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2–LOCAL LIE ISOMORPHISMS OF NEST ALGEBRAS

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Abstract. Let $\mathscr{N}$ and $\mathscr{M}$ be nests on a separable complex Hilbert space $\mathscr{H}$ of dimension greater than 2, and $\operatorname{Alg}\mathscr{N}$ and $\operatorname{Alg}\mathscr{M}$ be the associated nest algebras. We show that every additive 2-local Lie isomorphism Φ of $\operatorname{Alg}\mathscr{N}$ onto $\operatorname{Alg}\mathscr{M}$ has the form $\Phi = \phi + \tau$, where ϕ is an isomorphism or a negative of an anti-isomorphism of $\operatorname{Alg}\mathscr{N}$ onto $\operatorname{Alg}\mathscr{M}$, and τ is a linear map from $\operatorname{Alg}\mathscr{N}$ into $\mathbb{C}I$ vanishing on a sum of commutators.

1. Introduction and preliminaries

Let $\mathscr{A}$ and $\mathscr{B}$ be two associative algebras. Recall that a linear bijection $\phi : \mathscr{A} \to \mathscr{B}$ is called a Lie isomorphism if $\phi([A,B]) = [\phi(A), \phi(B)]$ for all $A, B \in \mathscr{A}$, where $[A,B] = AB - BA$ is the usual Lie product of A and B. The study of Lie isomorphism between associative algebras or operator algebras, primarily focusing upon their relations to associative (anti-)isomorphisms, has a long history. See [2, 4, 16, 17, 18] and the references therein. In [16], Marcoux and Sourour proved that every Lie isomorphism between nest algebras $\operatorname{Alg}\mathscr{N}$ and $\operatorname{Alg}\mathscr{M}$ on a separable complex Hilbert space has the form $\Phi = \phi + \tau$, where ϕ is an isomorphism or a negative of an anti-isomorphism of $\operatorname{Alg}\mathscr{N}$ onto $\operatorname{Alg}\mathscr{M}$, and τ is a linear map from $\operatorname{Alg}\mathscr{N}$ into $\mathbb{C}I$ vanishing on every commutator.

A well-known and active direction in the study of the local action of maps is the local map problem, which was initiated by Kadison [20] and Larson and Sourour [14] in 1990. Recall that a linear map θ of an algebra $\mathscr{A}$ is called a local isomorphism (respectively, local derivation) if for each $A \in \mathscr{A}$, there exists an isomorphism (respectively, a derivation) θ_A, depending on A such that $\theta(A) = \theta_A(A)$. There is a vast literature on local isomorphisms and local derivations, see for example [5, 11, 12, 19] and the references therein.

In 1997, Šemrl [23] introduced the notion of 2-local maps. A map δ on an algebra $\mathscr{A}$ (not necessarily linear) is called a 2-local isomorphism (respectively, 2-local derivation) if for each $A, B \in \mathscr{A}$, there exists an isomorphism (respectively, a derivation) $\delta_{A,B}$ such that $\delta(A) = \delta_{A,B}(A)$ and $\delta(B) = \delta_{A,B}(B)$. 2-local maps have been studied on different operator algebras by many authors [23, 1, 13, 15]. In [23], Šemrl studied 2-local isomorphisms and 2-local derivations on the algebra of all bounded linear operators on an infinite dimensional separable Hilbert space. A similar study for the finite dimensional case appeared in [13].

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Obviously, we can define 2-local Lie isomorphisms in a natural way. Let $\mathscr{A}$ and $\mathscr{B}$ be two associative algebras. We say that a map $\phi : \mathscr{A} \to \mathscr{B}$ is a 2-local Lie isomorphism if for each $A, B \in \mathscr{A}$, there exists a Lie isomorphism $\delta_{A,B} : \mathscr{A} \to \mathscr{B}$ such that $\delta(A) = \delta_{A,B}(A)$ and $\delta(B) = \delta_{A,B}(B)$. In the previous paper [10], Huang and the second author characterized 2-local Lie isomorphism between operator algebras on Banach spaces. Let X and Y be complex Banach spaces of dimension greater than 2. They proved that every 2-local Lie isomorphism Φ of $B(X)$ onto $B(Y)$ has the form $\Phi = \phi + \tau$, where ϕ is an isomorphism or a negative of an anti-isomorphism of $B(X)$ onto $B(Y)$, and τ is a homogeneous map from $B(X)$ into $\mathbb{C}I$ vanishing on every sum of commutators. In this paper, as the continuity of the previous work, we study 2-local Lie isomorphisms between nest algebras on Hilbert spaces.

Nest algebras, introduced in 1965 by Ringrose [21], are the most important subclass in the class of non-self-adjoint algebras, as von Neumann algebras are in the class of self-adjoint algebras. Let $\mathscr{H}$ be a Hilbert space over the complex field $\mathbb{C}$. Denote by $B(\mathscr{H})$ the algebra of all bounded linear operators on $\mathscr{H}$. A nest $\mathscr{N}$ on $\mathscr{H}$ is a chain of closed subspaces of $\mathscr{H}$ which contains 0 and $\mathscr{H}$ and is closed under the formation of arbitrary closed linear span (denoted by $\vee$) and intersection (denoted by $\wedge$). The nest algebra Alg$\mathscr{N}$ associated to the nest $\mathscr{N}$ is the set of all operators on X leaving every subspace in $\mathscr{N}$ invariant, that is, $\mathrm{Alg}\mathscr{N} = \{A \in B(\mathscr{H}) : AN \subseteq N, \forall N \in \mathscr{N}\}$. We refer the readers to [6] as a basic text on the theory of nest algebras.

We close this section with two well known results (see [16]).

PROPOSITION 1.1. *Let $\mathscr{N}$ and $\mathscr{M}$ be nests on a separable complex Hilbert space $\mathscr{H}$, and Alg$\mathscr{N}$ and Alg$\mathscr{M}$ be the associated nest algebras. Suppose that Φ : Alg$\mathscr{N}$ $\to$Alg$\mathscr{M}$ is a Lie isomorphism. Then one of the following holds.*

(1) There exist an invertible operator $T \in B(\mathscr{H})$ satisfying $T(\mathscr{M}) = \mathscr{N}$ and a linear map τ from Alg$\mathscr{N}$ into $\mathbb{C}I$ vanishing on each commutator such that $\Phi(A) = T^{-1}AT + \tau(A)$ for all $A \in$ Alg$\mathscr{N}$.

(2) There exist an invertible operator $S \in B(\mathscr{H})$ satisfying $S(\mathscr{M}) = \mathscr{N}^{\perp}$ and a linear map τ from Alg$\mathscr{N}$ into $\mathbb{C}I$ vanishing on each commutator such that $\Phi(A) = -S^{-1}JA^{*}JS + \tau(A)$ for all $A \in$ Alg$\mathscr{N}$, where J is the conjugate linear involution on $\mathscr{H}$ such that $J(\mathscr{N}^{\perp}) = \mathscr{N}^{\perp}$.

LEMMA 1.2. *Let $\mathscr{N}$ be a nest on a complex Hilbert space $\mathscr{H}$ and Alg$\mathscr{N}$ be the associated nest algebra. Let $A \in$Alg$\mathscr{N}$. Then*

(1) A is the sum of a scalar and an idempotent if and only if $[A,[A,[A,T]]] = [A,T]$ for every $T \in$Alg$\mathscr{N}$.

(2) A is the sum of a scalar and an idempotent whose range belongs to $\mathscr{N}$ if and only if $[A,[A,T]] = [A,T]$ for every $T \in$Alg$\mathscr{N}$.

2. Main result and its proof

The following is our main result.

THEOREM 2.1. *Let $\mathscr{N}$ and $\mathscr{M}$ be nests on a separable complex Hilbert space $\mathscr{H}$ of dimension greater than 2, and Alg$\mathscr{N}$ and Alg$\mathscr{M}$ be the associated nest algebras. Suppose that Φ : Alg$\mathscr{N}$ $\rightarrow$Alg$\mathscr{M}$ is an additive surjective 2-local Lie isomorphism. Then one of the following holds.*

(1) $\Phi = \phi + \tau$, where ϕ is an isomorphism from Alg$\mathscr{N}$ onto Alg$\mathscr{M}$, and τ is a linear map from Alg$\mathscr{N}$ into $\mathbb{C}I$ vanishing on every sum of commutators.

(2) $\Phi = -\phi + \tau$, where ϕ is an anti-isomorphism from Alg$\mathscr{N}$ onto Alg$\mathscr{M}$, and τ is a linear map from Alg$\mathscr{N}$ into $\mathbb{C}I$ vanishing on every sum of commutators.

The proof will be organized in a series of lemmas. In the following, for $A,B \in \mathrm{Alg}\mathscr{N}$, the symbol $\Phi_{A,B}$ stands for a Lie isomorphism from Alg$\mathscr{N}$ onto Alg$\mathscr{M}$ such that $\Phi(A) = \Phi_{A,B}(A)$ and $\Phi(B) = \Phi_{A,B}(B)$.

LEMMA 2.2. *Let Φ be an additive surjective 2-local Lie isomorphism from Alg$\mathscr{N}$ onto Alg$\mathscr{M}$. Then*

(1) Φ is bijective and linear;

(2) Φ^{-1} is also an additive 2-local Lie isomorphism;

(3) $\Phi(\mathbb{C}I) = \mathbb{C}I$;

(4) Φ preserves the commutativity.

Proof. (1) We only need show that Φ is homogeneous and injective. Let $\lambda \in \mathbb{C}$ and $A \in \mathrm{Alg}\mathscr{N}$. Then

$$\Phi(\lambda A) = \Phi_{A,\lambda A}(\lambda A) = \lambda \Phi_{A,\lambda A}(A) = \lambda \Phi(A).$$

Hence Φ is homogeneous. If $\Phi(A) = 0$, then $\Phi_{A,A}(A) = 0$ and $A = 0$. Hence Φ is injective.

(2) For $C,D \in \mathrm{Alg}\mathscr{M}$, there exist $A,B \in \mathrm{Alg}\mathscr{N}$ such that $\Phi(A) = C$ and $\Phi(B) = D$. Then there is a Lie isomorphism $\Phi_{A,B}$: $\mathrm{Alg}\mathscr{N} \rightarrow \mathrm{Alg}\mathscr{M}$ such that $C = \Phi(A) = \Phi_{A,B}(A)$ and $D = \Phi(B) = \Phi_{A,B}(B)$. Hence we have $A = \Phi^{-1}(C) = \Phi_{A,B}^{-1}(C)$ and $B = \Phi^{-1}(D) = \Phi_{A,B}^{-1}(D)$. Note that $\Phi_{A,B}^{-1}$ is a Lie isomorphism from Alg$\mathscr{M}$ onto Alg$\mathscr{N}$. Hence Φ^{-1} is also a 2-local Lie isomorphism.

(3) Let $\lambda \in \mathbb{C}$. For any $A \in \mathrm{Alg}\mathscr{N}$, we have

$$\begin{aligned}[\Phi(\lambda I), \Phi(A)] &= [\Phi_{\lambda I,A}(\lambda I), \Phi_{\lambda I,A}(A)] \\ &= \Phi_{\lambda I,A}([\lambda I, A]) = 0.\end{aligned}$$

Since Φ is surjective, we have $\Phi(\lambda I)C = C\Phi(\lambda I)$ for any $C \in \mathrm{Alg}\mathscr{M}$. By Corollary 19.5 in [6], we have $\Phi(\lambda I) \in \mathbb{C}I$, which implies $\Phi(\mathbb{C}I) \subseteq \mathbb{C}I$. Similarly, by (2), we can show that $\Phi^{-1}(\mathbb{C}I) \subseteq \mathbb{C}I$. Hence $\Phi(\mathbb{C}I) = \mathbb{C}I$.

(4) Let $A,B \in \mathrm{Alg}\mathcal{N}$ and $AB = BA$. Then

$$\begin{aligned} 0 = \Phi_{A,B}[A,B] &= [\Phi_{A,B}(A), \Phi_{A,B}(B)] \\ &= [\Phi(A), \Phi(B)], \end{aligned}$$

which implies the commutativity preservation of Φ. □

By Lemma 1.2, the following lemma is obvious.

LEMMA 2.3. *(1) If E is an idempotent operator in $\mathrm{Alg}\mathcal{N}$, then $\Phi(E) = F + \lambda_E I$, where $\lambda_E \in \mathbb{C}$ and F is an idempotent operator in $\mathrm{Alg}\mathcal{N}$. Furthermore, if $\mathrm{ran}E \in \mathcal{N}$, then $\mathrm{ran}F \in \mathcal{M}$. If $0 \neq E \neq I$, then both the scalar λ_E and the idempotent F occurring above are uniquely determined.*

(2) If F is an idempotent operator in $\mathrm{Alg}\mathcal{M}$, then $\Phi^{-1}(F) = E + \lambda_F I$, where $\lambda_F \in \mathbb{C}$ and E is an idempotent operator in $\mathrm{Alg}\mathcal{M}$. Furthermore, if $\mathrm{ran}F \in \mathcal{M}$, then $\mathrm{ran}E \in \mathcal{N}$. If $0 \neq F \neq I$, then both the scalar λ_F and the idempotent E occurring above are uniquely determined.

By Lemma 2.3, if $\mathcal{N}$ is trivial, i.e., $\mathcal{N} = \{0, \mathcal{H}\}$, then $\mathcal{M} = \{0, \mathcal{H}\}$, and then $\mathrm{Alg}\mathcal{N}$ =$\mathrm{Alg}\mathcal{M} = B(\mathcal{H})$. It follows from the main result of [10] that we can obtain Theorem 2.1. In the foregoing, we always assume that $\mathcal{N}$ is nontrivial, which implies the existence of a non-trivial subspace $N \in \mathcal{N}$ and its associated projection P belongs to $\mathrm{Alg}\mathcal{N}$. N is a fixed subspace and both N and its associated projection P are going to be crucial to obtain Theorem 2.1, concretely in the proof of Lemma 2.10. By Lemma 2.3, we have $\Phi(P) = Q + \lambda_P I$, where $\lambda_P \in \mathbb{C}$ and Q is an idempotent operator in $\mathrm{Alg}\mathcal{M}$ with $\mathrm{ran}\,Q \in \mathcal{M}$.

Let $\mathcal{E}(\mathcal{N})$ and $\mathcal{E}(\mathcal{M})$ denote the sets of all idempotents in $\mathrm{Alg}\mathcal{N}$ and $\mathrm{Alg}\mathcal{M}$, respectively. By Lemma 2.3, we can define a map $\hat{\Phi} : \mathcal{E}(\mathcal{N})\backslash\{0,I\} \to \mathcal{E}(\mathcal{M})\backslash\{0,I\}$ by $\hat{\Phi}(E) = \Phi(E) - \lambda_E I$. Since Φ and Φ^{-1} are both surjective 2-local Lie isomorphisms, it is easy to prove that $\hat{\Phi}$ is bijective.

Let $E_1, E_2 \in \mathrm{Alg}\mathcal{N}$ be any two idempotents. We say that $E_1 \leqslant E_2$ if $E_1E_2 = E_1 = E_2E_1$, or equivalently, E_1 and E_2 commute and $\mathrm{ran}E_1 \subseteq \mathrm{ran}E_2$. We say that $E_1 < E_2$ if $E_1 \leqslant E_2$ and $E_1 \neq E_2$. The proofs of the following two lemmas are taken from the proofs of Lemma 3.8 and Lemma 3.9 in [16]. We include them for completeness reasons.

LEMMA 2.4. *Let $E_1, E_2 \in \mathrm{Alg}\mathcal{N}$ be two idempotents with $0 < E_1 < E_2 < I$. Set $\hat{\Phi}(E_i) = F_i, i = 1,2$. Then either $0 < F_1 < F_2 < I$ or $0 < F_2 < F_1 < I$.*

Proof. Since $E_1E_2 = E_2E_1$, we have

$$\begin{aligned} 0 = \Phi_{E_1,E_2}([E_1,E_2]) &= [\Phi_{E_1,E_2}(E_1), \Phi_{E_1,E_2}(E_2)] \\ &= [\Phi(E_1), \Phi(E_2)] = [\hat{\Phi}(E_1), \hat{\Phi}(E_2)] = [F_1, F_2], \end{aligned}$$

that is $F_1F_2 = F_2F_1$. Now $E_1, E_2, E_1 - E_2 \notin \mathbb{C}I$ implies $F_1, F_2, F_1 - F_2 \notin \mathbb{C}I$, and in particular, $F_1 \neq F_2$. We may chose a Hamel basis that diagonalizes F_1 and F_2 simultaneously. Now if F_1 and F_2 are not comparable, then $\{-1,1\} \subseteq \sigma(F_1 - F_2)$. But

$F_1 - F_2 = \Phi(E_1) - \lambda_{E_1}I - (\Phi(E_2) - \lambda_{E_2}I) = \Phi(E_1 - E_2) + (\lambda_{E_2} - \lambda_{E_1})I = \Phi(E_1 - E_2 - I) - \Phi(I) + (\lambda + \lambda_{E_2} - \lambda_{E_1})I \in \mathscr{E}(\mathscr{M}) + \mathbb{C}I$. Thus there exists a scalar λ such that $\sigma(F_1 - F_2) = \{\lambda, \lambda + 1\}$, a contradiction. The proof is complete. □

LEMMA 2.5. *Let $E_1, E_2, E_3 \in Alg\mathscr{N}$ be idempotents with $0 < E_1 < E_2 < E_3 < I$. Set $\hat{\Phi}(E_i) = F_i, i = 1, 2, 3$.*

(1) If $F_1 < F_2$, then $F_1 < F_2 < F_3$.

(2) If $F_2 < F_1$, then $F_1 > F_2 > F_3$.

Proof. (1) Assume $F_1 < F_2$. By Lemma 2.4, F_1, F_2, F_3 are distinct and mutually comparable. Since $E_1 + E_3 - E_2 \in \mathscr{E}(\mathscr{N})$, we have $F_1 + F_3 - F_2 \in \mathscr{E}(\mathscr{M}) + \mathbb{C}I$, and so $\sigma(F_1 + F_3 - F_2) = \{\lambda, \lambda + 1\}$ for some $\lambda \in \mathbb{C}$. If $F_1 < F_3 < F_2$ or $F_3 < F_1 < F_2$, then $(F_1 + F_3 - F_2)^3 = F_1 + F_3 - F_2$. It follows that $\sigma(F_1 + F_3 - F_2) = \{0, 1, -1\}$, a contradiction. So we have $F_1 < F_2 < F_3$. Similarly, we can show that (2) holds. □

LEMMA 2.6. *(1) If there exists an idempotent $E_1 \in Alg\mathscr{N}$ such that $E_1 < P$ and $\hat{\Phi}(E_1) < \hat{\Phi}(P) = Q$ (or $E_1 > P$ and $\hat{\Phi}(E_1) > \hat{\Phi}(P) = Q$), then for any idempotent $E \in Alg\mathscr{N}$, $E < P$ implies $\hat{\Phi}(E) < Q$ and $E > P$ implies $\hat{\Phi}(E) > Q$.*

(2) If there exists an idempotent $E_1 \in Alg\mathscr{N}$ such that $E_1 < P$ and $\hat{\Phi}(E_1) > \hat{\Phi}(P) = Q$ (or $E_1 > P$ and $\hat{\Phi}(E_1) < \hat{\Phi}(P) = Q$), then for any idempotent $E \in Alg\mathscr{N}$, $E < P$ implies $\hat{\Phi}(E) > Q$ and $E > P$ implies $\hat{\Phi}(E) < Q$.

Proof. We shall prove (1). Part (2) is similar. Assume that there is an idempotent $E_1 \in \mathrm{Alg}\mathscr{N}$ such that $E_1 < P$ and $\hat{\Phi}(E_1) < \hat{\Phi}(P) = Q$. Clearly, given an idempotent $E \in \mathrm{Alg}\mathscr{N}$ satisfying $E > P$, we have $E_1 < P < E$. By Lemma 2.5, we have $\hat{\Phi}(E_1) < Q < \hat{\Phi}(E)$. Now assume that $E < P$. By Lemma 2.4, either $\hat{\Phi}(E) < Q$ or $\hat{\Phi}(E) > Q$. If $\hat{\Phi}(E) > Q$ occurs, then we have $\hat{\Phi}(E) > Q > \hat{\Phi}(E_1)$. Applying Lemma 2.4 and 2.5 to Φ^{-1}, we get as a contradiction that $E_1 < P < E$. Hence we must have $\hat{\Phi}(E) < Q$. The case that there exists an idempotent $E_1 \in \mathrm{Alg}\mathscr{N}$ such that $E_1 > P$ and $\hat{\Phi}(E_1) > \hat{\Phi}(P) = Q$ is dealt with in the same way. □

By Lemma 2.6, we may extend the definition of $\hat{\Phi}$ to the whole set of idempotents, $\mathscr{E}(\mathscr{N})$, by $\hat{\Phi}(0) = 0$, $\hat{\Phi}(I) = I$ if $\hat{\Phi}$ satisfies Lemma 2.6(1), and $\hat{\Phi}(0) = I$, $\hat{\Phi}(I) = 0$ if $\hat{\Phi}$ satisfies Lemma 2.6(2).

Up to now, we have proved that, if Φ satisfies the assumption in Theorem 2.1, then either Lemma 2.6(1) or Lemma 2.6(2) occurs. In the rest of this section, we deal with these two cases respectively.

Case 1. If the case of Lemma 2.6(1) occurs, then $\Phi = \phi + \tau$, where ϕ is an isomorphism from $\mathrm{Alg}\mathscr{N}$ onto $\mathrm{Alg}\mathscr{M}$, and τ is a linear map from $\mathrm{Alg}\mathscr{N}$ into $\mathbb{C}I$ vanishing on every sum of commutators.

We will complete the proof of Case 1 by checking some lemmas. Now, let $\mathscr{A}_{11} = P(\mathrm{Alg}\mathscr{N})P$, $\mathscr{A}_{12} = P(\mathrm{Alg}\mathscr{N})(I-P)$, $\mathscr{A}_{22} = (I-P)(\mathrm{Alg}\mathscr{N})(I-P)$, $\mathscr{B}_{11} = Q(\mathrm{Alg}\mathscr{M})Q$, $\mathscr{B}_{12} = Q(\mathrm{Alg}\mathscr{M})(I-Q)$, $\mathscr{B}_{22} = (I-Q)(\mathrm{Alg}\mathscr{M})(I-Q)$. Then $\mathrm{Alg}\mathscr{N} = \mathscr{A}_{11} \oplus \mathscr{A}_{12} \oplus \mathscr{A}_{22}$, $\mathrm{Alg}\mathscr{M} = \mathscr{B}_{11} \oplus \mathscr{B}_{12} \oplus \mathscr{B}_{22}$.

LEMMA 2.7. *Let Φ be an additive surjective 2-local Lie isomorphism satisfying Lemma 2.6(1). Then $\Phi(\mathscr{A}_{12}) = \mathscr{B}_{12}$.*

Proof. For any $A_{12} \in \mathscr{A}_{12}$, we have

$$\begin{aligned}\Phi(A_{12}) &= \Phi_{P,A_{12}}(A_{12}) = \Phi_{P,A_{12}}([P,A_{12}]) = [\Phi_{P,A_{12}}(P), \Phi_{P,A_{12}}(A_{12})] \\ &= [\Phi(P), \Phi(A_{12})] = [Q, \Phi(A_{12})] = Q\Phi(A_{12})(I-Q),\end{aligned}$$

which implies that $\Phi(\mathscr{A}_{12}) \subseteq \mathscr{B}_{12}$.

On the other hand, applying the same argument to Φ^{-1}, one can prove that $\Phi(\mathscr{A}_{12}) \supseteq \mathscr{B}_{12}$. $\square$

LEMMA 2.8. *Let Φ be an additive surjective 2-local Lie isomorphism satisfying Lemma 2.6(1). Then there exists a linear map $f_i : \mathscr{A}_{ii} \to \mathbb{C}$ such that $\Phi(A_{ii}) - f_i(A_{ii})I \in \mathscr{B}_{ii}$ for all $A_{ii} \in \mathscr{A}_{ii}, i = 1,2$. Moreover, for each $B_{ii} \in \mathscr{B}_{ii}$ there is $A_{ii} \in \mathscr{A}_{ii}$ such that $\Phi(A_{ii}) = B_{ii} + f_i(A_{ii})I$*

Proof. We only consider the case $i = 1$. The proof for the other case is analogous. For any $A_{11} \in \mathscr{A}_{11}$, let $\Phi(A_{11}) = T_{11} + T_{12} + T_{22}$. Then we have

$$\begin{aligned}0 &= \Phi([P,A_{11}]) = \Phi_{P,A_{11}}([P,A_{11}]) = [\Phi_{P,A_{11}}(P), \Phi_{P,A_{11}}(A_{11})] \\ &= [\Phi(P), \Phi(A_{11})] = [Q, \Phi(A_{11})],\end{aligned}$$

which implies that $T_{12} = 0$. Let $E \in \mathscr{A}_{22}$ be any idempotent with $E \neq I - P$. It is clear that $E < I - P$, equivalently, $I - E > P$. Since Φ meets Lemma 2.6(1), we have $\hat{\Phi}(I-E) = I - \hat{\Phi}(E) > Q$, i.e., $\hat{\Phi}(E) < I - Q$. Then

$$\begin{aligned}0 &= \Phi_{A_{11},E}([A_{11},E]) = [\Phi_{A_{11},E}(A_{11}), \Phi_{A_{11},E}(E)] = [\Phi(A_{11}), \Phi(E)] \\ &= [T_{11} + T_{22}, \hat{\Phi}(E)] = [T_{22}, \hat{\Phi}(E)].\end{aligned}$$

Since E is arbitrary, T_{22} commutes with every idempotent in $\mathscr{B}_{22}$. By Lemma 2.3 of [8] or Lemma 3.2 of [9], T_{22} commutes with every finite rank operator of $\mathscr{B}_{22}$. Hence $T_{22} \in \mathbb{C}(I-Q)$ (see [7]), that is $T_{22} = f_1(A_{11})(I-Q)$ for some $f_1(A_{11}) \in \mathbb{C}$. Thus

$$\Phi(A_{11}) = T_{11} + f_1(A_{11})(I-Q) = T_{11} - f_1(A_{11})Q + f_1(A_{11})I.$$

From this, we see that $\Phi(A_{11}) - f_1(A_{11})I \in \mathscr{B}_{11}$

To see that f_1 is linear, let A_{11}, B_{11} be two elements in $\mathscr{A}_{11}$ and λ be a scalar. Then

$$\Phi(A_{11}) - f_1(A_{11})I \in \mathscr{B}_{11}, \Phi(\lambda A_{11}) - f_1(\lambda A_{11})I \in \mathscr{B}_{11}$$

and

$$\Phi(A_{11}+B_{11}) - f_1(A_{11}+B_{11})I \in \mathscr{B}_{11}, \Phi(A_{11}) - f_1(A_{11})I \in \mathscr{B}_{11}, \Phi(B_{11}) - f_1(B_{11})I \in \mathscr{B}_{11}.$$

It follows from the linearity of Φ that $(f_1(\lambda A_{11}) - \lambda f_1(A_{11}))I \in \mathscr{B}_{11}$ and $(f_1(A_{11} + B_{11}) - f_1(A_{11}) - f_1(B_{11}))I \in \mathscr{B}_{11}$. This forces that $f_1(\lambda A_{11}) = \lambda f_1(A_{11})$ and $f_1(A_{11} + B_{11}) = f_1(A_{11}) + f_1(B_{11})$.

Finally, let $B_{11} \in \mathscr{B}_{11}$. Applying the preceding result to Φ^{-1}, there exist an $A_{11} \in \mathscr{A}_{11}$ and a scalar $\lambda \in \mathbb{C}$ such that $\Phi(A_{11} + \lambda I) = B_{11}$. Then $\Phi(A_{11}) = B_{11} + \mu I$ for some $\mu \in \mathbb{C}$. This implies $\Phi(A_{11}) - \mu I \in \mathscr{B}_{11}$. So $\mu = f_1(A_{11})$, completing the proof. $\square$

Let Φ be an additive surjective 2-local Lie isomorphism satisfying Lemma 2.6(1). Now we define the mapping Ψ: Alg$\mathscr{N} \to$ Alg$\mathscr{M}$ given by $\Psi(A) = \Phi(A) - (f_1(PAP) + f_2((I-P)A(I-P)))I$ for each $A \in$ Alg$\mathscr{N}$. Having in mind Lemma 2.7 and Lemma 2.8, Ψ satisfies the following properties.

LEMMA 2.9. *Let $A_{ij} \in \mathscr{A}_{ij}, 1 \leqslant i \leqslant j \leqslant 2$. Then*

(1) $\Psi(\mathscr{A}_{ij}) = \mathscr{B}_{ij}, 1 \leqslant i \leqslant j \leqslant 2$;

(2) $\Psi(A_{12}) = \Phi(A_{12})$;

(3) $\Psi(P) = Q, \Psi(I-P) = I - Q$;

(4) Ψ is linear and bijective;

(5) Ψ preserves the commutativity;

(6) For any idempotent P in $\mathscr{A}_{11}$ or $\mathscr{A}_{22}$, we have $\Psi(P) = \hat{\Phi}(P)$.

LEMMA 2.10. *There exist an isomorphism ϕ :Alg$\mathscr{N} \to$Alg$\mathscr{M}$ and a linear map τ_1 :Alg$\mathscr{N} \to \mathbb{C}I$ such that $\Psi = \phi + \tau_1$.*

Proof. Since Ψ is a bijective linear map preserving the commutativity, it follows from Corollary 5.4 in [3] that

$$\Psi = \alpha\phi + \tau_1,$$

where α is a non-zero scalar, ϕ is an isomorphism or anti-isomorphism of Alg$\mathscr{N}$ onto Alg$\mathscr{M}$, and τ_1 is a linear map from Alg$\mathscr{N}$ into $\mathbb{C}I$. We will finish the proof by showing that $\alpha = 1$ and ϕ is an isomorphism. We have that

$$Q = \Psi(P) = \alpha\phi(P) + \beta I$$

for some $\beta \in \mathbb{C}$. Since both Q and $\phi(P)$ are idempotents, we get

$$\alpha\phi(P) + \beta I = (\alpha^2 + 2\alpha\beta)\phi(P) + \beta^2 I.$$

Since $\phi(P) \notin \mathbb{C}I$, we have that $\alpha^2 + 2\alpha\beta - \alpha = 0$ and $\beta - \beta^2 = 0$. Hence either $\alpha = 1, \beta = 0$ or $\alpha = -1, \beta = 1$. Let $A \in \mathscr{A}_{22}$ be any non-zero idempotent with $A < I - P$ (equivalently, $P < I - A$). Since Φ meets Lemma 2.6(1), we have

$$\Psi(P) = \hat{\Phi}(P) < \hat{\Phi}(I-A) = I - \hat{\Phi}(A).$$

We claim that $\alpha = 1, \beta = 0$. Otherwise, we have that $\alpha = -1, \beta = 1$, it follows from Lemma 2.9 that

$$\Psi(A) = \hat{\Phi}(A) < I - \Psi(P) = \phi(P) = -\phi(I-P) + I.$$

Therefore

$$(-\phi(A) + \gamma I)(-\phi(I-P) + I) = -\phi(A) + \gamma I$$

for some $\gamma \in \mathbb{C}$, which implies that $\phi(A) = \phi(\gamma(I-P))$, and hence $A = \gamma(I-P)$, which is a contradiction. Let A_{12} be a non-zero element in $\mathscr{A}_{12}$. Then $\Psi(A_{12}) = \phi(A_{12}) + \eta I$ for some scalar η. Since both $\Psi(A_{12})$ and $\phi(A_{12})$ are square-zero, it follows that $2\eta\phi(A_{12}) + \eta^2 I = 0$. Since $\phi(A_{12}) \notin \mathbb{C}I$, we get that $\eta = 0$, so $\Psi(A_{12}) = \phi(A_{12})$. Finally, if ϕ is anti-isomorphism, then we get that

$$\Psi(A_{12}) = Q\Psi(A_{12}) = \phi(P)\phi(A_{12}) = \phi(A_{12}P) = 0.$$

This contradiction shows that ϕ is an isomorphism. $\square$

LEMMA 2.11. *Every additive surjective 2-local Lie isomorphism Φ satisfying Lemma 2.6(1) decomposes as the sum $\phi + \tau$, where ϕ is an isomorphism from $\mathrm{Alg}\,\mathscr{N}$ onto $\mathrm{Alg}\,\mathscr{M}$, and τ is a linear map from $\mathrm{Alg}\,\mathscr{N}$ into $\mathbb{C}I$ vanishing on every sum of commutators.*

Proof. We define the linear map $\tau\colon \mathrm{Alg}\,\mathscr{N} \to \mathbb{C}I$ given by $\tau(A) = \Phi(A) - \phi(A)$ for every $A \in \mathrm{Alg}\,\mathscr{N}$. Then $\Phi = \phi + \tau$. Since each isomorphism from $\mathrm{Alg}\,\mathscr{N}$ onto $\mathrm{Alg}\,\mathscr{M}$ is spatially implemented [22], there exists an invertible operator $T \in B(\mathscr{H})$ such that $\Phi(A) = T^{-1}AT + \tau(A)$ for all $A \in \mathrm{Alg}\,\mathscr{N}$.

Since the dimension of $\mathscr{H}$ is greater than 2, there exist three non-trivial projections $P_1, P_2, P_3 \in \mathrm{Alg}\,\mathscr{N}$ such that $P_1 + P_2 + P_3 = I$ and $P_1P_2 = P_1P_3 = P_2P_3 = 0$. Indeed, if there exist two non-trivial elements $N_1, N_2 \in \mathscr{N}$ satisfying $N_1 < N_2$, then there exist three projections $P_1, P_2, P_3 \in \mathrm{Alg}\,\mathscr{N}$ such that $\mathrm{ran}\,P_1 = N_1$, $\mathrm{ran}\,P_2 = N_2 - N_1$ and $\mathrm{ran}\,P_3 = I - N_2$. So $P_1 + P_2 + P_3 = I$ and $P_1P_2 = P_1P_3 = P_2P_3 = 0$. If there only exists one non-trivial element $N \in \mathscr{N}$, either the dimension of N or the dimension of $N^{\perp}$ is at least 2, then we can find three projections $P_1, P_2, P_3 \in \mathrm{Alg}\,\mathscr{N}$ such that $P_1 + P_2 + P_3 = I$ and $P_1P_2 = P_1P_3 = P_2P_3 = 0$. Now let P_0 be $P_1 + 2P_2 + 4P_3$. Let $B \in \mathrm{Alg}\,\mathscr{N}$ be a sum of commutators. Then by Proposition 1.1, either

$$T^{-1}P_0T + \tau(P_0) = S_1^{-1}P_0S_1 + \lambda_1 I, T^{-1}BT + \tau(B) = S_1^{-1}BS_1$$

for some invertible operator $S_1 \in B(\mathscr{H})$ and scalar λ_1, or

$$T^{-1}P_0T + \tau(P_0) = -S_2^{-1}JP_0JS_2 + \lambda_2 I, T^{-1}BT + \tau(B) = -S_2^{-1}JB^*JS_2$$

for some invertible operator $S_2 \in B(\mathscr{H})$ and scalar λ_2. If the second case occurs, we have that $T^{-1}P_0T = -S_2^{-1}JP_0JS_2 + \mu I$ for some scalar μ. Taking the spectrum, we have $\sigma(P_0) = -\sigma(P_0) + \mu$, that is $\{1,2,4\} = \{-1+\mu, -2+\mu, -4+\mu\}$, a contradiction. So the first case holds. Then we have that $T^{-1}BT + \tau(B) = S_1^{-1}BS_1$, which implies that $\sigma(B) + \tau(B) = \sigma(B)$. Since the spectrum $\sigma(B)$ of B is a compact set, it follows that $\tau(B) = 0$. $\square$

Case 2. If the case of Lemma 2.6(2) occurs, then $\Phi = -\phi + \tau$, where ϕ is an anti-isomorphism from $\mathrm{Alg}\,\mathscr{N}$ onto $\mathrm{Alg}\,\mathscr{M}$, and τ is a linear map from $\mathrm{Alg}\,\mathscr{N}$ into $\mathbb{C}I$ vanishing on every sum of commutators.

Consider the map $\tilde{\Phi}: \mathrm{Alg}\,\mathscr{N} \to (\mathrm{Alg}\,\mathscr{M})^* = \mathrm{Alg}\,\mathscr{M}^{\perp}$ defined by $\tilde{\Phi}(A) = -J\Phi(A)^*J$ for all $A \in \mathrm{Alg}\,\mathscr{N}$, where J is the conjugate-linear involution on $\mathscr{H}$ such that $J(\mathscr{M}^{\perp}) = \mathscr{M}^{\perp}$.

LEMMA 2.12. *$\tilde{\Phi}$ is an additive and surjective 2-local Lie isomorphism and satisfies Lemma 2.6(1).*

Proof. For $A,B \in \mathrm{Alg}\mathscr{N}$, there exists a Lie isomorphism $\Phi_{A,B}:\mathrm{Alg}\mathscr{N} \to \mathrm{Alg}\mathscr{M}$ such that $\Phi(A) = \Phi_{A,B}(A)$ and $\Phi(B) = \Phi_{A,B}(B)$. Consider the map $\tilde{\Phi}_{A,B} : \mathrm{Alg}\mathscr{N} \to (\mathrm{Alg}\mathscr{M})^* = \mathrm{Alg}\mathscr{M}^\perp$ defined by $\tilde{\Phi}_{A,B}(A) = -J\Phi_{A,B}(A)^*J$ for all $A \in \mathrm{Alg}\mathscr{N}$. Then $\tilde{\Phi}(A) = \tilde{\Phi}_{A,B}(A), \tilde{\Phi}(B) = \tilde{\Phi}_{A,B}(B)$, and $\tilde{\Phi}$ is linear bijection. For every $C,D \in \mathrm{Alg}\mathscr{N}$, we have

$$\begin{aligned}\tilde{\Phi}_{A,B}([C,D]) &= -J\Phi_{A,B}([C,D])^*J = -J[\Phi_{A,B}(C),\Phi_{A,B}(D)]^*J \\ &= J(\Phi_{A,B}(D)\Phi_{A,B}(C) - \Phi_{A,B}(C)\Phi_{A,B}(D))^*J \\ &= (J\Phi_{A,B}(C)^*J)(J\Phi_{A,B}(D)^*J) - (J\Phi_{A,B}(D)^*J)(J\Phi_{A,B}(C)^*J) \\ &= [\tilde{\Phi}_{A,B}(C),\tilde{\Phi}_{A,B}(D)].\end{aligned}$$

Hence $\tilde{\Phi}$ is a 2-local Lie isomorphism. Since Φ is additive and surjective, $\tilde{\Phi}$ is an additive and surjective 2-local Lie isomorphism.

Finally, we prove that $\tilde{\Phi}$ meets Lemma 2.6(1). For any nontrivial idempotent $E \in \mathrm{Alg}\mathscr{N}$, $\Phi(E) = \hat{\Phi}(E) + \lambda_E I$ for some scalar λ_E, we have $\tilde{\Phi}(E) = -J\hat{\Phi}(E)^*J - \lambda_E I$. Now we define the map $\hat{\hat{\Phi}} : \mathscr{E}(\mathscr{N}) \to \mathscr{E}(\mathscr{M})^*$ by $\hat{\hat{\Phi}}(E) = I - \hat{\Phi}(E)^*$ for all idempotent $E \in \mathrm{Alg}\mathscr{N}$. Since Φ meets Lemma 2.6(2), for any idempotent $E \in \mathrm{Alg}\mathscr{N}$, if $E < P$, we have $\hat{\hat{\Phi}}(E) = I - \hat{\Phi}(E)^* < I - \hat{\Phi}(P)^* = \hat{\hat{\Phi}}(P)$; if $E > P$, we have $\hat{\hat{\Phi}}(E) = I - \hat{\Phi}(E)^* > I - \hat{\Phi}(P)^* = \hat{\hat{\Phi}}(P)$. Hence $\tilde{\Phi}$ meets Lemma 2.6(1). □

By Lemma 2.12, $\tilde{\Phi} : \mathrm{Alg}\mathscr{N} \to (\mathrm{Alg}\mathscr{M})^* = \mathrm{Alg}\mathscr{M}^\perp$ is an additive and surjective 2-local Lie isomorphism and meets Lemma 2.6(1). Thus the arguments given for Case 1 ensure that $\tilde{\Phi}$ is the sum of an isomorphism $\delta : \mathrm{Alg}\mathscr{N} \to \mathrm{Alg}\mathscr{M}^\perp$ and a linear map $\eta : \mathrm{Alg}\mathscr{N} \to \mathbb{C}I \subset \mathrm{Alg}\mathscr{M}^\perp$ vanishing on every sum of commutator. Now we define $\phi : \mathrm{Alg}\mathscr{N} \to \mathrm{Alg}\mathscr{M}$ by $\phi(A) = J\delta(A)^*J$ and $\tau : \mathrm{Alg}\mathscr{N} \to \mathbb{C}I$ by $\tau(A) = -J\eta(A)^*J$. Then $\Phi = -\phi + \tau$, where ϕ is an anti-isomorphism from $\mathrm{Alg}\mathscr{N}$ onto $\mathrm{Alg}\mathscr{M}$, and τ is a linear map from $\mathrm{Alg}\mathscr{N}$ into $\mathbb{C}I$ vanishing on every sum of commutators. This completes the proof of Case 2.

Combining Cases 1 and 2, the proof of Theorem 2.1 is finished.

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