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SINGULAR VALUE INEQUALITIES RELATED TO THE AUDENAERT–ZHAN INEQUALITY

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Abstract. In this paper, we refine the Heinz mean inequality for singular values and give some generalizations of Audenaert-Zhan inequality for singular values and Zhan's conjecture for the case of negative t. Among others, we show that if $A,B \in \mathbb{M}_n$ are positive semidefinite and f,g are real valued continuous functions on $[0,\infty)$ such that g is monotone and $f(g^{-1}(\sqrt{t}))^2$ is operator monotone on $[0,\infty)$, then

$$s_j(f(A)(g(A)^2+g(B)^2)f(B)) \leqslant \frac{1}{2}s_j(f(A)^2g(A)^2+f(B)^2g(B)^2)$$

for $j=1,\ldots,n$, where s_j are the singular values in decreasing order.

1. Introduction

A capital letter means an $n\times n$ matrix in the matrix algebra $\mathbb{M}_n$. Let A,B be Hermitian matrices in $\mathbb{M}_n$, then the order relation $A \geqslant B$ means, as usual, that $A-B$ is positive semidefinite. We always denote by $\lambda_j(A)$ and $s_j(A)$ its eigenvalues and singular values, respectively, arranged in non-increasing order, and denote by $|A|$ the absolute value operator of A, that is, $|A| = (A^*A)^{\frac{1}{2}}$, where A^* is the adjoint operator of A.

The arithmetic-geometric mean inequality was proved by Bhatia and Kittaneh [3] to hold for singular values of arbitrary matrices $A,B \in \mathbb{M}_n$:

$$2s_j(AB^*) \leqslant s_j(A^*A+B^*B) \quad \text{for } j=1,2,\cdots,n. \tag{1.1}$$

Afterwards Bhatia and Kittaneh [2] proved that for positive semidefinite $A,B \in \mathbb{M}_n$,

$$s_j(A^{\frac{1}{2}}B^{\frac{3}{2}}+A^{\frac{3}{2}}B^{\frac{1}{2}}) \leqslant \frac{1}{2}s_j((A+B)^2) \quad \text{for } j=1,\ldots,n. \tag{1.2}$$

In [1, Theorem 2], Audenaert showed a singular value inequality for Heinz means, which is the affirmative answer to Zhan's conjecture [6, Conjecture 4]:

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THEOREM A. *Let $A,B \in \mathbb{M}_n$ be positive semidefinite. Then for* $0 \leqslant r \leqslant 1$

$$s_j(A^rB^{1-r}+A^{1-r}B^r) \leqslant s_j(A+B) \quad \textit{for } j=1,2,\cdots,n. \tag{1.3}$$

Moreover, Zhan posed the following conjecture in [6, Conjecture 3] that if $A,B \in \mathbb{M}_n$ are positive semidefinite, then for each $1 \leqslant 2r \leqslant 3$ and $-2 < t \leqslant 2$,

$$s_j(A^rB^{2-r}+A^{2-r}B^r) \leqslant \frac{2}{t+2} s_j(A^2+tAB+B^2) \quad \text{for } j=1,\dots,n. \tag{1.4}$$

The inequality (1.4) has been proved to hold for $r=\frac{1}{2},1,\frac{3}{2}$ and all $-2<t\leqslant 2$ by Dumitru, Levanger and Visinescu [4].

Furthermore, it was shown that the function $f(t)=\frac{2}{t+2}\lambda_j(A^2+B^2+\frac{t}{2}AB+\frac{t}{2}BA)$ is non-increasing on $(-2,\infty)$.

In this viewpoint we are tempted to show general singular value inequality for Audenaert-Zhan inequality (1.3) and refine the Heinz mean inequality for singular values as well. Also, we give a partial affirmative answer to Zhan's conjecture (1.4).

2. Main results

In this section, we show a unified form of Heinz means inequalities for singular values. The following results due to Tao [5, Theorem 1] and Audenaert [1, Corollary 1] play an important role in what follows.

THEOREM B. (Tao) *Given any positive semidefinite block matrix* $\begin{pmatrix} M & K \\ K^* & N \end{pmatrix}$, *where* $M,N \in \mathbb{M}_n$. *Then*

$$2s_j(K) \leqslant s_j\begin{pmatrix} M & K \\ K^* & N \end{pmatrix} \quad \textit{for } j=1,2,\dots,n.$$

THEOREM C. (Audenaert) *If $A,B \in \mathbb{M}_n$ are positive semidefinite, then*

$$\frac{1}{2}\lambda_j((A+B)(f(A)+f(B)) \leqslant \lambda_j(Af(A)+Bf(B)) \quad \textit{for } j=1,\dots,n. \tag{2.1}$$

for any matrix monotone function f.

We need the following known fact [7, Theorem 2.8]:

LEMMA 2.1. *For any matrices $X,Y \in \mathbb{M}_n$, $\lambda_j(XY)=\lambda_j(YX)$ for $j=1,\dots,n$.*

Now we state our main theorem:

THEOREM 2.2. *Let $A,B \in \mathbb{M}_n$ be positive semidefinite and f,g be real valued continuous functions on $[0,\infty)$. Further suppose that f and g satisfy either of the following conditions:*

(i) *g is monotone on $[0,\infty)$ and $h_1(t) = f(g^{-1}(\sqrt{t}))^2$ is operator monotone.*

(ii) *f is monotone on $[0,\infty)$ and $h_2(t) = g(f^{-1}(\sqrt{t}))^2$ is operator monotone.*

Then

$$s_j(f(A)(g(A)^2 + g(B)^2)f(B)) \leqslant s_j(f(A)^2 g(A)^2 + f(B)^2 g(B)^2) \tag{2.2}$$

for $j = 1,2,\ldots,n$.

Proof. By symmetry of (2.2), it suffices to prove the case of (i) only. Let us define the matrices $T = \begin{pmatrix} f(A) \\ f(B) \end{pmatrix}$ and $S = \begin{pmatrix} g(A) & g(B) \end{pmatrix}$. Then

$$0 \leqslant (TS)(TS)^* = \begin{pmatrix} f(A)^2 g(A)^2 + f(A)g(B)^2 f(A) & f(A)(g(A)^2 + g(B)^2)f(B) \\ f(B)(g(A)^2 + g(B)^2)f(A) & f(B)^2 g(B)^2 + f(B)g(A)^2 f(B) \end{pmatrix}.$$

Hence it follows from Theorem B that for $j = 1,\ldots,n$

$$2s_j(f(A)(g(A)^2 + g(B)^2)f(B)) \leqslant s_j((TS)(TS)^*)$$

and Lemma 2.1 implies that

$$s_j((TS)(TS)^*) = \lambda_j(TSS^*T^*) = \lambda_j(SS^*T^*T) = \lambda_j((g(A)^2 + g(B)^2)(f(A)^2 + f(B)^2)).$$

We put $A_1 = g(A)^2$ and $B_1 = g(B)^2$. By Theorem C it follows from the operator monotonicity of h_1 that

$$\begin{aligned} \lambda_j((g(A)^2 + g(B)^2)(f(A)^2 + f(B)^2)) &= \lambda_j((A_1 + B_1)(h(A_1) + h(B_1))) \\ &\leqslant 2\lambda_j(A_1 h(A_1) + B_1 h(B_1)) \\ &= 2\lambda_j(f(A)^2 g(A)^2 + f(B)^2 g(B)^2) \\ &= 2s_j(f(A)^2 g(A)^2 + f(B)^2 g(B)^2). \end{aligned}$$

Combining the above, we have the desired singular value inequality (2.2). □

If we put $f(t) = t$ or $g(t) = t$ in Theorem 2.2, then we have the following corollary:

COROLLARY 2.3. *Let $A, B \in \mathbb{M}_n$ be positive semidefinite and $f(\sqrt{t})^2$ is an operator monotone function on $[0,\infty)$. Then*

$$s_j(f(A)[A^2 + B^2]f(B)) \leqslant s_j(A^2 f(A)^2 + B^2 f(B)^2) \tag{i}$$

$$s_j(A[f(A)^2 + f(B)^2]B) \leqslant s_j(A^2 f(A)^2 + B^2 f(B)^2) \tag{ii}$$

for $j = 1,\ldots,n$.

By Theorem 2.2 we have the generalized Heinz mean inequality for singular values, which is a generalization of Audenaert-Zhan inequality (1.3):

THEOREM 2.4. *Let $A,B \in \mathbb{M}_n$ be positive definite and $r,s \in \mathbb{R}$ such that $rs \geqslant 0$. Then*

$$s_j(A^{\frac{r}{2}}(A^s+B^s)B^{\frac{r}{2}}) \leqslant \frac{1}{2}\lambda_j((A^r+B^r)(A^s+B^s)) \leqslant s_j(A^{r+s}+B^{r+s}) \tag{2.3}$$

for $j=1,\ldots,n$.

Proof. Put $f(t)=t^r$ and $g(t)=t^s$ in Theorem 2.2. Then $h_1(t)=t^{r/s}$ is operator monotone if and only if $0 \leqslant r \leqslant s$ or $0 \geqslant r \geqslant s$, and $h_2(t)=t^{s/r}$ is operator monotone if and only if $0 \leqslant s \leqslant r$ or $0 \geqslant s \geqslant r$. Hence the case of $rs \geqslant 0$ implies (2.3) by Theorem 2.2. □

If we put $s=\frac{1}{2}-r$ in Theorem 2.4, then we have the Audenaert-Zhan inequality for singular values (1.3):

COROLLARY 2.5. *Let $A,B \in \mathbb{M}_n$ be positive semidefinite. Then for $0 \leqslant r \leqslant 1$*

$$s_j(A^rB^{1-r}+A^{1-r}B^r) \leqslant \frac{1}{2}\lambda_j((A^{2r_0}+B^{2r_0})(A^{1-2r_0}+B^{1-2r_0})) \leqslant s_j(A+B)$$

for $j=1,\ldots,n$, where $r_0=\min\{r,1-r\}$.

Proof. It suffices to prove it for $0 \leqslant r \leqslant \frac{1}{2}$. If we put $s=\frac{1}{2}-r$ in Theorem 2.4, then the condition $r(\frac{1}{2}-r) \geqslant 0$ implies $0 \leqslant r \leqslant \frac{1}{2}$ and Corollary 2.5 follows from Theorem 2.4. □

REMARK 2.6. If we put $r=\frac{1}{4}$ in Corollary 2.5 and replace A and B by A^2 and B^2 respectively, then we have the result (1.2) due to Bhatia-Kittaneh.

REMARK 2.7. For $r=\frac{1}{2}$ we can obtain the following equality for singular values:

$$s_j(A+B)=\frac{1}{2}s_j^2\begin{pmatrix} A^{\frac{1}{2}} & A^{\frac{1}{2}} \\ B^{\frac{1}{2}} & B^{\frac{1}{2}} \end{pmatrix} \quad \text{for } j=1,2,\cdots,n.$$

Note that 2×2 matrices $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ are unitarily similar, then take the Kronecker product with $A+B$, we have $\begin{pmatrix} A+B & A+B \\ A+B & A+B \end{pmatrix}$ and $\begin{pmatrix} 2(A+B) & 0 \\ 0 & 0 \end{pmatrix}$ are unitarily similar. And also

$$\begin{pmatrix} A^{\frac{1}{2}} & A^{\frac{1}{2}} \\ B^{\frac{1}{2}} & B^{\frac{1}{2}} \end{pmatrix}^* \begin{pmatrix} A^{\frac{1}{2}} & A^{\frac{1}{2}} \\ B^{\frac{1}{2}} & B^{\frac{1}{2}} \end{pmatrix} = \begin{pmatrix} A+B & A+B \\ A+B & A+B \end{pmatrix}.$$

If we put $s=1-r$ in Theorem 2.4, then we generalize Zhan's conjecture (1.4) for the case of negative t:

COROLLARY 2.8. *Let $A,B \in \mathbb{M}_n$ be positive semidefinite. Then for $0 \leqslant r \leqslant 2$ and $-2 < t \leqslant 0$*

$$\begin{aligned} s_j(A^rB^{2-r}+A^{2-r}B^r) &\leqslant \frac{1}{2}\lambda_j((A^{2r_1}+B^{2r_1})(A^{2-2r_1}+B^{2-2r_1})) \\ &\leqslant \frac{2}{2+t}s_j(A^2+tAB+B^2) \end{aligned}$$

for $j=1,\ldots,n$, where $r_1=\min\{r,2-r\}$.

Proof. If we put $s=1-r$ in Theorem 2.4, then the condition $r(1-r)\geqslant 0$ implies $0\leqslant r\leqslant 1$, and by Theorem 2.4 we have

$$s_j(A^rB^{2-r}+A^{2-r}B^r) \leqslant \frac{1}{2}\lambda_j((A^{2r}+B^{2r})(A^{2(1-r)}+B^{2(1-r)}) \leqslant s_j(A^2+B^2).$$

For the case of $1\leqslant r\leqslant 2$, since $0\leqslant 2-r\leqslant 1$, we have

$$\begin{aligned} s_j(A^rB^{2-r}+A^{2-r}B^r) &= s_j(A^{2-(2-r)}B^{2-r}+A^{2-r}B^{2-(2-r)}) \\ &\leqslant \frac{1}{2}\lambda_j((A^{2(2-r)}+B^{2(2-r)})(A^{2r-2}+B^{2r-2})) \\ &\leqslant s_j(A^2+B^2) \end{aligned}$$

for $j=1,\ldots,n$, and we have the first inequality of Corollary 2.8.

The second inequality follows from non-increase of $f(t)=\frac{2}{2+t}\lambda_j(A^2+B^2+\frac{t}{2}AB+\frac{t}{2}BA)$ and for $-2<t\leqslant 0$

$$s_j(A^2+B^2+\frac{t}{2}AB+\frac{t}{2}BA)=\lambda_j(A^2+B^2+\frac{t}{2}AB+\frac{t}{2}BA)\leqslant s_j(A^2+tAB+B^2),$$

also see [4, Theorem 4.1] □

3. Generalization of Bhatia-Kittaneh inequality

In [5], Tao proved the following generalization of Bhatia-Kittaneh inequality (1.2): If A and B are positive semidefinite and m is a positive integer, then

$$2s_j(A^{\frac{1}{2}}(A+B)^{m-1}B^{\frac{1}{2}})\leqslant s_j((A+B)^m) \quad \text{for } j=1,\ldots,n. \tag{3.1}$$

In this section, based on Tao's technique, we show a variant of Tao' inequality (3.1):

THEOREM 3.1. *Let $A,B\in\mathbb{M}_n$ be positive definite and $r,s\in\mathbb{R}$. Then for $j=1,2,\cdots,n$,*

$$\begin{aligned} &2s_j(A^r(A^{r+s}+B^{r+s})^{m-1}(A^{2s}+B^{2s})(A^{r+s}+B^{r+s})^{m-1}B^r) \\ &\leqslant \lambda_j((A^{r+s}+B^{r+s})^{m-1}(A^{2s}+B^{2s})(A^{r+s}+B^{r+s})^{m-1}(A^{2r}+B^{2r})),\ \ m=1,2,\cdots \end{aligned}$$

Proof. Let us define the matrices $T = \begin{pmatrix} A^r \\ B^r \end{pmatrix}$ and $S = \begin{pmatrix} A^s & B^s \end{pmatrix}$, then

$$TS = \begin{pmatrix} A^{r+s} & A^r B^s \\ B^r A^s & B \end{pmatrix} \quad \text{and} \quad (ST)^m = (A^{r+s} + B^{r+s})^m.$$

Note that

$$(TS)^m = T(ST)^{m-1}S = \begin{pmatrix} A^r(A^{r+s}+B^{r+s})^{m-1}A^s & A^r(A^{r+s}+B^{r+s})^{m-1}B^s \\ B^r(A^{r+s}+B^{r+s})^{m-1}A^s & B^r(A^{r+s}+B^{r+s})^{m-1}B^s \end{pmatrix}.$$

Hence we have

$$(TS)^m((TS)^m)^* = \begin{pmatrix} Y & P \\ P^* & Z \end{pmatrix} \geqslant 0,$$

where $P = A^r(A^{r+s}+B^{r+s})^{m-1}(A^{2s}+B^{2s})(A^{r+s}+B^{r+s})^{m-1}B^r$, the exact forms of Y and Z is not needed. Put $X = (A^{r+s}+B^{r+s})^{m-1}(A^{2r}+B^{2r})(A^{r+s}+B^{r+s})^{m-1}$. Then Theorem B and Lemma 2.1 imply that for $m = 1, 2, \cdots$

$$\begin{aligned} & 2s_j(A^r(A^{r+s}+B^{r+s})^{m-1}(A^{2s}+B^{2s})(A^{r+s}+B^{r+s})^{m-1}B^r) \\ \leqslant\ & s_j((TS)^m((TS)^m)^*) = s_j(((TS)^m)^*(TS)^m) \\ =\ & s_j(S^*XS) = \lambda_j(XSS^*) \\ =\ & \lambda_j((A^{r+s}+B^{r+s})^{m-1}(A^{2r}+B^{2r})(A^{r+s}+B^{r+s})^{m-1}(A^{2s}+B^{2s})) \end{aligned}$$

for $j = 1, 2, \cdots, n$. □

REMARK 3.2. (i) If we put $m = 1$ in Theorem 3.1 and replace A and B by $A^{1/2}$ and $B^{1/2}$ respectively, then we get the first inequality in Theorem 2.4 for all $r, s \in \mathbb{R}$.

(ii) If we put $r = s = \frac{1}{2}$ in Theorem 3.1, then we have $T^* = S$ and this implies Tao' inequality (3.1) because $(TT^*)^m$ is positive semidefinite.

(iii) If we moreover put $r = s = \frac{1}{2}$ and $m = 1$ in Theorem 3.1, then we have Bhatia-Kittaneh inequality (1.2).

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