Operators and Matrices
Volume 10, Number 2 (2016), 345–377
doi:10.7153/oam-10-19

MULTIPLICATIVE LIE HIGHER DERIVATIONS OF UNITAL ALGEBRAS WITH IDEMPOTENTS

DONG HAN AND FENG WEI

(*Communicated by M. Omladič*)

Abstract. Let $\mathscr{R}$ be a commutative ring with identity and $\mathscr{A}$ be a unital algebra with nontrivial idempotent e over $\mathscr{R}$. Motivated by Benkovič's systematic and powerful work [2, 3, 4, 5, 6, 7, 8], we will study multiplicative Lie higher derivations (i.e. those Lie higher derivations without additivity assumption) on $\mathscr{A}$ in this article. Let $D = \{L_k\}_{k\in\mathbb{N}}$ be a multiplicative Lie higher derivation on $\mathscr{A}$. It is shown that under suitable assumptions, $D = \{L_k\}_{k\in\mathbb{N}}$ is of standard form; i.e. each component L_k $(k \geqslant 1)$ can be expressed through an additive higher derivation and a central mapping vanishing on all commutators of $\mathscr{A}$.

1. Introduction

Throughout this paper, $\mathscr{R}$ always denotes a commutative ring with an identity and $\mathscr{A}$ is a unital algebra over $\mathscr{R}$ which is 2 torsion-free. Suppose that $\mathscr{A}$ contains an idempotent $e \neq 0,1$. We denote the idempotent $1-e$ by f. According to the well-known Peirce decomposition formula, $\mathscr{A}$ can be represented in the following form

$$\mathscr{A} = e\mathscr{A}e + e\mathscr{A}f + f\mathscr{A}e + f\mathscr{A}f, \tag{1.1}$$

where $e\mathscr{A}e$ and $f\mathscr{A}f$ are subalgebras with unitary elements e and f, respectively. The subalgebra $e\mathscr{A}f$ is an $(e\mathscr{A}e, f\mathscr{A}f)$-bimodule and another subalgebra $f\mathscr{A}e$ is a $(f\mathscr{A}f, e\mathscr{A}e)$-bimodule. We will assume that $\mathscr{A}$ satisfies

$$\begin{aligned} &exe \cdot e\mathscr{A}f = \{0\} = f\mathscr{A}e \cdot exe \text{ implies } exe = 0,\\ &e\mathscr{A}f \cdot fxf = \{0\} = fxf \cdot f\mathscr{A}e \text{ implies } fxf = 0 \end{aligned} \tag{1.2}$$

for all $x \in \mathscr{A}$. In particular, if $f\mathscr{A}e = \{0\}$, then $\mathscr{A}$ degenerates to a triangular algebra. Some unital algebras with nontrivial idempotents share the property (1.2), such as triangular algebras, full matrix algebras, (semi-)simple algebras, prime algebras with nontrivial idempotents.

Mathematics subject classification (2010): 15A78, 15A86, 16W10.

Keywords and phrases: Lie higher derivation, higher derivation, additive derivation, unital algebra, generalized matrix algebra.

This work of the first author is supported by the Doctor Foundation of Henan Polytechnic University (B2010-21), the Natural Science Research Program of Education Department of Henan Province (16A110031 and 15A110026) and the National Natural Science Foundation of China (No. 11301155).

Let us first recall some basic facts related to Lie higher derivations of associative algebras. Various higher derivations, which consist of a family of some additive mappings, frequently appear in commutative and noncommutative contexts, see [15, 16, 21, 22, 23, 28, 29, 33, 34, 35, 36, 40, 41, 42, 46] and the references therein. Let $\mathscr{A}$ be a unital associative algebra over a commutative ring $\mathscr{R}$. Let $\mathbb{N}$ be the set of all non-negative integers and $D = \{L_k\}_{k\in\mathbb{N}}$ be a family of $\mathscr{R}$-linear mappings of $\mathscr{A}$ such that $L_0 = id_{\mathscr{A}}$. D is called:

(i) a *higher derivation* if
$$L_k(xy) = \sum_{i+j=k} L_i(x)L_j(y)$$
for all $x, y \in \mathscr{A}$ and for each $k \in \mathbb{N}$;

(ii) a *Lie higher derivation* if
$$L_k([x,y]) = \sum_{i+j=k} [L_i(x), L_j(y)]$$
for all $x, y \in \mathscr{A}$ and for each $k \in \mathbb{N}$.

If the additivity assumption in the above definition is removed, then the corresponding higher derivation (resp. Lie higher derivation) is said to be *multiplicative higher derivation* (resp. *multiplicative Lie higher derivation*). Note that L_1 is always a Lie derivation if $D = \{L_k\}_{k\in\mathbb{N}}$ is a Lie higher derivation. Obviously, every higher derivation is a Lie higher derivation. But the converse statements are in general not true. Assume that $\{d_k\}_{k\in\mathbb{N}}$ is a higher derivation (resp. multiplicative higher derivation) on $\mathscr{A}$. We can construct a sequence of $\mathscr{R}$-linear (resp. multiplicative) mappings
$$L_k = d_k + f_k, \qquad (\spadesuit)$$
where $f_k (k \in \mathbb{N})$ is a mapping from $\mathscr{A}$ into its center vanishing all commutators of $\mathscr{A}$. It is not difficult to see that $\{L_k\}_{k\in\mathbb{N}}$ is a Lie higher derivation (resp. multiplicative Lie higher derivation) on $\mathscr{A}$, but not a higher derivation (resp. multiplicative higher derivation) if $f_k \neq 0$ for some $k \in \mathbb{N}$. A Lie higher derivation (resp. multiplicative Lie higher derivation) $D = \{L_k\}_{k\in\mathbb{N}}$ is said to be *standard* if it has the property $(\spadesuit)$. Correspondingly, the $f_k (k \in \mathbb{N})$ is called a *linear functional*.

Many authors have made essential contributions to the related topics, see [2, 3, 4, 5, 6, 9, 10, 11, 14, 20, 24, 30, 31, 32, 35, 36, 37, 38, 39, 42, 43, 44, 45] and their references. Cheung in [11] presented sufficient conditions such that every Lie derivation on a triangular algebra $\mathscr{T} = \left[\begin{smallmatrix} A & M \\ O & B \end{smallmatrix}\right]$ is the sum of a derivation on $\mathscr{T}$ and a linear mapping from $\mathscr{T}$ into its center. Chen and Zhang [9] studied multiplicative Lie derivations of upper triangular matrix algebras and showed that each multiplicative Lie derivation from the $k \times k$ upper triangular matrix algebra $T_k(\mathscr{R})$ into the full matrix algebra $M_k(\mathscr{R})$ is of the form $T \longrightarrow TS - ST + T_\phi + f(T)I$ for all $T \in T_k(\mathscr{R})$, where $S \in M_k(\mathscr{R})$, ϕ is an additive derivation of $\mathscr{R}$, T_ϕ is the image of T under ϕ applied entrywise, f is a nonlinear mapping from $T_k(\mathscr{R})$ into $\mathscr{R}$ with $f([U,W]) = 0$ for all $U, W \in T_k(\mathscr{R})$. Yu and Zhang [44] extended this result to much more common triangular algebras and proved that under certain conditions every multiplicative Lie derivation on triangular algebras is the sum of an additive derivation and a mapping into its center vanishing on all commutators. Ji et al. [24] characterized multiplicative Lie triple derivations of 2 torsion-free triangular algebras and showed that every multiplicative Lie triple derivation on a 2 torsion-free triangular algebra $\mathscr{T} = \left[\begin{smallmatrix} A & M \\ O & B \end{smallmatrix}\right]$ is the sum of an additive derivation and a mapping into its center vanishing on all second commutators. Benkovič and Eremita [6] proved that each multiplicative Lie k-derivation on a triangular algebra $\mathscr{T} = \left[\begin{smallmatrix} A & M \\ O & B \end{smallmatrix}\right]$ has standard form if $\mathscr{T}$ satisfies the usual condition on projections of its center and if either A or B does not have nonzero central inner derivations. Wang and Wang [39] provided a characterization of multiplicative Lie k-derivations for a certain class of generalized matrix algebra. As a consequence, multiplicative Lie k-derivations of full matrix algebras are determined, which solves a conjecture due to Benkovič and Eremita in [6]. Benkovič [5] described Lie triple derivations of unital algebras with idempotents. Let $\mathscr{A}$ be a unital algebra with a nontrivial idempotent e over a unital commutative ring $\mathscr{R}$. He showed that under suitable assumptions, every Lie triple derivation Δ on $\mathscr{A}$ is of the form $\Delta = d + \delta + \gamma$, where d is a derivation of $\mathscr{A}$, δ is a singular Jordan derivation of $\mathscr{A}$ and γ is a linear mapping from $\mathscr{A}$ to its centre $\mathscr{Z}(\mathscr{A})$ that vanishes on $[\mathscr{A},[\mathscr{A},\mathscr{A}]]$. This result is further extended to Lie k-derivations of unital algebras with idempotents [38]. Fošner, Wei and Xiao [17] jointly investigated multiplicative Lie-type derivations of von Neumann algebras. Let $\mathscr{A}$ be a von Neumann algebra without abelian central summands of type I_1. It is shown that every multiplicative Lie k-derivation of $\mathscr{A}$ has the standard form, that is, can be expressed as a sum of an additive derivation and a central-valued mapping which annihilates each $(k-1)$-th commutator of $\mathscr{A}$.

In [40] Wei and Xiao addressed the question of when every higher derivation on a triangular algebra $\mathscr{T} = \left[\begin{smallmatrix} A & M \\ O & B \end{smallmatrix}\right]$ is an inner higher derivation. They also gave some characterizations on (generalized-)Jordan (triple-)higher derivations of triangular algebras. Xaio and Wei [42] extended the above mentioned Yu and Zhang's result [44] to multiplicative Lie higher derivation on triangular algebras and proved that under mild conditions, every multiplicative Lie high derivation on triangular algebras is of the standard form. Qi and Hou [36] described Lie higher derivations on nest algebras and obtained the following result. Let $\mathscr{N}$ be a nest on a Banach space X over the real or complex field $\mathbb{F}$. If there exists a non-trivial element in $\mathscr{N}$ which is complemented in X, then $D = \{L_k\}_{k\in\mathbb{N}}$ is a Lie higher derivation of the nest algebra $\tau(\mathscr{N})$ if and only if each L_k has the form $L_k(T) = d_k(T) + f_k(T)I$ for all $T \in \tau(\mathscr{N})$, where $\{d_k\}_{k\in\mathbb{N}}$ is a higher derivation and $\{f_k\}_{k\in\mathbb{N}}$ is a sequence of linear functionals satisfying the condition $f_k([U,W]) = 0$ for all $U,W \in \tau(\mathscr{N})$ and for each $k \in \mathbb{N}$. Qi [35] generalized this result to the case of more general triangular algebras. Li and Guo [28] proved that every Jordan higher derivation on a commutative subspace lattice algebra $\mathscr{A}$ is a higher derivation.

This article is aimed at studying multiplicative Lie higher derivations of a unital algebra $\mathscr{A}$ with the property (1.2). The organization of our paper is as follows. The second section is to present some basic properties and examples of unital algebras with nontrivial idempotents. We describe the form of multiplicative Lie derivations of $\mathscr{A}$ in section three. In last section, we give our main theorem which states that under certain conditions, every multiplicative Lie higher derivation on $\mathscr{A}$ is of the standard form $(\spadesuit)$.

2. Unital algebras with idempotents and examples

Let $\mathscr{A} = e\mathscr{A}e + e\mathscr{A}f + f\mathscr{A}e + f\mathscr{A}f$ be a unital algebra with nontrivial idempotents e and $f = 1 - e$, which satisfies the condition (1.2). It should be pointed out that the above more general condition of faithfulness was introduced by Benkovič in [8]. If we set $A = e\mathscr{A}e$, $M = e\mathscr{A}f$, $N = f\mathscr{A}e$ and $B = f\mathscr{A}f$, then

$$\mathscr{A} = e\mathscr{A}e + e\mathscr{A}f + f\mathscr{A}e + f\mathscr{A}f = A + M + N + B.$$

Thus an arbitrary element $x \in \mathscr{A}$ can be represented in the form

$$x = exe + exf + fxe + fxf = a + m + n + b,$$

where $a = exe \in A$, $m = exf \in M$, $n = fxe \in N$ and $b = fxf \in B$. Since $e^2 = e$, $f^2 = f$ and $ef = 0 = fe$, we know that

$$an = ab = ma = mm' = nn' = nb = ba = bm = 0$$

and

$$aa', mn \in A, am, mb \in M, na, bn \in N, bb', nm \in B$$

for all $a, a' \in A$, $m, m' \in M$, $n, n' \in N$ and $b, b' \in B$.

The center $\mathscr{Z}(\mathscr{A})$ of $\mathscr{A}$ is of the form given below.

$$\mathscr{Z}(\mathscr{A}) = \{\, a + b \in A + B \mid am = mb, na = bn \text{ for all } m \in M, n \in N \,\}.$$

Indeed, it follows from [27, Lemma 1] that the center $\mathscr{Z}(\mathscr{A})$ consists of all elements $a + b$ where $a \in \mathscr{Z}(A)$, $b \in \mathscr{Z}(B)$ and $am = mb$, $na = bn$ for all $m \in M, n \in N$. However, we must indicate that in our situation of assumption (1.2), one can easily conclude that $a \in \mathscr{Z}(A)$ and $b \in \mathscr{Z}(B)$ can be deleted. Now if $am = mb, na = bn$ for all $m \in M, n \in N$, then for any $a' \in A$, we get

$$(aa' - a'a)m = a(a'm) - a'(am) = (a'm)b - a'(mb) = 0$$

and

$$n(aa' - a'a) = (na)a' - (na')a = (bn)a' - b(na') = 0.$$

By (1.2) we have $aa' - a'a = 0$ and hence $a \in \mathscr{Z}(A)$. Likewise, we also have $b \in \mathscr{Z}(B)$.

Let us define two natural $\mathscr{R}$-linear projections $\pi_A : \mathscr{A} \to A$ and $\pi_B : \mathscr{A} \to B$ by

$$\pi_A : a + m + n + b \longmapsto a \quad \text{and} \quad \pi_B : a + m + n + b \longmapsto b.$$

It is easy to see that $\pi_A(\mathscr{Z}(\mathscr{A}))$ is a subalgebra of $\mathscr{Z}(A)$ and that $\pi_B(\mathscr{Z}(\mathscr{A}))$ is a subalgebra of $\mathscr{Z}(B)$. Give an element $a \in \pi_A(\mathscr{Z}(\mathscr{A}))$. If $a + b$, $a + b' \in \mathscr{Z}(\mathscr{A})$, then we have $am = mb = mb'$ and $na = bn = b'n$ for all $m \in M, n \in N$. So $b = b'$ by the assumption (1.2). That implies there exists a unique $b \in \pi_B(\mathscr{Z}(\mathscr{A}))$, which is denoted by $\sigma(a)$, such that $a + b \in \mathscr{Z}(\mathscr{A})$. It is easy to verify that the map $\sigma : \pi_A(\mathscr{Z}(\mathscr{A})) \to \pi_B(\mathscr{Z}(\mathscr{A}))$ is an algebraic isomorphism such that $am = m\sigma(a)$ and $na = \sigma(a)n$ for all $a \in \pi_A(\mathscr{Z}(\mathscr{A}))$, $m \in M, n \in N$.

Benkovič has provided some nice examples of unital algebras with nontrivial idempotents in [5]. We would like to add some further examples which belong to this category.

(i) There are a number of operators algebras which are unital algebras with idempotents and satisfy the condition (1.2). A standard example is $\mathscr{B}(X)$, the algebra of all bounded linear operators of a Banach space X. Each infinite von Neumann-algebra factor [25, Definition 6.3.1 and Lemma 6.3.3] is covered in this case. The Cuntz C^*-algebra $\mathscr{O}_k$ [12] is an algebra of this kind whenever $2 \leqslant k \leqslant \infty$. In addition, certain Cuntz-Krieger algebra [13] and some graph C^*-algebras [19] also satisfy the condition (1.2).

(ii) Each simple unital algebra $\mathscr{A}$ with a nontrivial idempotent e satisfies the property (1.2). It should be remarked that $\mathscr{I} = e\mathscr{A}f \cdot f\mathscr{A}e + e\mathscr{A}f + f\mathscr{A}e + f\mathscr{A}e \cdot e\mathscr{A}f$ is an ideal of the algebra $\mathscr{A}$. This implies that $\mathscr{I} = \mathscr{A}$, which is due to the simplicity of $\mathscr{A}$. It follows from the fact $exe \cdot e\mathscr{A}f = \{0\} = f\mathscr{A}e \cdot exe$ that $exe \cdot \mathscr{I} = exe \cdot \mathscr{A} = \{0\}$. Let us take $1 \in \mathscr{A}$. One immediately gets that $exe = 0$. Similarly, we can verify that the second assumption from (1.2) is also fulfilled.

(iii) Let A be a unital algebra over a commutative ring $\mathscr{R}$ and $\mathscr{A} = \mathscr{M}_k(A)$ $(k \geqslant 2)$ be the algebra of all $k \times k$ matrices over A. Let $\{E_{ij} | i, j = 1, 2, \cdots, k\}$ be the system of matrix units of $\mathscr{A}$ and 1 be the identity of $\mathscr{A}$. We denote the idempotent $e = E_{11}$ and $f = 1 - e$. We assert that $\mathscr{A}$ and e satisfy the condition (1.2). Note that the subalgebra $e\mathscr{A}e$ is isomorphic to A and that the subalgebra $f\mathscr{A}f$ is isomorphic to the matrix algebra $\mathscr{M}_{k-1}(A)$. Clearly, $(e\mathscr{A}e, f\mathscr{A}f)$-bimodule $e\mathscr{A}f \cong \mathscr{M}_{1\times(k-1)}(A)$ is faithful as a left $e\mathscr{A}e$-module and as a right $f\mathscr{A}f$-module. Another example mentioned is the so-called inflated algebras, which is actually isomorphic to the $\mathscr{M}_k(A)$ [26, Lemma 4.1].

(iv) Let $\mathscr{A}$ be a unital prime algebra with a nontrivial idempotent e. Let us take the idempotents e and $f = 1 - e$. In light of the primeness of $\mathscr{A}$, it follows from $exe \cdot e\mathscr{A}f = \{0\} = (exe)\mathscr{A}f$ that $exe = 0$. Similarly, applying the fact $e\mathscr{A}f \cdot fxf = \{0\} = e\mathscr{A}(fxf)$ yields $fxf = 0$ for all $x \in \mathscr{A}$. This shows that the algebra $\mathscr{A}$ satisfies the condition (1.2). Apart from the algebra $\mathscr{B}(X)$ of all bounded linear operators of a Banach space X, we would like to provide two additional wonderful examples of this case. Let G be a compact p-adic analytic group without non-trivial finite normal subgroups. Then the Iwasawa algebras Λ_G and Ω_G are both unital prime algebras with nontrivial idempotents [1, Theorem 4.2]. The graded Hecke algebras [18, Proposition 4.1.6] are also prime and share the property (1.2).

(v) Let $\mathscr{A}$ be a unital algebra with a nontrivial idempotent e. Suppose that $f\mathscr{A}e = \{0\}$ and that the bimodule $e\mathscr{A}f$ is faithful as a left $e\mathscr{A}e$-module and also as a right $f\mathscr{A}f$-module. Then the algebra $\mathscr{A}$ is a triangular algebra. In fact, every triangular algebra is an example satisfying the condition (1.2). The standard examples of triangular algebras include upper triangular matrix algebras, block upper triangular matrix algebras and Hilbert space nest algebras.

3. Multiplicative Lie derivations of unital algebras with idempotents

Let $\mathscr{A}$ be a unital algebra with a nontrivial idempotent e over a unital commutative ring $\mathscr{R}$. In this section, we show that under mild assumptions, every multiplicative Lie derivation of $\mathscr{A}$ has the standard form.

We are ready to state the first main result of this article.

THEOREM 3.1. *Let $\mathscr{A}$ be a 2 torsion-free unital algebra with a notrivial idempotent e and $L\colon \mathscr{A} \longrightarrow \mathscr{A}$ be a multiplicative Lie derivation. Suppose that*

(1) *$\mathscr{A}$ satisfies (1.2);*

(2) *$\pi_A(\mathscr{Z}(\mathscr{A})) = \mathscr{Z}(A)$ and $\pi_B(\mathscr{Z}(\mathscr{A})) = \mathscr{Z}(B)$;*

(3) *either A or B does not contain nonzero central ideals;*

(4) *for each $n \in N$, the condition $nM = 0$ or $Mn = 0$ implies $n = 0$;*

(5) *for each $m \in M$, the condition $mN = 0$ or $Nm = 0$ implies $m = 0$.*

Then $L = d + g$, where $d\colon \mathscr{A} \longrightarrow \mathscr{A}$ is an additive derivation and $g\colon \mathscr{A} \longrightarrow \mathscr{Z}(\mathscr{A})$ is a central mapping vanishing on all commutators.

REMARK 3.2. Let us first define a mapping $\nu\colon \mathscr{A} \longrightarrow \mathscr{A}$ by

$$\nu(x) = L(x) - [L(f), x].$$

One can easily check that $[L(f), x]$ is an inner derivation of $\mathscr{A}$ and that ν is a multiplicative Lie derivation of $\mathscr{A}$. Moreover,

$$\nu(f) = L(f) - [L(f), f] = L(f) - eL(f)f + fL(f)e.$$

Left-multiplication by e and right-multiplication by f in the above equality yields that $e\nu(f)f = 0$. Therefore, without loss of generality we can assume that $eL(f)f = 0$.

The proof of Theorem 3.1 can be reached via a series of lemmas.

LEMMA 3.3. *$L(0) = 0$ and $eL(x)f = 0$ for all $x \in A + B$.*

Proof. First we have

$$L(0) = L([0,0]) = [L(0),0] + [0,L(0)] = 0.$$

Note that $[x,f] = 0$ for all $x \in A+B$. Therefore,

$$\begin{aligned} 0 &= L([x,f]) \\ &= [L(x),f] + [x,L(f)] \\ &= eL(x)f - fL(x)e + [x,L(f)] \end{aligned} \tag{3.1}$$

for all $x \in A+B$. Left-multiplication by e and right-multiplication by f in (3.1) gives that

$$\begin{aligned} 0 &= eL(x)f + e[x,L(f)]f \\ &= eL(x)f + exL(f)f - eL(f)xf \end{aligned} \tag{3.2}$$

for all $x \in A+B$. Applying the fact $eL(f)f = 0$ to (3.2) yields $eL(x)f = 0$ for all $x \in A+B$. □

REMARK 3.4. Now we establish a new mapping $\xi : \mathscr{A} \longrightarrow \mathscr{A}$ as follows.

$$\xi(x) = L(x) - [L(e),x].$$

It is not difficult to verify that $[L(e),x]$ is an inner derivation of $\mathscr{A}$ and that ξ is a multiplicative Lie derivation of $\mathscr{A}$. Note that $eL(x)f = 0$ for all $x \in A+B$. Hence

$$e\xi(x)f = eL(x)f - e[L(e),x]f = 0$$

for all $x \in A+B$. In particular, we have

$$\xi(e) = L(e) - [L(e),e] = L(e) - fL(e)e + eL(e)f.$$

Left-multiplication by f and right-multiplication by e in the above equality yields $f\xi(e)e = 0$. Replacing L by ξ we may further assume that $fL(e)e = 0$.

LEMMA 3.5. *$fL(x)e = 0$ and $L(x) = eL(x)e + fL(x)f$ for all $x \in A+B$.*

Proof. By symmetry and Lemma 3.3 we get $fL(x)e = 0$ for all $x \in A+B$. The relation $eL(x)f = 0$ together with $fL(x)e = 0$ gives $L(x) = eL(x)e + fL(x)f$ for all $x \in A+B$. □

LEMMA 3.6. *$L(e), L(f) \in \mathscr{Z}(\mathscr{A})$, $L(m) = eL(m)f$ and $L(n) = fL(n)e$ for all $m \in M$, $n \in N$.*

Proof. For each $m \in M$, we have

$$\begin{aligned} L(m) &= L([e,m]) \\ &= [L(e),m] + [e,L(m)] \\ &= eL(m) - L(m)e + L(e)m - mL(e) \\ &= -fL(m)e + eL(m)f + L(e)m - mL(e). \end{aligned}$$

Multiplying by e from the left and by f from the right hand side and using Lemma 3.5 we see that $mL(e)-L(e)m=0$ for all $m\in M$. Analogously, we can prove that $L(e)n-nL(e)=0$ for all $n\in N$. These imply that $L(e)=eL(e)e+fL(e)f\in\mathscr{Z}(\mathscr{A})$ and

$$L(m)=-fL(m)e+eL(m)f,\quad L(n)=-eL(n)f+fL(n)e$$

for all $m\in M$, $n\in N$. Since $\mathscr{A}$ is 2 torsion-free, we have $fL(m)e=eL(n)f=0$ for all $m\in M$, $n\in N$. Therefore $L(m)=eL(m)f$, $L(n)=fL(n)e$ for all $m\in M$, $n\in N$. Then $L(f)\in\mathscr{Z}(\mathscr{A})$ is obvious by symmetry. □

REMARK 3.7. Let us set $\mathscr{T}=A+M+B$, then $\mathscr{T}$ is a subalgebra of $\mathscr{A}$. Trivially, $\mathscr{Z}(\mathscr{A})\subseteq\mathscr{Z}(\mathscr{T})$. Note that here $\mathscr{T}$ is not a triangular algebra, since M is not a faithful left A-module or a faithful right B-module. Similarly, we denote $A+N+B$ by $\mathscr{T}'$.

LEMMA 3.8. *$L(exf)=eL(x)f$ and $fL(x)e=0$ for all $x\in\mathscr{T}$; $L(fxe)=fL(x)e$ and $eL(x)f=0$ for all $x\in\mathscr{T}'$.*

Proof. For an arbitrary $x\in\mathscr{T}$, we have

$$L(exf)=L([x,f])=L(x)f-fL(x)=eL(x)f-fL(x)e.$$

Since $L(exf)\in M$ for all $x\in\mathscr{T}$ by Lemma 3.6, we assert from the above relation that $L(exf)=eL(x)f$ and so $fL(x)e=0$. By symmetry, $L(fxe)=fL(x)e$ and $eL(x)f=0$ for all $x\in\mathscr{T}'$. □

LEMMA 3.9. *For any $a\in A$, $n\in N$, $m\in M$ and $b\in B$, we have*

$$\begin{aligned}
&L(a+m)-L(a)-L(m)\in\mathscr{Z}(\mathscr{A}),\\
&L(b+m)-L(b)-L(m)\in\mathscr{Z}(\mathscr{A}),\\
&L(a+n)-L(a)-L(n)\in\mathscr{Z}(\mathscr{A}),\\
&L(b+n)-L(b)-L(n)\in\mathscr{Z}(\mathscr{A}).
\end{aligned}$$

Proof. We first claim that $[L(a+m)-L(a)-L(m),m']=0$ for all $a\in A$, $m,m'\in M$. In light of Lemma 3.6 we arrive at

$$\begin{aligned}
L(am')&=L([a+m,m'])\\
&=[L(a+m),m']+[a+m,L(m')]\\
&=[L(a+m),m']+[a,L(m')]
\end{aligned}$$

for all $a\in A,m,m'\in M$. On the other hand, we obtain

$$L(am')=L([a,m'])=[L(a),m']+[a,L(m')]$$

for all $a\in A$, $m'\in M$. Combining the above two relations, we get

$$[L(a+m)-L(a)-L(m),m']=0$$

for all $a \in A,\ m,m' \in M$.

Now let us prove that $[L(a+m)-L(a)-L(m),b]=0$ for all $a \in A,\ m \in M$ and $b \in B$. It is easy to check that

$$\begin{aligned} L(mb) &= L([a+m,b]) \\ &= [L(a+m),b]+[a+m,L(b)] \\ &= L(a+m)b-bL(a+m)+aL(b)+mL(b)-L(b)a-L(b)m \end{aligned}$$

and

$$L(mb)=L(m)b+mL(b)-bL(m)-L(b)m$$

for all $a \in A,\ m \in M$ and $b \in B$. Combining the last two relations we have

$$L(a+m)b-bL(a+m)+aL(b)-L(b)a=L(m)b-bL(m)$$

for all $a \in A,\ m \in M$ and $b \in B$. That is,

$$L(a+m)b+[a,L(b)]-L(m)b=bL(a+m)-bL(m)$$

for all $a \in A,\ m \in M$ and $b \in B$. This implies that

$$L(a+m)b-L(m)b-L(a)b=bL(a+m)-bL(m)-bL(a),$$

and hence $[L(a+m)-L(a)-L(m),b]=0$ for all $a \in A,\ m \in M$ and $b \in B$.

For convenience, let us denote $L(a+m)-L(a)-L(m)$ by x. Thus $x \in A+B$ by Lemma 3.8. Let us take $m' \in M$. Then for each $n \in N$, we get

$$(xn-nx)m'=xnm'-nxm'=nm'x-nm'x=0,$$

which leads to $xn-nx=0$ by the assumption. It follows that

$$[L(a+m)-L(a)-L(m),n]=[x,n]=0$$

for all $a \in A,\ m \in M$ and $n \in N$. Now one can see that $L(a+m)-L(a)-L(m) \in \mathscr{Z}(\mathscr{A})$ for all $a \in A,\ m \in M$. The rest follows by symmetry. □

LEMMA 3.10. *L is additive on M and N.*

Proof. Considering Lemma 3.6 and Lemma 3.9, we arrive at

$$\begin{aligned} L(m+m') &= L([e+m,f+m']) \\ &= [L(e+m),f+m']+[e+m,L(f+m')] \\ &= [L(e)+L(m),f+m']+[e+m,L(f)+L(m')] \\ &= [L(m),f]+[e,L(m')] \\ &= L(m)+L(m') \end{aligned}$$

for all $m',m \in M$. Similarly, we can also prove that $L(n+n')=L(n)+L(n')$ for all $n',n \in N$. □

LEMMA 3.11. *For any $a \in A$, $m \in M$, $n \in N$ and $b \in B$, we have*

$$L(a+m+b) - L(a) - L(b) - L(m) \in \mathscr{Z}(\mathscr{A}),$$
$$L(a+n+b) - L(a) - L(b) - L(n) \in \mathscr{Z}(\mathscr{A}).$$

Proof. By Lemma 3.6 we compute that

$$\begin{aligned} L([m',a+m+b]) &= [L(m'),a+m+b] + [m',L(a+m+b)] \\ &= [L(m'),a] + [L(m'),b] + [m',L(a+m+b)] \end{aligned}$$

for all $a \in A$, $m,m' \in M$ and $b \in B$. However, due to Lemma 3.10 we can also get

$$\begin{aligned} L([m',a+m+b]) &= L([m',a] + [m',b]) \\ &= L([m',a]) + L([m',b]) \\ &= [L(m'),a] + [m',L(a)] + [L(m'),b] + [m',L(b)] \end{aligned}$$

for all $a \in A$, $m,m' \in M$ and $b \in B$. Hence we have

$$[m',L(a+m+b) - L(a) - L(b)] = 0$$

for all $a \in A$, $m,m' \in M$ and $b \in B$. This implies that

$$[m',L(a+m+b) - L(a) - L(b) - L(m)] = 0$$

for all $a \in A$, $m,m' \in M$ and $b \in B$.

Now let us show that $[L(a+m+b) - L(a) - L(m) - L(b), b'] = 0$ for all $a \in A$, $m \in M$ and $b,b' \in B$. By a simple computation we have

$$\begin{aligned} L(mb' + [b,b']) &= L([a+m+b,b']) \\ &= [L(a+m+b),b'] + [a+m+b,L(b')] \\ &= L(a+m+b)b' - b'L(a+m+b) + aL(b') + mL(b') \\ &\quad + bL(b') - L(b')a - L(b')m - L(b')b. \end{aligned}$$

On the other hand, by Lemma 3.9 we also get

$$\begin{aligned} L(mb' + [b,b']) &= L(mb') + L([b,b']) + x \\ &= L(mb') + [L(b),b'] + [b,L(b')] + x \\ &= L(m)b' + mL(b') - b'L(m) - L(b')m + L(b)b' \\ &\quad - b'L(b) + bL(b') - L(b')b + x \end{aligned}$$

for all $m \in M$, $b,b' \in B$ and some $x \in \mathscr{Z}(\mathscr{A})$. Comparing the last two relations give that

$$\begin{aligned} &L(m)b' - b'L(m) + L(b)b' - b'L(b) + x \\ &= L(a+m+b)b' - b'L(a+m+b) + aL(b') - L(b')a \\ &= L(a+m+b)b' - b'L(a+m+b) + b'L(a) - L(a)b' \end{aligned}$$

for all $a \in A$, $m \in M$ and $b, b' \in B$. That is,

$$\begin{aligned} & b'L(a+m+b) - b'L(m) - b'L(b) - b'L(a) + x \\ & = L(a+m+b)b' - L(m)b' - L(b)b' - L(a)b', \end{aligned}$$

which can be rewritten as

$$[b', L(a+m+b) - L(a) - L(b) - L(m)] \in \mathscr{Z}(\mathscr{A})$$

for all $a \in A$, $m \in M$ and $b, b' \in B$. By Lemma 3.8 we know that $L(a+m+b) - L(a) - L(b) - L(m) \in A + B$. Note that

$$[b', L(a+m+b) - L(a) - L(b) - L(m)] \in B,$$

hence we have

$$[b', L(a+m+b) - L(a) - L(b) - L(m)] = 0$$

for all $a \in A$, $m \in M$ and $b, b' \in B$. The rest of the proof is similar to the proof of Lemma 3.9. The other assertion can also be reached in a similar way. □

LEMMA 3.12. *For any $a \in A$, $m \in M$, $n \in N$ and $b \in B$, we have $L(a+m+n+b) - L(a) - L(b) - L(m) - L(n) \in \mathscr{Z}(\mathscr{A})$.*

Proof. In view of Lemma 3.6, we routinely compute that

$$\begin{aligned} L([m'', [m', a+m+n+b]]) &= [L(m''), [m', a+m+n+b]] \\ &\quad + [m'', [L(m'), a+m+n+b]] \\ &\quad + [m'', [m', L(a+m+n+b)]] \\ &= [L(m''), [m', n]] + [m'', [L(m'), n]] \\ &\quad + [m'', [m', L(a+m+n+b)]] \end{aligned}$$

and

$$L([m'', [m', a+m+n+b]]) = [L(m''), [m', n]] + [m'', [L(m'), n]] + [m'', [m', L(n)]]$$

for all $a \in A$, $m, m', m'' \in M$, $n \in N$ and $b \in B$. Hence we have

$$[m'', [m', L(a+m+n+b) - L(n)]] = 0,$$

or

$$[m'', [m', fL(a+m+n+b)e - L(n)]] = 0 \tag{3.3}$$

for all $a \in A$, $m, m', m'' \in M$, $n \in N$ and $b \in B$.

By a simple calculation we can also get

$$\begin{aligned} L([b', [m', a+m+n+b]]) &= [L(b'), [m', a+m+n+b]] \\ &\quad + [b', [L(m'), a+m+n+b]] \\ &\quad + [b', [m', L(a+m+n+b)]] \end{aligned}$$

and

$$
\begin{aligned}
L([b',[m',a+m+n+b]]) &= L([b',[m',a+n+b]]) \\
&= [L(b'),[m',a+n+b]]+[b',[L(m'),a+n+b]] \\
&\quad +[b',[m',L(a+n+b)]]
\end{aligned}
$$

for all $a\in A$, $m,m'\in M$, $n\in N$ and $b,b'\in B$. Comparing the last two relations, we obtain

$$[b',[m',L(a+m+n+b)-L(a+n+b)]]=0$$

for all $a\in A$, $m,m'\in M$, $n\in N$ and $b,b'\in B$. Thus

$$[b',[m',fL(a+m+n+b)e-fL(a+n+b)e]]=0$$

for all $a\in A$, $m,m'\in M$, $n\in N$ and $b,b'\in B$. In view of Lemma 3.8 we can further get

$$[b',[m',fL(a+m+n+b)e-L(n)]]=0$$

for all $a\in A$, $m,m'\in M$, $n\in N$ and $b,b'\in B$. Applying the same arguments as in Lemma 3.9 we can show that

$$[n',[m',fL(a+m+n+b)e-L(n)]]=0 \tag{3.4}$$

for all $a\in A$, $m,m'\in M$, $n,n'\in N$ and $b\in B$. Combining (3.3) with (3.4), we obtain

$$[m',fL(a+m+n+b)e-L(n)]\in\mathscr{Z}(\mathscr{A})$$

for all $a\in A$, $m,m'\in M$, $n\in N$ and $b\in B$. Hence, $(fL(a+m+n+b)e-L(n))M\subseteq\mathscr{Z}(B)$ and $M(fL(a+m+n+b)e-L(n))\subseteq\mathscr{Z}(A)$ by the condition (2) of Theorem 3.1. Without loss of generality, we may assume that A does not contain nonzero central ideals. Since $M(fL(a+m+n+b)e-L(n))$ is a central ideal of A, we arrive at $M(fL(a+m+n+b)e-L(n))=0$. Applying the condition (4) yields

$$fL(a+m+n+b)e-L(n)=0 \tag{3.5}$$

for all $a\in A$, $m\in M$, $n\in N$ and $b\in B$.

Repeating the same procedure we can also prove that

$$eL(a+m+n+b)f-L(m)=0 \tag{3.6}$$

for all $a\in A$, $m\in M$, $n\in N$ and $b\in B$.

By a calculation we obtain

$$
\begin{aligned}
L([m',a+m+n+b]) &= [L(m'),a+m+n+b]+[m',L(a+m+n+b)] \\
&= [L(m'),a+n+b]+[m',L(a+m+n+b)]
\end{aligned}
$$

and

$$L([m',a+m+n+b])=[L(m'),a+n+b]+[m',L(a+n+b)]$$

for all $a \in A$, $m, m' \in M$, $n \in N$ and $b \in B$. The previous two equalities mean that

$$[m', L(a+m+n+b) - L(a+n+b)] = 0$$

for all $a \in A$, $m, m' \in M$, $n \in N$ and $b \in B$. Furthermore, Lemma 3.11 implies that

$$[m', L(a+m+n+b) - L(a) - L(b) - L(m) - L(n)] = 0 \tag{3.7}$$

for all $a \in A$, $m, m' \in M$, $n \in N$ and $b \in B$. Likewise, we can also prove that

$$[n', L(a+m+n+b) - L(a) - L(b) - L(m) - L(n)] = 0 \tag{3.8}$$

for all $a \in A$, $m \in M$, $n, n' \in N$ and $b \in B$. Equalities (3.5), (3.6), (3.7) and (3.8) jointly lead to

$$L(a+m+n+b) - L(a) - L(b) - L(m) - L(n) \in \mathscr{Z}(\mathscr{A})$$

for all $a \in A$, $m \in M$, $n \in N$ and $b \in B$. $\square$

LEMMA 3.13. *$fL(a)f \in \mathscr{Z}(B)$ for all $a \in A$ and $eL(b)e \in \mathscr{Z}(A)$ for all $b \in B$.*

Proof. It follows from Lemma 3.5 that

$$L(a) = eL(a)e + fL(a)f \text{ and } L(b) = eL(b)e + fL(b)f \tag{3.9}$$

for all $a \in A$ and $b \in B$. On the other hand

$$L(0) = L([a,b]) = L(a)b - bL(a) + aL(b) - L(b)a$$

for all $a \in A$ and $b \in B$. This together with (3.9) gives that

$$fL(a)fb - bfL(a)f + aeL(b)e - eL(b)ea = 0,$$

and hence

$$fL(a)fb - bfL(a)f = aeL(b)e - eL(b)ea = 0$$

for all $a \in A$ and $b \in B$. We complete the proof. $\square$

REMARK 3.14. Lemma 3.13 and assumption (2) of Theorem 3.1 imply that we can define a mapping

$$g'(x) = fL(exe)f + \sigma^{-1}(fL(exe)f) + eL(fxf)e + \sigma(eL(fxf)e)$$

so that $g'(x) \in \mathscr{Z}(\mathscr{A})$ for all $x \in \mathscr{A}$. It is easy to check that $g'([x,y]) = 0$ for all $x, y \in \mathscr{T}$. Then the mapping $\chi : \mathscr{A} \to \mathscr{A}$ defined by $\chi(x) = L(x) - g'(x)$ is a multiplicative Lie derivation of $\mathscr{T}$ or $\mathscr{T}'$.

LEMMA 3.15. *Let χ be as in Remark 3.14. Then*

$$\chi(am) = \chi(a)m + a\chi(m),\ \chi(mb) = \chi(m)b + m\chi(b),$$
$$\chi(na) = \chi(n)a + n\chi(a),\ \chi(bn) = \chi(b)n + b\chi(n)$$

for all $a \in A$, $m \in M$, $n \in N$ and $b \in B$.

Proof. Taking into account the fact that $\chi(a) = L(a) - g'(a) \in A$, we have

$$\chi(am) = [\chi(a), m] + [a, \chi(m)] = \chi(a)m + a\chi(m)$$

for all $a \in A$ and $m \in M$. Similarly, we can show that the rest of the relations hold. □

REMARK 3.16. Let $x \in \mathscr{A}$. We define $d\colon \mathscr{A} \to \mathscr{A}$ by

$$d(x) = \chi(exe) + \chi(exf) + \chi(fxe) + \chi(fxf)$$

Then $L(x) = d(x) + g'(x) + \chi(x) - d(x) = d(x) + g(x)$, where $g(x) = g'(x) + \chi(x) - d(x)$ is a mapping from $\mathscr{A}$ to its center by Lemma 3.12 and Remark 3.14.

Observe that on A, M, N and B the map d coincides with χ, and on M and N the map χ coincides with L.

LEMMA 3.17. *Let d be as in Remark 3.16. Then d is an additive derivation on A and B.*

Proof. First we have

$$d(b) = \chi(b) = L(b) - g'(b) \in B,$$
$$d(a) = \chi(a) = L(a) - g'(a) \in A$$

for all $a \in A$ and $b \in B$.

Let $a, c \in A$ and $m \in M$. By Lemma 3.10 and Lemma 3.15 we know that

$$d(acm) = d(ac)m + acd(m),$$
$$d(acm) = d(a)cm + ad(cm) = d(a)cm + ad(c)m + acd(m),$$
$$d((a+c)m) = d(a+c)m + (a+c)d(m),$$
$$d((a+c)m) = d(am) + d(cm) = d(a)m + ad(m) + d(c)m + cd(m).$$

It follows that

$$(d(ac) - d(a)c - ad(c))m = 0 \text{ and } (d(a+c) - d(a) - d(c))m = 0. \tag{3.10}$$

Let $a, c \in A$ and $n \in N$. Similarly, we have

$$n(d(ac) - d(a)c - ad(c)) = 0 \text{ and } n(d(a+c) - d(a) - d(c)) = 0. \tag{3.11}$$

Combining (3.10) with (3.11), we obtain

$$d(ac) - d(a)c - ad(c) = 0 \text{ and } d(a+c) - d(a) - d(c) = 0.$$

Hence d is an additive derivation on A. In the same way, we can show that d is also an additive derivation on B. The proof is completed. □

LEMMA 3.18. *For any $a \in A$, $m \in M$, $n \in N$ and $b \in B$, we have*

$$d(a + m + n + b) = d(a) + d(m) + d(n) + d(b).$$

Proof. According to Remark 3.16, we can compute that

$$\begin{aligned} ed(x)e &= e(\chi(exe)+\chi(exf)+\chi(fxe)+\chi(fxf))e = d(exe),\\ ed(x)f &= e(\chi(exe)+\chi(exf)+\chi(fxe)+\chi(fxf))f = d(exf),\\ fd(x)e &= f(\chi(exe)+\chi(exf)+\chi(fxe)+\chi(fxf))e = d(fxe),\\ fd(x)f &= f(\chi(exe)+\chi(exf)+\chi(fxe)+\chi(fxf))f = d(fxf) \end{aligned}$$

for all $x \in \mathscr{A}$. It follows that

$$d(a+m+n+b) = d(a)+d(m)+d(n)+d(b)$$

for all $a \in A$, $m \in M$, $n \in N$ and $b \in B$. □

LEMMA 3.19. *d is an additive derivation of $\mathscr{A}$.*

Proof. Let $x = x_a + x_m + x_n + x_b$, $y = y_a + y_m + y_n + y_b \in \mathscr{A}$. In view of Lemma 3.10, Lemma 3.17 and Lemma 3.18, we compute that

$$\begin{aligned} d(x+y) &= d(x_a+x_m+x_n+x_b+y_a+y_m+y_n+y_b)\\ &= d(x_a+y_a)+d(x_m+y_m)+d(x_n+y_n)+d(x_b+y_b)\\ &= d(x_a)+d(x_m)+d(x_n)+d(x_b)+d(y_a)+d(y_m)+d(y_n)+d(y_b)\\ &= d(x_a+x_m+x_n+x_b)+d(y_a+y_m+y_n+y_b)\\ &= d(x)+d(y). \end{aligned} \tag{3.12}$$

Considering Lemma 3.15, we have

$$\begin{aligned} &d(am) = d(a)m+ad(m),\ d(mb) = d(m)b+md(b),\\ &d(na) = d(n)a+nd(a),\ d(bn) = d(b)n+bd(n) \end{aligned} \tag{3.13}$$

for all $a \in A$, $m \in M$, $n \in N$ and $b \in B$.

Next let us show that $d(mn) = d(m)n+md(n)$ and $d(nm) = d(n)m+nd(m)$. For any m, $m_0 \in M$ and $n \in N$, we have

$$\begin{aligned} d([[m,n],m_0]) &= [[L(m),n],m_0]+[[m,L(n)],m_0]+[[m,n],L(m_0)]-g([[m,n],m_0])\\ &= [[d(m),n],m_0]+[[m,d(n)],m_0]+[[m,n],d(m_0)]-g([[m,n],m_0]) \end{aligned}$$

and

$$\begin{aligned} d([[m,n],m_0]) &= L([mn-nm,m_0])-g([[m,n],m_0])\\ &= [L(mn-nm),m_0]+[mn-nm,L(m_0)]-g([[m,n],m_0])\\ &= [d(mn-nm),m_0]+[mn-nm,L(m_0)]-g([[m,n],m_0])\\ &= [d(mn)-d(nm),m_0]+[mn-nm,d(m_0)]-g([[m,n],m_0]), \end{aligned}$$

where we used (3.12) in the last step. Therefore,

$$[d(mn)-d(nm)-[d(m),n]-[m,d(n)],m_0] = 0 \tag{3.14}$$

for all $m, m_0 \in M$ and $n \in N$. Repeating the same computational process, we can also get

$$[d(mn) - d(nm) - [d(m), n] - [m, d(n)], n_0] = 0 \tag{3.15}$$

for all $m \in M$ and $n,\ n_0 \in N$. It follows from (3.14) and (3.15) that

$$(d(mn) - d(m)n - md(n)) - (d(nm) - d(n)m - nd(m)) \in \mathscr{Z}(\mathscr{A}) \tag{3.16}$$

for all $m \in M$ and $n \in N$. We may assume that A does not contain nonzero central ideals. Set

$$\varepsilon(m,n) = d(mn) - d(m)n - md(n).$$

Let $a \in A$. Then by (3.13) and Lemma 3.17 we have

$$\begin{aligned}\varepsilon(am,n) &= d(amn) - d(am)n - amd(n)\\ &= d(a)mn + ad(mn) - d(a)mn - ad(m)n - amd(n)\\ &= a\varepsilon(m,n)\end{aligned}$$

for all $m \in M$ and $n \in N$. Note that $\varepsilon(m,n) \in \mathscr{Z}(A)$ for all $m \in M, n \in N$. So $A\varepsilon(m,n)$ is a central ideal of A. Consequently, $\varepsilon(m,n) = 0$ for all $m \in M$ and $n \in N$. This gives

$$d(mn) - d(m)n - md(n) = 0 \tag{3.17}$$

for all $m \in M$ and $n \in N$. This fact together with (3.16) leads to

$$d(nm) - d(n)m - nd(m) = 0 \tag{3.18}$$

for all $m \in M$ and $n \in N$.

It follows from the equalities (3.12), (3.13), (3.17), (3.18) and Lemma 3.17 that

$$\begin{aligned}d(xy) &= d(x_ay_a + x_ax_m + x_by_b + x_by_n + x_my_b + x_my_n + x_ny_a + x_ny_m)\\ &= d(x_ay_a) + d(x_ay_m) + d(x_by_b) + d(x_by_n) + d(x_my_b) + d(x_my_n) + d(x_ny_a) + d(x_ny_m)\\ &= d(x_a)y_a + d(x_a)y_m + d(x_b)y_b + d(x_b)y_n + d(x_m)y_b + d(x_m)y_n + d(x_n)y_a + d(x_n)y_m\\ &\quad + x_ad(y_a) + x_ad(y_m) + x_bd(y_b) + x_bd(y_n) + x_md(y_b) + x_md(y_n) + x_nd(y_a) + x_nd(y_m)\\ &= d(x)y + xd(y),\end{aligned}$$

and hence d is an additive derivation of $\mathscr{A}$. □

Now we are in a position to prove the main theorem of this section.

Proof of Theorem 3.1. First we have $L = d + g$. By Lemma 3.19 we know that d is an additive derivation of $\mathscr{A}$. Note that $g(x) \in \mathscr{Z}(\mathscr{A})$ for all $x \in \mathscr{A}$. It remains to show that $g([x,y]) = 0$ for all $x, y \in \mathscr{A}$.

$$\begin{aligned}g([x,y]) &= L([x,y]) - d([x,y])\\ &= [L(x),y] + [x,L(y)] - [d(x),y] - [x,d(y)]\\ &= [d(x),y] + [x,d(y)] - [d(x),y] - [x,d(y)] = 0\end{aligned}$$

for all $x, y \in \mathscr{A}$. □

4. Multiplicative Lie higher derivations of unital algebras with idempotents

THEOREM 4.1. *Let $\mathscr{A}$ be a 2 torsion-free unital algebra with a notrivial idempotent e and $\{L_k\}_{k\in\mathbb{N}}$ be a multiplicative Lie higher derivation of $\mathscr{A}$. If the following conditions are satisfied:*

(1) *$\mathscr{A}$ satisfies (1.2);*

(2) *$\pi_A(\mathscr{Z}(\mathscr{A})) = \mathscr{Z}(A)$ and $\pi_B(\mathscr{Z}(\mathscr{B})) = \mathscr{Z}(B)$;*

(3) *either A or B does not contain nonzero central ideals;*

(4) *for each $n \in N$, the condition $nM = 0$ or $Mn = 0$ implies $n = 0$;*

(5) *for each $m \in M$, the condition $mN = 0$ or $Nm = 0$ implies $m = 0$;*

(6) *for each $k \in \mathbb{N}$, $eL_k(e)f = 0$ and $fL_k(e)e = 0$,*

then $L_k = d_k + g_k$, where $\{d_k\}_{k\in\mathbb{N}}$ is an additive higher derivation of $\mathscr{A}$ and $\{g_k\}_{k\in\mathbb{N}}$ is a sequence of central mapping such that $g_k[x,y] = 0$ for all $x,y \in \mathscr{A}$.

REMARK 4.2. Notice that the condition (6) in the above Theorem is equivalent to the fact that $[L_k(e),e] = 0$, and so $[L_k(e),f] = 0$. Taking $k = 1$, this together with $L_1([e,f]) = 0$ further implies that $L_1(f)$ also commutes with e and consequently f.

In order to obtain this theorem, we will use an induction method for the component index k. When $k = 1$, L_1 is clearly a multiplicative Lie derivation on $\mathscr{A}$. It follows from Theorem 3.1 that there exist an additive derivation d_1 and a central mapping g_1 vanishing on all commutators such that $L_1 = d_1 + g_1$. By Remark 3.2 and Remark 3.4 we know that

$$\begin{aligned}\xi(x) &= \nu(x) - [\nu(e),x]\\ &= L_1(x) - [L_1(f),x] - [L_1(e) - [L_1(f),e],x]\\ &= L_1(x) - [L_1(f),x] - [L_1(e),x] + [[L_1(f),e],x]\end{aligned}$$

for all $x \in \mathscr{A}$. Using Remark 4.2, we can easily get $\xi(f) = L_1(f)$ and $\xi(e) = L_1(e)$. It follow from the proof of Theorem 3.1 that $L_1(f) \in \mathscr{Z}(\mathscr{A})$ and $L_1(e) \in \mathscr{Z}(\mathscr{A})$. Correspondingly, we have $\xi(x) = L_1(x)$ for all $x \in \mathscr{A}$. By the previous facts it is not difficult to see that L_1 satisfies the following properties:

$$\begin{aligned}&L_1(0) = 0,\ L_1(e) \in \mathscr{Z}(\mathscr{A}),\ \ L_1(f) \in \mathscr{Z}(\mathscr{A}),\\ &L_1(a) \in A + \mathscr{Z}(\mathscr{A}),\ \ L_1(b) \in B + \mathscr{Z}(\mathscr{A}),\\ &L_1(m) \in M,\ \ L_1(n) \in N\end{aligned}$$

for all $a \in A$, $m \in M$, $n \in N$ and $b \in B$. By Lemma 3.17 and Remark 3.16 we know that

$$\begin{aligned}&d_1(e) = 0,\ \ d_1(f) = 0,\\ &d_1(a) \in A,\ \ d_1(b) \in B,\\ &d_1(m) \in M,\ \ d_1(n) \in N\end{aligned}$$

for all $a \in A$, $m \in M$, $n \in N$ and $b \in B$.

Now let $s \in \mathbb{N}$ with $k \geqslant 1$ and we assume that Theorem 4.1 holds for all $s < k$. This implies there exist an additive mapping $\{d_i\}_{i=0}^{s}$ and a nonlinear central mapping g_s vanishing on all commutators such that $L_s(x) = d_s(x) + g_s(x)$ for all $x \in \mathscr{A}$. The sequence $d_s (s < k)$ is an additive higher derivation of order s and the mappings L_s and d_s satisfy the following properties:

$$\begin{aligned}
&L_s(0) = 0,\ L_s(e) \in \mathscr{Z}(\mathscr{A}),\ \ L_s(f) \in \mathscr{Z}(\mathscr{A}),\\
&L_s(a) \in A + \mathscr{Z}(\mathscr{A}),\ \ L_s(b) \in B + \mathscr{Z}(\mathscr{A}),\\
&L_s(m) \in M,\ \ L_s(n) \in N
\end{aligned}$$

and

$$\begin{aligned}
&d_s(e) = 0,\ \ d_s(f) = 0,\\
&d_s(a) \in A,\ \ d_s(b) \in B,\\
&d_s(m) \in M,\ \ d_s(n) \in N
\end{aligned}$$

for all $a \in A$, $m \in M$, $n \in N$ and $b \in B$.

The induction process can be realized through a series of lemmas. We must indicate that the proofs in the induction step are essentially the same as those in Section 3, but they are done in a slightly different way.

LEMMA 4.3. $L_k(0) = 0$.

Proof. By the induction hypothesis we have

$$L_k(0) = L_k([0,0]) = \sum_{\substack{i+j=k\\ 0<i,j<k}} [L_i(0), L_j(0)] = 0. \quad \square$$

LEMMA 4.4. *For each $k \in \mathbb{N}$, we have $eL_k(f)f = 0$ and $fL_k(f)e = 0$.*

Proof. Using the induction hypothesis and assumption (6), we obtain

$$\begin{aligned}
L_k([e,f]) &= [L_k(e), f] + \sum_{\substack{i+j=k\\ 0<i,j<k}} [L_i(e), L_j(f)] + [e, L_k(f)]\\
&= eL_k(f) - L_k(f)e = eL_k(f)f - fL_k(f)e = 0.
\end{aligned}$$

So $eL_k(f)f = 0$ and $fL_k(f)e = 0$. $\square$

LEMMA 4.5. *$L_k(e) \in \mathscr{Z}(\mathscr{A})$, $L_k(f) \in \mathscr{Z}(\mathscr{A})$, $L_k(m) \in M$ and $L_k(n) \in N$ for all $m \in M$ and $n \in N$.*

Proof. First we have

$$\begin{aligned}
L_k([e,m]) &= [L_k(e), m] + \sum_{\substack{i+j=k\\ 0<i,j<k}} [L_i(e), L_j(m)] + [e, L_k(m)],\\
L_k([n,e]) &= [L_k(n), e] + \sum_{\substack{i+j=k\\ 0<i,j<k}} [L_i(n), L_j(e)] + [n, L_k(e)]
\end{aligned}$$

for all $m \in M$ and $n \in N$. According to the induction hypothesis we get

$$L_k(m) = L_k(e)m - mL_k(e) + eL_k(m) - L_k(m)e,$$
$$L_k(n) = L_k(n)e - eL_k(n) + nL_k(e) - L_k(e)n$$

for all $m \in M$ and $n \in N$. Furthermore, we obtain

$$eL_k(e)em = mfL_k(e)f, \; neL_k(e)e = fL_k(e)fn$$

and

$$eL_k(m)e = 0, \quad fL_k(m)f = 0, \quad fL_k(m)e = 0,$$
$$eL_k(n)e = 0, \quad fL_k(n)f = 0, \quad eL_k(n)f = 0$$

for all $m \in M$ and $n \in N$. These facts together with assumption (6) and Lemma 4.4 imply that $L_k(e) \in \mathscr{Z}(\mathscr{A})$, $L_k(m) \in M$ and $L_k(n) \in N$ for all $m \in M$ and $n \in N$. Similarly, $L_k(f) \in \mathscr{Z}(\mathscr{A})$. □

LEMMA 4.6. *$L_k(a) \in A + \mathscr{Z}(\mathscr{A})$, $L_k(b) \in B + \mathscr{Z}(\mathscr{A})$ for all $a \in A$ and $b \in B$.*

Proof. In view of Lemma 4.3 we have

$$L_k([a,b]) = [L_k(a),b] + \sum_{\substack{i+j=k \\ 0<i,j<k}} [L_i(a),L_j(b)] + [a,L_k(b)] = 0$$

for all $a \in A$ and $b \in B$. Applying the induction hypothesis yields

$$[L_k(a),b] + [a,L_k(b)] = 0,$$

or

$$[fL_k(a)f + eL_k(a)f + fL_k(a)e, b] + [a, eL_k(b)e + eL_k(b)f + fL_k(b)e] = 0 \tag{4.1}$$

for all $a \in A$ and $b \in B$. This means that $[fL_k(a)f, b] = [a, eL_k(b)e] = 0$. Replacing b with f in (4.1) and using Lemma 4.4, we obtain

$$eL_k(a)f = fL_k(a)e = 0$$

for all $a \in A$. Note that there exists a unique algebraic isomorphism $\sigma\colon e\mathscr{Z}(\mathscr{A})e \to f\mathscr{Z}(\mathscr{A})f$ such that $a + \sigma(a) \in \mathscr{Z}(\mathscr{A})$ for all $a \in e\mathscr{Z}(\mathscr{A})e$, so

$$L_k(a) = eL_k(a)e - \sigma^{-1}(fL_k(a)f) + fL_k(a)f + \sigma^{-1}(fL_k(a)f)$$

for all $a \in A$. Therefore, $L_k(a) \in A + \mathscr{Z}(\mathscr{A})$ for all $a \in A$. Similarly, one can show that $L_k(b) \in B + \mathscr{Z}(\mathscr{A})$ for all $b \in B$. □

REMARK 4.7. For any $x \in \mathscr{A}$, we define

$$g'_k(x) := fL_k(exe)f + \sigma^{-1}(fL_k(exe)f) + eL_k(fxf)e + \sigma(eL_k(fxf)e).$$

Obviously, $g_k'(x) \in \mathscr{Z}(\mathscr{A})$. Now a new mapping $\chi_k : \mathscr{A} \to \mathscr{A}$ from $\mathscr{A}$ to itself can be defined by $\chi_k(x) = L_k(x) - g_k'(x)$ for all $x \in \mathscr{A}$. By Lemma 4.5 and Lemma 4.6 we obtain

$$\begin{aligned}
\chi_k(m) &= L_k(m) - g_k'(m) = L_k(m) \in M,\\
\chi_k(n) &= L_k(n) - g_k'(n) = L_k(n) \in N,\\
\chi_k(a) &= L_k(a) - g_k'(a) = eL_k(a)e - \sigma^{-1}(fL_k(a)f) \in A,\\
\chi_k(b) &= L_k(b) - g_k'(b) = fL_k(b)f - \sigma(eL_k(b)e) \in B
\end{aligned} \tag{4.2}$$

for all $a \in A, m \in M, n \in N$ and $b \in B$.

LEMMA 4.8. *For any $a \in A, m \in M, n \in N$ and $b \in B$, we have*

$$\begin{aligned}
\chi_k(am) &= \chi_k(a)m + a\chi_k(m) + \sum_{\substack{i+j=k\\0<i,j<k}} d_i(a)d_j(m),\\
\chi_k(bn) &= \chi_k(b)n + b\chi_k(n) + \sum_{\substack{i+j=k\\0<i,j<k}} d_i(b)d_j(n),\\
\chi_k(mb) &= \chi_k(m)b + m\chi_k(b) + \sum_{\substack{i+j=k\\0<i,j<k}} d_i(m)d_j(b),\\
\chi_k(na) &= \chi_k(n)a + n\chi_k(a) + \sum_{\substack{i+j=k\\0<i,j<k}} d_i(n)d_j(a).
\end{aligned}$$

Proof. For any $a \in A$ and $m \in M$, applying the induction hypothesis yields

$$\begin{aligned}
\chi_k(am) = L_k([a,m]) &= [L_k(a), m] + \sum_{\substack{i+j=k\\0<i,j<k}} [L_i(a), L_j(m)] + [a, L_k(m)]\\
&= [\chi_k(a), m] + \sum_{\substack{i+j=k\\0<i,j<k}} [d_i(a), d_j(m)] + [a, \chi_k(m)]\\
&= \chi_k(a)m + \sum_{\substack{i+j=k\\0<i,j<k}} d_i(a)d_j(m) + a\chi_k(m).
\end{aligned}$$

The rest can be proved in an analogous manner. □

LEMMA 4.9. *For any $a_1, a_2 \in A$ and $b_1, b_2 \in B$ we have*

$$\begin{aligned}
\chi_k(a_1a_2) &= \chi_k(a_1)a_2 + a_1\chi_k(a_2) + \sum_{\substack{i+j=k\\0<i,j<k}} d_i(a_1)d_j(a_2),\\
\chi_k(b_1b_2) &= \chi_k(b_1)b_2 + b_1\chi_k(b_2) + \sum_{\substack{i+j=k\\0<i,j<k}} d_i(b_1)d_j(b_2).
\end{aligned}$$

Proof. For any $a_1, a_2 \in A$ and $m \in M$, we deduce from Lemma 4.8 that

$$\begin{aligned}\chi_k(a_1a_2m) &= \chi_k(a_1a_2)m + \sum_{\substack{i+j=k\\0<i,j<k}} d_i(a_1a_2)d_j(m) + a_1a_2\chi_k(m)\\ &= \chi_k(a_1a_2)m + \sum_{\substack{i+j+l=k\\0<l<k}} d_i(a_1)d_j(a_2)d_l(m) + a_1a_2\chi_k(m).\end{aligned}$$

By Lemma 4.8 again we obtain

$$\begin{aligned}\chi_k(a_1a_2m) &= \chi_k(a_1)a_2m + \sum_{\substack{i+j=k\\0<i,j<k}} d_i(a_1)d_j(a_2m) + a_1\chi_k(a_2m)\\ &= \chi_k(a_1)a_2m + \sum_{\substack{i+j+l=k\\0<i<k}} d_i(a_1)d_j(a_2)d_l(m)\\ &\quad + a_1\chi_k(a_2)m + \sum_{\substack{i+j=k\\0<i,j<k}} a_1d_i(a_2)d_j(m) + a_1a_2\chi_k(m)\\ &= \chi_k(a_1)a_2m + \sum_{\substack{i+j+l=k\\0<l<k}} d_i(a_1)d_j(a_2)d_l(m)\\ &\quad + a_1\chi_k(a_2)m + \sum_{\substack{i+j=k\\0<i<k}} d_i(a_1)d_j(a_2)m + a_1a_2\chi_k(m)\end{aligned}$$

for all $a_1, a_2 \in A$ and $m \in M$.

Combining the last two relations, we arrive at

$$\chi_k(a_1a_2)m = (\chi_k(a_1)a_2 + a_1\chi_k(a_2) + \sum_{\substack{i+j=k\\0<i,j<k}} d_i(a_1)d_j(a_2))m \tag{4.3}$$

for all $a_1, a_2 \in A$ and $m \in M$. In an analogous way one can show that

$$n\chi_k(a_1a_2) = n(\chi_k(a_1)a_2 + a_1\chi_k(a_2) + \sum_{\substack{i+j=k\\0<i,j<k}} d_i(a_1)d_j(a_2)) \tag{4.4}$$

for all $a_1, a_2 \in A$ and $n \in N$. Therefore, equalities (4.3), (4.4) and assumption (1.2) imply that

$$\chi_k(a_1a_2) = \chi_k(a_1)a_2 + a_1\chi_k(a_2) + \sum_{\substack{i+j=k\\0<i,j<k}} d_i(a_1)d_j(a_2)$$

for all $a_1, a_2 \in A$. Similarly, we can show

$$\chi_k(b_1b_2) = b_1\chi_k(b_2) + \chi_k(b_1)b_2 + \sum_{\substack{i+j=k\\0<i,j<k}} d_i(b_1)d_j(b_2)$$

for all $b_1, b_2 \in A$. $\square$

LEMMA 4.10. *For any $a \in A, m \in M$, $n \in N$ and $b \in B$, we have*

$$\begin{aligned}
&\chi_k(a+m) - \chi_k(a) - \chi_k(m) \in \mathscr{Z}(\mathscr{A}),\\
&\chi_k(b+m) - \chi_k(b) - \chi_k(m) \in \mathscr{Z}(\mathscr{A}),\\
&\chi_k(a+n) - \chi_k(a) - \chi_k(n) \in \mathscr{Z}(\mathscr{A}),\\
&\chi_k(b+n) - \chi_k(b) - \chi_k(n) \in \mathscr{Z}(\mathscr{A}).
\end{aligned}$$

Proof. Let $a \in A$ and $m, m' \in M$. Considering the induction hypothesis, we get

$$\begin{aligned}
\chi_k(am') &= L_k([a+m, m'])\\
&= [L_k(a+m), m'] + [a+m, L_k(m')] + \sum_{\substack{i+j=k\\0<i,j<k}} [L_i(a+m), L_j(m')]\\
&= [\chi_k(a+m), m'] + [a+m, \chi_k(m')] + \sum_{\substack{i+j=k\\0<i,j<k}} [d_i(a+m), d_j(m')]\\
&= [\chi_k(a+m), m'] + a\chi_k(m') + \sum_{\substack{i+j=k\\0<i,j<k}} [d_i(a), d_j(m')].
\end{aligned}$$

Applying Lemma 4.8, we can further get

$$[\chi_k(a+m) - \chi_k(a), m'] = 0$$

for all $a \in A$ and $m, m' \in M$.

Let $a \in A$ and $m \in M$. To determine the term $e(\chi_k(a+m) - \chi_k(a))f$ and $f(\chi_k(a+m) - \chi_k(a))e$, we do the following computation using Lemma 4.5 and Lemma 4.6.

$$\begin{aligned}
&[e, \chi_k(a+m) - \chi_k(a)]\\
&= [e, L_k(a+m)] - [e, L_k(a)]\\
&= L_k([e, a+m]) - [L_k(e), a+m] - \sum_{\substack{i+j=k\\0<i,j<k}} [L_i(e), L_j(a+m)]\\
&= L_k([e,m]) = \chi_k(m).
\end{aligned}$$

This implies that

$$e(\chi_k(a+m) - \chi_k(a))f = \chi_k(m), f(\chi_k(a+m) - \chi_k(a))e = 0 \tag{4.5}$$

for all $a \in A$ and $m \in M$.

Next we will prove that $[\chi_k(a+m) - \chi_k(a) - \chi_k(m), b] = 0$ for all $a \in A$, $m \in M$

and $b \in B$. In view of the induction hypothesis, we routinely compute that

$$\begin{aligned}\chi_k(mb) &= L_k([a+m,b]) \\ &= [L_k(a+m),b]+[a+m,L_k(b)]+\sum_{\substack{i+j=k \\ 0<i,j<k}}[d_i(a+m),d_j(b)] \\ &= [\chi_k(a+m),b]+[a+m,\chi_k(b)]+\sum_{\substack{i+j=k \\ 0<i,j<k}}[d_i(a+m),d_j(b)] \\ &= [\chi_k(a+m),b]+[m,\chi_k(b)]+\sum_{\substack{i+j=k \\ 0<i,j<k}}d_i(m)d_j(b).\end{aligned}$$

Applying Lemma 4.8 again, we obtain

$$[\chi_k(a+m)-\chi_k(m),b]=0$$

for all $a \in A$, $m \in M$ and $b \in B$. Then by (4.2) we have

$$[\chi_k(a+m)-\chi_k(m)-\chi_k(a),b]=0$$

for all $a \in A$, $m \in M$ and $b \in B$.

Now let us denote $\chi_k(a+m)-\chi_k(m)-\chi_k(a)$ by x. Then due to the relation (4.5) we know that $x \in A+B$. Let us choose $m' \in M$. Then for arbitrary element $n \in N$, we have

$$(xn-nx)m' = xnm'-nxm' = nm'x-nm'x = 0.$$

This leads to $xn-nx=[x,n]=0$ for $n \in N$ by the assumption. Now we can see that $\chi_k(a+m)-\chi_k(m)-\chi_k(a) \in \mathscr{Z}(\mathscr{A})$ for all $a \in A$ and $m \in M$. We can also show the rest in a similar way. □

LEMMA 4.11. *χ_k is additive on A, M, N and B, respectively.*

Proof. By the induction hypothesis and applying Lemma 4.10 we immediately compute that

$$\begin{aligned}\chi_k(m+m') &= \chi_k([e+m,f+m']) \\ &= [\chi_k(e+m),f+m']+[e+m,\chi_k(f+m')] \\ &\quad +\sum_{\substack{i+j=k \\ 0<i,j<k}}[d_i(e+m),d_j(f+m')] \\ &= [\chi_k(e)+\chi_k(m),f+m']+[e+m,\chi_k(f)+\chi_k(m')] \\ &= \chi_k(m)+\chi_k(m')\end{aligned} \tag{4.6}$$

and

$$\begin{aligned}\chi_k(n+n') &= \chi_k([f+n,e+n']) \\ &= [\chi_k(f+n),e+n'] + [f+n,\chi_k(e+n')] \\ &\quad + \sum_{\substack{i+j=k \\ 0<i,j<k}} [d_i(f+n),d_j(e+n')] \\ &= [\chi_k(f)+\chi_k(n),e+n'] + [f+n,\chi_k(e)+\chi_k(n')] \\ &= \chi_k(n)+\chi_k(n')\end{aligned} \tag{4.7}$$

for all $m,m' \in M$ and $n,n' \in N$.

In view of Lemma 4.8 and (4.6) one can get

$$\begin{aligned}\chi_k((a_1+a_2)m) &= \chi_k(a_1m)+\chi_k(a_2m) \\ &= \chi_k(a_1)m + a_1\chi_k(m) + \sum_{\substack{i+j=k \\ 0<i,j<k}} d_i(a_1)d_j(m) \\ &\quad + \chi_k(a_2)m + a_2\chi_k(m) + \sum_{\substack{i+j=k \\ 0<i,j<k}} d_i(a_2)d_j(m)\end{aligned} \tag{4.8}$$

for all $a_1,a_2 \in A$ and $m \in M$. On the other hand, using Lemma 4.8 again, we arrive at

$$\chi_k((a_1+a_2)m) = \chi_k(a_1+a_2)m + (a_1+a_2)\chi_k(m) + \sum_{\substack{i+j=k \\ 0<i,j<k}} d_i(a_1+a_2)d_j(m) \tag{4.9}$$

for all $a_1,a_2 \in A$ and $m \in M$. Combining (4.8) with (4.9) gives

$$\chi_k(a_1+a_2)m = \chi_k(a_1)m + \chi_k(a_2)m$$

for all $a_1,a_2 \in A$ and $m \in M$. Likewise, we can also get

$$n\chi_k(a_1+a_2) = n\chi_k(a_1) + n\chi_k(a_2)$$

for all $a_1,a_2 \in A$ and $n \in N$. The last two relations jointly imply that

$$\chi_k(a_1+a_2) - \chi_k(a_1) - \chi_k(a_2) = 0$$

for all $a_1,a_2 \in A$, which is the desired result. Similarly, we also have the additivity of χ_k on B. □

LEMMA 4.12. *For any $a \in A, m \in M, n \in N$ and $b \in B$, we have*

$$\chi_k(a+m+b) - \chi_k(a) - \chi_k(b) - \chi_k(m) \in \mathscr{Z}(\mathscr{A}),$$
$$\chi_k(a+n+b) - \chi_k(a) - \chi_k(b) - \chi_k(n) \in \mathscr{Z}(\mathscr{A}).$$

Proof. In light of Lemma 4.8 and Lemma 4.11 we have

$$\begin{aligned}\chi_k(am'-m'b) &= \chi_k(am')-\chi_k(m'b)\\ &= \chi_k(a)m' + \sum_{\substack{i+j=k\\0<i,j<k}} d_i(a)d_j(m') + a\chi_k(m')\\ &\quad - \chi_k(m')b - \sum_{\substack{i+j=k\\0<i,j<k}} d_i(m')d_j(b) - m'\chi_k(b)\end{aligned}$$

for all $a\in A$, $m'\in M$ and $b\in B$. On the other hand,

$$\begin{aligned}\chi_k(am'-m'b) &= L_k([a+m+b,m'])\\ &= [\chi_k(a+m+b),m'] + \sum_{\substack{i+j=k\\0<i,j<k}} [d_i(a+m+b),d_j(m')]\\ &\quad + [a+m+b,\chi_k(m')]\\ &= [\chi_k(a+m+b),m'] + \sum_{\substack{i+j=k\\0<i,j<k}} d_i(a)d_j(m')\\ &\quad - \sum_{\substack{i+j=k\\0<i,j<k}} d_j(m')d_i(b) + a\chi_k(m') - \chi_k(m')b\end{aligned}$$

for all $a\in A$, $m,m'\in M$ and $b\in B$. Comparing the last two relations we assert

$$[\chi_k(a+m+b)-\chi_k(a)-\chi_k(b),m'] = 0 \tag{4.10}$$

for all $a\in A$, $m,m'\in M$ and $b\in B$.

Taking into account Lemma 4.5 and (4.2), we obtain

$$\begin{aligned}&[e,\chi_k(a+m+b)-\chi_k(a)-\chi_k(b)]\\ &= L_k([e,a+m+b]) - [L_k(e),a+m+b] - \sum_{\substack{i+j=k\\0<i,j<k}} [L_i(e),L_j(a+m+b)]\\ &= L_k([e,m]) = \chi_k(m)\end{aligned}$$

for all $a\in A$, $m\in M$ and $b\in B$. Hence we have

$$\begin{aligned}e(\chi_k(a+m+b)-\chi_k(a)-\chi_k(b))f &= \chi_k(m),\\ f(\chi_k(a+m+b)-\chi_k(a)-\chi_k(b))e &= 0\end{aligned} \tag{4.11}$$

for all $a\in A$, $m\in M$ and $b\in B$.

Next we will prove that $[\chi_k(a+m+b)-\chi_k(a)-\chi_k(m)-\chi_k(b),b]=0$ for all

$a \in A$, $m \in M$ and $b \in B$. It is easy to compute that

$$\begin{aligned}\chi_k(mb' + [b,b']) &= L_k([a+m+b,b']) - g'_k(mb' + [b,b'])\\ &= [L_k(a+m+b),b'] + [a+m+b,L_k(b')]\\ &\quad + \sum_{\substack{i+j=k\\0<i,j<k}} [L_i(a+m+b),L_j(b')] - g'_k(mb' + [b,b'])\\ &= \chi_k(a+m+b)b' - b'\chi_k(a+m+b)\\ &\quad + m\chi_k(b') + b\chi_k(b') - \chi_k(b')b\\ &\quad + \sum_{\substack{i+j=k\\0<i,j<k}} [d_i(a+m+b),d_j(b')] - g'_k([b,b'])\end{aligned}$$

for all $a \in A$, $m \in M$ and $b,b' \in B$. On the other hand, by Lemma 4.10 and Lemma 4.8 we know that

$$\begin{aligned}\chi_k(mb' + [b,b']) &= \chi_k(mb') + \chi_k([b,b']) + x\\ &= \chi_k(mb') + [\chi_k(b),b'] + [b,\chi_k(b')] + \sum_{\substack{i+j=k\\0<i,j<k}} [d_i(b),d_j(b')] - g'_k([b,b']) + x\\ &= \chi_k(m)b' + m\chi_k(b') + \sum_{\substack{i+j=k\\0<i,j<k}} [d_i(m),d_j(b')] + \chi_k(b)b' - b'\chi_k(b)\\ &\quad + b\chi_k(b') - \chi_k(b')b + \sum_{\substack{i+j=k\\0<i,j<k}} [d_i(b),d_j(b')] - g'_k([b,b']) + x\end{aligned}$$

for all $m \in M$ and $b,b' \in B$ and some $x \in \mathscr{Z}(\mathscr{A})$. Comparing the last two relations we have

$$\begin{aligned}&\chi_k(m)b' + \chi_k(b)b' - b'\chi_k(b) + x\\ &= \chi_k(a+m+b)b' - b'\chi_k(a+m+b)\end{aligned}$$

for all $a \in A$, $m \in M$ and $b,b' \in B$. Furthermore, the above relation can be rewritten as

$$\begin{aligned}&b'\chi_k(a+m+b) - b'\chi_k(m) - b'\chi_k(b) - b'\chi_k(a) + x\\ &= \chi_k(a+m+b)b' - \chi_k(m)b' - \chi_k(b)b' - \chi_k(a)b'\end{aligned}$$

for all $a \in A$, $m \in M$ and $b,b' \in B$. That is,

$$[b', \chi_k(a+m+b) - \chi_k(a) - \chi_k(b) - \chi_k(m)] \in \mathscr{Z}(\mathscr{A}),$$

for all $a \in A$, $m \in M$ and $b,b' \in B$. By (4.11) we can see that $\chi_k(a+m+b) - \chi_k(a) - \chi_k(b) - \chi_k(m) \in A+B$, so we can further get

$$[b', \chi_k(a+m+b) - \chi_k(a) - \chi_k(b) - \chi_k(m)] = 0 \tag{4.12}$$

for all $a \in A$, $m \in M$ and $b,b' \in B$. Taking into account (4.10) and (4.12) and repeating the same procedure in Lemma 3.9 we can prove that $\chi_k(a+m+b) - \chi_k(a) - \chi_k(b) - \chi_k(m) \in \mathscr{Z}(\mathscr{A})$. Similarly, we can also get $\chi_k(a+n+b) - \chi_k(a) - \chi_k(b) - \chi_k(n) \in \mathscr{Z}(\mathscr{A})$. $\square$

LEMMA 4.13. *For any $a \in A, m \in M, n \in N$ and $b \in B$, we have*

$$\chi_k(a+m+n+b) - \chi_k(a) - \chi_k(b) - \chi_k(m) - \chi_k(n) \in \mathscr{Z}(\mathscr{A}).$$

Proof. By the induction hypothesis we obtain

$$\begin{aligned} \chi_k([m',a+m+n+b]) &= [\chi_k(m'),a+m+n+b] + [m',\chi_k(a+m+n+b)] \\ &\quad + \sum_{\substack{i+j=k \\ 0<i,j<k}} [d_i(m'),d_j(a+m+n+b)] - g'_k([m',a+m+n+b]) \\ &= [\chi_k(m'),a+n+b] + [m',\chi_k(a+m+n+b)] \\ &\quad + \sum_{\substack{i+j=k \\ 0<i,j<k}} [d_i(m'),d_j(a+n+b)] - g'_k([m',a+m+n+b]) \end{aligned}$$

for all $a \in A$, $m,m' \in M$, $n \in N$ and $b \in B$. However, we can also get

$$\begin{aligned} \chi_k([m',a+m+n+b]) &= [\chi_k(m'),a+n+b] + [m',\chi_k(a+n+b)] \\ &\quad + \sum_{\substack{i+j=k \\ 0<i,j<k}} [d_i(m'),d_j(a+n+b)] - g'_k([m',a+m+n+b]) \end{aligned}$$

for all $a \in A$, $m,m' \in M$, $n \in N$ and $b \in B$. The last two relations imply that

$$[m',\chi_k(a+m+n+b) - \chi_k(a+n+b)] = 0$$

for all $a \in A$, $m,m' \in M$, $n \in N$ and $b \in B$. In view of Lemma 4.12 the above relation can be rewritten as

$$[m',\chi_k(a+m+n+b) - \chi_k(a) - \chi_k(b) - \chi_k(m) - \chi_k(n)] = 0 \tag{4.13}$$

for all $a \in A$, $m,m' \in M$, $n \in N$ and $b \in B$. Because of symmetry we also have

$$[n',\chi_k(a+m+n+b) - \chi_k(a) - \chi_k(b) - \chi_k(m) - \chi_k(n)] = 0 \tag{4.14}$$

for all $a \in A$, $m \in M$, $n,n' \in N$ and $b \in B$. From equalities (4.13), (4.14) and assumptions (4) and (5) we can see that

$$\begin{aligned} f\chi_k(a+m+n+b)e - \chi_k(n) = 0, \\ e\chi_k(a+m+n+b)f - \chi_k(m) = 0 \end{aligned} \tag{4.15}$$

for all $a \in A$, $m \in M$, $n \in N$ and $b \in B$. Equalities (4.13), (4.14) and (4.15) jointly lead to

$$\chi_k(a+m+n+b) - \chi_k(a) - \chi_k(b) - \chi_k(m) - \chi_k(n) \in \mathscr{Z}(\mathscr{A})$$

for all $a \in A$, $m \in M$, $n \in N$ and $b \in B$. $\square$

LEMMA 4.14. *For any $a \in A, m \in M, n \in N$ and $b \in B$, we have*

$$\chi_k(mn) = \chi_k(m)n + m\chi_k(n) + \sum_{\substack{i+j=k \\ 0<i,j<k}} d_i(m)d_j(n),$$

$$\chi_k(nm) = \chi_k(n)m + n\chi_k(m) + \sum_{\substack{i+j=k \\ 0<i,j<k}} d_i(n)d_j(m).$$

Proof. Let us choose arbitrary elements $m, m_0 \in M$ and $n \in N$. We routinely compute that

$$\begin{aligned} \chi_k([[m,n],m_0]) &= [L_k[m,n],m_0] + [[m,n],L_k(m_0)] \\ &\quad + \sum_{\substack{i+j=k \\ 0<i,j<k}} [d_i([m,n]),d_j(m_0)] - g'_k([[m,n],m_0]) \\ &= [[L_k(m),n] + [m,L_k(n)],m_0] \\ &\quad + \sum_{\substack{i+j=k \\ 0<i,j<k}} [[d_i(m),d_j(n)],m_0] + [[m,n],L_k(m_0)] \\ &\quad + \sum_{\substack{i+j=k \\ 0<i,j<k}} [d_i([m,n]),d_j(m_0)] - g'_k([[m,n],m_0]) \\ &= [[\chi_k(m),n],m_0] + [[m,\chi_k(n)],m_0] + [[m,n],\chi_k(m_0)] \\ &\quad + \sum_{\substack{i+j=k \\ 0<i,j<k}} [[d_i(m),d_j(n)],m_0] + \sum_{\substack{i+j=k \\ 0<i,j<k}} [d_i([m,n]),d_j(m_0)] \end{aligned} \tag{4.16}$$

and

$$\begin{aligned} \chi_k([[m,n],m_0]) &= L_k([mn-nm,m_0]) - g'_k([[m,n],m_0]) \\ &= [L_k(mn-nm),m_0] + [mn-nm,L_k(m_0)] \\ &\quad + \sum_{\substack{i+j=k \\ 0<i,j<k}} [d_i([m,n]),d_j(m_0)] \\ &= [\chi_k(mn-nm),m_0] + [mn-nm,L_k(m_0)] \\ &\quad + \sum_{\substack{i+j=k \\ 0<i,j<k}} [d_i([m,n]),d_j(m_0)] \\ &= [\chi_k(mn) - \chi_k(nm),m_0] + [mn-nm,\chi_k(m_0)] \\ &\quad + \sum_{\substack{i+j=k \\ 0<i,j<k}} [d_i(mn-nm),d_j(m_0)], \end{aligned} \tag{4.17}$$

where Lemma 4.11 and Lemma 4.13 are used in the second calculation. From the last two relations we have

$$[\chi_k(mn) - \chi_k(nm) - [\chi_k(m),n] - [m,\chi_k(n)],m_0] - \sum_{\substack{i+j=k \\ 0<i,j<k}} [[d_i(m),d_j(n)],m_0] = 0 \tag{4.18}$$

for all $m, m_0 \in M$ and $n \in N$. In the same way we also obtain

$$[\chi_k(mn) - \chi_k(nm) - [\chi_k(m), n] - [m, \chi_k(n)], n_0] - \sum_{\substack{i+j=k\\0<i,j<k}} [[d_i(m), d_j(n)], n_0] = 0 \quad (4.19)$$

for all $m \in M$ and $n, n_0 \in N$. It follows from (4.18) and (4.19) that

$$(\chi_k(mn) - \chi_k(m)n - m\chi_k(n)) - (\chi_k(nm) - \chi_k(n)m - n\chi_k(m)) - \sum_{\substack{i+j=k\\0<i,j<k}} [d_i(m), d_j(n)] \in \mathscr{Z}(\mathscr{A}) \quad (4.20)$$

for all $m \in M$ and $n \in N$. We might as will assume that A does not contain nonzero central ideals. Set

$$\varepsilon(m,n) = \chi_k(mn) - \chi_k(m)n - m\chi_k(n) - \sum_{\substack{i+j=k\\0<i,j<k}} d_i(m)d_j(n)$$

for all $m \in M$ and $n \in N$. Applying Lemma 4.8 and Lemma 4.9, we can get

$$\begin{aligned}
\varepsilon(am,n) &= \chi_k(amn) - \chi_k(am)n - am\chi_k(n) - \sum_{\substack{i+j=k\\0<i,j<k}} d_i(am)d_j(n) \\
&= \chi_k(a)mn + a\chi_k(mn) + \sum_{\substack{i+j=k\\0<i,j<k}} d_i(a)d_j(mn) - \chi_k(a)mn - a\chi_k(m)n \\
&\quad - \sum_{\substack{i+j=k\\0<i,j<k}} d_i(a)d_j(m)n - am\chi_k(n) - \sum_{\substack{i+j=k\\0<i,j<k}} d_i(am)d_j(n) \\
&= a\chi_k(mn) - a\chi_k(m)n - am\chi_k(n) \\
&\quad - \sum_{\substack{i+j=k\\0<i,j<k}} d_i(a)d_j(m)n - \sum_{\substack{i+j=k\\0<i,j<k}} d_i(am)d_j(n) + \sum_{\substack{i+j=k\\0<i,j<k}} d_i(a)d_j(mn) \\
&= a\chi_k(mn) - a\chi_k(m)n - am\chi_k(n) \\
&\quad - \sum_{\substack{i+j=k\\0<i,j<k}} d_i(a)d_j(m)n - \sum_{\substack{s+t+j=k\\0<j<k}} d_s(a)d_t(m)d_j(n) + \sum_{\substack{i+s+t=k\\0<i<k}} d_i(a)d_s(m)d_t(n) \\
&= a\chi_k(mn) - a\chi_k(m)n - am\chi_k(n) \\
&\quad - \sum_{\substack{i+j=k\\0<i,j<k}} ad_i(m)d_j(n) - \sum_{\substack{s+t+j=k\\0<s<k}} d_s(a)d_t(m)d_j(n) + \sum_{\substack{i+s+t=k\\0<i<k}} d_i(a)d_s(m)d_t(n) \\
&= a\varepsilon(m,n)
\end{aligned}$$

for all $m \in M$, $n \in N$ and $a \in A$. Note that $\varepsilon(m,n) \in \mathscr{Z}(A)$ for all $m \in M$ and $n \in N$. So $A\varepsilon(m,n)$ is a central ideal of A. Consequently, $\varepsilon(m,n) = 0$ for all $m \in M$ and $n \in N$. This gives the fact that

$$\chi_k(mn) - \chi_k(m)n - m\chi_k(n) - \sum_{\substack{i+j=k\\0<i,j<k}} d_i(m)d_j(n) = 0 \quad (4.21)$$

for all $m \in M$ and $n \in N$. By the previous fact and (4.21) we obtain

$$\chi_k(nm) - \chi_k(n)m - n\chi_k(m) - \sum_{\substack{i+j=k \\ 0<i,j<k}} d_i(n)d_j(m) = 0$$

for all $m \in M$ and $n \in N$. $\square$

REMARK 4.15. Let χ_k be as in Remark 4.7. Let us define

$$d_k(a+m+n+b) = \chi_k(a) + \chi_k(m) + \chi_k(n) + \chi_k(b)$$

for all $a \in A$, $m \in M$, $n \in N$ and $b \in B$.

Proof of Theorem 4.1. It follows from the definitions of g'_k and d_k that

$$L_k = d_k + g'_k + \chi_k - d_k = d_k + g_k.$$

Then Lemma 4.13 implies that $g_k = g'_k + \chi_k - d_k$ is a mapping from $\mathscr{A}$ into its centre $\mathscr{Z}(\mathscr{A})$.

Let us choose arbitrary elements $x, y \in \mathscr{A}$. Assume that $x = x_a + x_b + x_m + x_n$ and $y = y_a + y_b + y_m + y_n$, where $x_a, y_a \in A$, $x_b, y_b \in B$, $x_m, y_m \in M$ and $x_n, y_n \in N$. In view of Lemma 4.11 we have

$$\begin{aligned} d_k(x+y) &= \chi_k(x_a+y_a) + \chi_k(x_m+y_m) + \chi_k(x_n+y_n) + \chi_k(x_b+y_b) \\ &= \chi_k(x_a) + \chi_k(x_m) + \chi_k(x_n) + \chi_k(x_b) \\ &\quad + \chi_k(y_a) + \chi_k(y_m) + \chi_k(y_n) + \chi_k(y_b) \\ &= d_k(x) + d_k(y). \end{aligned}$$

Thus d_k is additive.

Now let us show that $\{d_s\}_{s=0}^{k}$ is a higher derivation of order k. In view of Lemma 4.8, 4.9, 4.14 and Remark 4.15, we can do the following computation.

$$\begin{aligned} d_k(xy) &= d_k(x_a y_a + x_a x_m + x_b y_b + x_b y_n + x_m y_b + x_m y_n + x_n y_a + x_n y_m) \\ &= d_k(x_a y_a) + d_k(x_a y_m) + d_k(x_b y_b) + d_k(x_b y_n) \\ &\quad + d_k(x_m y_b) + d_k(x_m y_n) + d_k(x_n y_a) + d_k(x_n y_m) \\ &= \chi_k(x_a y_a) + \chi_k(x_a y_m) + \chi_k(x_b y_b) + \chi_k(x_b y_n) \\ &\quad + \chi_k(x_m y_b) + \chi_k(x_m y_n) + \chi_k(x_n y_a) + \chi_k(x_n y_m) \\ &= \chi_k(x_a)y_a + x_a\chi_k(y_a) + \sum_{\substack{i+j=k \\ 0<i,j<k}} d_i(x_a)d_j(y_a) \\ &\quad + \cdots \end{aligned}$$

On the other hand, we observe that

$$
\begin{aligned}
& d_k(x)y + xd_k(y) + \sum_{\substack{i+j=k \\ 0<i,j<k}} d_i(x)d_j(y) \\
&= d_k(x_a+x_b+x_m+x_n)y + xd_k(y_a+y_b+y_m+y_n) + \sum_{\substack{i+j=k \\ 0<i,j<k}} d_i(x)d_j(y) \\
&= (d_k(x_a)+d_k(x_b)+d_k(x_m)+d_k(x_n))y \\
&\quad + x(d_k(y_a)+d_k(y_b)+d_k(y_m)+d_k(y_n)) + \sum_{\substack{i+j=k \\ 0<i,j<k}} d_i(x)d_j(y) \\
&= (\chi_k(x_a)+\chi_k(x_b)+\chi_k(x_m)+\chi_k(x_n))y \\
&\quad + x(\chi_k(y_a)+\chi_k(y_b)+\chi_k(y_m)+\chi_k(y_n)) + \sum_{\substack{i+j=k \\ 0<i,j<k}} d_i(x)d_j(y) \\
&= \chi_k(x_a)y_a + \chi_k(x_a)y_m + \chi_k(x_b)y_b + \chi_k(x_b)y_n + \chi_k(x_m)y_n + \chi_k(x_m)y_b \\
&\quad + \chi_k(x_n)y_a + \chi_k(x_n)y_m + x_a\chi_k(y_a) + x_a\chi_k(y_m) + x_b\chi_k(y_b) \\
&\quad + x_b\chi_k(y_n) + x_m\chi_k(y_n) + x_m\chi_k(y_b) + x_n\chi_k(y_a) + x_n\chi_k(y_m) \\
&\quad + \sum_{\substack{i+j=k \\ 0<i,j<k}} d_i(x_a)d_j(y_a) + \sum_{\substack{i+j=k \\ 0<i,j<k}} d_i(x_a)d_j(y_m) \\
&\quad + \sum_{\substack{i+j=k \\ 0<i,j<k}} d_i(x_b)d_j(y_b) + \sum_{\substack{i+j=k \\ 0<i,j<k}} d_i(x_b)d_j(y_n) \\
&\quad + \sum_{\substack{i+j=k \\ 0<i,j<k}} d_i(x_m)d_j(y_n) + \sum_{\substack{i+j=k \\ 0<i,j<k}} d_i(x_m)d_j(y_b) \\
&\quad + \sum_{\substack{i+j=k \\ 0<i,j<k}} d_i(x_n)d_j(y_a) + \sum_{\substack{i+j=k \\ 0<i,j<k}} d_i(x_n)d_j(y_m).
\end{aligned}
$$

Now we have

$$
d_k(xy) = d_k(x)y + xd_k(y) + \sum_{\substack{i+j=k \\ 0<i,j<k}} d_i(x)d_j(y)
$$

for all $x, y \in \mathscr{A}$.

It remains to show that $g_k([x,y]) = 0$ for all $x, y \in \mathscr{A}$.

$$
\begin{aligned}
g_k([x,y]) &= L_k([x,y]) - d_k([x,y]) \\
&= [L_k(x),y] + [x,L_k(y)] \\
&\quad + \sum_{\substack{i+j=k \\ 0<i,j<k}} [L_i(x),L_j(y)] - [d_k(x),y] - [x,d_k(y)] - \sum_{\substack{i+j=k \\ 0<i,j<k}} [d_i(x),d_j(y)] \\
&= [d_k(x),y] + [x,d_k(y)] - [d_k(x),y] - [x,d_k(y)] \\
&= 0
\end{aligned}
$$

for all $x, y \in \mathscr{A}$.

Finally, the other properties of d_k follow from what was already proved. □

Acknowledgements. The authors express their sincere thanks to the referees for their careful reading of the article and valuable suggestions and comments which help to improve the presentation of this paper.

REFERENCES

[1] K. ARDAKOV AND K. A. BROWN, *Ring-theoretic properties of Iwasawa algebras: a survey*, Doc. Math., (2006), Extra Volume, 7–33.
[2] D. BENKOVIČ, *Lie derivations on triangular matrices*, Linear Multilinear Algebra, **55** (2007), 619–626.
[3] D. BENKOVIČ, *Generalized Lie derivations on triangular algebras*, Linear Algebra Appl., **434** (2011), 1532–1544.
[4] D. BENKOVIČ, *Lie triple derivations on triangular matrices*, Algebra Colloq., **18** (2011), Special Issue No. 1, 819–826.
[5] D. BENKOVIČ, *Lie triple derivations of unital algebras with idempotents*, Linear Multilinear Algebra, **63** (2015), 141–165.
[6] D. BENKOVIČ AND D. EREMITA, *Multiplicative Lie n-derivations of triangular rings*, Linear Algebra Appl., **436** (2012), 4223–4240.
[7] D. BENKOVIČ AND M. GRAŠIČ, *Generalized derivations on unital algebras determined by action on zero products*, Linear Algebra Appl., **445** (2014), 347–368.
[8] D. BENKOVIČ AND N. ŠIROVNIK, *Jordan derivations of unital algebras with idempotents*, Linear Algebra Appl., **437** (2012), 2271–2284.
[9] L. CHEN AND J.-H. ZHANG, *Nonlinear Lie derivations on upper triangular matrices*, Linear Multilinear Algebra, **56** (2008), 725–730.
[10] Z.-X. CHEN AND Z.-K. XIAO, *Nonlinear Lie triple derivations on parabolic subalgebras of finite-dimensional simple Lie algebras*, Linear Multilinear Algebra, **60** (2012), 645–656.
[11] W.-S. CHEUNG, *Lie derivations of triangular algebras*, Linear Multilinear Algebra, **51** (2003), 299–310.
[12] J. CUNTZ, *Simple C^*-algebras generated by isometries*, Commun. math. Phys., **57** (1977), 173–185.
[13] J. CUNTZ AND W. KRIEGER, *A class of C^*-algebras and topological Markov chains*, Invent. Math., **56** (1980), 251–268.
[14] Y.-Q. DU AND Y. WANG, *Lie derivations of generalized matrix algebras*, Linear Algebra Appl., **437** (2012), 2719–2726.
[15] M. FERRERO AND C. HAETINGER, *Higher derivations and a theorem by Herstein*, Quaest. Math., **25** (2002), 249–257.
[16] M. FERRERO AND C. HAETINGER, *Higher derivations of semiprime rings*, Comm. Algebra, **30** (2002), 2321–2333.
[17] A. FOŠNER, F. WEI AND Z.-K. XIAO, *Nonlinear Lie-type derivations of von Neumann algebras and related topics*, Colloq. Math., **132** (2013), 53–71.
[18] K. E. GEHLES, *Properties of Cherednik algebras and graded Hecke algebras*, Ph. D. Thesis, University of Glasgow, 2006.
[19] P. GOLDSTEIN, *On graph C^*-algebras*, J. Aust. Math. Soc., **72** (2002), 153–160.
[20] D. HAN, *Lie-type higher derivations on operator algebras*, Bull. Iran Math. Soc., **40** (2014), 1169–1194.
[21] H. HASSE AND F. K. SCHMIDT, *Noch eine Begründung der Theorie der höheren Differentialquotienten in einem algebraischen Funktionenkörper einer Unbestimmten*, (German), J. Reine Angew. Math., **177** (1937), 215–237.
[22] N. HEEREMA, *Higher derivations and automorphisms of complete local rings*, Bull. Amer. Math. Soc., **76** (1970), 1212–1225.
[23] P. N. JEWELL, *Continuity of module and higher derivations*, Pacific J. Math., **68** (1977), 91–98.

[24] P.-S. JI, R.-R. LIU AND Y.-Z. ZHAO, *Nonlinear Lie triple derivations of triangular algebras*, Linear Multilinear Algebra, **60** (2012), 1155–1164.
[25] R. V. KADISON AND J. R. RINGROSE, *Fundamentals of the theory of operator algebras*, Vol. I–II. Academic Press, San Diego, 1983–1986.
[26] S. KÖNIG AND C.-C. XI, *A characteristic free approach to Brauer algebras*, Trans. Amer. Math. Soc., **353** (2000), 1489–1505.
[27] P. A. KRYLOV, *Isomorphism of generalized matrix rings*, Algebra and Logic, **47** (2008), 258–262.
[28] J. LI AND J. GUO, *Characterizations of higher derivations and Jordan higher derivations on CSL algebras*, Bull. Aust. Math. Soc., **83** (2011), 486–499.
[29] R. J. LOY, *Continuity of higher derivations*, Proc. Amer. Math. Soc., **37** (1973), 505–510.
[30] F. LU, *Lie triple derivations on nest algebras*, Math. Nachr., **280** (2007), 882–887.
[31] C. R. MIERS, *Lie derivations of von Neumann algebras*, Duke Math. J., **40** (1973), 403–409.
[32] C. R. MIERS, *Lie triple derivations of von Neumann algebras*, Proc. Amer. Math. Soc., **71** (1978), 57–61.
[33] J. B. MILLER, *Homomorphisms, higher derivations and derivations on associative algebras*, Acta Sci. Math. (Sezeged), **28** (1967), 221–231.
[34] J. B. MILLER, *Higher derivations on Banach algebras*, Amer. J. Math., **92** (1970), 301–331.
[35] X.-F. QI, *Characterization of Lie higher derivations on triangular algebras*, Acta Math. Sinica, **29** (2013), 1007–1018.
[36] X.-F. QI AND J.-C. HOU, *Lie higher derivations on nest algebras*, Commun. Math. Res., **26** (2010), 131–143.
[37] S.-L. SUN AND X.-F. MA, *Lie triple derivations of nest algebras on Banach spaces*, Linear Algebra Appl., **436** (2012), 3443–3462.
[38] Y. WANG, *Lie n-derivations of unital algebras with idempotents*, Linear Algebra Appl., **458** (2014), 512–525.
[39] Y. WANG AND Y. WANG, *Multiplicative Lie n-derivations of generalized matrix algebras*, Linear and Multilinear Algebra, **438** (2013), 2599–2616.
[40] F. WEI AND Z.-K. XIAO, *Higher derivations of triangular algebras and its applications*, Linear Algebra Appl., **435** (2011), 1034–1054.
[41] Z.-K. XIAO AND F. WEI, *Jordan higher derivations on triangular algebras*, Linear Algebra Appl., **432** (2010), 2615–2622.
[42] Z.-K. XIAO AND F. WEI, *Nonlinear Lie higher derivations on triangular algebras*, Linear and Multilinear Algebra, **60** (2012), 979–994.
[43] Z.-K. XIAO AND F. WEI, *Lie triple derivations of triangular algebras*, Linear Algebra Appl., **437** (2012), 1234–1249.
[44] W.-Y. YU AND J.-H. ZHANG, *Nonlinear Lie derivations of triangular algebras*, Linear Algebra Appl., **432** (2010), 2953–2960.
[45] J.-H. ZHANG, B.-W. WU AND H.-X. CAO, *Lie triple derivations of nest algebras*, Linear Algebra Appl., **416** (2006), 559–567.
[46] X. ZHANG, R.-L. AN AND J.-C. HOU, *Characterization of Higher derivations on CSL algebras*, Expo. Math., **31** (2013), 392–404.

(Received January 13, 2015)

Dong Han
School of Mathematics and Information Science
Henan Polytechnic University
Jiaozuo, 454000, P. R. China
e-mail: lishe@hpu.edu.cn

Feng Wei (corresponding author)
School of Mathematics and Statistics
Beijing Institute of Technology
Beijing, 100081, P. R. China
e-mail: daoshuo@hotmail.com, daoshuo@bit.edu.cn