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NONLINEAR MAPS PRESERVING HIGHER–DIMENSIONAL NUMERICAL RANGE OF SKEW LIE PRODUCT OF OPERATORS

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Abstract. Let k be a positive integer. Let H and K be complex Hilbert spaces of dimensions greater than k. By $W_k(A)$ denote the k-dimensional numerical range of an operator A. In this paper we prove that a surjective map $\phi : B(H) \to B(K)$ satisfies $W_k(AB - BA^*) = W_k(\phi(A)\phi(B) - \phi(B)\phi(A)^*)$ for all $A, B \in B(H)$ if and only if there exists a unitary operator $U \in B(H,K)$ such that $\phi(A) = \gamma UAU^*$ for all $A \in B(H)$, where $\gamma \in \{-1, 1\}$.

1. Introduction

Let H be a complex Hilbert space with the product $\langle \cdot, \cdot \rangle$ and denote by $B(H)$ the algebra of all bounded linear operators on H. A projection P on H is an operator in $B(H)$ which is self-adjoint and idempotent. For non-zero vectors $x, y \in H$, the rank-1 operator $x \otimes y$ is defined by the map $z \mapsto \langle z, y \rangle x$ for $z \in H$. For a finite rank operator A, we use $\mathrm{tr}(A)$ to denote its trace.

Recall that the numerical range of an operator $A \in B(H)$ is defined by

$$W(A) = \{\langle Ax, x \rangle : x \in H, \|x\| = 1\}.$$

This is useful in studying operators and matrices; for example, see [12]. Motivated by theory and applications, there are many generalizations of the numerical range [12]. Among others, Halmos introduced the higher-dimensional numerical range, which have been studied by many authors [9, 15, 20, 21, 13, 1, 2, 17, 11]. Let k be a positive integer which is strictly smaller than the dimension of H. For $A \in B(H)$, the k-dimensional numerical range $W_k(A)$ of A is defined by

$$W_k(A) = \{\tfrac{1}{k}\mathrm{tr}(PAP) : P \text{ is projection on } H \text{ with rank } k\},$$

which is equivalent to

$$W_k(A) = \left\{ \frac{1}{k} \sum_{j=1}^{k} \langle Ax_j, x_j \rangle : x_1, \cdots, x_k \text{ are orthonormal vectors} \right\}.$$

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It is obvious that $W_1(A) = W(A)$. Generally, to compute the higher-dimensional numerical range is more difficult than to do the numerical range. Moreover, from the viewpoint of operator theory, the closure of $W_k(A)$ does not necessarily contain the spectrum of A and the condition $W_k(A) \subseteq [0,+\infty]$ does not imply that A is positive.

There has been a great deal of interest in studying preservers of a given generalized numerical range, i.e., maps which leave invariant the given generalized numerical ranges, see [15]. Pierce and Watkins [21] characterized linear operators preserving k-dimensional numerical range on $\mathbb{C}_{n\times n}$ with $n \neq 2k$. C. K. Li [14] completed the work of Pierce and Watkins and characterized the unital linear operators on matrix spaces that preserve higher-dimensional numerical radius. In [20], Omladič considered the surjective linear maps between the algebras $B(H)$ and $B(K)$ that preserve higher-dimensional numerical range.

The purpose of this paper is to characterize nonlinear maps preserving higher-dimensional numerical range of skew Lie product of operators. For any $A, B \in B(H)$, the product $AB - BA^*$ is called the skew Lie product of A and B. This product is playing a more and more important role in some research topics, and its study has recently attracted many authors' attention (see, for example, [3,4,5,6,8,22,24,23,18]). In this paper, we will study the map ϕ that satisfies

$$W_k(AB - BA^*) = W_k(\phi(A)\phi(B) - \phi(B)\phi(A)^*)$$

for all A, B in the domain. We will show that such a map is a C^*-isomorphism. This was obtained in [7] for the numerical range setting; however our approach is very different from that because of the difference between the numerical range and the higher-dimensional numerical range.

2. Preliminaries

Throughout this section, k is a positive integer and H is a complex Hilbert space of dimension greater than k. By $[x_1, x_2, \ldots, x_n]$ denote the subspace spanned by vectors $x_1, x_2, \ldots, x_n$ in H. The symbol i will stand for the imaginary unit.

We first recall some basic properties on higher-dimensional numerical range. One may see [9, 13] for more information.

PROPOSITION 2.1. ([13]) *Let $A \in B(H)$.*

(1) $W_k(A)$ is convex.

(2) If U is unitary, then $W_k(UAU^*) = W_k(A)$; if U is conjugate unitary, then $W_k(UAU^*) = \mathfrak{Con}(W_k(A))$, where $\mathfrak{Con}(W_k(A)) = \{\bar{\lambda} : \lambda \in W_k(A)\}$.

(3) $W_k(\lambda A) = \lambda W_k(A)$ for any $\lambda \in \mathbb{C}$.

(4) $W_k(\lambda I + A) = \lambda + W_k(A)$ for any $\lambda \in \mathbb{C}$.

The following proposition is well-known; however we can't find the proof. For the sake of completeness, we include a proof.

PROPOSITION 2.2. *Let $A \in B(H)$ and $\lambda \in \mathbb{C}$.*

(1) $W_k(A) = \{\lambda\}$ if and only if $A = \lambda I$.

(2) $W_k(A) \subseteq \mathbb{R}$ if and only if A is self-adjoint.

Proof. (1) The sufficiency is obvious. We verify the necessity as follows.

Fix orthonormal vectors $e_1, e_2, \cdots, e_{k-1}$. For a unit vector $x \in [e_1, \cdots, e_{k-1}]^{\perp}$, we have

$$\frac{1}{k}(\langle Ax, x\rangle + \langle Ae_1, e_1\rangle + \cdots + \langle Ae_{k-1}, e_{k-1}\rangle) = \lambda.$$

So $\langle Ax, x\rangle = c$ for all unit vectors $x \in [e_1, \cdots, e_{k-1}]^{\perp}$, where $c := k\lambda - \sum_{j=1}^{k-1} \langle Ae_j, e_j\rangle$ is a constant.

Let y in H be a unit vector. Since the dimension of $[e_1, \cdots, e_{k-1}]$ is $k-1$, there exist $k-2$ orthonormal vectors $x_1, \cdots, x_{k-2} \in [e_1, \cdots, e_{k-1}] \cap [y]^{\perp}$. Since the dimension of $[e_1, \cdots, e_{k-1}]^{\perp}$ is at least 2, we can take a unit vector $x_{k-1} \in [e_1, \cdots, e_{k-1}]^{\perp} \cap [y]^{\perp}$, and then take a unit vector $x_k \in [e_1, \cdots, e_{k-1}]^{\perp} \cap [x_{k-1}]^{\perp}$. Thus $x_1, \cdots, x_k$ as well as $x_1, \cdots, x_{k-1}, y$ are orthonormal. Now we have

$$\lambda = \frac{1}{k}\sum_{j=1}^{k} \langle Ax_j, x_j\rangle \text{ and } \lambda = \frac{1}{k}\left(\sum_{j=1}^{k-1} \langle Ax_j, x_j\rangle + \langle Ay, y\rangle\right).$$

The former equation together with the previous result yields that

$$\sum_{j=1}^{k-2} \langle Ax_j, x_j\rangle = k\lambda - \langle Ax_{k-1}, x_{k-1}\rangle - \langle Ax_k, x_k\rangle = k\lambda - 2c.$$

Hence

$$\langle Ay, y\rangle = k\lambda - \sum_{j=1}^{k-1} \langle Ax_j, x_j\rangle = k\lambda - \sum_{j=1}^{k-2} \langle Ax_j, x_j\rangle - \langle Ax_{k-1}, x_{k-1}\rangle = c$$

for all unit vectors y. This implies that $A = cI$. Since $W_k(A) = \{\lambda\}$, we have $c = \lambda$, showing (1).

(2) If A is self-adjoint, by the definition, $W_k(A) \subseteq \mathbb{R}$.

Now suppose that $W_k(A) \subseteq \mathbb{R}$. Decompose $A = A_1 + iA_2$, where A_1 and A_2 are self-adjoint. Since $W_k(A) \subseteq \mathbb{R}$, we have $W_k(A_2) = 0$. Thus $A_2 = 0$ by (1). So A is self-adjoint. $\square$

It is not difficult to compute the higher-dimensional numerical range of a projection. It is surprising that the higher-dimensional numerical range can determine the rank of a projection.

PROPOSITION 2.3. *Let P be a projection in $B(H)$ with rank r.*

(1) If $r < k$, then the biggest in $W_k(P)$ is $\frac{r}{k}$.

(2) If $r \geqslant k$, then the biggest in $W_k(P)$ is 1.

PROPOSITION 2.4. *Let $x \in H$ and $A \in B(H)$. Then the center of the rectangular box from the vertical and horizontal support lines of $W_k(Ax \otimes x + x \otimes xA)$ is $\frac{\langle Ax,x\rangle}{k}$.*

Proof. Without loss of generality, we may assume $\|x\| = 1$. For simplicity, we write $T = Ax \otimes x + x \otimes xA$. Then T has rank two at most.

First suppose that A is self-adjoint. Then T is self-adjoint. If Ax and x are linearly dependent, then $Ax = \langle Ax, x\rangle x$. A simple computation gives that $T = 2\langle Ax,x\rangle x \otimes x$. It follows that $W_k(T) = [0, \frac{2\langle Ax,x\rangle}{k}]$, whose mid-point is obviously $\frac{\langle Ax,x\rangle}{k}$. We now assume that Ax and x are linearly independent. Then T only has two non-zero eigenvalues $\mu_1 = a + \sqrt{a^2+b^2}$ and $\mu_2 = a - \sqrt{a^2+b^2}$, where $a = \langle Ax, x\rangle$ and $b = \|Ax - ax\|$. Since $b \neq 0$, we have $\mu_2 < 0 < \mu_1$. Let e_j be the normalized eigenvector of T corresponding to μ_j, $j = 1,2$. Then under the decomposition $H = [e_1] \oplus [e_2] \oplus [e_1,e_2]^{\perp}$,

$$T = \begin{bmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Let $\{f_\lambda\}_{\lambda \in \Lambda}$ be an orthonormal basis for $[e_1,e_2]^{\perp}$. For orthonormal vectors $x_1, x_2, \ldots, x_k$, write

$$x_j = \alpha_{j1} e_1 + \alpha_{j2} e_2 + \sum_{\lambda \in \Lambda} \beta_{j\lambda} f_\lambda, \quad j = 1,2,\ldots,k.$$

Then with $r = 1,2$,

$$\sum_{j=1}^{k} |\alpha_{jr}|^2 = \sum_{j=1}^{k} |\langle x_j, e_r\rangle|^2 \leqslant \|e_r\|^2 = 1. \tag{2.1}$$

A computation shows that

$$\sum_{j=1}^{k} \langle Tx_j, x_j\rangle = \mu_1 \sum_{j=1}^{k} |\alpha_{j1}|^2 + \mu_2 \sum_{j=1}^{k} |\alpha_{j2}|^2.$$

This together with (2.1) gives

$$\mu_2 \leqslant \sum_{j=1}^{k} \langle Tx_j, x_j\rangle \leqslant \mu_1.$$

(Recall that $\mu_2 < 0 < \mu_1$.) On the other hand, if $x_1 = e_r, x_2, \ldots x_k \in [e_1,e_2]^{\perp}$, then $\sum_{j=1}^{k} \langle Tx_j, x_j\rangle = \mu_r, r = 1,2$. So by the convexity, $W_k(T) = [\frac{\mu_2}{k}, \frac{\mu_1}{k}]$. The mid-point of this interval is $\frac{\mu_1+\mu_2}{2k} = \frac{\langle Ax,x\rangle}{k}$.

Now for the general A, write $A = A_1 + iA_2$, where A_1 and A_2 are self-adjoint. By the previous result, the mid-point of $W_k(A_j x \otimes x + x \otimes xA_j)$ is $\frac{\langle A_j x,x\rangle}{k}$ for $j = 1,2$. So the center of the rectangular box from the vertical and horizontal support lines of $W_k(Ax \otimes x + x \otimes xA)$ is $\frac{\langle A_1x,x\rangle}{k} + \frac{i\langle A_2x,x\rangle}{k} = \frac{\langle Ax,x\rangle}{k}$. $\square$

COROLLARY 2.5. *Let A and B be in $B(H)$ and suppose that $W_k(Ax\otimes x+x\otimes xA)=W_k(Bx\otimes x+x\otimes xB)$ for all $x\in H$. Then $A=B$. In particular, if $W_k(CA-AC^*)=W_k(CB-BC^*)$ for all $C\in B(H)$, then $A=B$.*

Proof. For $x\in H$, putting $C=ix\otimes x$, we get

$$iW_k(Ax\otimes x+x\otimes xA)=iW_k(Bx\otimes x+x\otimes xB).$$

By Proposition 2.4, the centers of the rectangular box from by the vertical and horizontal support lines of $W_k(Ax\otimes x+x\otimes xA)$ and $W_k(Bx\otimes x+x\otimes xB)$ are $\frac{\langle Ax,x\rangle}{k}$ and $\frac{\langle Bx,x\rangle}{k}$, respectively. So $\langle Ax,x\rangle=\langle Bx,x\rangle$ for all $x\in H$. This implies that $A=B$. □

PROPOSITION 2.6. *Let A and B be in $B(H)$ and suppose that $\|Ax-\langle Ax,x\rangle x\|=\|Bx-\langle Bx,x\rangle x\|$ for all unit vectors x in H. Then $A=\mu I+\nu B$ or $A=\mu I+\nu B^*$ for $\mu,\nu\in\mathbb{C}$ with $|\nu|=1$.*

Proof. For a unit vector x, from $\|Ax-\langle Ax,x\rangle x\|=\|Bx-\langle Bx,x\rangle x\|$ we get that

$$\|Ax\|^2-|\langle Ax,x\rangle|^2=\|Bx\|^2-|\langle Bx,x\rangle|^2.$$

Then for a pair of orthonormal vectors x,y in H and a scalar λ with $|\lambda|=1$, by replacing x with $\frac{\sqrt{2}}{2}(x+\lambda y)$, we have

$$\begin{aligned}&2\|Ax+\lambda Ay\|^2-|\langle Ax,x\rangle+\langle Ay,y\rangle+\bar{\lambda}\langle Ax,y\rangle+\lambda\langle Ay,x\rangle|^2\\&=2\|Bx+\lambda By\|^2-|\langle Bx,x\rangle+\langle By,y\rangle+\bar{\lambda}\langle Bx,y\rangle+\lambda\langle By,x\rangle|^2.\end{aligned}$$

Since this equation also holds for $-\lambda$, it follows from the parallelogram law that

$$\begin{aligned}&2\|Ax\|^2+2\|Ay\|^2-|\langle Ax,x\rangle+\langle Ay,y\rangle|^2-|\bar{\lambda}\langle Ax,y\rangle+\lambda\langle Ay,x\rangle|^2\\&=2\|Bx\|^2+2\|By\|^2-|\langle Bx,x\rangle+\langle By,y\rangle|^2-|\bar{\lambda}\langle Bx,y\rangle+\lambda\langle By,x\rangle|^2.\end{aligned}$$

By replacing λ with $i\lambda$ and by the parallelogram law, we have

$$\begin{aligned}&2\|Ax\|^2+2\|Ay\|^2-|\langle Ax,x\rangle+\langle Ay,y\rangle|^2-|\langle Ax,y\rangle|^2-|\langle Ay,x\rangle|^2\\&=2\|Bx\|^2+2\|By\|^2-|\langle Bx,x\rangle+\langle By,y\rangle|^2-|\langle Bx,y\rangle|^2-|\langle By,x\rangle|^2.\end{aligned}$$

Comparing the last two equations, we get that

$$\mathrm{Re}\lambda^2\langle Ay,x\rangle\overline{\langle Ax,y\rangle}=\mathrm{Re}\lambda^2\langle By,x\rangle\overline{\langle Bx,y\rangle}$$

for all $\lambda\in\mathbb{C}$ with $|\lambda|=1$. So $\langle Ay,x\rangle\overline{\langle Ax,y\rangle}=\langle By,x\rangle\overline{\langle Bx,y\rangle}$ for all pairs of orthonormal vectors x,y. Now the desired result follows from [16, Proposition 2.3]. □

We will close this section by computing the higher-dimensional numerical radius of a rank-2 operator. For $A\in B(H)$, the k-dimensional numerical radius of A is defined as

$$w_k(A)=\sup\{|\lambda|:\lambda\in W_k(A)\}.$$

EXAMPLE 2.7. Let A be a self-adjoint operator in $B(H)$ and x is a unit vector in H. Then $w_k(x\otimes xA - Ax\otimes x) = \frac{1}{k}\|Ax - \langle Ax, x\rangle x\|$.

Proof. A simple computation shows that

$$x\otimes xA - Ax\otimes x = b(x\otimes y - y\otimes x),$$

where $b = \|Ax - \langle Ax, x\rangle x\|$ and y is a unit vector which is a multiple of $Ax - \langle Ax, x\rangle x$ and is orthogonal to x.

Let P be a projection in $B(H)$. Then since $\langle Px, Py\rangle + \langle P^{\perp}x, P^{\perp}y\rangle = \langle x, y\rangle = 0$, we have that

$$\mathrm{tr}(P(x\otimes y - y\otimes x)P) = \langle Px, Py\rangle - \langle Py, Px\rangle = \langle P^{\perp}y, P^{\perp}x\rangle - \langle P^{\perp}x, P^{\perp}y\rangle.$$

If $\|Px\|^2 + \|Py\|^2 \leqslant 1$, then

$$|\mathrm{tr}(P(x\otimes y - y\otimes x)P)| \leqslant 2|\langle Px, Py\rangle| \leqslant 1;$$

if $\|Px\|^2 + \|Py\|^2 \geqslant 1$, then $\|P^{\perp}x\|^2 + \|P^{\perp}y\|^2 = \|x\|^2 + \|y\|^2 - (\|Px\|^2 + \|Py\|^2) \leqslant 1$ and hence

$$|\mathrm{tr}(P(x\otimes y - y\otimes x)P)| \leqslant 2|\langle P^{\perp}x, P^{\perp}y\rangle| \leqslant 1.$$

Consequently, $w_k(x\otimes y - y\otimes x) \leqslant \frac{1}{k}$.

On the other hand, with $e_1 = \frac{\sqrt{2}}{2}(ix + y)$ and $e_2, \ldots, e_k$ being orthonormal vectors in $[x, y]^{\perp}$, we have

$$\sum_{j=1}^{k}\langle (x\otimes y - y\otimes x)e_j, e_j\rangle = -i.$$

So $w_k(x\otimes y - y\otimes x) \geqslant \frac{1}{k}$. Thus, $w_k(x\otimes y - y\otimes x) = \frac{1}{k}$ and then $w_k(x\otimes xA - Ax\otimes x) = \frac{b}{k}$. $\square$

3. Higher-dimensional numerical range preservers

In this section, we characterize nonlinear maps which preserve the higher-numerical range of skew Lie product of operators. The main result is as follows.

THEOREM 3.1. *Let k be a positive integer. Let H and K be two complex Hilbert spaces with the dimensions greater than k. Suppose that $\phi : B(H) \to B(K)$ is a surjective map. Then ϕ satisfies*

$$W_k(AB - BA^*) = W_k(\phi(A)\phi(B) - \phi(B)\phi(A)^*) \tag{3.2}$$

for all $A, B \in B(H)$ if and only if there is a real number $\gamma \in \{-1, 1\}$ and a unitary operator $U \in B(H, K)$ such that $\phi(A) = \gamma UAU^*$ for all $A \in B(H)$.

The sufficiency part is clear. To prove the necessity part, we need several lemmas, in which we will keep the notations in the statement of the theorem.

LEMMA 3.2. *ϕ is injective.*

Proof. Let $A,\, B \in B(H)$ and suppose that $\phi(A)=\phi(B)$. Then for any $C \in B(H)$, we have

$$\begin{aligned} W_k(CA-AC^*) &= W_k(\phi(C)\phi(A)-\phi(A)\phi(C)^*) \\ &= W_k(\phi(C)\phi(B)-\phi(B)\phi(C)^*) \\ &= W_k(CB-BC^*). \end{aligned}$$

It follows from Corollary 2.5 that $A=B$. □

LEMMA 3.3. *ϕ is homogeneous.*

Proof. Let $A \in B(H)$ and $\lambda \in \mathbb{C}$. Then for all $B \in B(H)$, we have that

$$\begin{aligned} &W_k(\phi(B)\phi(\lambda A)-\phi(\lambda A)\phi(B)^*) = \lambda W_k(BA-AB^*) \\ &= \lambda W_k(\phi(B)\phi(A)-\phi(A)\phi(B)^*) = W_k(\phi(B)(\lambda\phi(A))-(\lambda\phi(A))\phi(B)^*). \end{aligned}$$

Since ϕ is surjective, it follows from Corollary 2.5 that $\phi(\lambda A)=\lambda\phi(A)$. □

LEMMA 3.4. *The following statements are true.*

(1) ϕ preserves self-adjoint operators in both directions and preserves the commutativity of self-adjoint operators.

(2) Either $\phi(I)=I$ and $W_k(\phi(A))=W_k(A)$ for all $A \in B(H)$, or $\phi(I)=-I$ and $W_k(\phi(A))=-W_k(A)$ for all $A \in B(H)$.

Proof. From

$$W_k(\phi(I)\phi(A)-\phi(A)\phi(I)^*) = W_k(IA-AI^*) = \{0\},$$

it follows that

$$\phi(I)\phi(A)=\phi(A)\phi(I)^* \tag{3.3}$$

for all $A \in B(H)$. Since ϕ is surjective, this implies that $\phi(I)=\lambda I$ for some non-zero real number λ. Now, if A is self-adjoint, from (3.2) we see that $\phi(A)$ is also self-adjoint. Further, if A and B are commuting self-adjoint operators, by (3.2) we know that $\phi(A)$ and $\phi(B)$ commute. This proves (1).

To prove (2), we notice that $\phi(iI)=i\lambda I$ by Lemma 3.3. Then we have that

$$\{2i\} = W_k((iI)I-I(iI)^*) = W_k((i\lambda I)(\lambda I)+(\lambda I)(i\lambda I)) = \{2i\lambda^2\}.$$

So $\lambda=\pm 1$ and then $\phi(I)=\pm I$. If $\phi(I)=I$, then for $A \in B(H)$, there holds

$$\begin{aligned} 2iW_k(A) &= W_k((iI)A-A(iI)^*) \\ &= W_k(\phi(iI)\phi(A)-\phi(A)\phi(iI)^*) \\ &= W_k((iI)\phi(A)-\phi(A)(iI)^*) \\ &= 2iW_k(\phi(A)). \end{aligned}$$

It follows that $W_k(\phi(A)) = W_k(A)$. Similarly, if $\phi(I) = -I$, then $W_k(\phi(A)) = -W_k(A)$ for all $A \in B(H)$. $\square$

LEMMA 3.5. *Suppose that $\phi(I) = I$. Let P be a non-zero projection in $B(H)$. Then $\phi(P)$ is a projection in $B(K)$.*

Proof. Since

$$\begin{aligned} 2iW_k(P) &= W_k((iP)P - P(iP)^*) = W_k(\phi(iP)\phi(P) - \phi(P)\phi(iP)^*) \\ &= W_k(i\phi(P)\phi(P) + i\phi(P)\phi(P)) = 2iW_k(\phi(P)^2), \end{aligned}$$

we have that $W_k(\phi(P)^2) = W_k(P) = W_k(\phi(P))$.

Since

$$\begin{aligned} \{0\} &= W_k((iP)(I-P) - (I-P)(iP)^*) \\ &= W_k(\phi(iP)\phi(I-P) - \phi(I-P)\phi(iP)^*) \\ &= iW_k(\phi(P)\phi(I-P) + \phi(I-P)\phi(P)), \end{aligned}$$

we have that $\phi(P)\phi(I-P) + \phi(I-P)\phi(P) = 0$. Hence, since $\phi(P)\phi(I-P) = \phi(I-P)\phi(P)$ by Lemma 3.4, we have that $\phi(P)\phi(I-P) = 0$.

For any $A \in B(H)$, we have

$$\begin{aligned} &W_k(\phi(P-I)\phi(A) - \phi(A)\phi(P-I)) = W_k((P-I)A - A(P-I)) \\ &= W_k(PA - AP) = W_k(\phi(P)\phi(A) - \phi(A)\phi(P)). \end{aligned}$$

In particular, for any unit vector $y \in K$, we have

$$W_k(\phi(P-I)y\otimes y - y\otimes y\phi(P-I)) = W_k(\phi(P)y\otimes y - y\otimes y\phi(P)).$$

By Proposition 2.6 and Example 2.7, there are scalars μ and ν with $\nu \in \{-1,1\}$ such that

$$\phi(P-I) = \mu I + \nu\phi(P).$$

It follows from $\phi(P)\phi(P-I) = 0$ that $\phi(P)^2 = \lambda\phi(P)$ for some scalar λ. Hence, since $W_k(\phi(P)^2) = W_k(\phi(P))$ and $\phi(P) \neq 0$, we have that $\lambda = 1$. So $\phi(P)^2 = \phi(P)$. $\square$

We are now in a position to prove the main result.

Proof of Theorem. It was showed in [7] in the case $k = 1$ and the dimension of H is greater than 3. In the sequel, we assume that $k > 1$ or $k = 1$ and the dimension is equal to 2. Furthermore, we will suppose that $\phi(I) = I$; otherwise, consider $-\phi$ instead of ϕ.

Thus by Lemmas 3.4, 3.5 and Proposition 2.3, for $P \in B(H)$, $\phi(P)$ is a projection in $B(K)$ with rank one if and only if P is a projection in $B(H)$ with rank one.

Our first aim is to show that ϕ is additive. For this, let A and B be in $B(H)$. For a unit vector y in K, there is a unit vector x in H such that $\phi(x\otimes x) = y\otimes y$. Then we have that

$$iW_k(y\otimes y\phi(A+B) + \phi(A+B)y\otimes y) = iW_k(x\otimes x(A+B) + (A+B)x\otimes x),$$

$$iW_k(y\otimes y\phi(A)+\phi(A)y\otimes y)=iW_k(x\otimes xA+Ax\otimes x),$$

and

$$iW_k(y\otimes y\phi(B)+\phi(B)y\otimes y)=iW_k(x\otimes xB+Bx\otimes x).$$

By Proposition 2.4, we have that $\langle\phi(A+B)y,y\rangle=\langle(A+B)x,x\rangle$, $\langle\phi(A)y,y\rangle=\langle Ax,x\rangle$ and $\langle\phi(B)y,y\rangle=\langle Bx,x\rangle$. Consequently, $\langle\phi(A+B)y,y\rangle=\langle(\phi(A)+\phi(B))y,y\rangle$ for all unit vectors y in K. It follows that $\phi(A+B)=\phi(A)+\phi(B)$.

So far, taking into account Lemmas 3.3, we know that ϕ is a linear bijection from $B(H)$ onto $B(K)$ which preserves the k-dimensional numerical range. Now by the main result in [20] (Theorems 3.5, 4.2, 4.4 and the remark followed theorem 3.5), one of the following holds:

(1) there is a unitary U from H onto K such that $\phi(A)=UAU^*$ for all $A\in B(H)$.

(2) there is a conjugate unitary U from H onto K such that $\phi(A)=UA^*U^*$ for all $A\in B(H)$; or in the case the dimension of H is equal to $2k$, $\phi(A)=\frac{1}{k}\mathrm{tr}(A)I-UA^*U^*$.

(3) $k>1$ and the dimension of H is equal to $2k$, there is a unitary U from H onto K such that $\phi(A)=\frac{1}{k}\mathrm{tr}(A)I-UAU^*$ for all $A\in B(H)$.

(4) $k>1$ and the dimension of H is equal to $2k$, there is a conjugate unitary U from H onto K such that $\phi(A)=\frac{1}{k}\mathrm{tr}(A)I-UA^*U^*$ for all $A\in B(H)$.

Possibilities (3) and (4) are impossible. This is because that $\phi(P)$ would be not a projection with rank-1 for a projection P with rank-1. To complete the proof, we have to show that possibility (2) is impossible too. Suppose on the contrary that (2) holds. Then for all $A,B\in B(H)$, we have that

$$\begin{aligned}W_k(AB-BA^*)&=W_k(UA^*B^*U^*-UB^*AU^*)\\&=\mathfrak{Con}(W_k(A^*B^*-B^*A))=W_k(BA-A^*B).\end{aligned}$$

This is not true; for example, by taking $A=x\otimes y$ and $B=y\otimes z$ for orthonormal vectors x,y,z in H, it would yield that $x\otimes z=0$. □

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