

## FINITE RANK PERTURBATIONS AND SOLUTIONS TO THE OPERATOR RICCATI EQUATION

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*Abstract.* We consider an off-diagonal self-adjoint finite rank perturbation of a self-adjoint operator in a complex separable Hilbert space  $\mathfrak{H}_0 \oplus \mathfrak{H}_1$ , where  $\mathfrak{H}_1$  is finite dimensional. We describe the singular spectrum of the perturbed operator and establish a connection with solutions to the operator Riccati equation. In particular, we prove existence results for solutions in the case where the whole Hilbert space is finite dimensional.

### 1. Introduction

In the present article we analyse a special class of finite rank perturbations of a self-adjoint operator on a complex separable Hilbert space  $\mathfrak{H}$  and show how this is related to the existence of solutions to the so-called *operator Riccati equation*. This generalises results by Kostyrykin and Makarov in [7] where they considered rank one perturbations.

Let  $\mathbf{A}$  be a bounded self-adjoint operator on the Hilbert space  $\mathfrak{H}$  and  $\mathfrak{H}_0 \subset \mathfrak{H}$  be a closed  $\mathbf{A}$ -invariant subspace. We choose  $\mathfrak{H}_1 = \mathfrak{H}_0^\perp$  and define the self-adjoint operators  $A_i := \mathbf{A}|_{\mathfrak{H}_i}$  for  $i = 0, 1$ . Assume that the perturbation  $\mathbf{V} : \mathfrak{H} \rightarrow \mathfrak{H}$  is off-diagonal with respect to the orthogonal decomposition  $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_1$ , that means  $\text{Ran}(\mathbf{V}|_{\mathfrak{H}_0}) \subset \mathfrak{H}_1$  and  $\text{Ran}(\mathbf{V}|_{\mathfrak{H}_1}) \subset \mathfrak{H}_0$ . Consider then the perturbed self-adjoint operator

$$\mathbf{B} := \mathbf{A} + \mathbf{V} = \begin{pmatrix} A_0 & V \\ V^* & A_1 \end{pmatrix} \quad \text{with } \mathbf{V} = \begin{pmatrix} 0 & V \\ V^* & 0 \end{pmatrix},$$

where  $V : \mathfrak{H}_1 \rightarrow \mathfrak{H}_0$  is a bounded operator. We will study the *operator Riccati equation* associated with the operators above

$$A_1 X - X A_0 - X V X + V^* = 0, \tag{1}$$

where the *solution*  $X$  is a densely defined operator from  $\mathfrak{H}_0$  to  $\mathfrak{H}_1$ . The name bears analogy to the familiar *Riccati equation* as an ordinary differential equation and honours the Italian mathematician *Jacopo Francesco Riccati* (1676 – 1754).

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It is well-known (see, e.g., [8, Theorem 4.4]) that the graph of a densely defined operator  $X : \mathfrak{H}_0 \supset \text{Dom}(X) \rightarrow \mathfrak{H}_1$  is invariant for  $\mathbf{B}$  if and only if  $X$  is a strong solution to the Riccati equation (1) in the sense of Definition 4.1 below.

There are sufficient conditions which assure the existence of a solution to the Riccati equation. If the spectra of the self-adjoint operators  $A_0$  and  $A_1$  are separated and the operator norm of the perturbation  $V$  is sufficiently small, then there is a bounded solution to (1). For details about the smallness of the perturbation see [13, Theorem 3.3] in combination with [8].

If the spectra of  $A_0$  and  $A_1$  are even subordinated, i.e.

$$\sup \text{spec}(A_0) \leq \inf \text{spec}(A_1),$$

then a contractive solution to the Riccati equation exists regardless of the norm of the perturbation  $V$ , see [9]. Similar results can be found in [2] and [6].

Note that in this work it is not assumed that the spectra of the operators  $A_0$  and  $A_1$  are separated. Instead, we require that the Hilbert space  $\mathfrak{H}_1$  is finite dimensional. Under this assumption, we prove existence results for the Riccati equation. We are mainly interested in bounded solutions, but we also prove some statements about unbounded solutions. In particular, all these results hold under the assumption that the whole Hilbert space  $\mathfrak{H}$  is finite dimensional.

Our main results are the following theorem and the deduced corollary.

**THEOREM 1.1.** *Assume that  $\mathfrak{H}_1$  is finite dimensional with  $n := \dim \mathfrak{H}_1$ , and suppose that  $\text{Ran } V$  is a cyclic generating subspace for the operator  $A_0$ , i.e.*

$$\overline{\text{lin span} \left\{ A_0^k v \mid k \in \mathbb{N}_0, v \in \text{Ran } V \right\}} = \mathfrak{H}_0.$$

Then one has:

- (i) *The multiplicity of the spectrum of  $\mathbf{B}$  is at most  $n$ . If there is an eigenvalue of multiplicity  $n$ , then there is a bounded solution to the Riccati equation (1).*
- (ii) *Assume that  $\mathbf{B}$  has at least  $n$  eigenvalues outside the point spectrum of  $A_0$ , counted with multiplicities, and let  $U$  be the space spanned by the associated eigenvectors. Furthermore, suppose that*

$$P_{\mathfrak{H}_1} U = \mathfrak{H}_1,$$

where  $P_{\mathfrak{H}_1} : \mathfrak{H} \rightarrow \mathfrak{H}$  is the orthogonal projection onto  $\mathfrak{H}_1$ . Then the Riccati equation (1) has a bounded solution.

**COROLLARY 1.2.** *Let  $\mathfrak{H}$  be finite dimensional and assume that the spectra of  $\mathbf{B}$  and  $A_0$  are disjoint. Then there exists at least one solution to the Riccati equation (1).*

This is an interesting existence result for the matrix Riccati equation and it seems to us that this result has not been previously known. However, in [10, Example 7.1.1]

it has been shown that there is no general existence theorem for a matrix Riccati equation that is associated with a *non-self-adjoint* block operator  $\mathbf{B}$ , that means in (1) the matrix  $V^*$  is substituted by an arbitrary matrix  $W$ . Still, for the self-adjoint case, as it is considered in this paper, neither such a counterexample nor a general proven existence result is known. Corollary 1.2 states the existence for solutions under the mild assumption that the whole spectrum of  $A_0$  is changed under the perturbation  $\mathbf{V}$ .

Now we give an infinite dimensional example for illustrating the conditions of Theorem 1.1:

EXAMPLE 1.3. Let  $\mathfrak{H}_0 = L^2([-1, 1])$  and  $\mathfrak{H}_1 = \mathbb{C}^2$ . We set  $v(t) := 1$  for all  $t \in [-1, 1]$  and choose  $A_0$  as the multiplication operator

$$(A_0x)(t) = tx(t) \quad \text{almost everywhere}$$

for  $x \in \mathfrak{H}_0$ . It is obvious that the linear span of  $v$  is a cyclic generating subspace for  $A_0$ . The operator  $A_1$  is chosen as the null matrix and the perturbation  $V$  is given with respect to the standard basis  $(e_1, e_2)$  of  $\mathbb{C}^2$  by  $Ve_1 = v$  and  $Ve_2 = v$ . Then we can simply write:

$$\mathbf{B} = \left( \begin{array}{c|cc} A_0 & v & v \\ \hline \langle v, \cdot \rangle & 0 & 0 \\ \langle v, \cdot \rangle & 0 & 0 \end{array} \right).$$

Note that  $\lambda_1 = 0$  is still an eigenvalue of  $\mathbf{B}$  because  $0 \oplus (1, -1)^T \in \mathfrak{H}_0 \oplus \mathfrak{H}_1$  lies in the kernel. Since  $\|\mathbf{B}\| > 1$  and the absolutely continuous spectrum is invariant under finite rank perturbation (see [5]), there has to be an eigenvalue outside of  $[-1, 1]$ , denoted by  $\lambda_2$  from now on. One can show that  $x \oplus (1, 1)^T \in \mathfrak{H}_0 \oplus \mathfrak{H}_1$  with suitable  $x \in \mathfrak{H}_0$  is an eigenvector for  $\lambda_2$ . This is not written out in full here since we will have more effective methods for calculating eigenvalues after section 3.

If  $U$  denotes the linear space spanned by all eigenvectors of  $\mathbf{B}$ , we know by the eigenvectors of  $\lambda_1$  and  $\lambda_2$  from above that

$$P_{\mathfrak{H}_1}U \supset \text{lin span}\{(1, -1)^T, (1, 1)^T\} = \mathfrak{H}_1$$

holds. Therefore condition (ii) of Theorem 1.1 is fulfilled and the Riccati equation has a bounded solution. In section 4 we will see how one can explicitly construct this solution.

There is a large literature concerning the spectrum of the block operator  $\mathbf{B}$  and the Riccati equation, both in the finite dimensional matrix case, see, e.g., [10], [17], and in the infinite dimensional case, see, e.g., [11], [8], [12]. In the second case it is possible to expand the operator Riccati equation to unbounded coefficients which is especially interesting in the study of perturbation theory, see [16], [3] and [14] for applications of the Riccati equation in this field of research. The last three citations all portrait the fruitful connection between invariant subspaces of operators and solutions to the Riccati equation. We will extensively use this in the last section for the construction of these solutions but in a more explicit way than it is usually seen.

## 2. Preliminaries

In this section we want to fix some notations and explain facts about the concepts that we will use in the following and need to prove Theorem 1.1. Mainly, we present facts and proofs for readers that are not familiar with Herglotz functions, multiplicity of spectra and the decomposition of the spectrum into an absolutely continuous and singular part.

We will use the notation  $\mathcal{L}(\mathfrak{H}, \mathfrak{K})$  for the set of bounded linear operators from a Hilbert space  $\mathfrak{H}$  to a Hilbert space  $\mathfrak{K}$ . Moreover, we will write  $\mathcal{L}(\mathfrak{H})$  instead of  $\mathcal{L}(\mathfrak{H}, \mathfrak{H})$ . All considered Hilbert spaces are complex and separable. The spectrum of a bounded linear operator  $T : \mathfrak{H} \rightarrow \mathfrak{H}$  is denoted by  $\text{spec}(T)$  and the point spectrum, i.e. the set of all eigenvalues, is denoted by  $\text{spec}_p(T)$ . Moreover, we will use the following notion of multiplicity of spectra, cf. [1].

DEFINITION 2.1. For a self-adjoint operator  $T \in \mathcal{L}(\mathfrak{H})$  in a Hilbert space  $\mathfrak{H}$ , we call the minimal dimension of all cyclic generating subspaces the *multiplicity of the spectrum of  $T$* . Here, a subspace  $U \subset \mathfrak{H}$  is called a cyclic generating subspace for the operator  $T$  if

$$\text{lin span} \left\{ T^k u \mid k \in \mathbb{N}_0, u \in U \right\}$$

is dense in  $\mathfrak{H}$ .

With this definition the spectrum of an operator has multiplicity 1 and is called *simple* if and only if there is a cyclic vector for this operator. If we consider finite dimensional Hilbert spaces, the multiplicity of the spectrum above coincides with the maximal multiplicity of the eigenvalues of  $T$ . In infinite dimensional Hilbert spaces it is possible for the spectrum to have infinite multiplicity, e.g., the spectrum of the identity.

Now we will explain how so-called *Herglotz functions* can be used to describe self-adjoint operators and their spectra if the multiplicity is finite.

We will always write  $\mathbb{C}_+ := \{z \in \mathbb{C} \mid \text{Im} z > 0\}$  for the upper complex half-plane and also use the following notion from [4]:

DEFINITION 2.2. (i) A holomorphic function  $m : \mathbb{C}_+ \rightarrow \mathbb{C}$  is called a *scalar Herglotz function* if  $\text{Im} m(z) \geq 0$  for all  $z \in \mathbb{C}_+$ .

(ii) An analytic function  $M : \mathbb{C}_+ \rightarrow \mathbb{C}^{n \times n}$  or  $M : \mathbb{C}_+ \rightarrow \mathcal{L}(\mathbb{C}^n)$  with  $n \in \mathbb{N}$  is called a *matrix-valued Herglotz function* or *Herglotz matrix* if

$$\text{Im} M(z) := \frac{1}{2i} (M(z) - M(z)^*) \geq 0$$

for all  $z \in \mathbb{C}_+$ .

A classical result in this theory is that every matrix-valued Herglotz function has a unique integral representation, see [4, Theorem 5.4]. Therefore, there is a characteristic example of a Herglotz function if a matrix-valued measure  $\Omega$  is given. We call a map

on the Borel sets of  $\mathbb{R}$ , denoted by  $\mathcal{B}(\mathbb{R})$ , with  $\Omega : \mathcal{B}(\mathbb{R}) \rightarrow \mathbb{C}^{n \times n}$  a *matrix-valued measure* if

$$\Omega_{y,x} : \mathcal{B}(\mathbb{R}) \rightarrow \mathbb{C}, \quad \Delta \mapsto \langle y, \Omega(\Delta)x \rangle_{\mathbb{C}^n}$$

is a (finite) complex measure for all  $x, y \in \mathbb{C}^n$ . If we demand  $\Omega(\Delta) \geq 0$  for all Borel sets  $\Delta \subset \mathbb{R}$ , the map  $\Omega_{x,x}$  is a positive measure for all  $x \in \mathbb{C}^n$ .

EXAMPLE 2.3. For each matrix-valued measure  $\Omega$  with  $\Omega(\Delta) \geq 0$  for all Borel sets  $\Delta$  the map

$$M : z \mapsto \int_{\mathbb{R}} \frac{1}{t - z} d\Omega(t)$$

defines a matrix-valued Herglotz function on  $\mathbb{C}_+$ .

Lebesgue's decomposition theorem for ordinary  $\sigma$ -finite positive measures can easily be generalised to complex measures, see [15, Theorem 6.10], and to matrix-valued measures. So for each matrix-valued measure  $\Omega$  there is a unique decomposition into an absolutely continuous and a singular measure with respect to the Lebesgue measure. By separating the atoms of the measure, the singular part can additionally split into a singularly continuous part and a pure point part:

$$\Omega = \Omega_{ac} + \Omega_s = \Omega_{ac} + \Omega_{sc} + \Omega_{pp}.$$

With regard to the example of a Herglotz matrix above, we want to analyse these parts of the measure and it turns out that an ordinary positive measure is sufficient for that task:

PROPOSITION 2.4. Assume  $\Omega : \mathcal{B}(\mathbb{R}) \rightarrow \mathbb{C}^{n \times n}$  is a matrix-valued measure with  $\Omega(\Delta) \geq 0$  for all Borel sets  $\Delta \subset \mathbb{R}$ . Then the positive measure  $\omega(\Delta) := \text{tr}(\Omega(\Delta))$  is equivalent to  $\Omega$ , i.e. they have precisely the same null sets.

*Proof.* A null set for  $\Omega$  is clearly a null set for  $\omega$ . Conversely, if we choose a Borel set  $\Delta$  with  $\omega(\Delta) = 0$ , we can calculate

$$2 |\Omega_{jk}(\Delta)| \leq \Omega_{jj}(\Delta) + \Omega_{kk}(\Delta) \leq 2 \text{tr}(\Omega(\Delta)) = 0$$

for all  $1 \leq j, k \leq n$ . The first inequality is a standard property for non-negative matrices. See for example [4, Lemma 5.1].  $\square$

In [4] the authors give describing sets for the parts of the measure and call them supports. We also use this terminology here and call a Borel set  $S \subset \mathbb{R}$  a *support* of a given Borel measure  $\mu$ , which can be positive, complex or matrix-valued, if  $\mu(\mathbb{R} \setminus S) = 0$ . We call a support  $S$  of  $\mu$  *minimal* if  $S \setminus T$  has Lebesgue measure zero for any support  $T \subset S$ .

Since we will be merely interested in the singular part of the measure for analysing the Riccati equation, we just consider supports for the singular and the pure point part.

PROPOSITION 2.5. Let  $\Omega : \mathcal{B}(\mathbb{R}) \rightarrow \mathbb{C}^{n \times n}$  be a matrix-valued measure that fulfils  $\Omega(\Delta) \geq 0$  for all Borel sets  $\Delta \subset \mathbb{R}$  and  $M : \mathbb{C}_+ \rightarrow \mathbb{C}^{n \times n}$  the matrix-valued Herglotz function from Example 2.3. Then the set

$$S_{\Omega,s} := \left\{ \lambda \in \mathbb{R} \mid \lim_{\varepsilon \rightarrow 0^+} \operatorname{tr} \operatorname{Im} M(\lambda + i\varepsilon) = \infty \right\}$$

is a minimal support of the singular part  $\Omega_s$ . The set

$$S_{\Omega,pp} := \left\{ \lambda \in \mathbb{R} \mid \lim_{\varepsilon \rightarrow 0^+} \varepsilon \operatorname{tr} M(\lambda + i\varepsilon) \neq 0 \right\}$$

is the smallest support of the pure point part  $\Omega_{pp}$ .

*Proof.* By the equivalence of the measures  $\Omega$  and  $\operatorname{tr}\Omega$ , one can use the support theorem [4, Theorem 3.1] for scalar Herglotz functions or the support theorem [4, Theorem 6.1] for Herglotz matrices.  $\square$

In the next proposition we present a fundamental example of a Herglotz function in relation to a self-adjoint operator  $T \in \mathcal{L}(\mathfrak{H})$ , where we will write  $E_T$  for its projector-valued spectral measure. In section 3 we will concretise this example.

PROPOSITION 2.6. Let  $T \in \mathcal{L}(\mathfrak{H})$  be self-adjoint with multiplicity of the spectrum  $p \in \mathbb{N}$  and  $n \geq p$  be an integer number. Moreover, let  $V : \mathbb{C}^n \rightarrow \mathfrak{H}$  be a linear operator such that  $\operatorname{Ran} V$  is a cyclic generating subspace for  $T$ . Then

$$M : z \mapsto V^*(T - z)^{-1}V$$

is a matrix-valued Herglotz function for  $z \in \mathbb{C}_+$  which can be represented by a matrix-valued measure  $\Omega$  as

$$M(z) = \int_{\mathbb{R}} \frac{1}{t - z} d\Omega(t).$$

The measure  $\Omega$  is equivalent to the spectral measure  $E_T$ , i.e. they have the same null sets. The set

$$S_{pp} := \left\{ \lambda \in \mathbb{R} \mid \lim_{\varepsilon \rightarrow 0^+} \varepsilon \operatorname{tr} V^*(T - \lambda - i\varepsilon)^{-1}V \neq 0 \right\} \tag{2}$$

coincides with the point spectrum of  $T$ .

*Proof.* Since  $z \mapsto (T - z)^{-1}$  is analytic and because of the first resolvent identity, one can write

$$\operatorname{Im} M(z) = \operatorname{Im} z [V^*(T - \bar{z})^{-1}(T - z)^{-1}V].$$

Obviously  $\operatorname{Im} M(z) \geq 0$  holds and therefore  $M$  is a matrix-valued Herglotz function. If we now define

$$\Omega(\Delta) := V^* E_T(\Delta) V$$

for every Borel set  $\Delta \subset \mathbb{R}$ , we get a matrix-valued measure and the representation for  $M$  holds by the spectral theorem. Clearly, a null set of  $E_T$  is also a null set of  $\Omega$ . On the other hand, a Borel set  $\Delta$  with  $\Omega(\Delta) = 0$  fulfils via the polarisation identity

$$\langle v, E_T(\Delta')u \rangle = 0 \text{ for all } u, v \in \operatorname{Ran} V,$$

and for all Borel sets  $\Delta' \subset \Delta$ . Now we can use the spectral theorem for a measurable function  $f(t) := t^m \chi_\Delta(t) t^k$ , where  $\chi_\Delta$  is the characteristic function of  $\Delta$  and  $k, m$  non-negative integers:

$$0 = \int_{\Delta} t^{k+m} d\langle y, E_T(t)u \rangle = \langle y, f(T)u \rangle = \langle T^m v, \chi_\Delta(T) T^k u \rangle.$$

Since  $u, v \in \text{Ran } V$  and  $\text{Ran } V$  is a cyclic generating subspace for  $T$ , the following equation is true for all  $x, y \in \mathfrak{H}$ :

$$0 = \langle y, \chi_\Delta(T)x \rangle = \int_{\Delta} d\langle y, E_T(t)x \rangle = \langle y, E_T(\Delta)x \rangle.$$

That means that  $E_T(\Delta) = 0$  and the measures are equivalent.

The smallest support of  $\Omega_{pp}$  from Proposition 2.5 is therefore also a smallest support of the pure point part of the spectral measure  $E_T$  and it is well-known that the atoms of  $E_T$  are exactly the eigenvalues of  $T$ .  $\square$

In the next definition we will decompose the spectrum of a self-adjoint operator into three parts. Analogously to above, one can generalise Lebesgue's decomposition theorem even to a projector-valued measures like  $E_T$ . This is due to the fact that

$$\Delta \mapsto \langle y, E_T(\Delta)x \rangle$$

is a complex measure for every  $x, y \in \mathfrak{H}$  which has a unique Lebesgue decomposition.

**DEFINITION 2.7.** For a self-adjoint operator  $T \in \mathcal{L}(\mathfrak{H})$  with spectral measure  $E_T$  that has the Lebesgue decomposition

$$E_T = E_{T,ac} + E_{T,s} = E_{T,ac} + E_{T,sc} + E_{T,pp},$$

we define the following sets for  $w \in \{ac, s, sc, pp\}$

$$\text{spec}_w(T) := \{\lambda \in \mathbb{R} \mid \text{every open neighbourhood } U \text{ of } \lambda \text{ fulfils } E_{T,w}(U) \neq 0\}.$$

These closed sets  $\text{spec}_{ac}(T)$ ,  $\text{spec}_s(T)$ ,  $\text{spec}_{sc}(T)$  and  $\text{spec}_{pp}(T)$  are called the *absolutely continuous*, *singular*, *singularly continuous* and *pure point spectrum* of  $T$ , respectively.

Now it can be shown, cf. [5, Chapter 10], that for each self-adjoint operator  $T$  there is a decomposition of its spectrum into

$$\text{spec}(T) = \text{spec}_{ac}(T) \cup \text{spec}_{sc}(T) \cup \text{spec}_{pp}(T).$$

Admittedly, none of this unions has to be disjoint. Note also that the pure point spectrum is in general larger than the set of eigenvalues  $\text{spec}_p(T)$  since the latter does not have to be closed. However,

$$\text{spec}_{pp}(T) = \overline{\text{spec}_p(T)}$$

is always true.

### 3. Eigenvalues and singularly continuous spectrum of $\mathbf{B}$

Throughout this work we always assume the hypothesis below.

**HYPOTHESIS 3.1.** *Let  $\mathbf{B}$  be a bounded self-adjoint operator which is represented with respect to the orthogonal decomposition  $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_1$  as an operator block matrix*

$$\mathbf{B} := \begin{pmatrix} A_0 & V \\ V^* & A_1 \end{pmatrix},$$

where  $A_j \in \mathcal{L}(\mathfrak{H}_j)$  is self-adjoint for  $j = 0, 1$  and  $V \in \mathcal{L}(\mathfrak{H}_1, \mathfrak{H}_0)$ .

Assume in addition that the Hilbert-space  $\mathfrak{H}_1$  is finite dimensional and that  $\text{Ran } V$  is a cyclic generating subspace for the operator  $A_0$ , i.e.

$$\text{lin span} \left\{ A_0^k v \mid k \in \mathbb{N}_0, v \in \text{Ran } V \right\}$$

is dense in  $\mathfrak{H}_0$ .

Since this hypothesis claims that the multiplicity of the spectra  $A_0$  and  $A_1$  are not greater than  $\dim \mathfrak{H}_1$ , respectively, the multiplicity of the spectrum of  $\mathbf{B}$  is also restrained.

**LEMMA 3.2.** *Assume Hypothesis 3.1. Then  $\mathfrak{H}_1 \subset \mathfrak{H}$  is a cyclic generating subspace for  $\mathbf{B}$ . In particular, the multiplicity of the spectrum of  $\mathbf{B}$  cannot exceed  $\dim \mathfrak{H}_1$ .*

*Proof.* Set  $n := \dim \mathfrak{H}_1$  and choose a basis  $(e_i)_{i=1, \dots, n}$  of  $\mathfrak{H}_1$ . Since  $\text{Ran } V$  is a cyclic generating subspace for  $A_0$  by Hypothesis 3.1, one concludes that

$$\text{lin span} \{ Ve_i \oplus 0, 0 \oplus e_i \in \mathfrak{H}_0 \oplus \mathfrak{H}_1 \mid 1 \leq i \leq n \}$$

is a cyclic generating subspace for  $\mathbf{B}$ . If we substitute  $Ve_i \oplus 0$  with  $Ve_i \oplus A_1 e_i$ , the statement above will remain true. Since  $\mathbf{B}(0 \oplus e_i) = Ve_i \oplus A_1 e_i$  holds, the space

$$\text{lin span} \{ 0 \oplus e_i \in \mathfrak{H}_0 \oplus \mathfrak{H}_1 \mid 1 \leq i \leq n \} = \mathfrak{H}_1$$

is already a cyclic generating subspace for the operator  $\mathbf{B}$ .  $\square$

It follows from the lemma above that the spectrum of the self-adjoint operator  $\mathbf{A} := A_0 \oplus A_1$ , which could have the multiplicity  $2 \cdot \dim \mathfrak{H}_1$ , is always altered by the off-diagonal perturbation such that the multiplicity is at most only  $\dim \mathfrak{H}_1$ .

It is possible to classify the eigenvalues of  $\mathbf{B}$  into three distinct cases and it will turn out that this is necessary for finding solutions to the Riccati equation.

**LEMMA 3.3.** *Assume Hypothesis 3.1. A real number  $\lambda \in \mathbb{R}$  is an eigenvalue of the operator  $\mathbf{B}$  with multiplicity  $k$  if and only if there is a set of  $k$  linear independent vectors  $\{y_j\}_{j=1, \dots, k} \subset \mathfrak{H}_1$  with*

$$V y_j \in \text{Ran}(A_0 - \lambda), \quad j = 1, \dots, k,$$

and for each  $j$  one of the following statements holds:



(i)  $\lambda \notin \text{spec}_p(A_0)$  and

$$(A_1 - \lambda)y_j = V^*(A_0 - \lambda)^{-1}Vy_j.$$

(ii)  $\lambda \in \text{spec}_p(A_0)$  and

$$(A_1 - \lambda)y_j = \lim_{\varepsilon \rightarrow 0^+} V^*(A_0 - \lambda - i\varepsilon)^{-1}Vy_j.$$

(iii)  $\lambda \in \text{spec}_p(A_0)$  with an eigenvector  $x \in \mathfrak{H}_0$  and

$$(A_1 - \lambda)y_j = \lim_{\varepsilon \rightarrow 0^+} V^*(A_0 - \lambda - i\varepsilon)^{-1}Vy_j - V^*x.$$

Note that  $\text{Ran}(A_0 - \lambda) \subset (\text{Ran } E_{A_0}(\{\lambda\}))^\perp$  always holds and therefore the limit  $\lim_{\varepsilon \rightarrow 0^+} V^*(A_0 - \lambda - i\varepsilon)^{-1}Vy_j$  is well-defined by the spectral theorem. Here,  $E_{A_0}$  stands for the spectral measure of the self-adjoint operator  $A_0$ .

*Proof.* We will omit the proof of the multiplicity part because it is straightforward after having proved the following. We will just prove here that a real number  $\lambda$  is an eigenvalue of  $\mathbf{B}$  if and only if one of the three statements is fulfilled for a non-zero vector  $y_1 \in \mathfrak{H}_1$  with  $Vy_1 \in \text{Ran}(A_0 - \lambda)$ . We will start with the “only if” part.

Note that a number  $\lambda \in \mathbb{R}$  is an eigenvalue of  $\mathbf{B}$  if and only if the two equations

$$(A_0 - \lambda)y_0 = -Vy_1 \tag{3}$$

$$(A_1 - \lambda)y_1 = -V^*y_0 \tag{4}$$

are fulfilled for a non-zero vector  $(y_0, y_1) \in \mathfrak{H}_0 \oplus \mathfrak{H}_1$ .

*First case:* If  $\lambda \notin \text{spec}_p(A_0)$ , then  $(A_0 - \lambda)$  is injective and we immediately get the equation of statement (i) for  $y_1 \neq 0$ .

*Second case:* If  $\lambda \in \text{spec}_p(A_0)$ , the operator  $(A_0 - \lambda)$  is not injective and therefore we change equation (3) for an  $\varepsilon > 0$ :

$$V^*(A_0 - \lambda - i\varepsilon)^{-1}(A_0 - \lambda)y_0 = -V^*(A_0 - \lambda - i\varepsilon)^{-1}Vy_1. \tag{5}$$

By the spectral theorem we can calculate

$$\lim_{\varepsilon \rightarrow 0^+} V^*(A_0 - \lambda - i\varepsilon)^{-1}(A_0 - \lambda)y_0 = V^*(I_{\mathfrak{H}_0} - E_{A_0}(\{\lambda\}))y_0.$$

For  $y_0 \in \text{Ran } E_{A_0}(\{\lambda\})^\perp$  we get the equation of (ii) and for  $y_0 \notin \text{Ran } E_{A_0}(\{\lambda\})^\perp$  we get the equation of (iii). In both cases  $y_1 \neq 0$ .

To show the “if” part of the claim above, one has to construct an eigenvector  $(y_0, y_1) \in \mathfrak{H}_0 \oplus \mathfrak{H}_1$  for  $\mathbf{B}$ . Since  $y_1$  with  $Vy_1 \in \text{Ran}(A_0 - \lambda)$  is given, only  $y_0$  is to construct. In the case (i) and (ii) one can simply set  $y_0 \in \text{Ran } E_{A_0}(\{\lambda\})^\perp$  such that

$$(A_0 - \lambda)y_0 = -Vy_1$$

holds. In the third case (iii) one has to do a similar reasoning and choose the vector  $y'_0 \in \text{Ran } E_{A_0}(\{\lambda\})^\perp$  such that

$$(A_0 - \lambda)y'_0 = -Vy_1$$

holds. Then just set  $y_0 := y'_0 + x$ .  $\square$

REMARK 3.4. The characterisation of the eigenvalues of  $\mathbf{B}$  in Lemma 3.3 remains true in the case of infinite dimensional  $\mathfrak{H}_1$  if the (strong) limits are replaced with weak limits.

EXAMPLE 3.5. We consider the Hilbert space  $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_1$  with  $\mathfrak{H}_0 = \mathfrak{H}_1 = \mathbb{C}^2$  and the linear operator  $\mathbf{B} : \mathfrak{H} \rightarrow \mathfrak{H}$  given by:

$$\mathbf{B} = \begin{pmatrix} A_0 & V \\ V^* & A_1 \end{pmatrix} = \left( \begin{array}{c|cc} 1 & 1 & \\ \hline & 0 & 1 \ 1 \\ 1 & 1 & 0 \\ \hline & 1 & 0 \end{array} \right).$$

There are three eigenvalues of  $\mathbf{B}$  that belong to condition (i) of Lemma 3.3 and there is the eigenvalue 1 that fulfils condition (iii). By choosing  $y_1 = (0, 1)^T$  and  $x = (-1, 0)^T$  we see that

$$(A_1 - 1)y_1 = \lim_{\varepsilon \rightarrow 0^+} V^*(A_0 - 1 - i\varepsilon)^{-1}Vy_1 - V^*x$$

holds.

EXAMPLE 3.6. We quickly revisit Example 1.3 from the introduction. The operator  $\mathbf{A}$  is the orthogonal sum of the multiplication operator on  $L^2([-1, 1])$  and the zero operator on  $\mathbb{C}^2$ . While this operator has 0 as an eigenvalue of multiplicity 2, for  $\mathbf{B}$  this eigenvalue has to be simple since only case (i) of Lemma 3.3 is feasible and the kernel of  $V$  is only one-dimensional. Moreover, by the equation in case (i), we get for  $\lambda \notin [-1, 1]$  and  $y = (1, 1)^T$  the following

$$-\lambda = 2\langle v, (A_0 - \lambda)^{-1}v \rangle$$

and after an integration this becomes:

$$2 \log \left| \frac{\lambda + 1}{\lambda - 1} \right| = \lambda.$$

This equation has in fact a negative and positive solution and therefore three simple eigenvalues for the operator  $\mathbf{B}$  exist.

The singular and singularly continuous spectrum of  $\mathbf{B}$  can be described by the use of minimal supports of the spectral measure which we will do in the following. We write  $J_{\mathfrak{H}_1} : \mathfrak{H}_1 \rightarrow \mathfrak{H}$  for the inclusion map and in this case the adjoint satisfies  $J_{\mathfrak{H}_1}^*(x) = P_{\mathfrak{H}_1}(x)$  for all  $x \in \mathfrak{H}$ .

PROPOSITION 3.7. *Assume Hypothesis 3.1. The map  $M : \mathbb{C}_+ \rightarrow \mathcal{L}(\mathfrak{H}_1)$  defined by*

$$M(z) := J_{\mathfrak{H}_1}^* (\mathbf{B} - z)^{-1} J_{\mathfrak{H}_1}$$

*is a matrix-valued Herglotz function with*

$$M(z) = [(A_1 - z) - V^*(A_0 - z)^{-1}V]^{-1}. \quad (6)$$

*Proof.* Since  $z \mapsto (\mathbf{B} - z)^{-1}$  is analytic and because of the first resolvent identity,  $M$  is a matrix-valued Herglotz function, cf. [4]. Note that also Example 2.3 is applicable to prove this. The inverse of the Schur complement of  $(\mathbf{B} - z)$  shows equation (6), see [16, Proposition 1.6.2].  $\square$

The two propositions below yield a positive Borel measure which is equivalent to the spectral measure of  $\mathbf{B}$  and describe the singularly continuous spectrum and the pure point spectrum of the perturbed operator  $\mathbf{B}$ . This extends the results by Kostykin and Makarov in [7].

PROPOSITION 3.8. *Assume Hypothesis 3.1. The Herglotz function*

$$m(z) = \text{tr}(M(z))$$

*admits the representation*

$$m(z) = \int_{\mathbb{R}} \frac{1}{t - z} d\omega(t), \quad (7)$$

*where  $\omega$  is a positive Borel measure with compact support. Moreover,  $\omega$  and the spectral measure of  $\mathbf{B}$  are equivalent, i.e. the null sets coincide.*

*Proof.* From [4, Theorem 5.4] we know that  $m$  is a scalar Herglotz function. We define an operator-valued measure  $\Omega$  with values in  $\mathcal{L}(\mathfrak{H}_1)$  by

$$\Omega(\Delta) := J_{\mathfrak{H}_1}^* E_{\mathbf{B}}(\Delta) J_{\mathfrak{H}_1}$$

for every Borel set  $\Delta \subset \mathbb{R}$ . We easily see that

$$\int \frac{d\Omega(t)}{t - z} = J_{\mathfrak{H}_1}^* \int \frac{dE_{\mathbf{B}}(t)}{t - z} J_{\mathfrak{H}_1} = M(z) \quad \text{for all } z \in \mathbb{C}_+.$$

Hence,  $\omega(\Delta) := \text{tr}\Omega(\Delta)$  defines a positive measure with compact support, which satisfies equation (7).

Since by Lemma 3.2 the space  $\mathfrak{H}_1$  is a cyclic generating subspace for  $\mathbf{B}$ , the measure  $\Omega$  is equivalent to  $E_{\mathbf{B}}$  by Proposition 2.6. That  $\omega$  and  $\Omega$  are equivalent has been shown in Proposition 2.4.  $\square$

PROPOSITION 3.9. *Assume Hypothesis 3.1. The set*

$$S_s := \left\{ \lambda \in \mathbb{R} \mid \left\| [(A_1 - \lambda - i\varepsilon) - V^*(A_0 - \lambda - i\varepsilon)^{-1}V]^{-1} \right\| \xrightarrow{\varepsilon \rightarrow 0^+} \infty \right\}$$

is a minimal support of the singular part of the positive measure  $\omega$  from Proposition 3.8. The set

$$S_{pp} := \left\{ \lambda \in \mathbb{R} \mid \begin{array}{l} \text{There is } 0 \neq y \in \mathfrak{H}_1 \text{ with } Vy \in \text{Ran}(A_0 - \lambda) \text{ and there is} \\ x \in \text{Ran } E_{A_0}(\{\lambda\}) \text{ such that} \\ (A_1 - \lambda)y = \lim_{\varepsilon \rightarrow 0^+} V^*(A_0 - \lambda - i\varepsilon)^{-1}Vy - V^*x \end{array} \right\}$$

is the set of all atoms of  $\omega$ . In particular,  $S_{sc} := S_s \setminus S_{pp}$  is a minimal support for the singularly continuous part of  $\omega$ .

*Proof.* By [4, Theorem 6.1], which is formulated in Proposition 2.5, there is a minimal support of  $\Omega_s$ :

$$S_{\Omega_s} := \left\{ \lambda \in \mathbb{R} \mid \lim_{\varepsilon \rightarrow 0^+} \text{tr } \text{Im} M(\lambda + i\varepsilon) = \infty \right\}.$$

Of course this is by the equivalence of the measures, see Proposition 2.4, also a minimal support for  $\omega_s$ . Obviously,  $S_s \supset S_{\Omega_s}$  and therefore  $S_s$  is a support of  $\omega_s$  as well. It is minimal by [4, Theorem 5.4 (ii)].

The set  $S_{pp}$  coincides with all eigenvalues of  $\mathbf{B}$ . Note that we pushed the three cases of Lemma 3.3 into one formula here. By the equivalence of measures,  $S_{pp}$  is the set of all atoms of  $\omega$  and therefore the smallest support of  $\omega_{pp}$ .  $\square$

REMARK 3.10. The sets  $S_s$  and  $S_{pp}$  are connected to the spectrum of the perturbed operator  $\mathbf{B}$ . We already noted that  $S_{pp} = \text{spec}_p(\mathbf{B})$  but the relation to  $S_s$  is more subtle. In general neither  $S_s \supset \text{spec}_s(\mathbf{B})$  nor  $S_s \subset \text{spec}_s(\mathbf{B})$  is correct. However,  $\overline{S_s} \supset \text{spec}_s(\mathbf{B})$  is always true. Hence, if the singular spectrum of  $\mathbf{B}$  is non-empty, than  $S_s$  is also non-empty.

Now, we define two subsets  $K_{pp} \subset S_s$  and  $K_{sc} \subset S_s$  of the singular support of  $\omega$ , since not all points are suitable for the construction of a solution to the Riccati equation as one can see in next section. The proper subsets are given by:

$$K_{pp} := \{ \lambda \in \mathbb{R} \mid \Phi_{pp}(\lambda) \neq \emptyset \}$$

$$\text{where } \Phi_{pp}(\lambda) := \left\{ y \in \mathfrak{H}_1 \setminus \{0\} \mid \begin{array}{l} (A_1 - \lambda)y = \lim_{\varepsilon \rightarrow 0^+} V^*(A_0 - \lambda - i\varepsilon)^{-1}Vy \\ \text{and } \int \frac{1}{|t - \lambda|^2} d\langle Vy, E_{A_0}(t)Vy \rangle < \infty \end{array} \right\}$$

and

$$K_{sc} := \{ \lambda \in \mathbb{R} \mid \Phi_{sc}(\lambda) \neq \emptyset \}$$

$$\text{where } \Phi_{sc}(\lambda) := \left\{ y \in \mathfrak{H}_1 \setminus \{0\} \mid \begin{array}{l} (A_1 - \lambda)y = \lim_{\varepsilon \rightarrow 0^+} V^*(A_0 - \lambda - i\varepsilon)^{-1}Vy \\ \text{and } \int \frac{1}{|t - \lambda|^2} d\langle Vy, E_{A_0}(t)Vy \rangle = \infty \end{array} \right\}.$$

Note that  $K_{pp} \subset S_{pp}$  always holds and that we have equality if and only if there is no eigenvalue of  $\mathbf{B}$  which satisfies the condition (iii) of Lemma 3.3. In particular,  $K_{pp} = S_{pp}$  is fulfilled if the point spectra of  $A_0$  and  $\mathbf{B}$  are disjoint.

Furthermore, Kostrykin and Makarov have shown in [7, Theorem 3.4] that  $K_{pp} = S_{pp}$  and  $K_{sc} = S_{sc}$  hold if the Hilbert space  $\mathfrak{H}_1$  is one-dimensional. By using this result, they have constructed solutions to the Riccati equation for each  $\lambda \in S_s$  in the case that  $\dim \mathfrak{H}_1 = 1$ , see [7, Theorem 4.3]. In the following section we extend their results about solutions to the Riccati equation for an arbitrarily finite dimensional Hilbert space  $\mathfrak{H}_1$ .

#### 4. Solutions to the Riccati equation

The operator Riccati equation (1) a priori only makes sense as an operator identity if the solution  $X$  is bounded and  $\text{Dom}(X) = \mathfrak{H}_0$ . If one wants to include unbounded operators, a generalised definition of solutions is required. We will use the same notion of a so-called strong solution as in [7] and [8].

DEFINITION 4.1. A densely defined, not necessarily bounded or closable, linear operator  $X : \mathfrak{H}_0 \supset \text{Dom}(X) \rightarrow \mathfrak{H}_1$  is called a *strong solution* to the Riccati equation (1) if

$$\text{Ran}(A_0 + VX)|_{\text{Dom}(X)} \subset \text{Dom}(X)$$

and

$$A_1 Xx - X(A_0 + VX)x + V^*x = 0 \quad \text{for all } x \in \text{Dom}(X)$$

hold.

HYPOTHESIS 4.2. Assume Hypothesis 3.1. Suppose that  $K_{pp} \cup K_{sc}$  is not empty and that there are  $n := \dim \mathfrak{H}_1$  linear independent vectors  $y_1, \dots, y_n \in \mathfrak{H}_1$  which satisfy

$$(A_1 - \lambda_k)y_k = \lim_{\varepsilon \rightarrow 0^+} V^*(A_0 - \lambda_k - i\varepsilon)^{-1}Vy_k, \quad \lambda_k \in K_{pp} \cup K_{sc} \quad (8)$$

for  $k = 1, \dots, n$ . Moreover, denote  $\Lambda_{pp} := \{(\lambda_k, y_k) \mid y_k \in \Phi_{pp}(\lambda_k), k = 1, \dots, n\}$  and  $\Lambda_{sc} := \{(\lambda_k, y_k) \mid y_k \in \Phi_{sc}(\lambda_k), k = 1, \dots, n\}$  for the different parts. We will abbreviate this with  $\Lambda := \Lambda_{pp} \cup \Lambda_{sc}$ .

Under Hypothesis 4.2, we define for  $k = 1, \dots, n$  the, not necessarily orthogonal, projections  $P_{\Lambda, k} : \mathfrak{H}_1 \rightarrow \mathfrak{H}_1$  by

$$\begin{aligned} \text{Ran } P_{\Lambda, k} &= \text{lin span}\{y_k\}, \\ \text{Ker } P_{\Lambda, k} &= \text{lin span}\{y_j \mid j \neq k\}. \end{aligned}$$

We also define a possibly unbounded operator  $X_\Lambda : \mathfrak{H}_0 \supset \text{Dom}(X_\Lambda) \rightarrow \mathfrak{H}_1$  on the domain

$$\text{Dom}(X_\Lambda) := \left\{ x \in \mathfrak{H}_0 \mid \lim_{\varepsilon \rightarrow 0^+} \sum_{j=1}^n P_{\Lambda, j}^* V^*(A_0 - \lambda_j + i\varepsilon)^{-1} x \in \mathfrak{H}_1 \right\}$$

by

$$X_\Lambda x = \lim_{\varepsilon \rightarrow 0^+} \sum_{j=1}^n P_{\Lambda,j}^* V^* (A_0 - \lambda_j + i\varepsilon)^{-1} x, \tag{9}$$

which has the following properties.

PROPOSITION 4.3. *Assume Hypothesis 4.2 with a chosen  $\Lambda = \Lambda_{pp} \cup \Lambda_{sc}$ . Then:*

- (i) *The linear operator  $X_\Lambda$  is densely defined.*
- (ii) *If  $\Lambda_{sc}$  is not empty, then the operator  $X_\Lambda$  is unbounded and non-closable.*
- (iii) *If  $\Lambda_{sc}$  is empty, i.e.  $\{\lambda_1, \dots, \lambda_n\} \subset K_{pp}$ , then the operator  $X_\Lambda$  is bounded.*
- (iv)  *$A_0 x \in \text{Dom}(X_\Lambda)$  for all  $x \in \text{Dom}(X_\Lambda)$ .*
- (v)  *$X_\Lambda$  is a strong solution to the Riccati equation (1).*

*Proof.* A proof of the statement (i) for the case  $\dim \mathfrak{H}_1 = 1$  can be found in [7]. This proof has a straightforward generalisation to a finite dimensional  $\mathfrak{H}_1$ . With the same argument as in [7, Lemma 4.1] one can show that the limit

$$\lim_{\varepsilon \rightarrow 0^+} P_{\Lambda,j}^* V^* (A_0 - \lambda_j + i\varepsilon)^{-1} \varphi$$

exists for  $j \in \{1, \dots, n\}$  and  $\varphi \in \{p(A_0)u \mid p \text{ polynomial}, u \in \text{Ran } V\}$ . Since the latter set is dense in  $\mathfrak{H}_0$ , the operator  $X_\Lambda$  is densely defined.

To show (ii) we choose  $(\lambda_j, y_j) \in \Lambda_{sc}$  and define for all  $\varepsilon \in (0, 1]$  the bounded operators  $Y^\varepsilon \in \mathcal{L}(\mathfrak{H}_0, \mathfrak{H}_1)$  by

$$Y^\varepsilon x := P_{\Lambda,j}^* V^* (A_0 - \lambda_j + i\varepsilon)^{-1} x.$$

A short calculation with the spectral theorem shows that the operator norm is given by

$$\|Y^\varepsilon\|_{\mathfrak{H}_0 \rightarrow \mathfrak{H}_1} = |\alpha_j| \left( \int \frac{1}{|t - \lambda_j|^2 + \varepsilon^2} d\langle Vy_j, E_{A_0}(t) Vy_j \rangle \right)^{1/2} \tag{10}$$

where  $\alpha_j \in \mathbb{C}$  is a constant independent of  $\varepsilon$ . If  $X_\Lambda$  was bounded, the operator defined by  $Y := P_{\Lambda,j}^* X_\Lambda$  would also be bounded and therefore

$$\sup_{\varepsilon \in (0,1]} \|Y^\varepsilon x\|_{\mathfrak{H}_1} < \infty \quad \text{for all } x \in \mathfrak{H}_0.$$

Since the uniform boundedness principle claims that  $\sup_{\varepsilon \in (0,1]} \|Y^\varepsilon\|_{\mathfrak{H}_0 \rightarrow \mathfrak{H}_1}$  is finite and since that can be written by equation (10) and the monotone convergence theorem as

$$\int \frac{1}{|t - \lambda_j|^2} d\langle Vy_{\lambda_j}, E_{A_0}(t) Vy_{\lambda_j} \rangle < \infty,$$

there is a contradiction to  $(\lambda_j, y_j) \in \Lambda_{sc}$ .

To show (iii), assume that  $\{\lambda_1, \dots, \lambda_n\} \subset K_{pp}$  and that for all  $j = 1, \dots, n$  we have  $y_j \in \Phi_{pp}(\lambda_j)$ . Then define a bounded operator  $Z: \mathfrak{H}_1 \rightarrow \mathfrak{H}_0$  by

$$Zy := \text{w-lim}_{\varepsilon \rightarrow 0^+} \sum_{j=1}^n (A_0 - \lambda_j - i\varepsilon)^{-1} V P_{\Lambda, j} y, \quad y \in \mathfrak{H}_1.$$

Since all  $\lambda_j$  are eigenvalues of  $\mathbf{B}$  and  $V P_{\Lambda, j} y \in \text{Ran}(A_0 - \lambda_j)$  for all  $j = 1, \dots, n$  and  $y \in \mathfrak{H}_1$  by Lemma 3.3, the weak limit is well-defined. Choose  $x \in \text{Dom}(X_\Lambda)$  and  $y \in \mathfrak{H}_1$ . Then

$$\begin{aligned} \langle x, Zy \rangle_{\mathfrak{H}_0} &= \lim_{\varepsilon \rightarrow 0^+} \left\langle x, \sum_{j=1}^n (A_0 - \lambda_j - i\varepsilon)^{-1} V P_{\Lambda, j} y \right\rangle_{\mathfrak{H}_0} \\ &= \lim_{\varepsilon \rightarrow 0^+} \left\langle \sum_{j=1}^n P_{\Lambda, j}^* V^* (A_0 - \lambda_j + i\varepsilon)^{-1} x, y \right\rangle_{\mathfrak{H}_1} = \langle X_\Lambda x, y \rangle_{\mathfrak{H}_1}, \end{aligned}$$

so that  $Z^*$  is an extension of  $X_\Lambda$ . Hence,  $X_\Lambda$  is a closable operator of finite rank and therefore has to be bounded.

Statement (iv) is shown by applying the spectral theorem. For each  $j$  and all  $x \in \text{Dom}(X_\Lambda)$  one has

$$\lim_{\varepsilon \rightarrow 0^+} P_{\Lambda, j}^* V^* (A_0 - \lambda_j + i\varepsilon)^{-1} (A_0 - \lambda_j) x = P_{\Lambda, j}^* V^* x \quad (11)$$

because  $\mathfrak{H}_1$  is finite dimensional and  $\text{Ran}(V P_{\Lambda, j}) \subset (\text{Ran } E_{A_0}(\{\lambda_j\}))^\perp$ . Therefore, we have  $A_0 x \in \text{Dom}(X_\Lambda)$  for all  $x \in \text{Dom}(X_\Lambda)$ .

To show (v), we write the Riccati equation (1) in the form

$$\sum_{j=1}^n P_{\Lambda, j}^* (A_1 X - X A_0 - X V X + V^*) = 0$$

and consider each summand separately. We choose  $x \in \text{Dom}(X_\Lambda)$  and calculate by using the equations (8) and (9):

$$\begin{aligned} &P_{\Lambda, k}^* (A_1 X_\Lambda - X_\Lambda A_0 - X_\Lambda V X_\Lambda) x \\ &= P_{\Lambda, k}^* \left( A_1 X_\Lambda x - X_\Lambda A_0 x - \lim_{\varepsilon \rightarrow 0^+} (P_{\Lambda, k})^* V^* (A_0 - \lambda_k + i\varepsilon)^{-1} V X_\Lambda x \right) \\ &= P_{\Lambda, k}^* (A_1 - (A_1 - \lambda_k)) X_\Lambda x - P_{\Lambda, k}^* X_\Lambda A_0 x \\ &= P_{\Lambda, k}^* X_\Lambda (\lambda_k - A_0) x \\ &= \lim_{\varepsilon \rightarrow 0^+} P_{\Lambda, k}^* V^* (A_0 - \lambda_k - i\varepsilon)^{-1} (\lambda_k - A_0) x \\ &= -P_{\Lambda, k}^* V^* x. \end{aligned}$$

In the last step we used equation (11).  $\square$

Finally, we are able to prove our main results:

*Proof of Theorem 1.1.* By Lemma 3.2 the multiplicity of the spectrum of  $\mathbf{B}$  is at most  $n := \dim \mathfrak{H}_1$ . If there is an eigenvalue  $\lambda$  with multiplicity  $n$ , then Lemma 3.3 shows that there are vectors  $y_1, \dots, y_n \in \mathfrak{H}_1$  which span the Hilbert space  $\mathfrak{H}_1$ . Thus, also by Lemma 3.3 the inequality

$$\lim_{\varepsilon \rightarrow 0^+} |\operatorname{tr} V^*(A_0 - \lambda - i\varepsilon)^{-1}V| < \infty$$

holds and one concludes that  $\lambda \notin \operatorname{spec}_p(A_0)$ . This is due to Proposition 2.6, in particular equation (2), and the fact that  $\operatorname{Ran} V$  is a cyclic generating subspace for  $A_0$ . Eventually, we construct a bounded solution  $X_\Lambda$  to the Riccati equation with  $\Lambda = \{(\lambda, y_1), \dots, (\lambda, y_n)\}$  and Proposition 4.3. This proves (i).

Statement (ii) is formulated in such a way that there exists at least one  $\Lambda$  as in Hypothesis 4.2 such that Proposition 4.3 is applicable.  $\square$

*Proof of Corollary 1.2.* Since here it is not assumed that  $\operatorname{Ran} V$  is a cyclic generating subspace for  $A_0$ , we define

$$\mathfrak{K}_0 := \overline{\operatorname{lin span} \{A_0^k v \mid k \in \mathbb{N}_0, v \in \operatorname{Ran} V\}},$$

which is always a closed  $A_0$ -invariant subspace of  $\mathfrak{H}_0$ . One can choose  $X|_{\mathfrak{K}_0^\perp} = 0$  for a solution  $X$  to the Riccati equation (1), so that we can assume Hypothesis 3.1 without loss of generality.

As  $\mathfrak{H}$  is finite dimensional and spanned by the eigenvectors of  $\mathbf{B}$ , we always find a bounded solution  $X$  by Theorem 1.1 part (ii).  $\square$

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