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ADDITIVE MAPS PRESERVING m–NORMAL EIGENVALUES ON $\mathscr{B}(\mathscr{H})$

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Abstract. Let $\mathscr{H}$ be an infinite-dimensional complex Hilbert space and $\mathscr{B}(\mathscr{H})$ the algebra of all bounded linear operators on $\mathscr{H}$. For an operator $T \in \mathscr{B}(\mathscr{H})$ and a fixed non-negative integer m, an m-normal eigenvalue λ of T is the normal eigenvalue satisfying $\dim N(T-\lambda I) > m$. In this paper, we prove that, if an additive surjective map φ on $\mathscr{B}(\mathscr{H})$ preserves m as well as $m+1$-normal eigenvalues, then there is an invertible operator $A \in \mathscr{B}(\mathscr{H})$ such that $\varphi(T) = ATA^{-1}$ for all $T \in \mathscr{B}(\mathscr{H})$ or $\varphi(T) = AT^{tr}A^{-1}$ for all $T \in \mathscr{B}(\mathscr{H})$, where T^{tr} denotes the transpose of T with respect to an arbitrary but fixed orthonormal basis of $\mathscr{H}$.

1. Introduction

Linear or additive preserver problems are to characterize those linear or additive maps on operator algebras preserving certain properties, subsets or relations. Most important of all, we need to find certain properties which are isomorphism or anti-isomorphism invariants. The study of the problem has attracted the attention of many authors in the last decades [2, 3, 5, 6, 7, 11]. As we know, spectrum is a very fundamental and key concept in operator theory. Hence many authors have studied linear or additive maps preserving the spectrum as well as certain parts of the spectrum [1, 4, 9, 10]. For example, the author showed that additive maps on standard operator algebras preserving parts of the spectrum is either an isomorphism or anti-isomorphism in [4]. It is remarkable that various parts of the spectrum may be regarded as invariants of an automorphism or an anti-automorphism on the algebra of all bounded linear operators on a Banach (or Hilbert) space. It is known that certain parts of spectrum of operators are introduced to analyze the structure of operators. For example, the set of normal eigenvalues of an operator is given (cf. [8]). Note that the set of normal eigenvalues is at most countable and is a very "small" subset of spectrum in general. Thus how may the normal eigenvalues influence the structure of automorphisms on the algebra of all bounded linear operators on a Banach (or Hilbert) space? In this paper, we consider parts of the set of the normal eigenvalues as an invariant of an automorphism or an anti-automorphism on the algebra of all bounded linear operators on a complex infinite-dimensional Hilbert space.

Let $\mathscr{H}$ be a complex infinite-dimensional Hilbert space and $\mathscr{B}(\mathscr{H})$ the algebra of all bounded linear operators on $\mathscr{H}$. For $x, y \in \mathscr{H}$, we denote by $\langle x, y\rangle$ and $x \otimes y$ the

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inner product of x and y and the rank one operator defined by $(x\otimes y)z=\langle z,y\rangle x$, $\forall z\in\mathscr{H}$, respectively. The operator $x\otimes y$ is an idempotent if and only if $\langle x,y\rangle=1$. Let $T\in\mathscr{B}(\mathscr{H})$, we denote by $N(T)$ and $R(T)$ the null space and range of T respectively. For a subset M of $\mathscr{H}$, $\bigvee\{M\}$ denotes the closed subspace spanned by M. Let $\dim M$ (resp. codim$M$) is the dimension of M (resp. $M^{\perp}$, the orthogonal complement of M) if M is a closed subspace. Recall that an operator T is called Fredholm if it has closed range such that $\dim N(T)<\infty$ and codim$R(T)<\infty$. The index of a Fredholm operator $T\in\mathscr{B}(\mathscr{H})$ is given by $\mathrm{ind}(T)=\dim N(T)-\mathrm{codim}R(T)$. The ascent $asc(T)$ of T is the least non-negative integer n such that $N(T^n)=N(T^{n+1})$ and the descent $des(T)$ is the least non-negative integer n such that $R(T^n)=R(T^{n+1})$. An operator T is said to be Browder if it is Fredholm with finite ascent and descent. It is known that T is a Browder operator if and only if T is a Fredholm operator of index zero and $asc(T)<\infty$.

Let $T\in\mathscr{B}(\mathscr{H})$. If σ is a clopen subset of the spectrum $\sigma(T)$, then there exists an analytic Cauchy domain Ω such that $\sigma\subseteq\Omega$ and $[\sigma(T)\setminus\sigma]\cap\overline{\Omega}=\emptyset$. We let $E(\sigma;T)$ denote the Riesz idempotent of T corresponding to σ, that is,

$$E(\sigma;T)=\frac{1}{2\pi i}\int_{\Gamma}(\lambda-T)^{-1}d\lambda,$$

where $\Gamma=\partial\Omega$ is positively oriented with respect to Ω in the sense of complex variable theory. In this case, we denote $H(\sigma;T)=R(E(\sigma;T))$. If $\lambda\in iso\sigma(T)$, the isolate points of $\sigma(T)$, then $\{\lambda\}$ is a clopen subset of $\sigma(T)$ and we simply write $H(\lambda;T)$ instead of $H(\{\lambda\};T)$. If, in addition, $\dim H(\lambda;T)<\infty$, then λ is called a normal eigenvalue of T. The set of all normal eigenvalues of T will be denoted by $\sigma_0(T)$ (cf. [8]). Clearly, $\sigma_0(T)$ is contained in the point spectrum $\sigma_p(T)$. From Corollary 1.14 in [8], we can get

$$\begin{aligned}\sigma_0(T)&=\{\lambda\in\sigma(T):T-\lambda I\text{ is Browder}\}\\&=\{\lambda\in iso\sigma(T):T-\lambda I\text{ is Fredholm}\}.\end{aligned}$$

Given a non-negative integer m, we call an m-normal eigenvalue λ of T is the normal eigenvalue satisfying $\dim N(T-\lambda I)>m$. The set of all m-normal eigenvalues of T will be denoted by $\sigma_m(T)$.

That is

$$\sigma_m(T)=\{\lambda\in\sigma_0(T):\dim N(T-\lambda I)>m\}.$$

Then we obtain that

$$...\subseteq\sigma_m(T)...\subseteq\sigma_2(T)\subseteq\sigma_1(T)\subseteq\sigma_0(T).$$

In this paper, we characterize an additive surjective map φ on $\mathscr{B}(\mathscr{H})$ preserves m as well as $m+1$-normal eigenvalues for some fixed non-negative integer m, that is $\sigma_m(\varphi(T))=\sigma_m(T)$ and $\sigma_{m+1}(\varphi(T))=\sigma_{m+1}(T)$ for all $T\in\mathscr{B}(\mathscr{H})$. And we show that such a surjective map is an automorphism or an anti-automorphism.

2. Main results

We firstly need some auxiliary results.

LEMMA 1. *Let $m \geqslant 0$ and $T \in \mathscr{B}(\mathscr{H})$. If $0 \in \sigma_{m+1}(T)$, then for every rank one operator $F \in \mathscr{B}(\mathscr{H})$, either $0 \in \sigma_m(T+F)$ or $0 \in \sigma_m(T-F)$.*

Proof. Suppose that $0 \in \sigma_{m+1}(T)$. Then both $T+F$ and $T-F$ are Fredholm of index zero for every rank one operator F. Thus $\dim N(T+F) = \mathrm{codim} R(T+F)$ and $\dim N(T-F) = \mathrm{codim} R(T-F)$. Note that $R(T+F) \subseteq R(T)+R(F)$ and $\dim N(T) = \mathrm{codim} R(T) > m+1$. Then $\mathrm{codim} R(T+F) > m$, and $\dim N(T+F) > m$. Similarly, we have $\dim N(T-F) > m$. Since $asc(T) < \infty$, we have for every rank one operator F, either $asc(T+F) < \infty$ or $asc(T-F) < \infty$ by Proposition 2.7 in [12]. This implies that either $T+F$ or $T-F$ is Browder. Hence we get either $0 \in \sigma_m(T+F)$ or $0 \in \sigma_m(T-F)$. $\square$

PROPOSITION 1. *Let $k, m \geqslant 0$ and $T \in \mathscr{B}(\mathscr{H})$. If $\dim R(T) \geqslant 2$, then there exists an operator S satisfying $0 \in \sigma_k(S)$ such that $0 \notin \sigma_m(S+T)$ and $0 \notin \sigma_m(S-T)$.*

Proof. We will complete this proof by three cases:

Case (i) $\dim N(T) = \infty$.

Assume that $\dim R(T) \geqslant 2$. Then there exist two vectors x_0, y_0 such that Tx_0, Ty_0 are linearly independent. We can choose suitable vectors of $N(T)$ to perturb x_0, y_0, then there exist two vectors u_0, v_0 such that the vectors u_0, v_0, Tu_0, Tv_0 are linearly independent. Since $\dim N(T) = \infty$, we have $\{u_0, v_0, Tu_0, Tv_0\}^\perp \cap N(T)$ is infinite-dimensional. It follows that there is an orthonormal subset $\{u_i, v_i : i \geqslant 1\}$ of $\{u_0, v_0, Tu_0, Tv_0\}^\perp \cap N(T)$ with an infinite-dimensional orthogonal complement. Let $H_1 = \bigvee\{u_i, v_i : i \geqslant 0\}$ and $H_2 = \bigvee(\{Tu_0, Tv_0\} \cup \{u_i, v_i : i \geqslant 0\})$. We can choose an orthonormal subset $\{\xi_i\}_{i=0}^k \subseteq H_2^\perp$ such that $H_2^\perp = \bigvee\{\xi_i : i = 0, 1, \cdots, k\} \oplus M$, where M is an infinite-dimensional subspace. Take any two unit orthogonal vectors $\eta_1, \eta_2 \in H_1^\perp$ such that

$$H_1^\perp = \bigvee\{\eta_1, \eta_2\} \oplus H_2^\perp = \bigvee\{\eta_1, \eta_2\} \oplus M \oplus \bigvee\{\xi_i : i = 0, 1, \cdots, k\}.$$

We define an operator $S \in \mathscr{B}(\mathscr{H})$ by:

$$\begin{cases} Su_0 = -Tu_0,\ Sv_0 = Tv_0; \\ Su_{i+1} = u_i,\ Sv_{i+1} = v_i, \quad \forall\, i \geqslant 0; \\ S\xi_i = 0, \quad i = 0, 1, \cdots, k; \\ S: \bigvee\{\eta_1, \eta_2\} \oplus M \to M \text{ is a bounded invertible linear operator.} \end{cases}$$

It follows that $0 \in \sigma_k(S)$ and $(S+T)u_0 = (S-T)v_0 = 0$, $(S+T)^i u_i = u_0$, $(S-T)^i v_i = v_0$ for all $i \geqslant 0$. This implies that $asc(S+T) = asc(S-T) = \infty$, and thus $0 \notin \sigma_m(S+T)$ and $0 \notin \sigma_m(S-T)$.

Case (ii) $T = \lambda I + F$ for some non-zero complex number $\lambda \in \mathbb{C}$ and F is a finite rank operator.

Note that $\dim N(F) = \infty$. Then there exist two separable infinite-dimensional subspaces $H_1, H_2 \subseteq N(F)$ such that $H_1 \perp H_2$ and $(H_1 \oplus H_2)^{\perp}$ is also infinite-dimensional. Let $\{\omega_i\}_{i=0}^{k} \subseteq (H_1 \oplus H_2)^{\perp}$ be an orthonormal subset such that $(H_1 \oplus H_2)^{\perp} = \bigvee\{\omega_i : i = 0, 1, \cdots, k\} \oplus H_3$, where H_3 is an infinite-dimensional subspace. We define an operator $S \in \mathscr{B}(\mathscr{H})$ by:

$$\begin{cases} S|_{H_1} = -\lambda I_{H_1} + \frac{\lambda}{3} A_1, & \text{where } A_1 \text{ is a backward shift operator on } H_1; \\ S|_{H_2} = \lambda I_{H_2} + \frac{\lambda}{3} A_2, & \text{where } A_2 \text{ is a backward shift operator on } H_2; \\ S|_{H_3} = I; \\ S\omega_i = 0, i = 0, 1, 2, \cdots, k. \end{cases}$$

Then $0 \in \sigma_k(S)$. Moreover, $S + T|_{H_1} = \frac{\lambda}{3} A_1$, $S - T|_{H_2} = \frac{\lambda}{3} A_2$. Hence $asc(S + T) = asc(S - T) = \infty$. We obtain that $0 \notin \sigma_m(S + T)$ and $0 \notin \sigma_m(S - T)$.

Case (iii) $\dim N(T) < \infty$ and $T \neq \lambda I + F$ for any non-zero complex number $\lambda \in \mathbb{C}$ and for any finite rank operator F.

According to this hypothesis, we have for every closed subspace $N \subseteq H$ with finite codimension, there exists a vector $x \in N \cap T^{-1}N$ such that the vectors x, Tx are linearly independent. Without loss of generality, let $\|T\| \leqslant \frac{1}{2}$. Find a unit vector z_0 such that z_0 and Tz_0 are linearly independent. Let $H_0 = \{z_0, Tz_0\}^{\perp}$. Then there exists a unit vector $z_1 \in H_0 \cap T^{-1}H_0$ such that z_1 and Tz_1 are linearly independent. Let $H_1 = \{z_0, z_1, Tz_0, Tz_1\}^{\perp}$. Then we can choose a unit vector $z_2 \in H_1 \cap T^{-1}H_1$ such that z_2 and Tz_2 are linearly independent. Continuing this process, we can get a sequence of unit vectors $\{z_i\}_{i=0}^{\infty}$ such that z_i and Tz_i are linearly independent and $\{z_{i+1}, Tz_{i+1}\} \perp \{z_j, Tz_j : j = 0, 1, 2, ..., i\}$ for all $i \geqslant 0$. We can also assume that the orthogonal complement of $\bigvee\{z_i, Tz_i : i \geqslant 0\}$ is infinite-dimensional. Otherwise we can replace $\{z_i\}_{i=0}^{\infty}$ by $\{z_{2i}\}_{i=0}^{\infty}$. Let $\{y_i\}_{i=0}^{k} \cup \{x_i\}_{i=0}^{\infty}$ be an orthonormal sequence of $\bigvee\{z_i, Tz_i : i \geqslant 0\}^{\perp}$. For all $i \geqslant 0$, let $\zeta_i \in \bigvee\{z_i, Tz_i\}$ be a unit vector such that $\zeta_i \perp z_i$. Then $Tz_i = \alpha_i z_i + \beta_i \zeta_i$, where $\alpha_i, \beta_i \in \mathbb{C}$ satisfy $|\alpha_i| \leqslant \frac{1}{2}$, $0 < |\beta_i| \leqslant \frac{1}{2}$. Let $M_1 = \bigvee\{Tz_0, Tz_1, x_i, \zeta_{i+2} : i \geqslant 0\}$, $M_2 = \bigvee\{z_i : i \geqslant 0\}$, $M_3 = \bigvee\{y_i : i = 0, 1, \cdots, k\}$ and $M_4 = (M_1 \oplus M_2 \oplus M_3)^{\perp}$. We define an operator $S \in \mathscr{B}(\mathscr{H})$ by:

$$\begin{cases} Sz_0 = -Tz_0, \quad Sz_1 = Tz_1; \\ Sz_i = z_{i-2} - (-1)^i Tz_i, \quad \forall\, i \geqslant 2; \\ S\zeta_0 = x_0, \quad S\zeta_1 = x_1; \\ S\zeta_{i+2} = \zeta_{i+2}, \quad Sx_i = x_{i+2} \quad \forall\, i \geqslant 0; \\ Sy_i = 0, \quad \forall\, 0 \leqslant i \leqslant k; \\ S|_{M_4} = I. \end{cases}$$

We next prove that $S|_{M_1 \oplus M_2}$ is invertible. It is known that $S|_{M_1 \oplus M_2}$ is injective and $M_1 \subseteq R(S|_{M_1 \oplus M_2})$. Let P be the projection on M_2 and $B_1, B_2 \in \mathscr{B}(M_2)$ such that $B_1 z_0 = B_1 z_1 = 0$, $B_1 z_{i+2} = z_i$ and $B_2 z_i = (-1)^{i+1} \alpha_i z_i$ for all $i \geqslant 0$. Then $\|B_1\| = 1$ and $\|B_2\| \leqslant \frac{1}{2}$. Note that B_1 is surjective. So is $B_1 + B_2$. However, we now have $PS|_{M_2} = B_1 + B_2$. It follows that $S|_{M_1 \oplus M_2}$ is surjective. Moreover, $Sy_i = 0$ for $0 \leqslant i \leqslant k$ and $S|_{M_4} = I$. Then $0 \in \sigma_k(S)$. Also, $(S + T)z_0 = (S - T)z_1 = 0$, $(S + T)^i z_{2i} = z_0$, $(S - T)^i z_{2i+1} = z_1$ for all $i \geqslant 0$. This implies that $asc(S + T) = asc(S - T) = \infty$, and so $0 \notin \sigma_m(S + T)$ and $0 \notin \sigma_m(S - T)$.

According to the three cases, we get that there exists an operator S satisfying $0 \in \sigma_k(S)$ such that $0 \notin \sigma_m(S+T)$ and $0 \notin \sigma_m(S-T)$. $\square$

COROLLARY 1. *Let $m \geqslant 0$ and $T \in \mathscr{B}(\mathscr{H})$ be a non-zero operator. Then there exists an operator $S \in \mathscr{B}(\mathscr{H})$ such that $0 \in \sigma_m(S)$ but $0 \notin \sigma_m(S+T)$.*

Proof. It is sufficient to assume that $T = x \otimes y$ is a rank one operator by Proposition 1. Without loss of generality, we may assume that x and y are unit vectors. Note that $\sigma_m(\cdot)$ is similarity invariant. We now can assume that $x = y$ or $\langle x, y\rangle = 0$, that is, T is a rank one projection or a rank one nilpotent operator. Let $T = x \otimes x$ be a rank one projection. Put $\mathscr{H} = \bigvee\{x\} \oplus H_1 \oplus H_2$, where $\dim H_1 = m$. Let S be the projection on H_2. Then $0 \in \sigma_m(S)$ but $0 \notin \sigma_m(S+T)$. If $\langle x, y\rangle = 0$, then we let $\mathscr{H} = \bigvee\{x, y\} \oplus H_1 \oplus H_2$, where $\dim H_1 = m$. In this case, let P be the projection on H_2 and $S = y \otimes x + P$. Then S is what we require. $\square$

LEMMA 2. *Let $T = \sum_{i=0}^{m} e_i \otimes f_i$ be a rank-(m+1) operator and $\lambda \in \mathbb{C} - \{0\}$. If $\lambda \in \sigma_m(T)$, then $\langle e_i, f_j\rangle = \lambda \delta_{ij}$ for all $i, j = 0, 1, 2, \cdots, m$, where δ_{ij} is the Kronecker number.*

Proof. Suppose that $\lambda \in \sigma_m(T)$. Then we have $\dim N(T - \lambda I) > m$. It is known that $N(T - \lambda I) \subseteq R(T)$ and $\dim R(T) = m+1$. Hence, $\dim N(T - \lambda I) = m+1$, that is $N(T - \lambda I) = R(T) = \bigvee\{e_i : i = 0, 1, 2, \cdots, m\}$. Then $Te_j = \lambda e_j$ for every $j = 0, 1, 2, \cdots, m$. Now $Te_j = \sum_{i=0}^{m} \langle e_j, f_i\rangle e_i$. So $\sum_{i=0}^{m} \langle e_j, f_i\rangle e_i = \lambda e_j$, this implies that $\langle e_i, f_j\rangle = 0$ for $i \neq j$ and $\langle e_i, f_i\rangle = \lambda$, where $0 \leqslant i, j \leqslant m$. $\square$

LEMMA 3. *Let $m \geqslant 0$ and $A, B \in \mathscr{B}(\mathscr{H})$. If $\sigma_m(A+F) = \sigma_m(B+F)$ for all operator $F \in \mathscr{B}(\mathscr{H})$ with rank not greater than m+1, then $A = B$.*

Proof. Let $x \in \mathscr{H}$, fix a scalar $\alpha \in \mathbb{C}$ such that $|\alpha| > \|A\| + \|B\|$. We define an operator

$$F_x = \begin{cases} \|x\|^{-2}(A - \alpha I)x \otimes x, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$

Then we have $F_x x = Ax - \alpha x$. If $x \neq 0$, then $\alpha \in \sigma_p(A - F_x) \subseteq \sigma(A - F_x)$. It follows that $\alpha \in \sigma_0(A - F_x)$ from the fact that $\|A - F_x\| \geqslant |\alpha| > \|A\| \geqslant \|A\|_e = \|A - F_x\|_e$, where $\|A\|_e$ is the essential norm of A.

In the following, we will prove that $\alpha \in \sigma_0(B - F_x)$ if $x \neq 0$. There are two cases:

Case (1) $\dim N(A - F_x - \alpha I) > m$.

Now, we have $\alpha \in \sigma_m(A - F_x)$, so $\alpha \in \sigma_m(B - F_x) \subseteq \sigma_0(B - F_x)$.

Case (2) $\dim N(A - F_x - \alpha I) \leqslant m$.

Assume that $\alpha \notin \sigma_0(B - F_x)$. Note that $|\alpha| > \|B\| \geqslant \|B\|_e = \|B - F_x\|_e$. We obtain that $B - F_x - \alpha I$ is invertible. Choose $m+1$ orthogonal vectors $x_0 = x, x_1, x_2, \cdots, x_m$ and let $F_m = F_{x_0} + F_{x_1} + \cdots + F_{x_m}$. Then $(A - F_m)x_i = \alpha x_i$ for all $0 \leqslant i \leqslant m$. It implies that $\dim N(A-F_m-\alpha I)>m$. Now, $\|A-F_m\|\geqslant|\alpha|>\|A\|\geqslant\|A\|_e=\|A-F_m\|_e$. Hence $\alpha\in\sigma_m(A-F_m)$ and hence $\alpha\in\sigma_m(B-F_m)$. We know that

$$B-F_m-\alpha I=(B-F_{x_0}-\alpha I)-(F_{x_1}+\cdots+F_{x_m}).$$

Since $B-F_{x_0}-\alpha I$ is invertible and $F_{x_1}+\cdots+F_{x_m}$ is a rank-m operator, we have $\dim N(B-F_m-\alpha I)\leqslant m$. This is a contradiction. Therefore, we get that $\alpha\in\sigma_0(B-F_x)$.

It follows that there exists a non-zero vector $y_x\in\mathscr{H}$ such that $(B-F_x)y_x=\alpha y_x$ for any non-zero vector $x\in\mathscr{H}$. We claim that if there exist two vectors $y_1,y_2\in\mathscr{H}$ such that $(B-F_x)y_1=\alpha y_1$ and $(B-F_x)y_2=\alpha y_2$, then y_1 and y_2 are linearly dependent. According to the assumption, we have $(B-\alpha I)y_1=F_xy_1$ and $(B-\alpha I)y_2=F_xy_2$. Then F_xy_1 and F_xy_2 are linearly dependent since F_x is rank one. We may assume that $F_xy_1=\mu F_xy_2$ for some constant $\mu\in\mathbb{C}$. Then $(B-\alpha I)y_1=\mu(B-\alpha I)y_2$, that is, $(B-\alpha I)(y_1-\mu y_2)=0$. We know that y_1 and y_2 are linearly independent. Then $\alpha\in\sigma_p(B)$. But $B-\alpha I$ is invertible since $|\alpha|>\|B\|$. This is a contradiction. Thus y_1 and y_2 are linearly dependent.

Note that $(A-\alpha I)x=F_xx$ and $(B-\alpha I)y_x=F_xy_x$ for any non-zero vector $x\in\mathscr{H}$. Then there is an unique non-zero vector y_x such that $(A-\alpha I)x=F_xx=(B-\alpha I)y_x=F_xy_x$ for any nonzero $x\in\mathscr{H}$. We define $y_x=0$ if $x=0$. Thus, we can define a map T on $\mathscr{H}$ such that $Tx=y_x$. Moreover, we have that $(A-\alpha I)x=F_xx=(B-\alpha I)Tx=F_xTx$ for all $x\in\mathscr{H}$. This implies that $T=(B-\alpha I)^{-1}(A-\alpha I)$.

If $x\neq 0$, then $F_x=\|x\|^{-2}(A-\alpha I)x\otimes x$ and $F_xTx=F_xx$, thus

$$\|x\|^{-2}\langle Tx,x\rangle(A-\alpha I)x=(A-\alpha I)x.$$

We obtain that $\langle Tx,x\rangle=\|x\|^2=\langle x,x\rangle$. Note that if $x=0$, then $Tx=0$. So we get that

$$\langle Tx,x\rangle=\langle x,x\rangle,\qquad\forall x\in\mathscr{H}.$$

Therefore, $T=I$. That is, we have $(B-\alpha I)^{-1}(A-\alpha I)=I$, and so $A=B$. $\square$

THEOREM 1. *Let φ be a surjective additive map on $\mathscr{B}(\mathscr{H})$ and $m\geqslant 0$. If $\sigma_m(\varphi(T))=\sigma_m(T)$ and $\sigma_{m+1}(\varphi(T))=\sigma_{m+1}(T)$ for all $T\in\mathscr{B}(\mathscr{H})$, then there is an invertible operator $A\in\mathscr{B}(\mathscr{H})$ such that $\varphi(T)=ATA^{-1}$ for all $T\in\mathscr{B}(\mathscr{H})$ or $\varphi(T)=AT^{tr}A^{-1}$ for all $T\in\mathscr{B}(\mathscr{H})$, where T^{tr} denotes the transpose of T with respect to an arbitrary but fixed orthonormal basis of $\mathscr{H}$.*

Proof. We first show φ is injective. Let $\varphi(T)=0$. If $T\neq 0$, then by Corollary 1, there exists an operator S such that $0\in\sigma_m(S)$ but $0\notin\sigma_m(S+T)$. Note that

$$\sigma_m(S+T)=\sigma_m(\varphi(S+T))=\sigma_m(\varphi(S))=\sigma_m(S).$$

It is a contradiction. Thus, $T=0$.

Let $T\in\mathscr{B}(\mathscr{H})$ with $\dim R(T)\geqslant 2$. By Proposition 1, there exists an operator S satisfying $0\in\sigma_{m+1}(S)$ such that $0\notin\sigma_m(S+T)$ and $0\notin\sigma_m(S-T)$. Then

$0 \notin \sigma_m(\varphi(S)+\varphi(T))$ and $0 \notin \sigma_m(\varphi(S)-\varphi(T))$. We know that $0 \in \sigma_{m+1}(S) = \sigma_{m+1}(\varphi(S))$ by the assumption. Then by Lemma 1, we have that $\dim R(\varphi(T)) \geqslant 2$. Since φ is bijective and φ^{-1} has the same property as φ, it follows that φ preserves the set of operators of rank one in both directions.

We claim that φ preserves idempotents of rank one and their linear spans in both directions. That is $\varphi(\mathbb{C}P) \subseteq \mathbb{C}\varphi(P)$ for every idempotent of rank one P. Let $e_0 \otimes f_0$ be a rank one idempotent and let $P = \sum_{i=0}^{m} e_i \otimes f_i$ be a rank-$(m+1)$ idempotent. For any non-zero $\lambda \in \mathbb{C}$, then $\varphi(\lambda P) = \sum_{i=0}^{m} \varphi(\lambda e_i \otimes f_i)$ is also a rank-$(m+1)$ operator. Note that $\lambda \in \sigma_m(\lambda P)$. Then $\lambda \in \sigma_m(\varphi(\lambda P))$. It follows from Lemma 2 that $\varphi(\lambda e_0 \otimes f_0) = \lambda y_\lambda \otimes g_\lambda$, where $y_\lambda \otimes g_\lambda$ is a rank one idempotent. In particular, let $\lambda = 1$ and let $\varphi(e_0 \otimes f_0) = y \otimes g$, then we can get $y \otimes g$ is a rank one idempotent. Thus φ preserves idempotents of rank one in both directions. We claim that $\varphi(\mathbb{C}e_0 \otimes f_0) \subseteq \mathbb{C}\varphi(e_0 \otimes f_0)$.

Since $\varphi(e_0 \otimes f_0) = y \otimes g$ and $\langle y, g\rangle = 1$, then we can find two vectors $z, h \in \mathscr{H}$ such that $\langle z, g\rangle = 0$, $\langle y, h\rangle = 0$ and $\langle z, h\rangle = 1$. For $y \otimes h$ and $z \otimes g$, there exist two rank one operators $u \otimes k$ and $v \otimes p$ such that $\varphi(u \otimes k) = y \otimes h$ and $\varphi(v \otimes p) = z \otimes g$ as φ is surjective. We know that

$$\varphi(e_0 \otimes f_0 + u \otimes k) = y \otimes g + y \otimes h, \;\; \varphi(e_0 \otimes f_0 + v \otimes p) = y \otimes g + z \otimes g.$$

Then both $e_0 \otimes f_0 + u \otimes k$ and $e_0 \otimes f_0 + v \otimes p$ are rank one operators. It implies that $\lambda e_0 \otimes f_0 + u \otimes k$ and $\lambda e_0 \otimes f_0 + v \otimes p$ are also rank one operator for any non-zero $\lambda \in \mathbb{C}$. Fix a non-zero complex number λ, we have $\varphi(\lambda e_0 \otimes f_0) = \lambda y_\lambda \otimes g_\lambda$, where $y_\lambda \otimes g_\lambda$ is a rank one idempotent. Then $\varphi(\lambda e_0 \otimes f_0 + u \otimes k) = \lambda y_\lambda \otimes g_\lambda + y \otimes h$ is also rank one. We obtain that y_λ and y are linearly dependent or the same is true for g_λ and h.

We assert that y_λ and y are linearly dependent. Otherwise, we have g_λ and h are linearly dependent. Then there exists some non-zero $\alpha_\lambda \in \mathbb{C}$ such that $g_\lambda = \overline{\alpha_\lambda} h$. Thus

$$\varphi(\lambda e_0 \otimes f_0 + v \otimes p) = \alpha_\lambda \lambda y_\lambda \otimes h + z \otimes g.$$

Since h and g are linearly independent, there is some non-zero $\beta_\lambda \in \mathbb{C}$ such that $y_\lambda = \beta_\lambda z$, and so $\varphi(\lambda e_0 \otimes f_0) = \alpha_\lambda \beta_\lambda \lambda z \otimes h$, where $z \otimes h$ is a rank one idempotent. We also know that $\varphi(\lambda e_0 \otimes f_0) = \lambda y_\lambda \otimes g_\lambda$, where $y_\lambda \otimes g_\lambda$ is a rank one idempotent. It follows that $\alpha_\lambda \beta_\lambda \lambda = \lambda$. Therefore,

$$\varphi(\lambda e_0 \otimes f_0) = \lambda z \otimes h.$$

As φ is surjective, we can find two vectors $w, l \in \mathscr{H}$ such that $\varphi(w \otimes l) = z \otimes h$ and $\langle w, l\rangle = 1$. It is clear that $y \otimes g + z \otimes h$ is a projection of rank two. Then $e_0 \otimes f_0 + w \otimes l$ is a rank two operator, and then the operator $\lambda e_0 \otimes f_0 + w \otimes l$ is also rank two. Thus

$$\varphi(\lambda e_0 \otimes f_0 + w \otimes l) = (\lambda + 1) z \otimes h.$$

This is a contradiction since φ preserves the set of operators of rank one in both directions. Therefore, we get that y_λ and y are linearly dependent. Then there exists some non-zero $\gamma_\lambda \in \mathbb{C}$ such that $y_\lambda = \gamma_\lambda y$. Note that $\varphi(\lambda e_0 \otimes f_0 + v \otimes p) = \gamma_\lambda \lambda y \otimes g_\lambda + z \otimes g$, which is also rank one. Thus we can find some non-zero $\mu_\lambda \in \mathbb{C}$ such that $g_\lambda = \overline{\mu_\lambda} g$ because y and z are linearly independent. So $\varphi(\lambda e_0 \otimes f_0) = \gamma_\lambda \mu_\lambda \lambda y \otimes g$, where $y \otimes g$ is a rank one idempotent. We also know that $\varphi(\lambda e_0 \otimes f_0) = \lambda y_\lambda \otimes g_\lambda$, where $y_\lambda \otimes g_\lambda$ is also a rank-1 idempotent. Thus $\gamma_\lambda \mu_\lambda \lambda = \lambda$. Therefore,

$$\varphi(\lambda e_0 \otimes f_0) = \lambda y \otimes g = \lambda \varphi(e_0 \otimes f_0).$$

According to the above, we have that φ preserves idempotents of rank one and their linear spans in both directions. It follows from Theorem 4.4 in [13] that there is a bounded invertible linear or conjugate-linear operator A on $\mathscr{H}$ such that one of the following assertions holds.

(1) $\varphi(F) = AFA^{-1}$ for all finite rank operators $F \in \mathscr{B}(\mathscr{H})$;

(2) $\varphi(F) = AF^{tr}A^{-1}$ for all finite rank operators $F \in \mathscr{B}(\mathscr{H})$, where F^{tr} is the transpose of F with respect to an arbitrary but fixed orthonormal basis of $\mathscr{H}$. If A is conjugate-linear, then $\varphi(iP) = A(iP)A^{-1} = -i\varphi(P)$ or $\varphi(iP) = A(iP)^{tr}A^{-1} = -i\varphi(P)$ for any rank-$(m+1)$ idempotent P, which means that $\sigma_m(iP) = \{i\}$ while $\sigma_m(\varphi(iP)) = \{-i\}$. This is a contradiction. Thus A must be linear.

Assume that (1) holds. Let $T \in \mathscr{B}(\mathscr{H})$ and for any finite rank operator F, we have

$$\begin{aligned} \sigma_m(T+F) &= \sigma_m(\varphi(T) + \varphi(F)) \\ &= \sigma_m(\varphi(T) + AFA^{-1}) \\ &= \sigma_m(A(A^{-1}\varphi(T)A + F)A^{-1}) \\ &= \sigma_m(A^{-1}\varphi(T)A + F). \end{aligned}$$

Then we get that $T = A^{-1}\varphi(T)A$ by Lemma 3. Therefore, $\varphi(T) = ATA^{-1}$ for all $T \in \mathscr{B}(\mathscr{H})$.

If (2) holds, then we similarly have that $\varphi(T) = AT^{tr}A^{-1}$ for all $T \in \mathscr{B}(\mathscr{H})$. □

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