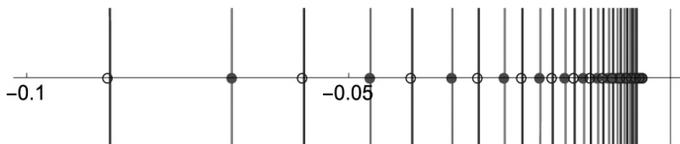

Figure 3: *Approximate eigenvalues of T_γ, $\gamma = 2.5$, as described in Theorem 4. The magenta (resp. white) circles correspond to $-\lambda_n^D$ (resp. $-\lambda_n^N$). The solid (resp. dashed) lines are the left-hand sides of (25) (resp. (26)) whose roots are the eigenvalues. Only curves and eigenvalues up to -0.01 are displayed.*

This immediately implies

COROLLARY 5. *As $n \to \infty$,*

$$\lambda_n^D - \lambda_n^N = O\left((\lambda_n^{\#})^{3/2}\right).$$

Some old papers found both in physical and mathematical literature addressed the problem of approximating the eigenvalues of Schrödinger operators with shifted Coulomb potentials, see e.g. [13, 26] and references therein. However they were considering somewhat different asymptotic limits, and to the best of our knowledge the two-terms asymptotics of Theorem 4 are new.

3. Sharp asymptotics of the eigenvalues of the non-self-ajdiont operator

Our main result is the following

THEOREM 6.   **(i)** *The eigenvalues of A_γ lie asymptotically on the curves*

$$|\operatorname{Im}\mu| = \Upsilon(\gamma)|\operatorname{Re}\mu|^{3/2} + O\left((\operatorname{Re}\mu)^2\right), \qquad \mu \to 0, \qquad (9)$$

where

$$\Upsilon(\gamma) = \frac{1}{\pi\gamma}\log\frac{q^2(\gamma)}{1+q^2(\gamma)} \qquad (10)$$

and

$$q(\gamma) := \pi\sqrt{\gamma}\left(J_0(2\sqrt{\gamma})J_1(2\sqrt{\gamma}) + Y_0(2\sqrt{\gamma})Y_1(2\sqrt{\gamma})\right). \qquad (11)$$

(ii) *More precisely, the eigenvalues $\{\mu\}_{n\in\mathbb{N}}$ of A_γ in the first quadrant (ordered by decreasing real part) are related to the absolute values $\lambda_n^{\#}$ of the eigenvalues of the self-adjoint operators $T_\gamma^{\#}$, $\# = D$ or N, by*

$$\mu_n = \lambda_n^D + \Upsilon^-(\gamma)(\lambda_n^D)^{3/2} + O\left((\lambda_n^D)^2\right) = \lambda_n^N + \Upsilon^+(\gamma)(\lambda_n^N)^{3/2} + O\left((\lambda_n^N)^2\right) \quad (12)$$

as $n \to \infty$, where

$$\Upsilon^{\mp}(\gamma) = \frac{4}{\pi\gamma}\arctan\left(\frac{1}{i \mp 2q(\gamma)}\right). \qquad (13)$$

The expressions for eigenvalues in the other quadrants are obtained by symmetries with respect to $\mathbb{R}$ and $i\mathbb{R}$.

Before proceeding to the proofs, we want to discuss the statements of Theorem 6 in more detail.

REMARK 7.   (a) It is immediately seen from (10) and (13) that

$$\Upsilon(\gamma) = \operatorname{Im}\Upsilon^-(\gamma) = \operatorname{Im}\Upsilon^+(\gamma).$$

(b) If we introduce two functions $\tau^{\mp} : \mathbb{R}_+ \to \mathbb{C}$ by

$$\tau^{\mp}(t) = t + \Upsilon^{\mp} t^{3/2},$$

then

$$\operatorname{Im} \tau^-(t) = \operatorname{Im} \tau^+(t) = \Upsilon t^{3/2},$$

which is just another way of writing (a). (We have dropped the dependence on γ here for clarity.)

(c) The statement (ii) of Theorem 6 contains in fact several results. Taking the imaginary parts of (12) leads immediately to (9) by Corollary 5. The equalities obtained by taking the real parts (12) are more intricate: they indicate that, up to the terms of order $(\lambda_n^\#)^2$, the values of $\operatorname{Re}(\tau^-(\lambda_n^D))$ and $\operatorname{Re}(\tau^+(\lambda_n^N))$ coincide.

In other words, we can construct the eigenvalues of the non-self-adjoint operator A_γ by perturbing either the anti-symmetric self-adjoint eigenvalues λ_n^D or the symmetric self-adjoint eigenvalues λ_n^N, and the different formulae still lead to the same result, modulo higher-order terms.

(d) The only previously known bound, see Proposition 1(g), implies only that for our potential V_γ, $|\operatorname{Im} \mu| < 2\gamma$.

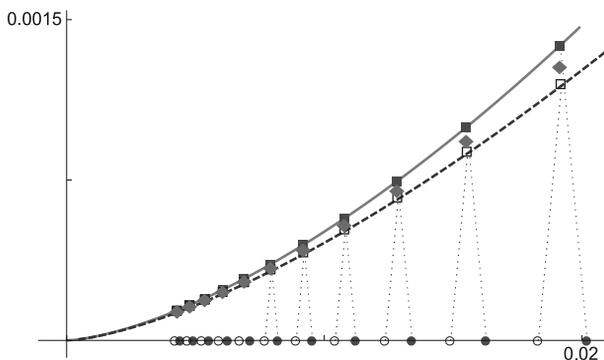The typical eigenvalue behaviour is illustrated in Figure 4.

Figure 4: *The red diamonds are numerically computed exact eigenvalues of A_γ, $\gamma = 2.5$, lying in the first quadrant. The complex parametric curves $\mu = \tau^-(t)$ (the magenta solid line) and $\mu = \tau^+(t)$ (the blue dashed line) are as in Remark 7(b). The approximated complex eigenvalues, computed by the first part of formula (12) are shown as magenta squares, and computed by the second part of formula (12) are shown as white squares. The absolute values of the eigenvalues of T_γ are marked on the real line in the same way as in Figure 3. The dotted arrows are to indicate which real eigenvalue "produces" the corresponding complex ones.*

4. Explicit form of the Jost solutions

4.1. Solutions of the equation on the half-line

Let f_ξ be a general solution of the differential equation

$$-\frac{d^2}{dy^2}f(y) - \frac{1}{y}f(y) = \xi f(y), \qquad y \in \mathbb{R}_+. \tag{14}$$

The change of variables

$$y = \gamma(x+1), \tag{15}$$

relates f_ξ with solutions $g_{\mu,\gamma}$ of the differential equation

$$-\frac{d^2}{dx^2}g(x) - \frac{\gamma}{1+x}g(x) = \mu\, g(x), \qquad x \in \mathbb{R}_+, \tag{16}$$

by

$$g_{\mu,\gamma}(x) = f_{\mu/\gamma^2}\left(\gamma(x+1)\right). \tag{17}$$

4.2. Explicit solutions of the differential equation (16)

Assuming $\xi \in \overline{\mathbb{C}}_+$, we will write

$$s := -i\sqrt{-\xi} \tag{18}$$

where we take the principal branch of $\sqrt{\cdot}$, i.e. the uniquely determined analytic branch that maps $\mathbb{R}_+$ into itself. Obviously $s^2 = \xi$.

With the ansatz

$$f(y) = ye^{-isy}h(y), \tag{19}$$

equation (14) can be reduced to

$$-y\frac{d^2}{dy^2}h(y) - (2 - 2isy)\frac{d}{dy}h(y) + (2is - 1)h(y) = 0. \tag{20}$$

With the additional change of variables

$$w = 2isy \tag{21}$$

we arrive at a particular case of the Kummer Hypergeometric Equation [23, Chapter 13.1]

$$w\frac{d^2}{dw^2}\tilde{h}(w) + (b - w)\frac{d}{dw}\tilde{h}(w) - a\tilde{h}(w) = 0 \tag{22}$$

with

$$a = 1 - \frac{1}{2is}, \qquad b = 2, \qquad w = 2isy, \qquad \tilde{h}(w) = h\left(\frac{w}{2is}\right).$$

The two linearly independent solutions of (22) are known as Kummer hypergeometric functions $M(a,b;w)$ and $U(a,b;w)$ so that

$$\tilde{h}(w) = C_1 U(a,b;w) + C_2 M(a,b;w), \qquad C_1, C_2 = \text{const}.$$

Hence the solutions $f_\xi(y)$ of (14) are of the form

$$f_\xi(y) = y e^{-isy} \left(C_1 U\left(1 - \frac{1}{2is}, 2; \, 2isy\right) + C_2 M\left(1 - \frac{1}{2is}, 2; \, 2isy\right)\right). \tag{23}$$

Thus, by (17), the solutions of (16) are of the form

$$g_{\mu,\gamma}(x) = \gamma(1+x) e^{-\sqrt{-\mu}\,(1+x)} \left(C_1 U\left(1 - \frac{\gamma}{2\sqrt{-\mu}}, 2; \, 2\sqrt{-\mu}\,(1+x)\right)\right.$$

$$\left. + C_2 M\left(1 - \frac{\gamma}{2\sqrt{-\mu}}, 2; \, 2\sqrt{-\mu}\,(1+x)\right)\right).$$

4.3. The Jost solutions of (16)

It is well known, see e.g. [23, 13.7.1 and 13.7.2], that the first order asymptotic behaviour of the Kummer Hypergeometric Functions is given, as $|w| \to \infty$, by

$$U(a,b;w) \sim w^{-a}, \qquad\qquad -\frac{3\pi}{2} < \arg(w) < \frac{3\pi}{2},$$

$$M(a,b;w) \sim \frac{e^w w^{b-a}}{\Gamma(a)} + \frac{e^{\pi i a} w^{-a}}{\Gamma(b-a)}, \qquad -\frac{\pi}{2} \leqslant \arg(w) < \frac{3\pi}{2}, \quad a, b-a \notin -\mathbb{N} \cup \{0\},$$

where $\Gamma(\cdot)$ stands for the usual Gamma function.

For $\mu \in \mathbb{C} \setminus \mathbb{R}_+$, we have $-\sqrt{-\mu} \subset \{z \in \mathbb{C} \mid \operatorname{Re} z < 0\}$, and therefore

$$U\left(1 - \frac{\gamma}{2\sqrt{-\mu}}, 2; \, 2y\sqrt{-\mu}\right) \sim (2y\sqrt{-\mu})^{\frac{\gamma}{2\sqrt{-\mu}} - 1},$$

and

$$M\left(1 - \frac{\gamma}{2\sqrt{-\mu}}, 2; \, 2y\sqrt{-\mu}\right) \sim \frac{(2y\sqrt{-\mu})^{\frac{\gamma}{2\sqrt{-\mu}} + 1}}{\Gamma\left(1 - \frac{\gamma}{2\sqrt{-\mu}}\right)} e^{2y\sqrt{-\mu}}$$

as $y \to \infty$.

This in turn implies that the f_ξ and g_μ defined above are $L^2(\mathbb{R}_+)$ if and only if $C_2 = 0$. For convenience, we choose further on the normalisation $C_1 = 1$.

The $L^2(\mathbb{R}_+)$ solutions of (16) are called the *Jost solutions*. We denote them by

$$\varphi_\mu(\gamma, x) := g_{\mu,\gamma}(x)\big|_{C_1=1, C_2=0} = \gamma(1+x) e^{-\sqrt{-\mu}\,(1+x)} U\left(1 - \frac{\gamma}{2\sqrt{-\mu}}, 2; \, 2(1+x)\sqrt{-\mu}\right). \tag{24}$$

5. Proof of Theorem 4

5.1. The characteristic equations for the self-adjoint problem

It is well known that T has a negative discrete spectrum accumulating to zero. We need a more detailed knowledge of the asymptotics of its eigenvalues and of the corresponding eigenfunctions.

It follows from the arguments of the previous sections that up to a scaling constant the eigenfunctions of the self-adjoint problem (16) are given, on $\mathbb{R}_+$, by

$$\psi_\lambda(\gamma,x) = \gamma(1+x)e^{-\sqrt{\lambda}(1+x)}\, U\left(1 - \frac{\gamma}{2\sqrt{\lambda}}, 2;\, 2(1+x)\sqrt{\lambda}\right).$$

The eigenvalues $-\lambda_n^D$ of the self-adjoint operator T_γ^D with Dirichlet boundary conditions at zero are thus given by the solutions of $\psi_\lambda(\gamma,0) = 0$, i.e.

$$\frac{\gamma e^{-\sqrt{\lambda}}}{2\sqrt{\lambda}} U\left(-\frac{\gamma}{2\sqrt{\lambda}}, 0, 2\sqrt{\lambda}\right) = 0 \tag{25}$$

The eigenvalues $-\lambda_n^N$ of the self-adjoint operator T_γ^N with Neumann boundary conditions at zero are given by the solutions of $\frac{d}{dx}\psi_\lambda(\gamma,x)\big|_{x=0} = 0$, i.e.

$$\frac{\gamma e^{-\sqrt{\lambda}}}{\lambda^2}\left((\gamma-2\sqrt{\lambda})U\left(-\frac{\gamma}{2\sqrt{\lambda}},-1,2\sqrt{\lambda}\right)+2\sqrt{\lambda}(\sqrt{\lambda}-1)U\left(-\frac{\gamma}{2\sqrt{\lambda}},0,2\sqrt{\lambda}\right)\right)=0. \tag{26}$$

The solutions of transcendental equations (25) and (26) can be computed numerically, although it is a non-trivial task as the left-hand sides of these equations oscillate wildly for small λ (cf. Figure 6). We instead use asymptotic techniques to approximate the Kummer functions as $\lambda \to 0$ and to control their oscillations. A quick took at (25) and (26) shows that we require asymptotic formulas, as $\lambda \to 0+$, for

$$U\left(-\frac{\gamma}{2\sqrt{\lambda}}, c; 2\sqrt{\lambda}\right), \quad c \in \{0,-1\}, \tag{27}$$

Unfortunately, it is a difficult task — the corresponding formulas, are not, in fact, in the standard references. We rely, instead, on the results from the forthcoming book [25] which we summarise and adapt in the Appendix.

5.2. Asymptotic solutions of a transcendental equation

A crucial element of our analysis is the investigation of the large κ-roots of the equation

$$\tan(\gamma\kappa) = G(\kappa,\gamma) \tag{28}$$

where γ is treated as a parameter, and where G depends analytically on κ in the vicinity of $\kappa = \infty$ and, to leading order, is of class $\mathscr{F}$ as a function of γ. The required results are summarised in the following

LEMMA 8. *Let $G(\kappa, \gamma)$ be an analytic function of κ around $\kappa = \infty$ such that*

$$G(\kappa, \gamma) = G_0(\gamma)\left(1 + O(\kappa^{-1})\right), \qquad as \qquad \kappa \to \infty,$$

$G_0 \in \mathcal{F}$, *and the O terms are regular in* γ. *Then the solutions* $\kappa_n(\gamma)$, *ordered increasingly, of the equation* (28), *are given, as* $n \to \infty$, *by*

$$\kappa_n(\gamma) = \frac{\pi n}{\gamma} + \frac{1}{\gamma}\Theta_{G_0}(\gamma) + O(n^{-1}). \tag{29}$$

The proof of Lemma 8 is in fact immediate as soon as we recall Definition 2 of Θ and the fact that tan is π-periodic.

Considering additional terms in the expansion of G one can get additional terms in the expansion of κ_n. This is in fact what we do in more detail in Section 6.2.

5.3. Approximation of Dirichlet eigenvalues

We can use the asymptotic approximation obtained in (A.8) to reduce (25) to the simpler form

$$\cos\left(\frac{\gamma\pi}{2\sqrt{\lambda}}\right)\left(J_1\left(2\sqrt{\gamma}\right) + O(\lambda)\right) + \sin\left(\frac{\gamma\pi}{2\sqrt{\lambda}}\right)\left(Y_1\left(2\sqrt{\gamma}\right) + O(\lambda)\right) = 0. \tag{30}$$

This in turn can be rewritten as

$$\tan\left(\frac{\gamma\pi}{2\sqrt{\lambda}}\right) = -\frac{J_1\left(2\sqrt{\gamma}\right)}{Y_1\left(2\sqrt{\gamma}\right)} + O(\lambda). \tag{31}$$

Applying Lemma 8 with

$$\kappa = \frac{1}{2\sqrt{\lambda}}, \qquad G_0(\gamma) = -\frac{J_1\left(2\sqrt{\gamma}\right)}{Y_1\left(2\sqrt{\gamma}\right)} = -R_1(\gamma),$$

we obtain, after a minor effort,

$$\lambda_n = \frac{\gamma^2\pi^2}{4}\left(n\pi - \Theta_{G_0}(\gamma)\right)^{-2} + O(n^{-4})$$

$$= \frac{\gamma^2}{4n^2}\left(1 + \frac{2}{\pi n}\Theta_{G_0}(\gamma) + O(n^{-2})\right) \tag{32}$$

$$= \frac{\gamma^2}{4n^2}\left(1 - \frac{2}{\pi n}\Theta_{R_1}(\gamma) + O(n^{-2})\right) \tag{33}$$

as $n \to +\infty$, thus proving the first part of Theorem 4.

5.4. Approximation of Neumann eigenvalues

The analysis for Neumann eigenvalues is slightly more complicated. Again we can use (A.8) to reduce (26) to

$$\tan\left(\frac{\gamma\pi}{2\sqrt{\lambda}}\right) = -\frac{P(\gamma,\lambda)}{Q(\gamma,\lambda)} \tag{34}$$

where

$$
\begin{aligned}
P(\gamma,\lambda) &:= \sqrt{\gamma}(5\sqrt{\lambda}-8)(\sqrt{\lambda}+1)J_1\left(2\sqrt{\gamma}\right) + (11\sqrt{\lambda}-8\gamma)(2\sqrt{\lambda}-\gamma)J_2\left(2\sqrt{\gamma}\right) \\
&\quad - 8\sqrt{\gamma}\sqrt{\lambda}(2\sqrt{\lambda}-\gamma)J_3\left(2\sqrt{\gamma}\right) + O(\lambda^{3/2}), \\
Q(\gamma,\lambda) &:= \sqrt{\gamma}(5\sqrt{\lambda}-8)(\sqrt{\lambda}+1)Y_1\left(2\sqrt{\gamma}\right) + (11\sqrt{\lambda}-8\gamma)(2\sqrt{\lambda}-\gamma)Y_2\left(2\sqrt{\gamma}\right) \\
&\quad - 8\sqrt{\gamma}\sqrt{\lambda}(2\sqrt{\lambda}-\gamma)Y_3\left(2\sqrt{\gamma}\right) + O(\lambda^{3/2}).
\end{aligned}
$$

Applying once again Lemma 8 with

$$\kappa = \frac{1}{2\sqrt{\lambda}}, \qquad G_0(\gamma) = -\frac{P(\gamma,0)}{Q(\gamma,0)} = -\frac{J_0\left(2\sqrt{\gamma}\right)}{Y_0\left(2\sqrt{\gamma}\right)} = -R_0(\gamma),$$

we quickly arrive at

$$\lambda_n = \frac{\gamma^2}{4n^2}\left(1 + \frac{2}{\pi n}\Theta_{G_0}(\gamma) + O(n^{-2})\right) = \frac{\gamma^2}{4n^2}\left(1 - \frac{2}{\pi n}\Theta_{R_0}(\gamma) + O(n^{-2})\right) \tag{35}$$

as $n \to \infty$, thus proving the second part of Theorem 4.

6. Proof of the asymptotic results of the non-self-adjoint operator

6.1. Eigenvalues and the Jost solutions

LEMMA 9. *The eigenvalues of* (5) *are the zeroes of the determinant*

$$M(\mu) = M_\gamma(\mu) = \varphi'_\mu(\gamma,0)\,\varphi_{-\mu}(\gamma,0) + \varphi'_{-\mu}(\gamma,0)\,\varphi_\mu(\gamma,0). \tag{36}$$

Proof. Suppose that $\mu \in \mathbb{C}$ is an eigenvalue of A_γ, and that $g_\mu(x) \in L^2(\mathbb{R})$ is a corresponding eigenfunction. Then g_μ solves the differential equation

$$-\frac{d^2}{dx^2}g_\mu(x) - \frac{\gamma}{1+|x|}g_\mu(x) = \text{sign}(x)\mu g_\mu(x).$$

If $g_\pm$ denote the restrictions of g_μ on $\mathbb{R}_+$ and $\mathbb{R}_-$, then by integrability we must have

$$g_+(x) = C_+\varphi_\mu(\gamma,x), \qquad g_-(-x) = C_-\varphi_{-\mu}(\gamma,x), \qquad x \in \mathbb{R}_+,$$

where $\varphi_\mu(\gamma,x)$ is the Jost solution (24).

As an eigenfunction should be continuously differentiable at zero, we obtain

$$\begin{cases} C_+\varphi_\mu(\gamma,0) - C_-\varphi_{-\mu}(\gamma,0) & = 0, \\ C_+\varphi'_\mu(\gamma,0) + C_-\varphi'_{-\mu}(\gamma,0) & = 0, \end{cases}$$

which has a non-trivial solution if and only if $M_\gamma(\mu) = 0$. □

REMARK 10.   (a) It follows from (24) that if μ is real, either φ_μ or $\varphi_{-\mu}$ is not square integrable, and therefore A_γ cannot have real eigenvalues.

(b) By [6, Proposition 4.6] one can instead look for the eigenvalues of (3) as the zeroes of the m-function

$$m_\gamma(\mu) = \frac{\varphi'_\mu(\gamma,0)}{\varphi_\mu(\gamma,0)} + \frac{\varphi'_{-\mu}(\gamma,0)}{\varphi_{-\mu}(\gamma,0)}. \tag{37}$$

The use of half-line m-functions is natural and has been already suggested elsewhere, in particular the statement of Lemma 9 may be found e.g. in [11, 17]. For some further developments concerning indefinite Sturm–Liovuille problems with turning point at zero see also [15] and references therein.

(c) In what follows we assume that μ is in the upper half plane $\mathbb{C}_+$ and look for the eigenvalues on the first quadrant. The final result will follow by symmetry (see Proposition 1(a) and Proposition 1(f)).

6.2. The determinant

We can use (24) and the known relations [23, §13.3] between Kummer hypergeometric functions to rewrite (36) as

$$M(\mu)$$
$$= \frac{\gamma^2\sqrt{-\mu}e^{-\sqrt{-\mu}-\sqrt{\mu}}}{8\mu^{5/2}}\left[(\gamma\sqrt{-\mu}+2\mu)\,U\left(-\frac{\gamma}{2\sqrt{-\mu}},-1;2\sqrt{-\mu}\right)U\left(-\frac{\gamma}{2\sqrt{\mu}},0;2\sqrt{\mu}\right) \right.$$
$$+ (2\mu - \gamma\sqrt{\mu})\,U\left(-\frac{\gamma}{2\sqrt{-\mu}},0;2\sqrt{-\mu}\right)U\left(-\frac{\gamma}{2\sqrt{\mu}},-1;2\sqrt{\mu}\right)$$
$$\left. + 2\mu\left(\sqrt{-\mu}+\sqrt{\mu}-2\right)U\left(-\frac{\gamma}{2\sqrt{-\mu}},0;2\sqrt{-\mu}\right)U\left(-\frac{\gamma}{2\sqrt{\mu}},0;2\sqrt{\mu}\right)\right].$$
$$\tag{38}$$

To find approximate solutions of $M(\mu) = 0$, we use the asymptotic formula (A.8).

Let us define for brevity

$$j_v := J_v(2\sqrt{\gamma}), \qquad\qquad y_v := Y_v(2\sqrt{\gamma}),$$

$$K_+ := \cos\left(\frac{\gamma\pi}{2\sqrt{\mu}}\right), \qquad\qquad K_- := \cosh\left(\frac{\gamma\pi}{2\sqrt{\mu}}\right),$$

$$S_+ := \sin\left(\frac{\gamma\pi}{2\sqrt{\mu}}\right), \qquad\qquad S_- := \sinh\left(\frac{\gamma\pi}{2\sqrt{\mu}}\right),$$

$$T_+ := \tan\left(\frac{\gamma\pi}{2\sqrt{\mu}}\right), \qquad\qquad T_- := \tanh\left(\frac{\gamma\pi}{2\sqrt{\mu}}\right).$$

With these abbreviations and with the use of asymptotic formulae (A.8), equation (36) becomes

$$
\begin{aligned}
\frac{i\sqrt{\gamma}}{\sqrt{\mu}}\Gamma\left(\frac{i\gamma}{2\sqrt{\mu}}+1\right)&\Gamma\left(\frac{\gamma}{2\sqrt{\mu}}+1\right)\left\{\left(1-\frac{5\sqrt{\mu}}{8\gamma}\right)(2\mu-i\gamma\sqrt{\mu})\,(j_1 K_+ + y_1 K_-)\right.\\
&\times\left[\left(1+\frac{11i\sqrt{\mu}}{8\gamma}\right)(j_2 S_+ + iy_2 S_-)+\frac{\sqrt{\mu}\,(y_3 S_- - ij_3 S_+)}{\sqrt{\gamma}}\right]\\
+i&\left(1+\frac{5i\sqrt{\mu}}{8\gamma}\right)(j_1 S_+ + iy_1 S_-)\left[(2\mu-\gamma\sqrt{\mu})\right.\\
&\times\left(\left(1-\frac{11\sqrt{\mu}}{8\gamma}\right)(j_2 K_+ + y_2 K_-)+\frac{\sqrt{\mu}\,(j_3 K_+ + y_3 K_-)}{\sqrt{\gamma}}\right)\\
&\left.\left.-\frac{\left(\frac{1}{8}-\frac{i}{8}\right)\left(\sqrt{\mu}-(1+i)\right)\sqrt{\mu}\,(8\gamma-5\sqrt{\mu})\,(j_1 K_+ + y_1 K_-)}{\sqrt{\gamma}}\right]\right\}=0,
\end{aligned}
\tag{39}
$$

where we have dropped the lower order terms.

Simplifying, writing $S_\pm = T_\pm K_\pm$, and collecting terms in $K_\pm$, we get

$$
\begin{aligned}
K_+&\left\{(1+i)j_1^2\sqrt{\gamma}(\sqrt{\mu}-(1+i))\left(64i\gamma^2-40(1+i)\gamma\sqrt{\mu}+25\mu\right)\right.\\
&+T_- y_1\left(-8i\gamma^2+(5+16i)\gamma\sqrt{\mu}-10\mu\right)\left(j_2(8\gamma-11\sqrt{\mu})+8j_3\sqrt{\gamma}\sqrt{\mu}\right)\\
&-j_1\left[-16ij_3\sqrt{\gamma}\sqrt{\mu}\left(4(1+i)\gamma^2-5\gamma\sqrt{\mu}+5(1-i)\mu\right)\right.\\
&+2j_2\left(64\gamma^3-128(1-i)\gamma^2\sqrt{\mu}-135i\gamma\mu+55(1+i)\mu^{3/2}\right)\\
&+T_-(8\gamma-5\sqrt{\mu})\left((1+i)\sqrt{\gamma}(\sqrt{\mu}-(1+i))(8\gamma+5i\sqrt{\mu})y_1\right.\\
&\left.\left.\left.+(\gamma+2i\sqrt{\mu})(8\sqrt{\gamma}\sqrt{\mu}y_3+y_2(-11\sqrt{\mu}+8i\gamma))\right)\right]\right\}\\
-K_-&\left\{y_1\left[8j_3\sqrt{\gamma}\sqrt{\mu}\left(-8i\gamma^2+(16+5i)\gamma\sqrt{\mu}-10\mu\right)\right.\right.\\
&+j_2\left(64\gamma^3-(40-216i)\gamma^2\sqrt{\mu}-(176+135i)\gamma\mu+110\mu^{3/2}\right)\\
&+(1+i)T_-\left(\sqrt{\gamma}(\sqrt{\mu}-(1+i))(64\gamma^2-40(1-i)\gamma\sqrt{\mu}-25i\mu)y_1\right.
\end{aligned}
\tag{40}
$$

$$+8(1-i)\sqrt{\gamma}\sqrt{\mu}y_3\left(4(1+i)\gamma^2-5\gamma\sqrt{\mu}+5(1-i)\mu\right)$$

$$+y_2\left(64(1+i)\gamma^3-256\gamma^2\sqrt{\mu}+135(1-i)\gamma\mu+110i\mu^{3/2}\right)\Big)\Big]$$

$$+j_1\left(8\gamma+5i\sqrt{\mu}\right)\Big((k-2\sqrt{\mu})(y_2(8\gamma-11\sqrt{\mu})+8\sqrt{\gamma}\sqrt{\mu}y_3)$$

$$+(1-i)\sqrt{\gamma}(\sqrt{\mu}-(1+i))(8\gamma-5\sqrt{\mu})y_1\Big)\Big\}=0.$$

In what follows, we essentially replicate the reasoning in Lemma 8, but working to a higher accuracy. Introduce in the equation (40) the ansatz

$$\mu=\lambda+v\lambda^{3/2}+\eta\lambda^2, \tag{41}$$

where λ is, as before, the absolute value of an eigenvalue of the self-adjoint operator with either Dirichlet or Neumann boundary conditions. Now replace back K_-, T_- and K_+ with the corresponding expressions. The next step — expanding the left-hand side of the resulting equation in the Taylor series with respect to λ around zero, — is delicate.

First of all, observe that as $\lambda \to 0$

$$\cos\left(\frac{\gamma\pi}{2\sqrt{\lambda+v\lambda^{3/2}+\eta\lambda^2}}\right)$$

$$=\cos\left(\frac{\gamma\pi}{2\sqrt{\lambda}}\left(1-\left(\frac{v\sqrt{\lambda}}{2}+\left(\frac{\eta}{2}-\frac{3v^2}{8}\right)\lambda+O(\lambda^{3/2})\right)\right)\right)$$

$$=\cos\left(\frac{\gamma\pi}{2\sqrt{\lambda}}\right)\cos\left(\frac{\gamma\pi}{4}\left(v+\left(\eta-\frac{3v^2}{4}\right)\sqrt{\lambda}\right)\right)$$

$$+\sin\left(\frac{\gamma\pi}{2\sqrt{\lambda}}\right)\sin\left(\frac{\gamma\pi}{4}\left(v+\left(\eta-\frac{3v^2}{4}\right)\sqrt{\lambda}\right)\right)+O(\lambda),$$

and similarly

$$\sin\left(\frac{\gamma\pi}{2\sqrt{\lambda+v\lambda^{3/2}+\eta\lambda^2}}\right)$$

$$=\sin\left(\frac{\gamma\pi}{2\sqrt{\lambda}}\right)\cos\left(\frac{\gamma\pi}{4}\left(v+\left(\eta-\frac{3v^2}{4}\right)\sqrt{\lambda}\right)\right)$$

$$-\cos\left(\frac{\gamma\pi}{2\sqrt{\lambda}}\right)\sin\left(\frac{\gamma\pi}{4}\left(v+\left(\eta-\frac{3v^2}{4}\right)\sqrt{\lambda}\right)\right)+O(\lambda).$$

We want to derive a similar expansion for $\tanh\left(\frac{\gamma\pi}{2\sqrt{\lambda+v\lambda^{3/2}+\eta\lambda^2}}\right)$. We use

$$\tanh(t_1-t_2)=\frac{\sinh(t_2)+\cosh(t_2)\tanh(t_1)}{\cosh(t_2)-\sinh(t_2)\tanh(t_1)}$$

with $t_1:=\frac{\gamma\pi}{2\sqrt{\lambda}}$ and $t_2:=\frac{\gamma\pi}{4}\left(v+\left(\eta-\frac{3v^2}{4}\right)\sqrt{\lambda}+O(\lambda)\right)$.

As $\tanh(\pi\gamma/2\sqrt{\lambda}) = 1$ for $\lambda \to 0$ modulo exponentially small terms, we get (again up to exponentially small errors)

$$\tanh\left(\frac{\gamma\pi}{2\sqrt{\lambda + \nu\lambda^{3/2} + \eta\lambda^2}}\right) = \frac{\sinh(t_2) + \cosh(t_2)}{\cosh(t_2) - \sinh(t_2)} = -1,$$

and (40) reduces to an equation involving only $\tan\left(\frac{\gamma\pi}{2\sqrt{\lambda}}\right)$, $\tan\left(\frac{\gamma\pi}{4}\left(\nu + \left(\eta - \frac{3\nu^2}{4}\right)\sqrt{\lambda}\right)\right)$ and powers of $\sqrt{\lambda}$. This is still, however, very hard to control.

6.3. Complex eigenvalue curves

We can now use our knowledge of the self-adjoint problem (see (31) and (34)) to simplify (40) further. For definiteness, suppose that $-\lambda$ is an eigenvalue of T^D.

Using the approximate identity (31), obtained for the eigenvalues of the Dirichlet self-adjoint problem on the half line, we can reduce the already simplified (40) to an equation involving only $\tan\left(\frac{\gamma\pi}{4}\left(\nu + \left(\eta - \frac{3\nu^2}{4}\right)\sqrt{\lambda}\right)\right)$ and powers of $\sqrt{\lambda}$. Collecting the tangent terms, after some cumbersome but straightforward simplifications we arrive at

$$\tan\left(\frac{\gamma\pi}{4}\left(\nu + \left(\eta - \frac{3\nu^2}{4}\right)\sqrt{\lambda}\right)\right) = \frac{\tilde{P}(\lambda, \gamma)}{\tilde{Q}(\lambda, \gamma)} \tag{42}$$

where

$$\tilde{P}(\lambda, \gamma) = -(4 + 4i)\gamma + (12 + 7i)\sqrt{\lambda},$$
$$\tilde{Q}(\lambda, \gamma) = i(4 + 4i)\gamma + (7 + 12i)\sqrt{\lambda}$$
$$+ 2\pi\left(4\sqrt{\gamma}\left((1 + i)\gamma - 3\sqrt{\lambda}\right)(j_0 j_1 + y_0 y_1) + 7\sqrt{\lambda}\left(j_1^2 + y_1^2\right)\right),$$

and we have dropped terms of order $O(\lambda)$. We have used the following standard relations [23, Ch. 13] for the Bessel functions,

$$J_{n+1}(2\sqrt{\gamma})Y_n(2\sqrt{\gamma}) - J_n(2\sqrt{\gamma})Y_{n+1}(2\sqrt{\gamma}) = \frac{1}{\pi\sqrt{\gamma}},$$
$$J_1(2\sqrt{\gamma}) - \sqrt{\gamma}J_2(2\sqrt{\gamma}) = \sqrt{\gamma}J_0(2\sqrt{\gamma}),$$
$$Y_1(2\sqrt{\gamma}) - \sqrt{\gamma}Y_2(2\sqrt{\gamma}) = \sqrt{\gamma}Y_0(2\sqrt{\gamma}),$$

in the simplifications.

Note that right-hand side of (42) does not depend on ν. We can now invert the tangent and solve with respect to ν, to find the coefficient for the $\lambda^{3/2}$ term in (41). Expanding in λ around 0 and taking the leading term we get

$$\nu =: \Upsilon^-(\gamma) = \frac{4}{\pi\gamma}\arctan\left(\frac{1}{i - 2\pi\sqrt{\gamma}\left(J_0(2\sqrt{\gamma})J_1(2\sqrt{\gamma}) + Y_0(2\sqrt{\gamma})Y_1(2\sqrt{\gamma})\right)}\right). \tag{43}$$

The contribution of the other terms in the expansion then forms a part of η, which we drop.

REMARK 11. We can repeat the same procedure using the relation (34) for the absolute value of a Neumann eigenvalue λ as a starting point. In this case we obtain

$$v =: \Upsilon^+(\gamma) = v = \frac{4}{\pi\gamma}\arctan\left(\frac{1}{i + 2\pi\sqrt{\gamma}\left(J_0(2\sqrt{\gamma})J_1(2\sqrt{\gamma}) + Y_0(2\sqrt{\gamma})Y_1(2\sqrt{\gamma})\right)}\right).$$
(44)

REMARK 12. One can use standard asymptotic formulas for Bessel functions (see [23, Chapter 10]) to observe

$$q(\gamma) \sim \begin{cases} -\frac{\log(\gamma)}{\pi} & \text{if } \gamma \to 0 \\ \frac{1}{4\sqrt{\gamma}} - \frac{3\cos(4\sqrt{\gamma})}{512\gamma} & \text{if } \gamma \to \infty \end{cases}.$$

Moreover, again using standard estimates and properties of Bessel functions and their zeroes one can observe that

$$|J_0(x)|, |J_1(x)|, |Y_0(x)|, |Y_1(x)| < \frac{1}{\sqrt{2}} \text{ for } x \geqslant 2,$$

and the four functions are monotone for $x < 2$ (J_0 increasing and bounded by 1, the other ones decreasing, in particular the Y are unbounded). In particular, for any $\gamma \in (0,1)$ neither $\Upsilon^-(\gamma)$ and $\Upsilon^+(\gamma)$ nor their real and imaginary parts, vanish. Moreover $\Upsilon^\mp(\gamma)$ define curves in the complex plane that diverge as $\gamma \to 0$ and converge to 0 as $\gamma \to +\infty$.

With $q(\gamma)$ as in (11) we get (13). Then the identity $\operatorname{Im}\Upsilon^-(\gamma) = \operatorname{Im}\Upsilon^+(\gamma)$ follows immediately using the standard relation between arctan and log (see Remark 7(a)), thus proving part (i) of Theorem 6.

Proving part (ii) of Theorem 6 requires some extra work. First of all observe that, up to the errors of order $O(n^{-4})$, we have

$$\begin{aligned}
\lambda_n^D - \lambda_n^N &= -\frac{\gamma^2}{2\pi n^3}\left(\Theta_{R_1}(\gamma) - \Theta_{R_0}(\gamma)\right) \\
&= -\frac{\gamma^2}{4\pi n^3}\left(i\log\left(\frac{(J_1(2\sqrt{\gamma}) - iY_1(2\sqrt{\gamma}))(J_0(2\sqrt{\gamma}) + iY_0(2\sqrt{\gamma}))}{(J_1(2\sqrt{\gamma}) + iY_1(2\sqrt{\gamma}))(J_0(2\sqrt{\gamma}) - iY_0(2\sqrt{\gamma}))}\right)\right) \\
&= -\frac{\gamma^2}{4\pi n^3}\left(i\log\left(\frac{q(\gamma) + i}{q(\gamma) - i}\right)\right),
\end{aligned}$$

where $R_k(\gamma)$ are defined in Theorem 4 and the first identity follows from the fact that zeroes of Y_0 and Y_1 are interlaced. Similarly

$$\Upsilon^-(\gamma) - \Upsilon^+(\gamma) = \frac{2i}{\pi\gamma}\log\left(\frac{2q(\gamma) - 2i}{2q(\gamma) + 2i}\right).$$

Despite the appearance of the complex unity i in the above formulae, all these expressions are in fact real!

To show that the two asymptotic formulae (12) for μ_n coincide up to the lower order terms, we use Theorem 4, to obtain, as $n \to \infty$,

$$\left(\lambda_n^D + \Upsilon_n^-(\gamma)(\lambda_n^D)^{3/2}\right) - \left(\lambda_n^N + \Upsilon_n^+(\gamma)(\lambda_n^N)^{3/2}\right) + O\left(\frac{1}{n^4}\right)$$

$$= -\frac{\gamma^2 i}{4\pi n^3} \log\left(\frac{q(\gamma)+i}{q(\gamma)-i}\right) + \frac{2i}{\pi\gamma} \log\left(\frac{2q(\gamma)+2i}{2q(\gamma)-2i}\right)\left(\frac{\gamma^2}{4n^2}\right)^{3/2} + O\left(\frac{1}{n^4}\right)$$

$$= \frac{\gamma^2 i}{4\pi n^3}\left(\log\left(\frac{q(\gamma)+i}{q(\gamma)-i}\right) - \log\left(\frac{q(\gamma)+i}{q(\gamma)-i}\right)\right) + O\left(\frac{1}{n^4}\right)$$

$$= O\left(\frac{1}{n^4}\right),$$

thus concluding the proof.

7. Generalizations and other remarks

The procedure used to prove Theorem 6 can be repeated in a completely similar way to obtain a result for the operator

$$A(\gamma_+,\gamma_-) = JT_V, \quad V(x) = \begin{cases} \frac{\gamma_+}{1+|x|} & \text{if } x > 0 \\ \frac{\gamma_-}{1+|x|} & \text{if } x < 0 \end{cases}, \quad \gamma_+,\gamma_- \in \mathbb{R}_+.$$

In this case the m-function is of the form

$$M(\lambda) = \frac{\varphi'_{\mu,\gamma_+}(0)}{\varphi_{\mu,\gamma_+}(0)} + \frac{\varphi'_{-\mu,\gamma_-}(0)}{\varphi_{-\mu,\gamma_-}(0)}.$$
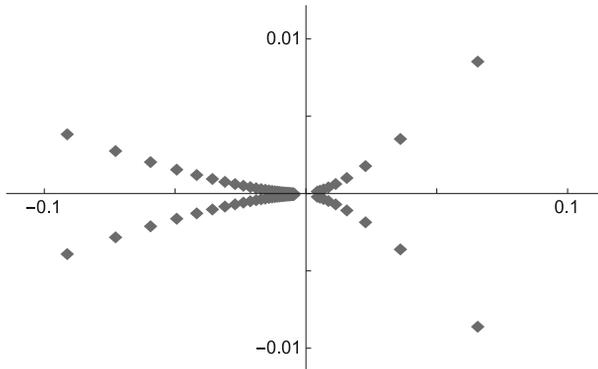

Figure 5: *Approximated eigenvalues of $A(\gamma_+,\gamma_-)$ for $\gamma_- = 1.5$, $\gamma_+ = 5$.*

The curves in the upper (resp. lower) half plane are no more symmetric w.r.t. $i\mathbb{R}$, however for the left quadrants and right quadrants we can extend Theorem 6. The only difference is that now the Υ^- and Υ^+ are now functions of both γ_+ and γ_-.

Let $v, \eta \in \mathbb{R}_+$. Set

$$f_-(v, \eta) := \frac{J_1^2(2\sqrt{\eta}) + J_0^2(2\sqrt{\eta})}{J_1^2(2\sqrt{v}) + J_0^2(2\sqrt{v})} \left(i - \pi\sqrt{v}\left(J_0(2\sqrt{v})J_1(2\sqrt{v}) + Y_0(2\sqrt{v})Y_1(2\sqrt{v})\right) \right)$$
$$- \pi\sqrt{\eta}\left(J_0(2\sqrt{\eta})J_1(2\sqrt{\eta}) + Y_0(2\sqrt{\eta})Y_1(2\sqrt{\eta})\right),$$

$$f_+(v, \eta) := \frac{J_1^2(2\sqrt{\eta}) + J_0^2(2\sqrt{\eta})}{J_1^2(2\sqrt{v}) + J_0^2(2\sqrt{v})} \left(i + \pi\sqrt{v}\left(J_0(2\sqrt{v})J_1(2\sqrt{v}) + Y_0(2\sqrt{v})Y_1(2\sqrt{v})\right) \right)$$
$$+ \pi\sqrt{\eta}\left(J_0(2\sqrt{\eta})J_1(2\sqrt{\eta}) + Y_0(2\sqrt{\eta})Y_1(2\sqrt{\eta})\right).$$

Then the two factor multiplying the term $\operatorname{Re}\mu^{3/2}$ are given by

$$\Upsilon^-(\gamma_+, \gamma_-) := \frac{4}{\pi} \begin{cases} \gamma_-^{-1}\arctan(1/f_-(\gamma_+, \gamma_-)) & \text{if } \operatorname{Re}\mu > 0 \\ \gamma_+^{-1}\arctan(1/f_-(\gamma_-, \gamma_+)) & \text{if } \operatorname{Re}\mu < 0 \end{cases},$$

$$\Upsilon^+(\gamma_+, \gamma_-) := \frac{4}{\pi} \begin{cases} \gamma_-^{-1}\arctan(1/f_+(\gamma_+, \gamma_-)) & \text{if } \operatorname{Re}\mu > 0 \\ \gamma_+^{-1}\arctan(1/f_+(\gamma_-, \gamma_+)) & \text{if } \operatorname{Re}\mu < 0 \end{cases}.$$

One can immediately see that the asymmetry appearing w.r.t. $i\mathbb{R}$ is reflected in the asymmetric dependence on γ_+ and γ_-.

It is interesting to observe that for $\operatorname{Re}\mu > 0$ the effect of γ_- is much stronger than the one of γ_+ (the latter appears only in the cotangent term, its contribution is bounded, while the former additionally appears as an inverse prefactor). The situation is opposite when $\operatorname{Re}\mu < 0$.

The expressions for $\Upsilon^\mp(\gamma_-, \gamma_+)$ are more involved than the ones for $\Upsilon^\mp(\gamma)$ but, as expected, they simplify to (43) and (44) for $\gamma_+ = \gamma_-$. As that case, it is possible to use the standard results on Bessel functions to show that the two constants have non-zero real and imaginary part for any $\gamma_\pm > 0$.

To answer the general question posed in [3] for a wider class of potentials one would need good estimates of the Jost functions in a complex half ball containing the origin and the positive and negative real axis. To our knowledge, the best result of this kind is contained in a paper by Yafaev [28]. In that work, however, the author needed to exclude two cones containing the real axis for his estimates to hold. Additionally he could get only the first term in the asymptotic expansion, whereas for our result we would need at least the first two.

A. Uniform asymptotic expansion of Kummer hypergeometric functions

We need to approximate

$$U\left(-\frac{\gamma}{2\sqrt{-\mu}}, c; 2\sqrt{-\mu}\right), \qquad U\left(-\frac{\gamma}{2\sqrt{\mu}}, c; 2\sqrt{\mu}\right), \quad c \in \{0, -1\}, \qquad \text{(A.1)}$$

in the limit $\mu \to 0$. We use the theory developed in [25, Chapter 27]. By formula [25, (27.4.85)], as $a \to \infty$, and with az bounded and $\operatorname{Re}(az) > 0$,

$$U(-a, c; az) \sim \beta^{1-c}\Gamma(a+1)e^{\frac{1}{2}az}\left(C_{c-1}(\zeta)\sum_{n=0}^{\infty}\frac{A_n}{a^n} + \beta C_{c-2}(\zeta)\sum_{n=0}^{\infty}\frac{B_n}{a^n}\right), \qquad \text{(A.2)}$$

where A_n and B_n are defined by an iterative procedure, $\zeta = 2\beta a$ and

$$C_\nu(\zeta) = \cos(\pi a)J_\nu(\zeta) + \sin(\pi a)Y_\nu(\zeta). \tag{A.3}$$

On can immediately see that in our case

$$a = a_\pm = \frac{\gamma}{2\sqrt{\pm\mu}}, \qquad z = z_\pm = \frac{\gamma}{a_\pm^2}. \tag{A.4}$$

We will drop the subscript $\pm$ for the rest of the discussion.

Additionally we need an expression for β. This is defined in [25, (27.4.36)] as

$$\beta = \frac{1}{2}(w_0 + \sinh(w_0))$$

where $w_0 = 2\operatorname{arcsinh}(\frac{1}{2}\sqrt{z})$, see [25, (27.4.33)]. Equation [25, (27.4.52)] gives an asymptotic expansion for β as $z \to 0$:

$$\beta^2 = z + \frac{1}{12}z^2 + O(z^3).$$

By Taylor expansion we get

$$\beta = \sqrt{z}\sqrt{1 + \frac{1}{12}z + O(z^2)} = \sqrt{z}\left(1 + \frac{1}{24}z + O(z^2)\right). \tag{A.5}$$

Therefore

$$\zeta = 2\beta a = 2\sqrt{\gamma}\left(1 + \frac{\gamma}{24}a^{-2} + O(a^{-4})\right)$$

and $\operatorname{Re}(az) = \operatorname{Re}(\gamma/a)$.

Observe that for $\gamma \in \mathbb{R}_+$, $\operatorname{Re}(az) > 0$ iff $\operatorname{Re}(a) > 0$.

The coefficients A_0 and B_0 also have explicit expressions that can be derived using some symmetry properties and L'Hôpital rule, see [25, (27.4.74)]):

$$A_0 = \left(\frac{\beta}{2\sin(\theta)}\right)^c \sqrt{\frac{2}{\beta}\tan\theta} \, \cos(c\theta)$$

$$B_0 = \left(\frac{\beta}{2\sin(\theta)}\right)^c \sqrt{\frac{2}{\beta}\tan\theta} \, \frac{\sin(c\theta)}{\beta}$$

where $\theta = -\frac{1}{2}iw_0$.

The computation of A_n and B_n for $n \geqslant 0$ is quite involved, however we will need only A_1. One can exploit the procedure to compute A_0 and B_0, and the recursive definition of the coefficients to get a Taylor approximation in negative powers of a for $c \in \{0, -1\}$. We get

$$\text{if } c = 0, \quad A_0^0 = 1 + O(a^{-2}), \quad A_1^0 = -\frac{5}{16} + O(a^{-2}), \quad \beta B_0^0 = 0, \tag{A.6}$$
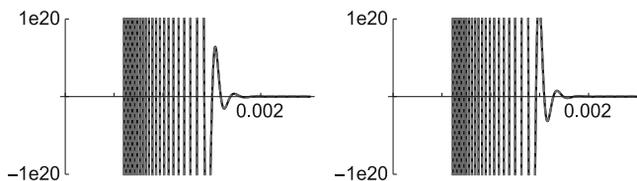
Figure 6: *Plot of real part (left) and imaginary part (right) of* $U\left(-\frac{\gamma}{2\sqrt{-\mu}}, c; 2\sqrt{-\mu}\right)$ *(black) and its approximation given by* (A.8) *(dashed red) for small values of* μ *and* $\gamma = 2.5$.

if $c = -1$, $\quad A_0^{-1} = 1 + O(a^{-4}), \quad A_1^{-1} = -\frac{11}{16} + O(a^{-2}), \quad \beta B_0^{-1} = -\frac{\sqrt{\gamma}}{2a} + O(a^{-3}).$

$$\tag{A.7}$$

With these, (A.2) can be re-written

$$U\left(-a, c; \frac{\gamma}{a}\right) \sim \left(\frac{\sqrt{\gamma}}{a}\right)^{\frac{1-c}{2}} \Gamma(a+1) e^{\frac{\gamma}{2a}} \left(\tilde{C}_{c-1}(a, \gamma)(\tilde{A}_0^c + \tilde{A}_1^c) + \tilde{C}_{c-2}(a, \gamma)\tilde{B}_0^c + O(a^{-2})\right)$$

$$\tag{A.8}$$

where

$$\tilde{C}_\nu(a, \gamma) := \cos(\pi a)J_\nu(2\sqrt{\gamma}) + \sin(\pi a)Y_\nu(2\sqrt{\gamma}) \tag{A.9}$$

and $\tilde{A}_0^c$, $\tilde{A}_1^c$ and $\tilde{B}_0^c$ are obtained dropping the error term in the appropriate coefficient in (A.6) and (A.7).

REMARK A.1. Here the error is in fact $O(\gamma/a^2)$, we may thus expect the improvement in the precision of the asymptotics when $\gamma \gg 1$.

REMARK A.2. (Validity of the expansion) If we define

$$t_1 = \beta + \pi i + \sqrt{(\beta + \pi i)^2 - \beta^2},$$

then the asymptotic formula (A.2) is valid for

$$-\arg t_1 - \frac{\pi}{2} + \delta \leqslant \arg a \leqslant \arg t_1 + \frac{\pi}{2} - \delta, \tag{A.10}$$

and the same applies to z (see [25, Chapter 26.4.2]).
In our case

$$t_1 \sim 2\pi i + 2\beta + O(\beta^2).$$

For $|a| \gg 1$, $\arg t_1$ is in the upper complex half plane. In particular this allows a and z to be in the closure of the first and fourth quadrant.

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