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NOTE ON SOME OPERATOR EQUATIONS AND LOCAL SPECTRAL PROPERTIES

IL JU AN AND EUNGIL KO

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Abstract. In this paper we define $\mathscr{S}_{k,j}$ by the set of solutions (A,B) of the operator equations $A^kB^{j+1}A^k = A^{2k+j}$ and $B^kA^{j+1}B^k = B^{2k+j}$. Then we observe the set $\mathscr{S}_{k,j}$ is increasing for all integers $k \geqslant 1$ and $j \geqslant 0$.

Now let a pair $(A,B) \in \mathscr{S}_{k,j} \cap \mathscr{S}_{j+1,k-1}$ for any integer $k \geqslant 1$ and $j \geqslant 0$. We show that if any one of the operators A, AB, BA, and B has Bishop's property (β), then all others have the same property. Furthermore, we prove that the operators A^{k+j}, A^kB^{j+1}, $A^{j+1}B^k$, $B^{j+1}A^k$, B^kA^{j+1} and B^{k+j} have the same spectra and spectral properties. Finally, we investigate their Weyl type theorems.

1. Introduction

Let $\mathscr{X}$ and $\mathscr{Y}$ be infinite dimensional Banach spaces and let $B(\mathscr{X},\mathscr{Y})$ denote the algebra of bounded linear operators from $\mathscr{X}$ to $\mathscr{Y}$, and abbreviate $B(\mathscr{X},\mathscr{X})$ to $B(\mathscr{X})$. Let $K(\mathscr{X})$ be the ideal of all compact operators in $B(\mathscr{X})$. If $T \in B(\mathscr{X})$, we shall write $N(T)$ and $R(T)$ for the null space and range of T. Also, let $\alpha(T) := \dim N(T)$, $\beta(T) := \dim N(T^*)$, and let $\sigma(T)$, $\sigma_p(T)$, $\sigma_a(T)$, $\sigma_r(T)$, $\sigma_c(T)$, $p_0(T)$, and $\pi_0(T)$ denote the spectrum, the point spectrum, the approximate point spectrum, the residual spectrum, and the continuous spectrum of T, respectively. For $T \in B(\mathscr{X})$, the smallest nonnegative integer p such that $N(T^p) = N(T^{p+1})$ is called the *ascent* of T and denoted by $p(T)$. If no such integer exists, we set $p(T) = \infty$. The smallest nonnegative integer q such that $R(T^q) = R(T^{q+1})$ is called the *descent* of T and denoted by $q(T)$. If no such integer exists, we set $q(T) = \infty$.

The famous "reversal of product" for inverse says that if $A \in B(\mathscr{X},\mathscr{Y})$ and $C \in B(\mathscr{Y},\mathscr{Z})$ are invertible, then so is $CA \in B(\mathscr{X},\mathscr{Z})$, with $(CA)^{-1} = A^{-1}C^{-1}$. In general, the sum of two invertible operators need not be invertible. However it is well known that if A and C are in $B(\mathscr{X})$, then

$$I - CA \text{ is invertible} \iff I - AC \text{ is invertible.}$$

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More generally, we consider that for any integers $k \geqslant 1$ and $j \geqslant 0$,

$$I - C^{j+1}A^k \text{ is invertible} \iff I - A^kC^{j+1} \text{ is invertible.}$$

Indeed, assume that $I - C^{j+1}A^k$ is surjective, that is, $(I - C^{j+1}A^k)\mathscr{X} = \mathscr{X}$. For arbitrary $y \in \mathscr{X}$, there exists $x \in \mathscr{X}$ such that $C^{j+1}y = (I - C^{j+1}A^k)x$. Thus $A^kC^{j+1}y = A^k(I - C^{j+1}A^k)x = (I - A^kC^{j+1})A^kx$, so that

$$\begin{aligned} y &= A^kC^{j+1}y + (I - A^kC^{j+1})y = (I - A^kC^{j+1})A^kx + (I - A^kC^{j+1})y \\ &= (I - A^kC^{j+1})(A^kx + y) \in (I - A^kC^{j+1})\mathscr{X}. \end{aligned}$$

Hence $(I - A^kC^{j+1})\mathscr{X} = \mathscr{X}$. Conversely, we have the similar method. It follows that for any integers $k \geqslant 1$ and $j \geqslant 0$,

$$I - C^{j+1}A^k \text{ is surjective} \iff I - A^kC^{j+1} \text{ is surjective.}$$

We now assume that $I - C^{j+1}A^k$ is injective. If $(I - A^kC^{j+1})x = 0$, then $x = A^kC^{j+1}x$. Hence $C^{j+1}x = C^{j+1}A^kC^{j+1}x$, so that $(I - C^{j+1}A^k)C^{j+1}x = 0$. Since $I - C^{j+1}A^k$ is injective, we have that $C^{j+1}x = 0$, which implies that $A^kC^{j+1}x = 0$. Since $(I - A^kC^{j+1})x = 0$, we get that $x = 0$. Thus $I - A^kC^{j+1}$ is injective. Conversely, we have the similar method. Therefore this means that for any integers $k \geqslant 1$ and $j \geqslant 0$,

$$I - C^{j+1}A^k \text{ is injective} \iff I - A^kC^{j+1} \text{ is injective.}$$

As mentioned in [2] we replace from $I - A^kC^{j+1}$ to certain $I - A^kB^{j+1}$ and specifically we will suppose that $A^kB^{j+1}A^k = A^kC^{j+1}A^k$. The special case is of interest to us, the case $A^kB^{j+1}A^k = A^{2k+j}$, in which $C^{j+1} = A^j$ for any integer $j \geqslant 0$.

Now we let a pair (A, B) be the solution of the operator equations

$$A^kB^{j+1}A^k = A^{2k+j} \text{ and } B^kA^{j+1}B^k = B^{2k+j}. \tag{1.1}$$

In particular, when $k = 1$ and $j = 0$, the operators A and B are solutions of the system of operator equations

$$ABA = A^2 \text{ and } BAB = B^2. \tag{1.2}$$

This means that if a pair (A, B) of Banach space operators is the solution of the operator equations (1.2), then so is this of the operator equations (1.1). In [10], I. Vidav proved that A and B are self-adjoint operators satisfying the operator equations (1.2) if and only if $A = PP^*$ and $B = P^*P$ for some idempotent operator P. Also, the common spectral properties of the operators A and B satisfying the operator equations (1.2) have been studied by C. Schmoeger [9]. In particular, it is possible to relate the various spectra, the single-valued extension property and Bishop's property (β) of A and B, which has been carried out by [5]. So we extend the previous results for the operator equations (1.2) to those for the operator equations (1.1). We start our program with the following section.

2. Preliminaries

An operator $T \in B(\mathscr{X})$ is called *upper semi-Fredholm* if it has closed range and finite dimensional null space and is called *lower semi-Fredholm* if it has closed range and its range has finite co-dimension. If $T \in B(\mathscr{X})$ is either upper or lower semi-Fredholm, then T is called *semi-Fredholm*, and *index of a semi-Fredholm operator* $T \in B(\mathscr{X})$ is defined by

$$i(T) := \alpha(T) - \beta(T).$$

If both $\alpha(T)$ and $\beta(T)$ are finite, then T is called *Fredholm*. $T \in B(\mathscr{X})$ is called *Weyl* if it is Fredholm of index zero. The essential spectrum $\sigma_e(T)$, the Weyl spectrum $\sigma_w(T)$ and the Browder spectrum $\sigma_b(T)$ of $T \in B(\mathscr{X})$ are defined as follows.

$$\sigma_e(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Fredholm}\},$$

$$\sigma_w(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\},$$

and

$$\sigma_b(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Browder}\},$$

respectively. Evidently

$$\sigma_e(T) \subseteq \sigma_w(T) \subseteq \sigma_b(T) = \sigma_e(T) \cup \operatorname{acc} \sigma(T),$$

where we write $\operatorname{acc} K$ for the accumulation points of $K \subseteq \mathbb{C}$.

By definition,

$$\sigma_{ea}(T) := \cap\{\sigma_a(T+K) : K \in K(\mathscr{X})\}$$

is the *essential approximate point spectrum*,

$$\sigma_{ab}(T) := \cap\{\sigma_a(T+K) : TK = KT \text{ and } K \in K(\mathscr{X})\}$$

is the *Browder essential approximate point spectrum*.

If we write $\operatorname{iso} K = K \setminus \operatorname{acc} K$, then we let

$$\pi_{00}(T) := \{\lambda \in \operatorname{iso} \sigma(T) : 0 < \alpha(T - \lambda) < \infty\},$$

$$\pi_{00}^a(T) := \{\lambda \in \operatorname{iso} \sigma_a(T) : 0 < \alpha(T - \lambda) < \infty\},$$

$$p_{00}(T) := \sigma(T) \setminus \sigma_b(T),$$

and

$$p_{00}^a(T) := \sigma_a(T) \setminus \sigma_{ab}(T).$$

We say that *Weyl's theorem holds for* $T \in B(\mathscr{X})$ if there is equality

$$\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T),$$

that *Browder's theorem holds for* $T \in B(\mathscr{X})$ if there is equality

$$\sigma(T) \setminus \sigma_w(T) = p_{00}(T),$$

that *a-Weyl's theorem holds for $T \in B(\mathscr{X})$* if there is equality

$$\sigma_a(T) \setminus \sigma_{ea}(T) = \pi_{00}^a(T),$$

and that *a-Browder's theorem holds for $T \in B(\mathscr{X})$* if there is equality

$$\sigma_a(T) \setminus \sigma_{ea}(T) = p_{00}^a(T).$$

It is known [3, 4, 6] that we have

$$a\text{-Weyl's theorem} \Rightarrow \text{Weyl's theorem} \Rightarrow \text{Browder's theorem};$$

$$a\text{-Weyl's theorem} \Rightarrow a\text{-Browder's theorem} \Rightarrow \text{Browder's theorem}.$$

Let $\mathscr{A}$ be a unital algebra. We say that an element $x \in \mathscr{A}$ is *Drazin invertible of degree k* if there exists an element $a \in \mathscr{A}$ such that

$$x^k ax = x^k, \ axa = a, \ \text{and} \ xa = ax.$$

Let $a \in \mathscr{A}$. Then the *Drazin spectrum* is defined by

$$\sigma_D(a) := \{\lambda \in \mathbb{C} : a - \lambda \text{ is not Drazin invertible}\}.$$

It is well known that T is Drazin invertible if and only if it has finite ascent and descent, which is also equivalent to the fact that

$$T = T_1 \oplus T_2, \text{ where } T_1 \text{ is invertible and } T_2 \text{ is nilpotent.}$$

In terms of local spectral theory ([1], [8]) recall the following definitions.

DEFINITION 2.1. Let $T \in B(\mathscr{X})$.

(1) An operator T has Bishop's property (β) if for every open subset U of $\mathbb{C}$ and every sequence of analytic functions $f_n : U \to \mathscr{X}$ with the property that $(T - \lambda)f_n(\lambda) \to 0$ as $n \to \infty$, uniformly on all compact subsets of U, it follows that $f_n(\lambda) \to 0$ as $n \to \infty$, uniformly on all compact subsets of U.

(2) An operator T has the single valued extension property at $\lambda_0 \in \mathbb{C}$, abbreviated T has SVEP at λ_0 if for every open neighborhood U of λ_0 the only analytic function $f : U \longrightarrow \mathscr{X}$ which satisfies the equation

$$(T - \lambda)f(\lambda) = 0$$

is the constant function $f \equiv 0$ on U. The operator T is said to have SVEP if T has SVEP at every $\lambda \in \mathbb{C}$.

In general, the following implications hold:

$$\text{Bishop's property } (\beta) \Longrightarrow \text{SVEP}.$$

Evidently, every operator T, as well as its dual T^*, has SVEP at every point of the boundary $\partial\sigma(T)$ of the spectrum $\sigma(T)$, in particular, at every isolated point of $\sigma(T)$. We also have (see [1, Theorem 3.8])

$$p(T-\lambda) < \infty \implies T \text{ has SVEP at } \lambda, \tag{2.1}$$

and dually

$$q(T-\lambda) < \infty \implies T^* \text{ has SVEP at } \lambda. \tag{2.2}$$

It is well known from [1] that if $T-\lambda$ is semi-Fredholm, then the implications (2.1) and (2.2) are equivalent.

3. Local spectral properties and some operator equations

Throughout this paper we define $\mathscr{S}_{k,j}$ by the set of solutions (A,B) of the operator equations

$$A^kB^{j+1}A^k = A^{2k+j} \ \textit{and} \ B^kA^{j+1}B^k = B^{2k+j}$$

for all integers $k \geqslant 1$ and $j \geqslant 0$. In particular, if a pair $(A,B) \in \mathscr{S}_{1,0}$, then $ABA = A^2$ and $BAB = B^2$. Then the following inclusions are satisfied.

PROPOSITION 3.1.
(1) $\mathscr{S}_{1,0} \subset \mathscr{S}_{k,j}$ for all integers $k \geqslant 1$ and a fixed integer $j \geqslant 0$.
(2) $\mathscr{S}_{k,j} \subset \mathscr{S}_{k+1,j}$ for every $k \in \mathbb{N}$ and all integer $j \geqslant 0$.

Proof. (1) Suppose that $(A,B) \in \mathscr{S}_{1,0}$. Then $A^nB = AB^n$ and $B^nA = BA^n$ for $n \geqslant 2$. Then for all integer $k, j \geqslant 1$,

$$A^kB^{j+1}A^k = A^{k-1}AB^{j+1}A^k = A^{k-1}A^{j+1}BA^k = A^{k+j-1}ABAA^{k-1} = A^{2k+j},$$

and

$$B^kA^{j+1}B^k = B^{k-1}BA^{j+1}B^k = B^{k-1}B^{j+1}AB^k = B^{k+j-1}BABB^{k-1} = B^{2k+j}.$$

Hence $(A,B) \in \mathscr{S}_{k,j}$, so that this inclusion is satisfied.

(2) Suppose that $(A,B) \in \mathscr{S}_{k,j}$ for every $k \in \mathbb{N}$ and all integer $j \geqslant 0$. We first fix $j \geqslant 0$. Then $A^kB^{j+1}A^k = A^{2k+j}$ and $B^kA^{j+1}B^k = B^{2k+j}$. Thus we have that

$$A^{k+1}B^{j+1}A^{k+1} = AA^kB^{j+1}A^kA = A^{2(k+1)+j},$$

and

$$B^{k+1}A^{j+1}B^{k+1} = BB^kA^{j+1}B^kB = B^{2(k+1)+j}.$$

Thus $(A,B) \in \mathscr{S}_{k+1,j}$ for every $k \in \mathbb{N}$ and a fixed integer $j \geqslant 0$. $\square$

From Proposition 3.1, it is obvious that the set $\mathscr{S}_{k,j}$ is nonempty. Moreover, we observe the following remark.

REMARK 3.2. Set $\mathscr{S}_{\infty,j} := \bigcup_{k=1}^{\infty} \mathscr{S}_{k,j}$ for a fixed integer $j \geqslant 0$. Then

$$\mathscr{S}_{1,j} \subset \mathscr{S}_{2,j} \subset \cdots \subset \mathscr{S}_{k,j} \subset \mathscr{S}_{k+1,j} \subset \cdots \subset \mathscr{S}_{\infty,j}.$$

However, the following example says that $\mathscr{S}_{1,j} \neq \mathscr{S}_{1,j+1}$ holds for integers $j \geqslant 0$.

EXAMPLE 3.3. If $A = \begin{pmatrix} \omega I & 0 \\ 0 & \overline{\omega} I \end{pmatrix}$ and $B = \begin{pmatrix} \overline{\omega} I & 0 \\ 0 & \omega I \end{pmatrix}$ are in $B(\mathscr{X} \oplus \mathscr{X})$, where $\omega^{2n+1} = 1$ and $\omega \in \mathbb{C} \setminus \{1\}$, then by the induction,

$$(A,B) \in \mathscr{S}_{1,n} \text{ for integers } n \geqslant 1.$$

But, $(A,B) \notin \mathscr{S}_{1,n-1} \cup \mathscr{S}_{1,n+1}$ for integers $n \geqslant 1$. In fact, assume that there exists some integer $m \geqslant 1$ such that $(A,B) \in \mathscr{S}_{1,m-1} \cup \mathscr{S}_{1,m+1}$. If $(A,B) \in \mathscr{S}_{1,m-1}$, then $AB^mA \neq A^{m+1}$ and $BA^mB \neq B^{m+1}$ since $\omega^{2m+1} = 1$. If $(A,B) \in \mathscr{S}_{1,m+1}$, then $AB^{m+2}A \neq A^{m+3}$ and $BA^{m+2}B \neq B^{m+3}$ since $\omega^{2m+1} = 1$. Hence we have a contradiction. Thus $(A,B) \in \mathscr{S}_{1,n-1} \cup \mathscr{S}_{1,n+1}$. Moreover, it follows that $\mathscr{S}_{1,j} \not\subset \mathscr{S}_{1,j+1}$ holds for all integers $j \geqslant 1$.

EXAMPLE 3.4. In general, if $(A,B) \in \mathscr{S}_{1,0}$, then it follows that the equality $A^nB = AB^n$ holds for $n \geqslant 2$. However, this equality is not satisfied when $(A,B) \in \mathscr{S}_{k,j}$, for any nonnegative integers k and j. Let's consider the operator matrices A and B defined in Example 3.3. Then a pair (A,B) is in $\mathscr{S}_{k,j}$ for $k = 1$ and any integer $j \geqslant 1$. By the straightforward calculation, we have that $A^nB \neq AB^n$ for $n \geqslant 2$.

From Example 3.4 we can guess the following proposition.

PROPOSITION 3.5. *If $(A,B) \in \mathscr{S}_{n+1,n}$ for any integer $n \geqslant 0$, then for $k \geqslant n+1$*

$$A^{k+2n+1}B^k = A^kB^{k+2n+1} \text{ and } B^{k+2n+1}A^k = B^kA^{k+2n+1}.$$

Proof. Suppose that $(A,B) \in \mathscr{S}_{n+1,n}$ for any integer $n \geqslant 0$. Then $A^{n+1}B^{n+1}A^{n+1} = A^{3n+2}$ and $B^{n+1}A^{n+1}B^{n+1} = B^{3n+2}$. We show that for every integer $k \geqslant n+1$, $A^{k+2n+1}B^k = A^kB^{k+2n+1}$ and $B^{k+2n+1}A^k = B^kA^{k+2n+1}$. If $k = n+l$ for $l \geqslant 1$, $A^{k+2n+1}B^k = A^{l-1}A^{3n+2}B^{n+l} = A^{l-1}A^{n+1}B^{n+1}A^{n+1}B^{n+1}B^{l-1} = A^{n+l}B^{3n+l+1} = A^kB^{k+2n+1}$, and similarly, $B^{k+2n+1}A^k = B^kA^{k+2n+1}$. This proof is complete. $\square$

Now we show that the operators A, AB, BA and B have the property (β) in common. For this we need the following lemma.

LEMMA 3.6. *Let $(A,B) \in \mathscr{S}_{k,j}$ for any integer $k \geqslant 1$ and $j \geqslant 0$. Then the followings are satisfied.*

(1) A^{k+j} has property (β) if and only if A^kB^{j+1} has property (β).

(2) B^{k+j} has property (β) if and only if B^kA^{j+1} has property (β).

Proof. (1) Suppose that A^kB^{j+1} has property (β) at $\mu \in \mathbb{C}$. Let $\mathscr{U}$ be an open neighborhood of μ and $f_n : \mathscr{U} \to \mathscr{X}$ be a sequence of analytic functions such that $(A^{k+j} - \lambda)f_n(\lambda) \to 0$ in $\mathscr{U}$.

Then $A^kB^{j+1}(A^{k+j} - \lambda)f_n(\lambda) \to 0$ and $(A^{2k+2j} - \lambda A^kB^{j+1})f_n(\lambda) \to 0$. Since $A^{2k+2j}f_n(\lambda) = \lambda^2 f_n(\lambda)$ in $\mathscr{U}$, we have that $(A^kB^{j+1} - \lambda)(-\lambda f_n(\lambda)) \to 0$. But, A^kB^{j+1} has property (β) at $\mu \in \mathbb{C}$, hence $-\lambda f_n(\lambda) \to 0$ so that $f_n(\lambda) \to 0$ for all λ in $\mathscr{U}$. Thus A^{k+j} has property (β) at μ.

Conversely, assume that A^{k+j} has property (β) at $\mu \in \mathbb{C}$. Let $g_n : \mathscr{U} \to \mathscr{X}$ be a sequence of analytic functions such that $(A^kB^{j+1} - \lambda)g_n(\lambda) \to 0$ in $\mathscr{U}$. Then $A^kB^{j+1}(A^kB^{j+1} - \lambda)g_n(\lambda) \to 0$, so that $(A^{2k+j}B^{j+1} - \lambda A^kB^{j+1})g_n(\lambda) \to 0$. Hence we have $(A^{k+j} - \lambda)(A^kB^{j+1}g_n(\lambda)) \to 0$. Since A^{k+j} has property (β) at μ, $A^kB^{j+1}g_n(\lambda) \to 0$. But, $(A^kB^{j+1} - \lambda)g_n(\lambda) \to 0$ in $\mathscr{U}$, thus $g_n(\lambda) \to 0$ for all λ in $\mathscr{U}$. So A^kB^{j+1} has property (β) at μ. Since μ is arbitrary in $\mathbb{C}$, this is complete.

(2) The proof is obvious by the similar process as above. □

As some applications of Lemma 3.6, we get the following theorem.

THEOREM 3.7. *Let $(A,B) \in \mathscr{S}_{k,j} \cap \mathscr{S}_{j+1,k-1}$ for any integer $k \geqslant 1$ and $j \geqslant 0$. Then the following statements are equivalent.*

(1) A has property (β).

(2) AB has property (β).

(3) BA has property (β).

(4) B has property (β).

Proof. Suppose that A has property (β). Then it follows from [1, Theorem 2.40] that A^{k+j} has property (β). So we first show that B^kA^{j+1} has property (β). Let $f_n : \mathscr{U} \to \mathscr{X}$ be a sequence of analytic functions for every open neighborhood $\mathscr{U}$ of $\lambda_0 \in \mathbb{C}$. Suppose that $(B^kA^{j+1} - \lambda)f_n(\lambda) \to 0$. Then $A^{j+1}(B^kA^{j+1} - \lambda)f_n(\lambda) \to 0$. Since $(A,B) \in \mathscr{S}_{j+1,k-1}$, we have $(A^{k+j} - \lambda)A^{j+1}f_n(\lambda) \to 0$. And then $A^{j+1}f_n(\lambda) \to 0$. But, $B^kA^{j+1}f_n(\lambda) = \lambda f_n(\lambda)$, hence $\lambda f_n(\lambda) \to 0$. Since λ is arbitrary in $\mathscr{U}$, we have $f_n(\lambda) \to 0$. Thus B^kA^{j+1} has property (β). So it follows from Lemma 3.6 that B^{k+j} has property (β). Therefore B has property (β).

Conversely, suppose that B has property (β). Then B^{k+j} has property (β). So we only need to prove that A^kB^{j+1} has property (β) by Lemma 3.6. Assume that $g : \mathscr{W} \to \mathscr{X}$ is the analytic function for every open neighborhood $\mathscr{W}$ of $\mu_0 \in \mathbb{C}$ and $(A^kB^{j+1} - \mu)f_n(\mu) \to 0$. Then $B^{j+1}(A^kB^{j+1} - \mu)f_n(\mu) \to 0$. Since $(A,B) \in \mathscr{S}_{j+1,k-1}$, we have $(B^{k+j} - \mu)B^{j+1}f_n(\mu) \to 0$. And then $B^{j+1}f_n(\lambda) \to 0$. But, $(A^kB^{j+1} - \mu)f_n(\mu) \to 0$, hence $\mu f_n(\mu) \to 0$. Since μ is arbitrary in $\mathscr{W}$, we have $f_n(\mu) \to 0$. Thus A^kB^{j+1} has property (β), so that A^{k+j} has. Consequently, A has property (β). □

REMARK 3.8. Similarly, Theorem 3.7 holds for the single-valued extension property. This means that if one of A, AB, BA, or B has SVEP whenever $(A,B) \in \mathscr{S}_{k,j} \cap \mathscr{S}_{j+1,k-1}$ for any integer $k \geqslant 1$ and $j \geqslant 0$, then all of the operators A, AB, BA, and B have SVEP.

If $(A,B) \in \mathscr{S}_{1,0}$, then the operators A, AB, BA, and B have the spectrum, point spectrum, approximate point spectrum, residual spectrum, essential spectrum, and Weyl spectrum in common (see, [5] and [9]). So we extend these to the operators A^{k+j}, A^kB^{j+1}, $A^{j+1}B^k$, $B^{j+1}A^k$, B^kA^{j+1}, and B^{k+j} when $(A,B) \in \mathscr{S}_{k,j} \cap \mathscr{S}_{j+1,k-1}$ for any integer $k \geqslant 1$ and $j \geqslant 0$. We shall require the following results.

LEMMA 3.9. *Let $(A,B) \in \mathscr{S}_{k,j}$ for any integer $k,j \geqslant 0$. Then the followings are satisfied for $\lambda \neq 0$.*
$N(A^{k+j}-\lambda I) = N(A^kB^{j+1}-\lambda I)$ and $N(B^{k+j}-\lambda I) = N(B^kA^{j+1}-\lambda I)$.

Proof. (1) Let $x \in N(A^{k+j}-\lambda I)$. Then $A^{k+j}x = \lambda x$, so that $A^kB^{j+1}A^{k+j}x = \lambda A^kB^{j+1}x$. Hence $A^{2k+2j}x = \lambda A^kB^{j+1}x$ and $\lambda A^kB^{j+1}x = A^{2k+2j}x = \lambda^2 x$. Since $\lambda \neq 0$, $A^kB^{j+1}x = \lambda x$. So $x \in N(A^kB^{j+1}-\lambda I)$. Thus $N(A^{k+j}-\lambda I) \subseteq N(A^kB^{j+1}-\lambda I)$.

Conversely, suppose that $x \in N(A^kB^{j+1}-\lambda I)$. Then $A^kB^{j+1}x = \lambda x$, so that $A^kB^{j+1}A^kB^{j+1}x = \lambda A^kB^{j+1}x$ and then $A^{k+j}A^kB^{j+1}x = \lambda A^kB^{j+1}x$. Thus $\lambda A^{k+j}x = \lambda^2 x$, which implies that $A^{k+j}x = \lambda x$. Hence $x \in N(A^{k+j}-\lambda I)$. Consequently, The first equality holds. From similar argument, the second statement can be proved. □

LEMMA 3.10. *Let $(A,B) \in \mathscr{S}_{k,j} \cap \mathscr{S}_{j+1,k-1}$ for any integer $k \geqslant 1$ and $j \geqslant 0$. Then the following properties hold for $\lambda \neq 0$.*
(1) $N(A^{k+j}-\lambda I) = N(A^kB^{j+1}-\lambda I) = N(A^{j+1}B^k-\lambda I)$.
(2) $N(B^{k+j}-\lambda I) = N(B^kA^{j+1}-\lambda I) = N(B^{j+1}A^k-\lambda I)$.

Proof. From Lemma 3.9 we only need to show that $N(A^{k+j}-\lambda I) = N(A^{j+1}B^k-\lambda I)$. Let $x \in N(A^{k+j}-\lambda I)$. Then $A^{k+j}x = \lambda x$, so that $\lambda^2 x = A^{2k+2j}x = A^{j+1}B^kA^{k+j}x = \lambda A^{j+1}B^kx$. Since $\lambda \neq 0$, we have that $A^{j+1}B^kx = \lambda x$. So $x \in N(A^{j+1}B^k-\lambda I)$. Thus $N(A^{k+j}-\lambda I) \subseteq N(A^{j+1}B^k-\lambda I)$.

Conversely, let $x \in N(A^{j+1}B^k-\lambda I)$. Then $A^{j+1}B^kx = \lambda x$, so that $\lambda A^{k+j}x = A^{k+2j+1}B^kx = A^{j+1}B^kA^{j+1}B^kx = \lambda^2 x$. Hence $N(A^{j+1}B^k-\lambda I) \subseteq N(A^{k+j}-\lambda I)$, which completes the proof. □

Lemmas 3.9 and 3.10 ensure that the following proposition holds.

PROPOSITION 3.11. *Let $(A,B) \in \mathscr{S}_{k,j} \cap \mathscr{S}_{j+1,k-1}$ for any integer $k \geqslant 1$ and $j \geqslant 0$. Then the followings are equivalent for $\lambda \neq 0$.*
(1) $N(A^{k+j}-\lambda I) = \{0\}$.
(2) $N(A^kB^{j+1}-\lambda I) = \{0\}$.
(3) $N(A^{j+1}B^k-\lambda I) = \{0\}$.
(4) $N(B^{j+1}A^k-\lambda I) = \{0\}$.
(5) $N(B^kA^{j+1}-\lambda I) = \{0\}$.
(6) $N(B^{k+j}-\lambda I) = \{0\}$.

Proof. We only need to show that (1)$\Rightarrow$ (4) and (6)$\Rightarrow$(3) from Lemma 3.10. Let $N(A^{k+j}-\lambda I) = \{0\}$. Assume that $(B^{j+1}A^k-\lambda)y = 0$ for some nonzero $y \in \mathscr{X}$. Then $0 = A^k(B^{j+1}A^k-\lambda)y = (A^{2k+j}-\lambda A^k)y = (A^{k+j}-\lambda)A^ky$. Hence $A^ky \in N(A^{k+j}-$

$\lambda I) = \{0\}$, so that $\lambda y = 0$. Since $\lambda \neq 0$, $y = 0$, but this is a contradiction. Thus $N(B^{j+1}A^k - \lambda I) = \{0\}$, which completes the implication (1)$\Rightarrow$ (4). By similar way, it is shown that (6)$\Rightarrow$(3). $\square$

LEMMA 3.12. *Let $(A,B) \in \mathscr{S}_{k,j} \cap \mathscr{S}_{j+1,k-1}$ for any integer $k \geqslant 1$ and $j \geqslant 0$. Then the followings hold for $\lambda \neq 0$.*

(1) $B^k(N(A^{k+j} - \lambda I)) = N(B^{k+j} - \lambda I)$.

(2) $A^k(N(B^{k+j} - \lambda I)) = N(A^{k+j} - \lambda I)$.

Proof. Let $x \in N(A^{k+j} - \lambda I)$. Then $A^{k+j}x = \lambda x$. So $B^{2k+j}A^{k+j}x = \lambda B^{2k+j}x$, and this implies that $B^kA^{j+1}B^kA^{j+1}A^{k-1}x = \lambda B^{2k+j}x$. Hence $B^kA^{2k+2j}x = \lambda B^{2k+j}x$, so that $\lambda^2 B^k x = \lambda B^{2k+j}x$. Since $\lambda \neq 0$, $B^{2k+j}x = \lambda B^k x$ and then $(B^{k+j} - \lambda I)B^k x = 0$. Thus $B^k x \in N(B^{k+j} - \lambda I)$, which implies that $B^k(N(A^{k+j} - \lambda I)) \subseteq N(B^{k+j} - \lambda I)$.

Conversely, if $x \in N(B^{k+j} - \lambda I)$, then $B^kA^{j+1}x = \lambda x$ by Lemma 3.9. So $A^{k+2j+1}x = A^{j+1}B^kA^{j+1}x = \lambda A^{j+1}x$, hence $(A^{k+j} - \lambda)A^{j+1}x = 0$. Thus $A^{j+1}x \in N(A^{k+j} - \lambda I)$, so that $\lambda x = B^kA^{j+1}x \in B^k(N(A^{k+j} - \lambda I))$. Since $\lambda \neq 0$, we get that $x \in B^k(N(A^{k+j} - \lambda I))$. Consequently, $B^k(N(A^{k+j} - \lambda I)) = N(B^{k+j} - \lambda I)$. By the similar way, it is shown that $A^k(N(B^{k+j} - \lambda I)) \subseteq N(A^{k+j} - \lambda I)$. Hence we complete our proof. $\square$

PROPOSITION 3.13. *Let $(A,B) \in \mathscr{S}_{k,j} \cap \mathscr{S}_{j+1,k-1}$ for any integer $k \geqslant 1$ and $j \geqslant 0$. Then we have the following equalities for $\lambda \neq 0$.*

$$\alpha(A^{k+j} - \lambda I) = \alpha(A^kB^{j+1} - \lambda I) = \alpha(A^{j+1}B^k - \lambda I)$$
$$= \alpha(B^{j+1}A^k - \lambda I) = \alpha(B^kA^{j+1} - \lambda I) = \alpha(B^{k+j} - \lambda I).$$

Proof. We first show that

$$N(A^k) \cap N(B^{k+j} - \lambda I) = \{0\} \text{ and } N(B^k) \cap N(A^{k+j} - \lambda I) = \{0\}. \tag{3.1}$$

Assume that there exists a nonzero $x \in \mathscr{X}$ such that $A^k x = 0$ and $B^{k+j}x = \lambda x$. Then we have that $\lambda^2 x = B^{2k+2j}x = B^{j+1}A^kB^{k+j}x = \lambda B^{j+1}A^k x = 0$. However, x is a nonzero element, so that $\lambda = 0$. This is a contradiction. Thus $N(A^k) \cap N(B^{k+j} - \lambda) = \{0\}$. Similarly, the second equality of (3.1) can be proved, so that the restrictions of A^k to $N(B^{k+j} - \lambda)$ and B^k to $N(A^{k+j} - \lambda)$ are injective. Therefore the proof follows from Lemmas 3.10 and 3.12. $\square$

LEMMA 3.14. *Let $(A,B) \in \mathscr{S}_{k,j}$ for any nonnegative integer k, j. Then we have the following equalities.*

(1) $\sigma_p(A^{k+j}) \setminus \{0\} = \sigma_p(A^kB^{j+1}) \setminus \{0\}$ and $\sigma_p(B^{k+j}) \setminus \{0\} = \sigma_p(B^kA^{j+1}) \setminus \{0\}$.

(2) $\sigma_a(A^{k+j}) \setminus \{0\} = \sigma_a(A^kB^{j+1}) \setminus \{0\}$ and $\sigma_a(B^{k+j}) \setminus \{0\} = \sigma_a(B^kA^{j+1}) \setminus \{0\}$.

Proof. (1) The proof is obvious from Lemma 3.9.

(2) Let $\lambda \in \sigma_a(A^{k+j}) \setminus \{0\}$. Then there exists a sequence $(x_n) \subset \mathscr{X}$ with $\|x_n\| = 1$ for all $n \in \mathbb{N}$ such that $(A^{k+j} - \lambda I)x_n \to 0$ as $n \to \infty$. Let $z_n := (A^{k+j} - \lambda I)x_n$. Then $A^{k+j}x_n = \lambda x_n + z_n$ and $z_n \to 0$ as $n \to \infty$. So

$$A^{2k+2j}x_n = \lambda A^{k+j}x_n + A^{k+j}z_n = \lambda^2 x_n + \lambda z_n + A^{k+j}z_n.$$

But, $A^{2k+2j}x_n = A^kB^{j+1}A^{k+j}x_n = \lambda A^kB^{j+1}x_n + A^kB^{j+1}z_n$. Thus

$$(\lambda A^kB^{j+1} - \lambda^2)x_n = (\lambda I + A^{k+j} - A^kB^{j+1})z_n \longrightarrow 0 \text{ as } n \to \infty. \tag{3.2}$$

For $\lambda \neq 0$, $(A^kB^{j+1} - \lambda)x_n \to 0$ as $n \to \infty$, so that $\lambda \in \sigma_a(A^kB^{j+1}) \setminus \{0\}$.

Conversely, let $\lambda \in \sigma_a(A^kB^{j+1}) \setminus \{0\}$. Then there exists a sequence $(y_n) \subset \mathscr{X}$ with $\|y_n\| = 1$ for all $n \in \mathbb{N}$ such that $(A^kB^{j+1} - \lambda I)y_n \to 0$ as $n \to \infty$. Let $w_n := (A^kB^{j+1} - \lambda I)y_n$. Then $A^kB^{j+1}y_n = \lambda y_n + w_n$ and $w_n \to 0$ as $n \to \infty$. So

$$A^{2k+j}B^{j+1}y_n = \lambda A^kB^{j+1}y_n + A^kB^{j+1}w_n = \lambda^2 y_n + \lambda w_n + A^kB^{j+1}w_n.$$

However, $A^{2k+j}B^{j+1}y_n = A^{k+j}(\lambda y_n + w_n) = \lambda A^{k+j}y_n + A^{k+j}w_n$. Thus

$$(\lambda A^{k+j} - \lambda^2 I)y_n = (\lambda I + A^kB^{j+1} - A^{k+j})w_n \longrightarrow 0 \text{ as } n \to \infty.$$

For $\lambda \neq 0$, $(A^{k+j} - \lambda I)y_n \to 0$ as $n \to \infty$, so that $\lambda \in \sigma_a(A^{k+j}) \setminus \{0\}$. Therefore $\sigma_a(A^{k+j}) \setminus \{0\} = \sigma_a(A^kB^{j+1}) \setminus \{0\}$. The second equality can be proved by similar process. $\square$

THEOREM 3.15. *Let $(A,B) \in \mathscr{S}_{k,j} \cap \mathscr{S}_{j+1,k-1}$ for any integer $k \geqslant 1$ and $j \geqslant 0$. Then the following statements hold.*

(1) $\sigma_p(A^{k+j}) \setminus \{0\} = \sigma_p(A^kB^{j+1}) \setminus \{0\} = \sigma_p(A^{j+1}B^k) \setminus \{0\} = \sigma_p(B^{j+1}A^k) \setminus \{0\} = \sigma_p(B^kA^{j+1}) \setminus \{0\} = \sigma_p(B^{k+j}) \setminus \{0\}$.

(2) $\sigma_a(A^{k+j}) \setminus \{0\} = \sigma_a(A^kB^{j+1}) \setminus \{0\} = \sigma_a(A^{j+1}B^k) \setminus \{0\} = \sigma_a(B^{j+1}A^k) \setminus \{0\} = \sigma_a(B^kA^{j+1}) \setminus \{0\} = \sigma_a(B^{k+j}) \setminus \{0\}$.

Proof. (1) The proof follows from Lemma 3.10.

(2) It is sufficient to show that $\sigma_a(A^{j+1}B^k) \setminus \{0\} = \sigma_a(A^{k+j}) \setminus \{0\} = \sigma_a(B^{k+j}) \setminus \{0\}$ by Lemma 3.14. To show the first equality, we let $\lambda \in \sigma_a(A^{j+1}B^k) \setminus \{0\}$. Then there exists a sequence $(x_n) \subset \mathscr{X}$ with $\|x_n\| = 1$ for all $n \in \mathbb{N}$ such that $(A^{j+1}B^k - \lambda I)x_n \to 0$ as $n \to \infty$. Let $z_n := (A^{j+1}B^k - \lambda I)x_n$. Then $A^{j+1}B^kx_n = \lambda x_n + z_n$ and $z_n \to 0$ as $n \to \infty$. So

$$A^{k+2j+1}B^kx_n = \lambda A^{j+1}B^kx_n + A^{j+1}B^kz_n = \lambda^2 x_n + \lambda z_n + A^{j+1}B^kz_n.$$

However, $A^{k+2j+1}B^kx_n = A^{k+j}(A^{j+1}B^kx_n) = \lambda A^{k+j}x_n + A^{k+j}z_n$. Thus

$$(\lambda A^{k+j} - \lambda^2 I)x_n = (\lambda I + A^{j+1}B^k - A^{k+j})z_n \longrightarrow 0 \text{ as } n \to \infty.$$

For $\lambda \neq 0$, $(A^{k+j} - \lambda I)x_n \to 0$ as $n \to \infty$, so that $\lambda \in \sigma_a(A^{k+j}) \setminus \{0\}$.

Conversely, let $\lambda \in \sigma_a(A^{k+j}) \setminus \{0\}$. Then there exists a sequence $(y_n) \subset \mathscr{X}$ with $\|y_n\| = 1$ for all $n \in \mathbb{N}$ such that $(A^{k+j} - \lambda I)y_n \to 0$ as $n \to \infty$. Let $w_n := (A^{k+j} - \lambda I)y_n$. Then $A^{k+j}y_n = \lambda y_n + w_n$ and $w_n \to 0$ as $n \to \infty$. So

$$A^{2k+2j}y_n = \lambda A^{k+j}y_n + A^{k+j}w_n = \lambda^2 y_n + \lambda w_n + A^{k+j}w_n.$$

But, $A^{2k+2j}y_n = A^{j+1}B^kA^{k+j}y_n = \lambda A^{j+1}B^ky_n + A^{j+1}B^kw_n$. Thus

$$(\lambda A^{j+1}B^k - \lambda^2)y_n = (\lambda I + A^{k+j} - A^{j+1}B^k)w_n \longrightarrow 0 \text{ as } n \to \infty.$$

For $\lambda \neq 0$, $(A^{j+1}B^k - \lambda)y_n \to 0$ as $n \to \infty$, so that $\lambda \in \sigma_a(A^{j+1}B^k)\setminus\{0\}$. Therefore $\sigma_a(A^{j+1}B^k)\setminus\{0\} = \sigma_a(A^{k+j})\setminus\{0\}$.

Now, we let $\lambda \in \sigma_a(A^{k+j})\setminus\{0\}$. From (3.2) in the proof of Lemma 3.14, we have that

$$(\lambda B^{k+2j+1} - \lambda^2 B^{j+1})x_n = u_n,$$

where $u_n := (\lambda B^{j+1} + B^{j+1}A^{k+j} - B^{k+2j+1})z_n \to 0$ as $n \to \infty$. Hence $\lambda(B^{k+j} - \lambda I)B^{j+1}x_n = u_n$, so that for $\lambda \neq 0$,

$$(B^{k+j} - \lambda I)B^{j+1}x_n = \frac{1}{\lambda}u_n. \tag{3.3}$$

From this, there is a positive integer m such that $B^{j+1}x_n \neq 0$ for $n \geqslant m$ and $\|B^{j+1}x_n\|^{-1}$ is bounded. Let $y_n := \|B^{j+1}x_n\|^{-1}B^{j+1}x_n$ for $n \geqslant m$. Then $\|y_n\| = 1$. It follows from (3.3) that for $n \geqslant m$,

$$\begin{aligned}(B^{k+j} - \lambda I)y_n &= \|B^{j+1}x_n\|^{-1}(B^{k+j} - \lambda I)B^{j+1}x_n \\ &= (\lambda\|B^{j+1}x_n\|^{-1})u_n.\end{aligned}$$

Therefore $(B^{k+j} - \lambda)y_n \to 0$ as $n \to \infty$. So $\lambda \in \sigma_a(B^{k+j})\setminus\{0\}$. Similarly, the opposite inclusion is satisfied. Consequently, this complete the proof by Lemma 3.14. $\square$

COROLLARY 3.16. *Let $(A,B) \in \mathscr{S}_{k,j} \cap \mathscr{S}_{j+1,k-1}$ for any integer $k \geqslant 1$ and $j \geqslant 0$. Then we have the following equalities.*

(1) $\sigma_p(A^{k+j}) = \sigma_p(B^kA^{j+1}) = \sigma_p(B^{j+1}A^k)$.
(2) $\sigma_p(B^{k+j}) = \sigma_p(A^kB^{j+1}) = \sigma_p(A^{j+1}B^k)$.
(3) $\sigma_a(A^{k+j}) = \sigma_a(B^kA^{j+1}) = \sigma_a(B^{j+1}A^k)$.
(4) $\sigma_a(B^{k+j}) = \sigma_a(A^kB^{j+1}) = \sigma_a(A^{j+1}B^k)$.

Proof. (1) We suppose that $N(A^{k+j} - \lambda I) = \{0\}$. It was already shown by Proposition 3.11 when $\lambda \neq 0$. So we assume that $\lambda = 0$. Then we have that for every $x \in \mathscr{X}$,

$$\begin{aligned}A^{k+j}(B^kA^{j+1} - A^{k+j})x &= (A^{k+j}B^kA^{j+1} - A^{2k+2j})x \\ &= (A^{k-1}A^{j+1}B^kA^{j+1} - A^{2k+2j})x = 0.\end{aligned}$$

Similarly, $A^{k+j}(B^{j+1}A^k - A^{k+j})x = 0$ for every $x \in \mathscr{X}$. Since A^{k+j} is injective, we have that $(B^kA^{j+1} - A^{k+j})x = 0$ and $(B^{j+1}A^k - A^{k+j})x = 0$ for every $x \in \mathscr{X}$. Thus $A^{k+j} = B^kA^{j+1} = B^{j+1}A^k$ for $k \geqslant 1$ and $j \geqslant 0$.

Therefore $\sigma_p(A^{k+j}) = \sigma_p(B^kA^{j+1}) = \sigma_p(B^{j+1}A^k)$ for $k \geqslant 1$ and $j \geqslant 0$.

(2) It is shown by similar process as the proof of (1).

(3) Suppose that $\lambda \in \sigma_a(A^{k+j})$. It was already shown by Theorem 3.15 when $\lambda \neq 0$. So we assume that $\lambda = 0$. Then A^{k+j} is bounded below. It follows from the proof in part (1) that these can be proved. $\square$

Theorem 3.15 shows that the only 0 can fail to be in the point spectrum and approximate point spectrum of the operators A^{k+j}, A^kB^{j+1}, $A^{j+1}B^k$, $B^{j+1}A^k$, B^kA^{j+1}, and B^{k+j}. Evidently, the operators have the same point spectrum and approximate point spectrum whenever $j = 0$ in Theorem 3.15.

COROLLARY 3.17. *Let $(A,B) \in \mathscr{S}_{k,j} \cap \mathscr{S}_{j+1,k-1}$ for any integer $k \geqslant 1$ and $j = 0$. Then the following equalities hold.*

(1) $\sigma_p(A^{k+j}) = \sigma_p(A^k B^{j+1}) = \sigma_p(A^{j+1}B^k) = \sigma_p(B^{j+1}A^k) = \sigma_p(B^k A^{j+1}) = \sigma_p(B^{k+j})$.

(2) $\sigma_a(A^{k+j}) = \sigma_a(A^k B^{j+1}) = \sigma_a(A^{j+1}B^k) = \sigma_a(B^{j+1}A^k) = \sigma_a(B^k A^{j+1}) = \sigma_a(B^{k+j})$.

Proof. (1) Suppose that $N(A^k B - \lambda I) = \{0\}$ for any integer $k \geqslant 1$. If $\lambda \neq 0$, then it is shown by Proposition 3.11. So we assume that $\lambda = 0$. Then $A^k B(A^k B - B^k) = A^k BA^k B - A^k B^{k+1} = 0$. Since $A^k B$ is injective, we have that $A^k B = B^k$ for any integer $k \geqslant 1$. Thus we get that for every $x \in \mathscr{X}$,

$$A^k B(B^{2k-1} - B^{2k-2})x = (A^k B^{2k} - A^k B^{2k-1})x = (A^k B^k AB^k - A^{2k} B^k)x = 0$$

Thus $B^{2k-1} = B^{2k-2}$ for any integer $k \geqslant 1$. Since $N(B) \subset N(B^k) = \{0\}$, it follows that $B = I$. So $B^k = I$ for all integer $k \geqslant 1$. It follows from the equation $B^k AB^k = B^{2k}$ that $A = I$, which implies that $A^k = I$ for all integer $k \geqslant 1$. This means that $A^{k+j} = A^k B^{j+1} = A^{j+1}B^k = B^{j+1}A^k = B^k A^{j+1} = B^{k+j} = I$. So the proof is complete.

(2) It is immediately shown by the proof in (1) and Theorem 3.15. $\square$

Under the similar conditions, we have more results for the residual spectrum, spectrum, and continuous spectrum of the operators A^{k+j}, $A^k B^{j+1}$, $A^{j+1}B^k$, $B^{j+1}A^k$, $B^k A^{j+1}$, and B^{k+j}.

PROPOSITION 3.18. *Let $(A,B) \in \mathscr{S}_{k,j} \cap \mathscr{S}_{j+1,k-1}$ for any $k \geqslant 1$ and $j \geqslant 0$. Then the following equalities hold.*

$$\begin{aligned}\sigma_r(A^{k+j}) \setminus \{0\} &= \sigma_r(A^k B^{j+1}) \setminus \{0\} = \sigma_r(A^{j+1}B^k) \setminus \{0\} \\ &= \sigma_r(B^{j+1}A^k) \setminus \{0\} = \sigma_r(B^k A^{j+1}) \setminus \{0\} = \sigma_r(B^{k+j}) \setminus \{0\}.\end{aligned}$$

Proof. It suffices to show that

$$\sigma_r(A^{k+j}) \setminus \{0\} \subseteq \sigma_r(A^k B^{j+1}) \setminus \{0\} \subseteq \sigma_r(A^{j+1}B^k) \setminus \{0\} \subseteq \sigma_r(B^{k+j}) \setminus \{0\}. \tag{3.4}$$

Let $\lambda \in \sigma_r(A^{k+j}) \setminus \{0\}$. Then $\lambda \notin \sigma_p(A^{k+j})$ and $\overline{R(A^{k+j} - \lambda I)} \neq \mathscr{X}$. Thus $N(A^{*k+j} - \lambda I^*) \neq \{0\}$. Since $(A^*, B^*) \in \mathscr{S}_{k,j} \cap \mathscr{S}_{j+1,k-1}$, by Proposition 3.11, $N(B^{*k+j} - \lambda I^*) \neq \{0\}$. However, it follows from Lemma 3.10 that

$$N(A^k B^{j+1} - \lambda I)^* = N(B^{*j+1}A^{*k} - \lambda I^*) = N(B^{*k+j} - \lambda I^*) \neq \{0\}.$$

Thus $\overline{R(A^k B^{j+1} - \lambda I)} \neq \mathscr{X}$. By Theorem 3.15, $\lambda \notin \sigma_p(A^k B^{j+1})$, so that $\lambda \in \sigma_r(A^k B^{j+1}) \setminus \{0\}$. Hence $\sigma_r(A^{k+j}) \setminus \{0\} \subseteq \sigma_r(A^k B^{j+1}) \setminus \{0\}$. Now, let $\lambda \in \sigma(A^k B^{j+1}) \setminus \{0\}$. Then $\lambda \notin \sigma_p(A^k B^{j+1})$ and $\overline{R(A^k B^{j+1} - \lambda I)} \neq \mathscr{X}$. So $N(A^k B^{j+1} - \lambda I)^* \neq \{0\}$. Since $(A^*, B^*) \in \mathscr{S}_{k,j} \cap \mathscr{S}_{j+1,k-1}$, by Lemma 3.10,

$$N(B^{*k}A^{*j+1} - \lambda I^*) = N(B^{*j+1}A^{*k} - \lambda I^*) = N(A^k B^{j+1} - \lambda I)^* \neq \{0\}.$$

Hence $\overline{R(A^{j+1}B^k - \lambda I)} \neq \mathscr{X}$. Since $\lambda \notin \sigma_p(A^{j+1}B^k)$ by Theorem 3.15, we have that $\lambda \in \sigma_r(A^{j+1}B^k) \setminus \{0\}$. Similarly, if $\lambda \in \sigma_r(A^{j+1}B^k) \setminus \{0\}$, then it follows that

$$N(B^{*k+j} - \lambda I^*) = N(B^{*k}A^{*j+1} - \lambda I^*) = N(A^{j+1}B^k - \lambda I)^* \neq \{0\},$$

and so $\overline{R(B^{k+j}-\lambda I)} \neq \mathscr{X}$. Since $\lambda \notin \sigma_p(B^{k+j})$, we have that $\lambda \in \sigma_r(B^{k+j}) \setminus \{0\}$. Therefore (3.4) is proved. □

COROLLARY 3.19. *Let $(A,B) \in \mathscr{S}_{k,j} \cap \mathscr{S}_{j+1,k-1}$ for any integer $k \geqslant 1$ and $j \geqslant 0$. Then we have the following equalities.*

(1) $\sigma_r(A^{k+j}) = \sigma_r(B^k A^{j+1}) = \sigma_r(B^{j+1}A^k)$.

(2) $\sigma_r(B^{k+j}) = \sigma_r(A^k B^{j+1}) = \sigma_r(A^{j+1}B^k)$.

Proof. The proof follows from Proposition 3.18 and Corollary 3.16. □

COROLLARY 3.20. *Let $(A,B) \in \mathscr{S}_{k,j} \cap \mathscr{S}_{j+1,k-1}$ for any integer $k \geqslant 1$ and $j = 0$. Then the following equalities hold.*

$$\sigma_r(A^{k+j}) = \sigma_r(A^k B^{j+1}) = \sigma_r(A^{j+1}B^k) = \sigma_r(B^{j+1}A^k) = \sigma_r(B^k A^{j+1}) = \sigma_r(B^{k+j}).$$

Proof. The proof is immediately shown by Corollary 3.17 and Proposition 3.18. □

We next find the following spectral relations.

THEOREM 3.21. *Let $(A,B) \in \mathscr{S}_{k,j} \cap \mathscr{S}_{j+1,k-1}$ for any $k \geqslant 1$ and $j \geqslant 0$. Then the following equalities hold.*

$$\sigma(A^{k+j}) = \sigma(A^k B^{j+1}) = \sigma(A^{j+1}B^k) = \sigma(B^{j+1}A^k) = \sigma(B^k A^{j+1}) = \sigma(B^{k+j}).$$

Proof. We first show that

$$\sigma(A^{k+j}) \setminus \{0\} \subseteq \sigma(A^k B^{j+1}) \setminus \{0\} \subseteq \sigma(A^{j+1}B^k) \setminus \{0\} \subseteq \sigma(B^{k+j}) \setminus \{0\}. \tag{3.5}$$

Let $\lambda \in \sigma(A^{k+j}) \setminus \{0\}$. Assume that $\lambda \in \rho(A^k B^{j+1})$. Then $\alpha(A^k B^{j+1} - \lambda I) = 0$ and $\lambda \notin \sigma_a(A^k B^{j+1})$. By Proposition 3.13 and Theorem 3.15, we have that $\alpha(A^{k+j} - \lambda I) = 0$ and $\lambda \notin \sigma_a(A^{k+j})$. Therefore

$$\gamma(A^{k+j} - \lambda I) = inf\{\|(A^{k+j} - \lambda I)x\| : x \in \mathscr{X} \text{ and } \|x\| = 1\} > 0,$$

so that $R(A^{k+j} - \lambda I)$ is closed. From this, $A^{k+j} - \lambda I$ is upper semi-Fredholm. Since $\lambda \in \rho(B^{*j+1}A^{*k})$, we have $\alpha(B^{*j+1}A^{*k} - \lambda I^*) = \{0\}$ and $\lambda \notin \sigma_a(B^{*j+1}A^{*k})$. But, $(A^*, B^*) \in \mathscr{S}_{k,j} \cap \mathscr{S}_{j+1,k-1}$, hence it follows that $\alpha(B^{*k+j} - \lambda I) = \{0\}$ and $\lambda \notin \sigma_a(B^{*k+j})$. Thus we get that

$$\beta(A^{k+j} - \lambda I) = \alpha(A^{*k+j} - \lambda I^*) = \alpha(B^{*k+j} - \lambda I^*) = 0.$$

Thus $\lambda \in \rho(A^{k+j})$, but this is a contradiction. Hence $\sigma(A^{k+j}) \setminus \{0\} \subseteq \sigma(A^k B^{j+1}) \setminus \{0\}$.

Now, let $\lambda \in \sigma(A^k B^{j+1}) \setminus \{0\}$ and assume that $\lambda \in \rho(B^{k+j})$. Then $\alpha(B^{k+j} - \lambda I) = 0$ and $\lambda \notin \sigma_a(B^{k+j})$. By Propositions 3.13 and Theorem 3.15, we have $\alpha(A^k B^{j+1} - \lambda I) = 0$ and $\lambda \notin \sigma_a(A^k B^{j+1})$. Therefore $\gamma(A^k B^{j+1} - \lambda I) > 0$, so that $A^k B^{j+1} - \lambda I$ is upper semi-Fredholm. Since $\lambda \in \rho(B^{*k+j})$, we have $\alpha(B^{*k+j} - \lambda I) = \{0\}$ and

$\lambda \notin \sigma_a(B^{*j+1}A^{*k})$. So it follows that $\beta(A^kB^{j+1}-\lambda I)=\alpha(B^{*j+1}A^{*k}-\lambda I)=0$. Thus $\lambda \in \rho(A^kB^{j+1})$, but this is a contradiction. Hence $\sigma(A^kB^{j+1})\setminus\{0\}\subseteq\sigma(B^{k+j})\setminus\{0\}$. Therefore (3.5) is proved. Now, we assume that A^{k+j} is invertible. Then it follows from the equation $A^kB^{j+1}A^k=A^{2k+j}$ that B^{j+1} is also invertible, so that $B^{j+1}A^k=A^{k+j}$. Since the equation $B^{j+1}A^kB^{j+1}=B^{k+2j+1}$ holds, we have that $A^kB^{j+1}=B^{j+1}A^k=B^{k+j}$. Also it follows from the equation $A^{j+1}B^kA^{j+1}=A^{k+2j+1}$ that $B^kA^{j+1}=A^{j+1}B^k=A^{k+j}$. Therefore the proof is completed. □

COROLLARY 3.22. *Let $(A,B)\in\mathscr{S}_{k,j}\cap\mathscr{S}_{j+1,k-1}$ for any $k\geqslant 1$ and $j\geqslant 0$. If A has the property (β) and $\sigma(A)$ has nonempty interior in $\mathbb{C}$, then B, AB, and BA have a nontrivial invariant subspace.*

Proof. From Theorems 3.7 and 3.21 we get that B, AB, and BA have the property (β) and their spectra have nonempty interior in $\mathbb{C}$. Hence the proof follows from [8]. □

COROLLARY 3.23. *Let $(A,B)\in\mathscr{S}_{k,j}\cap\mathscr{S}_{j+1,k-1}$. Then the following equalities hold for any integer $k\geqslant 1$ and $j\geqslant 0$.*

$$\begin{aligned}\sigma_c(A^{k+j})\setminus\{0\}&=\sigma_c(A^kB^{j+1})\setminus\{0\}=\sigma_c(A^{j+1}B^k)\setminus\{0\}\\&=\sigma_c(B^{j+1}A^k)\setminus\{0\}=\sigma_c(B^kA^{j+1})\setminus\{0\}=\sigma_c(B^{k+j})\setminus\{0\}.\end{aligned}$$

Moreover, for any integer $k\geqslant 1$ and $j=0$ we have that

$$\sigma_c(A^{k+j})=\sigma_c(A^kB^{j+1})=\sigma_c(A^{j+1}B^k)=\sigma_c(B^{j+1}A^k)=\sigma_c(B^kA^{j+1})=\sigma_c(B^{k+j}).$$

Let $\widehat{T}$ denote the coset in $B(\mathscr{X})/K(\mathscr{X})$. Then it is obvious that for $T\in B(\mathscr{X})$ we have $\sigma_e(T)=\sigma(\widehat{T})$. From this argument, we get the following corollary.

COROLLARY 3.24. *Let $(A,B)\in\mathscr{S}_{k,j}\cap\mathscr{S}_{j+1,k-1}$ for any $k\geqslant 1$ and $j\geqslant 0$. Then the following equalities hold.*

$$\sigma_e(A^{k+j})=\sigma_e(A^kB^{j+1})=\sigma_e(A^{j+1}B^k)=\sigma_e(B^{j+1}A^k)=\sigma_e(B^kA^{j+1})=\sigma_e(B^{k+j}).$$

Proof. Since $(\widehat{A},\widehat{B})\in\mathscr{S}_{k,j}\cap\mathscr{S}_{j+1,k-1}$, it follows from Theorem 3.21. □

We next study how Weyl type theorems hold for the operators A^{k+j}, A^kB^{j+1}, $A^{j+1}B^k$, $B^{j+1}A^k$, B^kA^{j+1}, and B^{k+j} in common. We first begin with the following lemma.

LEMMA 3.25. *Let $(A,B)\in\mathscr{S}_{k,j}\cap\mathscr{S}_{j+1,k-1}$ for any $k\geqslant 1$ and $j\geqslant 0$. If $\lambda\notin\sigma_e(A^{k+j})$, then the following equalities hold.*

$$\begin{aligned}ind(A^{k+j}-\lambda I)&=ind(A^kB^{j+1}-\lambda I)=ind(A^{j+1}B^k-\lambda I)\\&=ind(B^{j+1}A^k-\lambda I)=ind(B^kA^{j+1}-\lambda I)=ind(B^{k+j}-\lambda I).\end{aligned}\tag{3.6}$$

Proof. Suppose that $\lambda \neq 0$. Since $(A^*, B^*) \in \mathscr{S}_{k,j} \cap \mathscr{S}_{j+1,k-1}$ for any $k \geqslant 1$ and $j \geqslant 0$, it follows from Proposition 3.13 and Corollary 3.24 that

$$\begin{aligned}\beta(A^{k+j} - \lambda I) &= \beta(A^k B^{j+1} - \lambda I) = \beta(A^{j+1} B^k - \lambda I)\\ &= \beta(B^{j+1} A^k - \lambda I) = \beta(B^k A^{j+1} - \lambda I) = \beta(B^{k+j} - \lambda I),\end{aligned}$$

which implies that (3.6) holds for $\lambda \neq 0$. Now we suppose that $\lambda = 0$. Then A^{k+j} is Fredholm, so that $\widehat{A^{k+j}}$ is invertible. Since $(\widehat{A}, \widehat{B}) \in \mathscr{S}_{k,j} \cap \mathscr{S}_{j+1,k-1}$, we have that $\widehat{A^{k+j}} = \widehat{A^k B^{j+1}} = \widehat{A^{j+1} B^k} = \widehat{B^{j+1} A^k} = \widehat{B^k A^{j+1}} = \widehat{B^{k+j}}$ by the similar argument in the proof of Theorem 3.21. Hence we get that

$$\begin{aligned}\mathrm{ind}(A^{k+j}) &= \mathrm{ind}(A^k B^{j+1}) = \mathrm{ind}(A^{j+1} B^k)\\ &= \mathrm{ind}(B^{j+1} A^k) = \mathrm{ind}(B^k A^{j+1}) = \mathrm{ind}(B^{k+j}).\end{aligned}$$

Therefore the proof is complete. □

LEMMA 3.26. *Let $(A,B) \in \mathscr{S}_{k,j} \cap \mathscr{S}_{j+1,k-1}$ for any integer $k \geqslant 1$ and $j \geqslant 0$. Then the following equalities hold.*

(1) $\sigma_w(A^{k+j}) = \sigma_w(A^k B^{j+1}) = \sigma_w(A^{j+1} B^k) = \sigma_w(B^{j+1} A^k) = \sigma_w(B^k A^{j+1}) = \sigma_w(B^{k+j})$.

(2) $\sigma_b(A^{k+j}) = \sigma_b(A^k B^{j+1}) = \sigma_b(A^{j+1} B^k) = \sigma_b(B^{j+1} A^k) = \sigma_b(B^k A^{j+1}) = \sigma_b(B^{k+j})$.

Proof. (1) It follows from Corollary 3.24 and Lemma 3.25.

(2) It is well known that for an operator $T \in B(\mathscr{X})$, $T - \lambda I$ is Browder if and only if $T - \lambda I$ is Weyl and T has SEVP at λ (see [1]). Thus if one of the operators $A^{k+j} - \lambda I$, $A^k B^{j+1} - \lambda I$, $A^{j+1} B^k - \lambda I$, $B^{j+1} A^k - \lambda I$, $B^k A^{j+1} - \lambda I$, and $B^{k+j} - \lambda I$ is Browder, then all of them are Browder by part (1) and the proof in Theorem 3.7. □

The following proposition is obvious from Lemma 3.26.

PROPOSITION 3.27. *Let $(A,B) \in \mathscr{S}_{k,j} \cap \mathscr{S}_{j+1,k-1}$ for any integer $k \geqslant 1$ and $j \geqslant 0$. Then the followings are equivalent.*

(1) Browder's theorem holds for A^{k+j}.

(2) Browder's theorem holds for $A^k B^{j+1}$.

(3) Browder's theorem holds for $A^{j+1} B^k$.

(4) Browder's theorem holds for $B^{j+1} A^k$.

(5) Browder's theorem holds for $B^k A^{j+1}$.

(6) Browder's theorem holds for B^{k+j}.

Furthermore, we can easily prove from Theorem 3.21 and Proposition 3.13 that

$$\begin{aligned}\pi_{00}(A^{k+j}) \setminus \{0\} &= \pi_{00}(A^k B^{j+1}) \setminus \{0\} = \pi_{00}(A^{j+1} B^k) \setminus \{0\}\\ &= \pi_{00}(B^{j+1} A^k) \setminus \{0\} = \pi_{00}(B^k A^{j+1}) \setminus \{0\} = \pi_{00}(B^{k+j}) \setminus \{0\}.\end{aligned}$$

Hence we have the following results from these arguments.

THEOREM 3.28. *Let* $(A,B) \in \mathscr{S}_{k,j} \cap \mathscr{S}_{j+1,k-1}$ *for any integer* $k \geqslant 1$ *and* $j \geqslant 0$. *Suppose that the range of* T *is closed whenever* $0 \in iso\sigma(T)$, *where* $T \in \{A^{k+j}, A^kB^{j+1}, A^{j+1}B^k, B^{j+1}A^k, B^kA^{j+1}, B^{k+j}\}$. *Then the following statements are equivalent.*

(1) Weyl's theorem holds for A^{k+j}.

(2) Weyl's theorem holds for A^kB^{j+1}.

(3) Weyl's theorem holds for $A^{j+1}B^k$.

(4) Weyl's theorem holds for $B^{j+1}A^k$.

(5) Weyl's theorem holds for B^kA^{j+1}.

(6) Weyl's theorem holds for B^{k+j}.

Proof. Suppose that Weyl's theorem holds for A^{k+j}. It follows from Proposition 3.13, Theorem 3.21 and Lemma 3.26 that for $\lambda \neq 0$,

$$\lambda \in \sigma(A^kB^{j+1}) \setminus \sigma_w(A^kB^{j+1}) \Leftrightarrow \lambda \in \pi_{00}(A^kB^{j+1}).$$

So we only need to show that the above equivalence holds for $\lambda = 0$. Assume that $0 \in \sigma(A^kB^{j+1}) \setminus \sigma_w(A^kB^{j+1})$. Then $0 \in \sigma(A^{k+j}) \setminus \sigma_w(A^kB^{j+1})$. Since Weyl's theorem holds for A^{k+j}, we have that $0 \in \pi_{00}(A^{k+j})$. Thus $0 \in \mathrm{iso}\sigma(A^kB^{j+1})$ and $\alpha(A^kB^{j+1}) > 0$. Since A^kB^{j+1} is Weyl, $\alpha(A^kB^{j+1}) < \infty$. Hence $0 \in \pi_{00}(A^kB^{j+1})$. Now, suppose that $0 \in \pi_{00}(A^kB^{j+1})$. Then $0 \in \mathrm{iso}\sigma(A^kB^{j+1})$ and $0 < \alpha(A^kB^{j+1}) < \infty$. Since A^kB^{j+1} has closed range by hypothesis, A^kB^{j+1} is upper semi-Fredholm. Since $B^{*j+1}A^{*k}$ has SVEP at 0, we have $\beta(A^kB^{j+1}) \leqslant \alpha(A^kB^{j+1}) < \infty$. Hence A^kB^{j+1} is Fredholm. Also A^kB^{j+1} has SVEP at 0, hence $\mathrm{ind}(A^kB^{j+1}) = 0$. Thus A^kB^{j+1} is Weyl, so that $0 \in \sigma(A^kB^{j+1}) \setminus \sigma_w(A^kB^{j+1})$. Consequently, Weyl's theorem holds for A^kB^{j+1}. The rest of the equivalences can be proved by the similar process. $\square$

For an operator $T \in B(\mathscr{X})$, it is well known that $\sigma_{le}(T) = \sigma_a(\widehat{T})$ and $\sigma_{re}(T) = \sigma_a(\widehat{T}^*)$. So we have the following lemma.

LEMMA 3.29. *Let* $(A,B) \in \mathscr{S}_{k,j} \cap \mathscr{S}_{j+1,k-1}$ *for any integer* $k \geqslant 1$ *and* $j \geqslant 0$. *Then the following equalities hold.*

(1) $\sigma_{le}(A^{k+j}) \setminus \{0\} = \sigma_{le}(A^kB^{j+1}) \setminus \{0\} = \sigma_{le}(A^{j+1}B^k) \setminus \{0\} = \sigma_{le}(B^{j+1}A^k) \setminus \{0\} = \sigma_{le}(B^kA^{j+1}) \setminus \{0\} = \sigma_{le}(B^{k+j}) \setminus \{0\}$.

(2) $\sigma_{re}(A^{k+j}) \setminus \{0\} = \sigma_{re}(A^kB^{j+1}) \setminus \{0\} = \sigma_{re}(A^{j+1}B^k) \setminus \{0\} = \sigma_{re}(B^{j+1}A^k) \setminus \{0\} = \sigma_{re}(B^kA^{j+1}) \setminus \{0\} = \sigma_{re}(B^{k+j}) \setminus \{0\}$.

In particular, if $j = 0$ *then we have that*

(3) $\sigma_{le}(A^{k+j}) = \sigma_{le}(A^kB^{j+1}) = \sigma_{le}(A^{j+1}B^k) = \sigma_{le}(B^{j+1}A^k) = \sigma_{le}(B^kA^{j+1}) = \sigma_{le}(B^{k+j})$.

(4) $\sigma_{re}(A^{k+j}) = \sigma_{re}(A^kB^{j+1}) = \sigma_{re}(A^{j+1}B^k) = \sigma_{re}(B^{j+1}A^k) = \sigma_{re}(B^kA^{j+1}) = \sigma_{re}(B^{k+j})$.

Proof. (1) The proof follows from Theorem 3.15. Since $(\widehat{A}^*, \widehat{B}^*) \in \mathscr{S}_{k,j} \cap \mathscr{S}_{j+1,k-1}$ for any integer $k \geqslant 1$ and $j \geqslant 0$, (2) holds again from Theorem 3.15. Furthermore, (3) and (4) are immediately shown by Corollary 3.17. $\square$

THEOREM 3.30. *Let $(A,B) \in \mathscr{S}_{k,j} \cap \mathscr{S}_{j+1,k-1}$ for any integer $k \geqslant 1$ and $j = 0$. Then the following equalities hold.*

$$\begin{aligned}\sigma_{ea}(A^{k+j}) &= \sigma_{ea}(A^k B^{j+1}) = \sigma_{ea}(A^{j+1}B^k) \\ &= \sigma_{ea}(B^{j+1}A^k) = \sigma_{ea}(B^k A^{j+1}) = \sigma_{ea}(B^{k+j}).\end{aligned} \tag{3.7}$$

Furthermore, If the range of T is closed whenever $0 \in iso\sigma_a(T)$, where $T \in \{A^{k+j}, A^kB^{j+1}, A^{j+1}B^k, B^{j+1}A^k, B^kA^{j+1}, B^{k+j}\}$, then the followings are equivalent.

(1) a-Weyl's theorem holds for A^{k+j}.
(2) a-Weyl's theorem holds for A^kB^{j+1}.
(3) a-Weyl's theorem holds for $A^{j+1}B^k$.
(4) a-Weyl's theorem holds for $B^{j+1}A^k$.
(5) a-Weyl's theorem holds for B^kA^{j+1}.
(6) a-Weyl's theorem holds for B^{k+j}.

Proof. Suppose that $\lambda \notin \sigma_{ea}(A^{k+j})$. Then $A^{k+j} - \lambda I$ is upper semi-Fredholm and $\mathrm{ind}(A^{k+j} - \lambda I) \leqslant 0$. Since $A^kB^{j+1} - \lambda I$ is upper semi-Fredholm by Lemma 3.29 (3), we only need to show that $\mathrm{ind}(A^kB^{j+1} - \lambda I) \leqslant 0$. If $\beta(A^kB^{j+1} - \lambda I) = \infty$, then it is obvious. So we assume that $\beta(A^kB^{j+1} - \lambda I) < \infty$. Then $A^kB^{j+1} - \lambda I$ is Fredholm and hence it follows from Lemma 3.25 that $\mathrm{ind}(A^kB^{j+1} - \lambda I) = \mathrm{ind}(A^{k+j} - \lambda I) \leqslant 0$. Thus $\lambda \notin \sigma_{ea}(A^kB^{j+1})$. The same process can be applied to the rest, so that (3.7) is proved. Now we observe that from Corollary 3.17 and Proposition 3.13

$$\begin{aligned}\pi_{00}^a(A^{k+j}) \setminus \{0\} &= \pi_{00}^a(A^kB^{j+1}) \setminus \{0\} = \pi_{00}^a(A^{j+1}B^k) \setminus \{0\} \\ &= \pi_{00}^a(B^{j+1}A^k) \setminus \{0\} = \pi_{00}^a(B^kA^{j+1}) \setminus \{0\} = \pi_{00}^a(B^{k+j}) \setminus \{0\}.\end{aligned}$$

Suppose that a-Weyl's theorem holds for A^{k+j}. Then it is obvious that for $\lambda \neq 0$,

$$\lambda \in \sigma_a(A^kB^{j+1}) \setminus \sigma_{ea}(A^kB^{j+1}) \Leftrightarrow \lambda \in \pi_{00}^a(A^kB^{j+1}).$$

So we only need to prove that the above equivalence holds for $\lambda = 0$. Assume that $0 \in \sigma_a(A^kB^{j+1}) \setminus \sigma_{ea}(A^kB^{j+1})$. Then $0 \in \sigma_a(A^{k+j}) \setminus \sigma_{ea}(A^{k+j})$. Since a-Weyl's theorem holds for A^{k+j}, we have that $0 \in \pi_{00}^a(A^{k+j})$. Thus $0 \in \mathrm{iso}\sigma_a(A^kB^{j+1})$ and $\alpha(A^kB^{j+1}) > 0$. Since A^kB^{j+1} is upper semi-Fredholm, $\alpha(A^kB^{j+1}) < \infty$. Hence $0 \in \pi_{00}^a(A^kB^{j+1})$. Now, assume that $0 \in \pi_{00}^a(A^kB^{j+1})$. Then $0 \in \mathrm{iso}\sigma_a(A^kB^{j+1})$ and $0 < \alpha(A^kB^{j+1}) < \infty$. Since A^kB^{j+1} has closed range by hypothesis, A^kB^{j+1} is upper semi-Fredholm. Since A^kB^{j+1} has SVEP at 0, we have that $p(A^kB^{j+1}) < \infty$, so that $\mathrm{ind}(A^kB^{j+1}) \leqslant 0$. Thus $0 \in \sigma_a(A^kB^{j+1}) \setminus \sigma_{ea}(A^kB^{j+1})$. Consequently, a-Weyl's theorem holds for A^kB^{j+1}. The rest of the equivalences can be proved by similar process. $\square$

For an operator $T \in B(\mathscr{X})$, a *hole* in $\sigma_e(T)$ is a bounded component of $\mathbb{C} \setminus \sigma_e(T)$. A *pseudohole* in $\sigma_e(T)$ is a component of $\sigma_e(T) \setminus \sigma_{le}(T)$ or $\sigma_e(T) \setminus \sigma_{re}(T)$. The *spectral picture* of an operator $T \in B(\mathscr{X})$ (notation : $SP(T)$) is the structure consisting of the set $\sigma_e(T)$, the collection of holes and pseudoholes in $\sigma_e(T)$, and the indices associated with these holes and pseudoholes.

THEOREM 3.31. *Let* $(A,B) \in \mathscr{S}_{k,j} \cap \mathscr{S}_{j+1,k-1}$ *for any integer* $k \geqslant 1$ *and* $j = 0$. *Then the following equalities hold.*

$$SP(A^{k+j}) = SP(A^k B^{j+1}) = SP(A^{j+1} B^k) = SP(B^{j+1} A^k) = SP(B^k A^{j+1}) = SP(B^{k+j})$$

Proof. If λ belongs to a hole or pseudohole in $\sigma_e(A^{k+j})$, then all the indices of the operators $A^{k+j} - \lambda I$, $A^k B^{j+1} - \lambda I$, $A^{j+1} B^k - \lambda I$, $B^{j+1} A^k - \lambda I$, $B^k A^{j+1} - \lambda I$, and $B^{k+j} - \lambda I$ are equal by Lemma 3.25. Thus it follows from Corollary 3.24 and Lemma 3.29 that all of the operators A^{k+j}, $A^k B^{j+1}$, $A^{j+1} B^k$, $B^{j+1} A^k$, $B^k A^{j+1}$, and B^{k+j} have the same spectral picture, which completes the proof. $\square$

PROPOSITION 3.32. *If* $(A,B) \in \mathscr{S}_{k,j} \cap \mathscr{S}_{j+1,k-1}$ *for any* $k \geqslant 1$ *and* $j \geqslant 0$, *then the following properties hold.*

(1) $\sigma_{B^{k+j}}(B^{j+1}y) \subseteq \sigma_{A^k B^{j+1}}(y) \subseteq \sigma_{A^{k+j}}(x)$ *for* $y := A^k x$, *and* $\sigma_{A^{k+j}}(A^{j+1}z) \subseteq \sigma_{A^k B^{j+1}}(z) \subseteq \sigma_{B^{k+j}}(x)$ *for* $z := B^k x$ *for all* $x \in \mathscr{X}$.

(2) $B^{j+1} \mathscr{X}_{A^k B^{j+1}}(F) \subseteq \mathscr{X}_{B^{j+1}}(F)$, $A^k \mathscr{X}_{A^{k+j}}(F) \subseteq \mathscr{X}_{A^k B^{j+1}}(F)$, $A^{j+1} \mathscr{X}_{B^k A^{j+1}}(F) \subseteq \mathscr{X}_{A^{j+1}}(F)$, *and* $B^k \mathscr{X}_{B^{k+j}}(F) \subseteq \mathscr{X}_{B^k A^{j+1}}(F)$ *for any closed set* $F \in \mathbb{C}$.

Proof. (1) It suffices to show the first inclusions. Let $y := A^k x \in \mathscr{X}$ be given for each $x \in \mathscr{X}$ and let $\mu \in \rho_{A^k B^{j+1}}(y)$. Then we can choose a neighborhood D of μ and an analytic function $f : D \to \mathscr{X}$ such that $(A^k B^{j+1} - \lambda) f(\lambda) = y$ for all $\lambda \in D$. Since

$$\begin{aligned}(B^{k+j} - \lambda) B^{j+1} f(\lambda) &= (B^{k+2j+1} - \lambda B^{j+1}) f(\lambda) \\ &= (B^{j+1} A^k B^{j+1} - \lambda B^{j+1}) f(\lambda) = B^{j+1} y\end{aligned}$$

for all $\lambda \in D$, we obtain that $\mu \in \rho_{B^{k+j}}(B^{j+1}y)$. So $\rho_{A^k B^{j+1}}(y) \subseteq \rho_{B^{k+j}}(B^{j+1}y)$, that is, $\sigma_{B^{k+j}}(B^{j+1}y) \subseteq \sigma_{A^k B^{j+1}}(y)$. Similarly, let $\mu_0 \in \rho_{A^{k+j}}(x)$ for all $x \in \mathscr{X}$. Then we consider a neighborhood U of μ_0 and an analytic function $g : U \to \mathscr{X}$ such that $(A^{k+j} - \lambda_0) g(\lambda_0) = x$ for all $\lambda_0 \in U$. Since

$$\begin{aligned}(A^k B^{j+1} - \lambda_0) A^k g(\lambda_0) &= (A^k B^{j+1} A^k - \lambda_0 A^k) g(\lambda_0) \\ &= (A^{2k+j} - \lambda_0 A^k) g(\lambda_0) = A^k x = y\end{aligned}$$

for all $\lambda_0 \in U$, we have that $\mu_0 \in \rho_{A^k B^{j+1}}(y)$. Therefore $\sigma_{A^k B^{j+1}}(y) \subseteq \sigma_{A^{k+j}}(x)$ for all $x \in \mathscr{X}$.

(2) Let F be any closed set in $\mathbb{C}$. If $y \in \mathscr{X}_{A^k B^{j+1}}(F)$, then it follows from part (1) that $\sigma_{B^{j+1}}(B^{j+1}y) \subseteq \sigma_{A^k B^{j+1}}(y) \subseteq F$. Thus $B^{j+1} y \in \mathscr{X}_{B^{j+1}}(F)$, and so $B^{j+1} \mathscr{X}_{A^k B^{j+1}}(F) \subseteq \mathscr{X}_{B^{j+1}}(F)$. Similarly, if $x \in \mathscr{X}_{A^{k+j}}(F)$, then it follows from part (1) that $\sigma_{A^k B^{j+1}}(y) \subseteq \sigma_{A^{k+j}}(x) \subseteq F$. Thus $A^k x = y \in \mathscr{X}^{A^k B^{j+1}}(F)$, and so $A^k \mathscr{X}_{A^{k+j}}(F) \subseteq \mathscr{X}_{A^k B^{j+1}}(F)$. By symmetry, we have that $A^{j+1} \mathscr{X}_{B^k A^{j+1}}(F) \subseteq \mathscr{X}_{A^{j+1}}(F)$ and $B^k \mathscr{X}_{B^{k+j}}(F) \subseteq \mathscr{X}_{B^k A^{j+1}}(F)$. $\square$

COROLLARY 3.33. *Let* $(A,B) \in \mathscr{S}_{k,j} \cap \mathscr{S}_{j+1,k-1}$ *for any* $k \geqslant 1$ *and* $j \geqslant 0$. *Then the following statements hold. If* A *has SVEP, then*

$$\bigcup_{x \in \mathscr{X}} \sigma_{A^k B^{j+1}}(A^k x) \subseteq \bigcup_{x \in \mathscr{X}} \sigma_{A^{k+j}}(x) = \sigma_{su}(A^{k+j}) = \sigma(A^{k+j}) = \sigma(B^{k+j}).$$

LEMMA 3.34. *Let* $(A,B) \in \mathscr{S}_{k,j} \cap \mathscr{S}_{j+1,k-1}$ *for any* $k \geqslant 1$ *and* $j \geqslant 0$*. Then the following equality holds for* $\lambda \neq 0$.

$$
\begin{aligned}
p(A^{k+j}-\lambda) &= p(A^kB^{j+1}-\lambda) = p(A^{j+1}B^k-\lambda) \\
&= p(B^{j+1}A^k-\lambda I) = p(B^kA^{j+1}-\lambda) = p(B^{k+j}-\lambda).
\end{aligned}
$$

Proof. It suffices to show that

$$
p(B^{k+j}-\lambda) \leqslant p(B^kA^{j+1}-\lambda) \leqslant p(A^{k+j}-\lambda).
$$

Suppose that $p(B^{k+j}-\lambda) := n$ for any integer $n \geqslant 1$. Then $N(B^{k+j}-\lambda)^{n-1} \subsetneqq N(B^{k+j}-\lambda)^n = N(B^{k+j}-\lambda)^{n+1} = \cdots$. Thus we can suppose that $(B^{k+j}-\lambda)^n x = 0$ and $(B^{k+j}-\lambda)^{n-1}x \neq 0$ for some nonzero $x \in \mathscr{X}$ and some $n \geqslant 1$. Then it follows from $(A,B) \in \mathscr{S}_{k,j}$ that

$$
\begin{aligned}
(B^kA^{j+1}-\lambda)^n B^{k+j}x &= \left[\sum_{i=0}^{n} \binom{n}{i} (B^kA^{j+1})^i(-\lambda)^{n-i}\right] B^{k+j}x \\
&= B^{k+j}(B^{k+j}-\lambda)^n x = 0.
\end{aligned}
$$

Thus $B^{k+j}x \in N(B^kA^{j+1}-\lambda)^n$. Assume that $B^{k+j}x \in N(B^kA^{j+1}-\lambda)^{n-1}$. Then $(B^kA^{j+1}-\lambda)^{n-1}B^{k+j}x = 0$, so that $B^{k+j}(B^{k+j}-\lambda)^{n-1}x = 0$. Hence $B^{k+j}(B^{k+j}-\lambda)^{n-1}x - \lambda(B^{k+j}-\lambda)^{n-1}x = (B^{k+j}-\lambda)^n x = 0$. So $(B^{k+j}-\lambda)^{n-1}x = 0$ for $\lambda \neq 0$. This is a contradiction. Thus $p(B^kA^{j+1}-\lambda) \geqslant n = p(B^{k+j}-\lambda)$.

Now, suppose that $(B^kA^{j+1}-\lambda)^n x = 0$ and $(B^kA^{j+1}-\lambda)^{n-1}x \neq 0$ for some nonzero $x \in \mathscr{X}$ and some $n \geqslant 1$. Since $(A,B) \in \mathscr{S}_{j+1,k-1}$, we have

$$
\begin{aligned}
(A^{k+j}-\lambda)^n A^{j+1}x &= \left[\sum_{i=0}^{n} \binom{n}{i} (A^{k+j})^i(-\lambda)^{n-i}\right] A^{j+1}x \\
&= A^{j+1}(B^kA^{j+1}-\lambda)^n x = 0.
\end{aligned}
$$

Thus $A^{j+1}x \in N(A^{k+j}-\lambda)^n$. Assume that $A^{j+1}x \in N(A^{k+j}-\lambda)^{n-1}$. Then $(A^{k+j}-\lambda)^{n-1}A^{j+1}x = 0$. Since $(A,B) \in \mathscr{S}_{j+1,k-1}$, we have $A^{j+1}(B^kA^{j+1}-\lambda)^{n-1}x = 0$. So $B^kA^{j+1}(B^kA^{j+1}-\lambda)^{n-1}x - \lambda(B^kA^{j+1}-\lambda)^{n-1}x = (B^kA^{j+1}-\lambda)^n x = 0$. Hence $(B^kA^{j+1}-\lambda)^{n-1}x = 0$ for $\lambda \neq 0$. This is a contradiction. Therefore $p(A^{k+j}-\lambda) \geqslant n = p(B^kA^{j+1}-\lambda)$. $\square$

From Lemma 3.34 we have more result as follows.

THEOREM 3.35. *Let* $(A,B) \in \mathscr{S}_{k,j} \cap \mathscr{S}_{j+1,k-1}$ *for any integer* $k \geqslant 1$ *and* $j = 0$*. Then the following equalities hold.*

$$
\begin{aligned}
\sigma_{ab}(A^{k+j}) &= \sigma_{ab}(A^kB^{j+1}) = \sigma_{ab}(A^{j+1}B^k) \\
&= \sigma_{ab}(B^{j+1}A^k) = \sigma_{ab}(B^kA^{j+1}) = \sigma_{ab}(B^{k+j}).
\end{aligned}
\tag{3.8}
$$

Furthermore, the followings are equivalent.

(1) a-Browder's theorem holds for A^{k+j}.
(2) a-Browder's theorem holds for A^kB^{j+1}.
(3) a-Browder's theorem holds for $A^{j+1}B^k$.
(4) a-Browder's theorem holds for $B^{j+1}A^k$.
(5) a-Browder's theorem holds for B^kA^{j+1}.
(6) a-Browder's theorem holds for B^{k+j}.

Proof. Let $\lambda \in \sigma_{ab}(A^{k+j})$. Then $A^{k+j} - \lambda I$ is upper semi-Fredholm and $p(A^{k+j} - \lambda I) < \infty$. If $\lambda \neq 0$, then it is obvious by Lemmas 3.29 and 3.34 that $\lambda \in \sigma_{ab}(A^kB^{j+1})$. So assume that $\lambda = 0$. Then A^{k+j} has finite ascent, so that it has SVEP at 0. It follows from the proof of Theorem 3.7 that A^kB^{j+1} has SVEP at 0. Since A^kB^{j+1} is upper semi-Fredholm, it has finite ascent. Therefore $0 \in \sigma_{ab}(A^kB^{j+1})$. Throughout this similar way, (3.8) can be proved. Furthermore, we have that if a-Browder's theorem holds for one of the operators A^{k+j}, A^kB^{j+1}, $A^{j+1}B^k$, $B^{j+1}A^k$, B^kA^{j+1}, and B^{k+j}, then all of them satisfy a-Browder's theorem from (3.7) in Theorem 3.30. $\square$

Finally, the spectral mapping theorem for Drazin spectrum implies the following theorem.

THEOREM 3.36. *Let $(A,B) \in \mathscr{S}_{k,j} \cap \mathscr{S}_{j+1,k-1}$ for any $k \geqslant 1$ and $j \geqslant 0$. Then*

$$\begin{aligned}\sigma_D(A^{k+j}) &= \sigma_D(A^kB^{j+1}) = \sigma_D(A^{j+1}B^k) \\ &= \sigma_D(B^{j+1}A^k) = \sigma_D(B^kA^{j+1}) = \sigma_D(B^{k+j}).\end{aligned}$$

Proof. We observe that $(A^kB^{j+1})^2 = A^{2k+j}B^{j+1}$. Since $\sigma_D(TS) = \sigma_D(ST)$ for every operators T and S, we have $\sigma_D(A^{2k+j}B^{j+1}) = \sigma_D(A^kB^{j+1}A^{k+j})$. By the spectral mapping theorem of the Drazin spectrum,

$$\begin{aligned}\{\sigma_D(A^kB^{j+1})\}^2 &= \sigma_D[(A^kB^{j+1})^2] = \sigma_D(A^{2k+j}B^{j+1}) \\ &= \sigma_D(A^kB^{j+1}AA^{k+j}) = \sigma_D(A^{2k+2j}) = \{\sigma_D(A^{k+j})\}^2.\end{aligned}$$

Since $(A,B) \in \mathscr{S}_{j+1,k-1}$, it holds that $(A^kB^{j+1})^2 = A^kB^{k+2j+1}$. From this, we have that $\{\sigma_D(A^kB^{j+1})\}^2 = \{\sigma_D(B^{k+j})\}^2$. Similarly, it is obvious that $\{\sigma_D(B^kA^{j+1})\}^2 = \{\sigma_D(B^{k+j})\}^2$. Consequently, the proof is completed. $\square$

REFERENCES

[1] P. AIENA, *Fredholm and Local Spectral Theory, with Applications to Multipliers*, Kluwer Academic Publishers, 2004.

[2] G. CORACH, B. P. DUGGAL AND R. HARTE, *Extensions of Jacobson's lemma*, Communications in Algebra **41** (2013), 520–531.

[3] S. V. DJORDJEVIĆ AND D. S. DJORDJEVIĆ, *Weyl's theorem:Continuity of the spectrum and quasi-hyponormal operators*, Acta Sci. Math. (Szeged) **64** (1998), 259–269.

[4] S. V. DJORDJEVIĆ AND Y. M. HAN, *Browder's theorems and spectral continuity*, Glasgow Math. J. **42** (2000), 479–486.

[5] B. P. DUGGAL, *Operator equations $ABA = A^2$ and $BAB = B^2$*, Funct. Anal. Approx. Comput. **3** (2011), 9–18.

[6] R. E. HARTE AND W. Y. LEE, *Another note on Weyl's theorem*, Trans. Amer. Math. Soc. **349** (1997), 2115–2124.
[7] C. SCHMOEGER, *On the operator equations $ABA = A^2$ and $BAB = B^2$*, Publications de l'Institut Math. (N. S.) **78** (**92**) (2005), 127–133.
[8] K. B. LAURSEN AND M. M. NEUMANN, An Introduction to Local Spectral Theory, London Mathematical Society Monographs New Series 20, Clarendon Press, Oxford, 2000.
[9] C. SCHMOEGER, *Common spectral properties of linear operators A and B such that $ABA = A^2$ and $BAB = B^2$*, Publications de l'Institut Math. (N. S.) **79** (**93**) (2006), 109–114.
[10] I. VIDAV, *On idempotent operators in a Hilbert space*, Publ. Inst. Math. (Beograd) **4** (**18**) (1964), 157–163.

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Il Ju An
PDE Functional Analysis Research Center
Seoul National University
Seoul, 08826, Korea
e-mail: dlfwn79@khu.ac.kr

Eungil Ko
Department of Mathematics
Ewha Womans University
Seoul, 03760, Korea
e-mail: eiko@ewha.ac.kr