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LOG AND HARMONICALLY LOG–CONVEX FUNCTIONS RELATED TO MATRIX NORMS

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Abstract. In this article, we introduce the concept of harmonically log-convex functions, which seems to be strongly connected to unitarily invariant norms. Then, we prove Hermite-Hadamard inequalities for these functions. As an application, we present many inequalities for the trace operator and unitarily invariant norms.

1. Introduction

Let $\mathbb{M}_n$ be the space of all $n\times n$ complex matrices and $\mathbb{M}_n^+$ be the class of $\mathbb{M}_n$ consisting of positive semi-definite matrices. Inequalities involving quantities of the form $|||AXB|||$ have been of great interest in the literature. In this context, $A,B\in\mathbb{M}_n^+, X\in\mathbb{M}_n$ and $|||\ |||$ is any unitarily invariant norm. Recall that these are norms satisfying $|||UAV|||=|||A|||$ for all unitary matrices U and V. Among the most interesting inequalities are Hölder-Young and Heinz inequalities that state, respectively,

$$|||A^{\nu}XB^{1-\nu}|||\leqslant |||AX|||^{\nu}\ |||XB|||^{1-\nu}\leqslant \nu|||AX|||+(1-\nu)|||XB||| \tag{1.1}$$

and

$$2|||A^{1/2}XB^{1/2}|||\leqslant |||A^{\nu}XB^{1-\nu}+A^{1-\nu}XB^{\nu}|||\leqslant |||AX+XB|||. \tag{1.2}$$

for all $\nu\in[0,1]$. For proofs of these inequalities, we refer the reader to [11] and [1], respectively.

These inequalities have been studied thoroughly and many refinements and generalizations have been obtained in the literature. We refer the reader to [4], [6], [10] and [12] for such inequalities. Among the very recent generalizations of Young's inequality (1.1) is our result in [15]

$$|||A^pXB^q|||\leqslant |||A^{p+r}XB^{q-r}|||^{\frac{p-q+r}{p-q+2r}}\ |||A^{q-r}XB^{p+r}|||^{\frac{r}{p-q+2r}} \tag{1.3}$$

where $p\geqslant q\geqslant r\geqslant 0$. In [6], [10], [12] and [16] integral versions of inequality (1.2) have been obtained, where convexity of the function

$$\nu\to |||A^{\nu}XB^{1-\nu}+A^{1-\nu}XB^{\nu}|||$$

was the key to these studies. For example, in [12] the following theorem was proved for the Heinz means $f(\nu)=|||A^{\nu}XB^{1-\nu}+A^{1-\nu}XB^{\nu}|||$.

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THEOREM 1.1. *Let* $0 < \mu \leqslant \frac{1}{2}$, *then*

$$f(\mu) \leqslant f(\mu/2) \leqslant \frac{1}{\mu}\int_0^{\mu} f(v)dv \leqslant \frac{f(0)+f(\mu)}{2} \leqslant f(0), \tag{1.4}$$

and for $\frac{1}{2} \leqslant \mu \leqslant 1$, *we have*

$$f(\mu) \leqslant f\left(\frac{1+\mu}{2}\right) \leqslant \frac{1}{1-\mu}\int_{\mu}^{1} f(v)dv \leqslant \frac{f(1)+f(\mu)}{2} \leqslant f(1). \tag{1.5}$$

The proof was merely based on the well known Hermite-Hadamard inequalities that for a convex function f on $[a,b]$,

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{2}. \tag{1.6}$$

Then refinements and generalizations of inequalities of Theorem 1.1 have been obtained based on corresponding refinements of the Hermite-Hadamard inequalities (1.6). See for example [6] and [10].

In this paper, we introduce the concept of harmonically log-convex functions which, as shown, is strongly related to these norm quantities. Several relations between log-convex and harmonically log-convex functions are presented, implying new Hermite-Hadamard type inequalities for such functions.

Moreover, we investigate log-convexity of various functions like $f(t) = |||A^t|||$, $f(v) = |||A^vXB^{1-v}|||\ |||A^{1-v}XB^v|||$ and $f(p) = \|A\|_p$. Then we use these convexity results to obtain other inequalities.

This work is motivated by the extensive study of operator convex functions. We refer the reader to [3] for a comprehensive study of this topic.

2. Main Results

2.1. Harmonically log-convex functions

Harmonically convex functions were defined in [9] as follows.

DEFINITION 2.1. Let $I \subset \mathbb{R}\backslash\{0\}$ be an interval. A function $f : I \to \mathbb{R}$ is said to be harmonically convex on I if

$$f\left(\frac{t_1t_2}{\lambda t_1 + (1-\lambda)t_2}\right) \leqslant \lambda f(t_2) + (1-\lambda)f(t_1),$$

for all $t_1, t_2 \in I$ and $\lambda \in [0,1]$.

Simply speaking, $f : [a,b] \to \mathbb{R}$ is harmonically convex if the function $g : [\frac{1}{b}, \frac{1}{a}]$ defined by $g(t) = f(1/t)$ is convex.

Motivated by this definition, we define harmonically log-convex functions as follows.

DEFINITION 2.2. Let $I \subset \mathbb{R}^+ := (0,\infty)$ be an interval. A function $f : I \to \mathbb{R}^+$ is said to be harmonically log-convex on I if

$$f\left(\frac{t_1 t_2}{(1-\lambda)t_1+\lambda t_2}\right) \leqslant f^{\lambda}(t_1) f^{1-\lambda}(t_2),$$

for all $t_1, t_2 \in I$ and $\lambda \in [0,1]$.

One can easily prove the following characterization of harmonically log-convex functions.

PROPOSITION 2.3. *Let $I \subset \mathbb{R}^+$ be an interval. A function $f : I \to \mathbb{R}^+$ is harmonically log-convex on $I := [a,b]$ if and only if the function $g : \left[\frac{1}{b}, \frac{1}{a}\right] \to \mathbb{R}$ defined by $g(x) = f(1/x)$ is log-convex.*

Hence, the functions $f(t) = t^p, p > 0$ are harmonically log-convex.

PROPOSITION 2.4. *Let $I = [a,b] \subset \mathbb{R}^+$ be an interval. If $f : I \to \mathbb{R}^+$ is log-convex, then the function $g : \left[\frac{1}{b}, \frac{1}{a}\right] \to \mathbb{R}^+$ defined by*

$$g(t) = f^t(1/t)$$

is log-convex too.

Proof. Let $t_1, t_2 \in [a,b]$, $\lambda \in [0,1]$, and $t = \frac{1}{\lambda t_1^{-1}+(1-\lambda)t_2^{-1}}$. Then, clearly $t = \alpha t_1 + \beta t_2$ where $\alpha = \frac{\lambda t}{t_1}$ and $\beta = \frac{1-\lambda}{t_2} t$. Hence, noting that $\alpha + \beta = 1$ and recalling that f is log-convex, we get

$$\begin{aligned} g\left(\lambda t_1^{-1} + (1-\lambda)t_2^{-1}\right) &= g(1/t) \\ &= f^{1/t}(t) \\ &= f^{1/t}(\alpha t_1 + \beta t_2) \\ &\leqslant f^{\frac{\alpha}{t}}(t_1) f^{\frac{\beta}{t}}(t_2) \\ &= f^{\frac{\lambda}{t_1}}(t_1) f^{\frac{1-\lambda}{t_2}}(t_2) \\ &= g^{\lambda}(1/t_1) g^{1-\lambda}(1/t_2). \end{aligned}$$

This proves that g is log-convex on $[1/b, 1/a]$. □

Now propositions 2.3 and 2.4 imply the following corollary.

COROLLARY 2.5. *Let $I \subset \mathbb{R}^+$ be an interval. If $f : I \to \mathbb{R}^+$ is log-convex, then the function $g : I \to \mathbb{R}^+$ defined by*

$$g(t) = f^{1/t}(t)$$

is harmonically log-convex.

Moreover, we have

COROLLARY 2.6. *Let $I \subset \mathbb{R}^+$ be an interval. If $f: I \to \mathbb{R}^+$ is harmonically log-convex, then the function $g: I \to \mathbb{R}^+$ defined by*

$$g(t) = f^t(t)$$

is log-convex.

Proof. Since f is harmonically log-convex, $h(t) = f(1/t)$ is log-convex, by proposition 2.3. Then, by proposition 2.4, the function $g(t) = h^t(1/t)$ is log-convex. This implies the result. □

Now corollaries 2.5 and 2.6 imply the following corollary.

COROLLARY 2.7. *Let $I \subset \mathbb{R}^+$ be an interval. A function $f: I \to \mathbb{R}^+$ is log-convex if and only if the function $g: I \to \mathbb{R}^+$ defined by*

$$g(t) = f^{1/t}(t)$$

is harmonically log-convex.

The proof of the following composition relation is immediate from the definitions of log-convex and harmonically log-convex functions.

PROPOSITION 2.8. *Let $f: I_1 \to \mathbb{R}$ be harmonically convex and $g: I_2 \to \mathbb{R}$ be log-convex and increasing. If $f(I_1) \subseteq I_2$, then the composite function $g \circ f: I_1 \to \mathbb{R}$ is harmonically log-convex.*

Our next result treats log-convex functions on partitions of $[0,1]$.

THEOREM 2.9. *Let $f = f(\nu)$ be log-convex on $[0,1]$, and let $n \in \mathbb{N} \cup \{0\}$. Then if for some $k \in \{1, 2, \cdots, 2^n\}$, $\nu \in \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right]$, we have*

$$f(\nu) \leqslant \left(f\left(\frac{k-1}{2^n}\right)\right)^{k-2^n\nu} \left(f\left(\frac{k}{2^n}\right)\right)^{2^n\nu-k+1}. \tag{2.1}$$

Proof. Observe that when $\nu \in \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right]$, then for $\alpha = k - 2^n\nu$,

$$\nu = \alpha\frac{k-1}{2^n} + (1-\alpha)\frac{k}{2^n}.$$

Then using log-convexity of f, the result follows. □

THEOREM 2.10. *Let f be log-convex on $[0,1]$. For $n \in \mathbb{N} \cup \{0\}$ let*

$$I_{k,n} = \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right], \quad k = 1, \cdots, 2^n.$$

Define

$$g_n(\nu) = \left(f\left(\frac{k-1}{2^n}\right)\right)^{k-2^n\nu} \left(f\left(\frac{k}{2^n}\right)\right)^{2^n\nu-k+1}, \quad \nu \in I_{n,k}.$$

Then $g_{n+1}(\nu) \leqslant g_n(\nu)$ for all $\nu \in [0,1]$. Moreover $g_n \to f$ uniformly on $[0,1]$..

Proof. Let $\nu \in [0,1]$, then for each n, there exists $k \in \{1,2,\cdots,2^n\}$ such that $\nu \in \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right]$. But then $\nu \in I_{2k-1,n+1} \cup I_{2k,n+1}$. If $\nu \in I_{2k-1,n+1}$, we have

$$\begin{aligned}
g_{n+1}(\nu) &= \left(f\left(\frac{2k-2}{2^{n+1}}\right)\right)^{2k-1-2^{n+1}\nu} \left(f\left(\frac{2k-1}{2^{n+1}}\right)\right)^{2^{n+1}\nu-2k+2} \\
&= \left(f\left(\frac{k-1}{2^n}\right)\right)^{2k-1-2^{n+1}\nu} \left(f\left(\frac{1}{2}\frac{k-1}{2^n} + \frac{1}{2}\frac{k}{2^n}\right)\right)^{2^{n+1}\nu-2k+2} \\
&\leqslant \left(f\left(\frac{k-1}{2^n}\right)\right)^{2k-1-2^{n+1}\nu} \left\{\left(f\left(\frac{k-1}{2^n}\right)\right)^{1/2} \left(f\left(\frac{k}{2^n}\right)\right)^{1/2}\right\}^{2^{n+1}\nu-2k+2} \\
&= \left(f\left(\frac{k-1}{2^n}\right)\right)^{k-2^n\nu} \left(f\left(\frac{k}{2^n}\right)\right)^{2^n\nu-k+1} \\
&= g_n(\nu).
\end{aligned}$$

This completes the proof when $\nu \in I_{2k-1,n+1}$. If $\nu \in I_{2k,n+1}$, similar computations yield the result. The fact that $g_n \to f$ follows immediately noting that f is continuous. $\square$

We remark that in the recent paper [16], similar partition ideas have been proved for convex functions.

2.2. Hermite-Hadamard type inequalities

In [5] the following inequalities were proved for positive log-convex functions f on (a,b):

$$\begin{aligned}
f\left(\frac{a+b}{2}\right) &\leqslant \exp\left(\frac{1}{b-a}\int_a^b \ln f(x)\,dx\right) \\
&\leqslant \frac{1}{b-a}\int_a^b G(f(x), f(a+b-x))dx \leqslant \frac{1}{b-a}\int_a^b f(x)dx \\
&\leqslant L(f(a), f(b)) \leqslant \frac{f(a)+f(b)}{2},
\end{aligned} \tag{2.2}$$

where $G(p,q) = \sqrt{pq}$ and $L(p,q) = \frac{p-q}{\ln p - \ln q}$.

The following inequalities are the corresponding inequalities for harmonically log-convex functions.

THEOREM 2.11. *Let $f:[a,b]\subset\mathbb{R}^+\to\mathbb{R}^+$ be harmonically log-convex. Then*

$$\begin{aligned} f\left(\frac{2ab}{a+b}\right) &\leqslant \exp\left(\frac{ab}{b-a}\int_a^b \frac{\ln f(x)}{x^2}\,dx\right)\\ &\leqslant \frac{ab}{b-a}\int_a^b \frac{1}{x^2}G\left(f(x),f\left(\frac{abx}{bx+ax-ab}\right)\right)dx \leqslant \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\\ &\leqslant L(f(a),f(b)) \leqslant \frac{f(a)+f(b)}{2}, \end{aligned}$$

Proof. This follows immediately by applying inequalities (2.2) to the function $g(x)=f(1/x)$ defined on $[1/b,1/a]$, observing log-convexity of g. $\square$

Utilizing theorem 2.11 and corollary 2.5 we get the following inequalities for log-convex functions.

COROLLARY 2.12. *Let $f:[a,b]\subset\mathbb{R}^+\to\mathbb{R}^+$ be log-convex. Then*

$$\begin{aligned} f^{\frac{a+b}{2ab}}\left(\frac{2ab}{a+b}\right) &\leqslant \exp\left(\frac{ab}{b-a}\int_a^b \frac{\ln f(x)}{x^3}dx\right)\\ &\leqslant \frac{ab}{b-a}\int_a^b \frac{f^{\frac{1}{x}}(x)}{x^2}dx\\ &\leqslant \frac{f^{\frac{1}{a}}(a)-f^{\frac{1}{b}}(b)}{\frac{\ln f(a)}{a}-\frac{\ln f(b)}{b}}\\ &\leqslant \frac{f^{\frac{1}{a}}(a)+f^{\frac{1}{b}}(b)}{2}. \end{aligned}$$

Proof. Since f is log-convex, $g(x)=f^{1/x}(x)$ is harmonically log-convex, by corollary 2.5. Now apply theorem 2.11 to the function g, and simplify to get the result. $\square$

REMARK. The author was not aware of [13], where the definition of a harmonically-log convex function was introduced. However, after the paper has been published online, the paper [13] has been to the author's attention. We remark that the two papers treat the idea differently, but some integral results have minor similarities.

2.3. Applications and examples of log-convex and harmonically log-convex functions

The following lemma has been proved in [15].

LEMMA 2.13. *Let $A,B\in\mathbb{M}_n^+$ and $X\in\mathbb{M}_n$. Then the function $f:(0,\infty)\times(0,\infty)\to[0,\infty)$ defined by*

$$f(p,q)=|||A^pXB^q|||$$

is log-convex.

As an application of this log-convexity, we present the following trace inequalities.

THEOREM 2.14. *Let $A,B \in \mathbb{M}_n^+$, $X \in \mathbb{M}_n$ and $p \geqslant q > 0$. Then for $r \leqslant q$, we have*

$$\begin{aligned}\operatorname{tr}(A^pXB^qX^*) &\leqslant \left(\operatorname{tr}(A^{p+r}XB^{q-r}X^*)\right)^{\alpha}\left(\operatorname{tr}(A^{q-r}XB^{p+r}X^*)\right)^{\beta}\\ &\leqslant \alpha\operatorname{tr}(A^{p+r}XB^{q-r}X^*)+\beta\operatorname{tr}(A^{q-r}XB^{p+r}X^*)\end{aligned}$$

where $\alpha = \frac{p-q+r}{p-q+2r}$ and $\beta = 1-\alpha$.

Proof. By Lemma 2.13, it follows that the function $f(p,q) = \|A^pXB^q\|_2$ is log-convex on $(0,\infty)\times(0,\infty)$. Hence, for the mentioned α,β,

$$\begin{aligned}f(p,q) &= f\left(\alpha(p+r,q-r)+\beta(q-r,p+r)\right)\\ &\leqslant f^{\alpha}(p+r,q-r)f^{\beta}f(q-r,p+r)\\ &\leqslant \alpha f(p+r,q-r)+\beta f(q-r,p+r).\end{aligned}$$

Recalling that $f(p,q) = \|A^pXB^q\|_2$, we get

$$\begin{aligned}\|A^pXB^q\|_2^2 &\leqslant \left(\|A^{p+r}XB^{q-r}\|_2^2\right)^{\alpha}\ \left(\|A^{q-r}XB^{p+r}\|_2^2\right)^{\beta}\\ &\leqslant \alpha\|A^{p+r}XB^{q-r}\|_2^2+\beta\|A^{q-r}XB^{p+r}\|_2^2.\end{aligned}$$

But $\|T\|_2^2 = \operatorname{tr}(TT^*)$, hence the inequality

$$\|A^pXB^q\|_2^2 \leqslant \left(\|A^{p+r}XB^{q-r}\|_2^2\right)^{\alpha}\ \left(\|A^{q-r}XB^{p+r}\|_2^2\right)^{\beta}$$

becomes

$$\operatorname{tr}(A^{2p}XB^{2q}X^*) \leqslant \left\{\operatorname{tr}\left((A^2)^{p+r}X(B^2)^{q-r}X^*\right)\right\}^{\alpha}\left\{\operatorname{tr}\left((A^2)^{q-r}X(B^2)^{p+r}X^*\right)\right\}^{\beta}.$$

Since this is true for any $A,B \in \mathbb{M}_n^+$, we may replace A by $\sqrt{A}$ and B by $\sqrt{B}$, to get

$$\operatorname{tr}(A^pXB^qX^*) \leqslant \left(\operatorname{tr}(A^{p+r}XB^{q-r}X^*)\right)^{\alpha}\left(\operatorname{tr}(A^{q-r}XB^{p+r}X^*)\right)^{\beta}$$

which implies both inequalities of the theorem. □

The $\nu-$version of these inequalities can be stated as follows.

COROLLARY 2.15. *Let $A,B \in \mathbb{M}_n^+$ and $X \in \mathbb{M}_n$. Then for $0 \leqslant \nu \leqslant 1$, we have*

$$\begin{aligned}\operatorname{tr}(A^{\nu}XB^{1-\nu}X^*) &\leqslant \left\{\operatorname{tr}(A|X|^2)\right\}^{\nu}\left\{\operatorname{tr}(B|X^*|^2)\right\}^{1-\nu}\\ &\leqslant \nu\operatorname{tr}(A|X|^2)+(1-\nu)\operatorname{tr}(B|X^*|^2).\end{aligned}$$

In [14], it was proved that the function

$$f(r)=\frac{p-q+r}{p-q+2r}a^{p+r}b^{q-r}+\frac{r}{p-q+2r}a^{q-r}b^{p+r}$$

is increasing on $[0,q]$, for the positive numbers a,b. On the other hand, in [15] it has been proved that for any unitarily invariant norm the functions

$$f(r)=|||A^{p+r}XB^{q-r}|||^{\frac{p-q+r}{p-q+2r}}|||A^{q-r}XB^{p+r}|||^{\frac{r}{p-q+2r}}$$

and

$$g(r)=\frac{p-q+r}{p-q+2r}|||A^{p+r}XB^{q-r}|||+\frac{r}{p-q+2r}|||A^{q-r}XB^{p+r}|||$$

are increasing on $[0,q]$. Simulating the proofs in [15] we can easily prove the following.

PROPOSITION 2.16. *Let* $A,B\in\mathbb{M}_n^+$, $X\in\mathbb{M}_n$ *and* $p\geqslant q>0$. *Then the functions*

$$f(r)=\left\{\mathrm{tr}(A^{p+r}XB^{q-r}X^*)\right\}^{\frac{p-q+r}{p-q+2r}}\left\{\mathrm{tr}(A^{q-r}XB^{p+r}X^*)\right\}^{\frac{r}{p-q+2r}}$$

$$g(r)=\frac{p-q+r}{p-q+2r}\mathrm{tr}(A^{p+r}XB^{q-r}X^*)+\frac{r}{p-q+2r}\mathrm{tr}(A^{q-r}XB^{p+r}X^*)$$

are increasing on $[0,q]$.

What this proposition tells us is that $f(0)\leqslant f(r)\leqslant f(q)$ for $0\leqslant r\leqslant q$. Hence, for such r,

$$\begin{aligned}\mathrm{tr}(A^pXB^qX^*)&\leqslant\left\{\mathrm{tr}(A^{p+r}XB^{q-r}X^*)\right\}^{\frac{p-q+r}{p-q+2r}}\left\{\mathrm{tr}(A^{q-r}XB^{p+r}X^*)\right\}^{\frac{r}{p-q+2r}}\\&\leqslant\left\{\mathrm{tr}(A^{p+q}|X|^2)\right\}^{\frac{p}{p+q}}\left\{\mathrm{tr}(|X|^2B^{p+q})\right\}^{\frac{q}{p+q}},\end{aligned}\tag{2.3}$$

as for g we get

$$\begin{aligned}\mathrm{tr}(A^pXB^qX^*)&\leqslant\frac{p-q+r}{p-q+2r}\mathrm{tr}(A^{p+r}XB^{q-r}X^*)+\frac{r}{p-q+2r}\mathrm{tr}(A^{q-r}XB^{p+r}X^*)\\&\leqslant\frac{p}{p+q}\mathrm{tr}(A^{p+q}|X|^2)+\frac{q}{p+q}\mathrm{tr}(|X^*|^2B^{p+q}),\end{aligned}\tag{2.4}$$

introducing intermediate terms between

$$\mathrm{tr}(A^pXB^qX^*)\text{ and }\left\{\mathrm{tr}(A^{p+q}|X|^2)\right\}^{\frac{p}{p+q}}\left\{\mathrm{tr}(|X^*|^2B^{p+q})\right\}^{\frac{q}{p+q}}.$$

One last remark about these trace quantities is the following corollary.

COROLLARY 2.17. *Let* $A,B\in\mathbb{M}_n^+$, $X\in\mathbb{M}_n$. *The function* $f:(0,\infty)\times(0,\infty)\longrightarrow[0,\infty)$ *defined by* $f(p,q)=\mathrm{tr}(A^pXB^qX^*)$ *is log-convex.*

Proof. Since $(p,q) \longrightarrow \|A^p X B^q\|_2$ is log-convex, we have

$$(p,q) \longrightarrow g(p,q) := \{\operatorname{tr}(A^p X B^q B^q X^* A^p)\}^{1/2}$$

is log-convex. But then g^2 is log-convex. The result follows by replacing A and B with $\sqrt{A}$ and $\sqrt{B}$, respectively. □

This corollary must be compared with the well known Lieb's concavity theorem that

$$(A,B) \longrightarrow \operatorname{tr}(A^p X B^q X^*), \quad p+q \leqslant 1$$

is concave, and the well known Ando's convexity theorem that

$$(A,B) \longrightarrow \operatorname{tr}(A^p X B^{-q} X^*), \quad 1 \leqslant p \leqslant 2,\ 0 \leqslant q \leqslant 1,\ p-q \geqslant 1,$$

is convex. See [3] pages 118–119.

Observe that inequality (2.3) may be written in the form

$$\operatorname{tr}(AB) \leqslant (\operatorname{tr}A^p)^{1/p} \operatorname{tr}(B^q)^{1/q} \tag{2.5}$$

for the conjugate exponents p,q upon choosing $X = I$. This is the well known Hölder inequality for the trace operator. Recall that in general,

$$|||AB||| \leqslant |||A^p|||^{1/p} |||B^q|||^{1/q} \tag{2.6}$$

for any unitarily invariant norm and conjugate exponents p,q.

THEOREM 2.18. *Let $A \in \mathbb{M}_n$. Then, the function $f : [0,\infty) \to [0,\infty)$ defined by*

$$f(t) = |||A^t|||$$

is log-convex, hence is convex, for any unitarily invariant norm $|||\ \ |||$.

Proof. Let $t_1, t_2 \in [0,\infty)$ and $\alpha, \beta \geqslant 0$ be such that $\alpha + \beta = 1$. Then,

$$\begin{aligned} f(\alpha t_1 + \beta t_2) &= |||A^{\alpha t_1 + \beta t_2}||| \\ &= \left|\left|\left|A^{\alpha t_1} A^{\beta t_2}\right|\right|\right| \\ &\leqslant \left|\left|\left|A^{t_1}\right|\right|\right|^{\alpha} \left|\left|\left|A^{t_2}\right|\right|\right|^{\beta} \\ &= f^{\alpha}(t_1) f^{\beta}(t_2), \end{aligned}$$

where we have used (2.6) with $p = \frac{1}{\alpha}$ and $q = \frac{1}{\beta}$. This completes the proof. □

On the other hand, theorem 2.18 and corollary 2.5 imply the following.

THEOREM 2.19. *Let $A \in \mathbb{M}_n^+$. Then the function $f : (0,\infty) \to [0,\infty)$ defined by $f(t) = |||A^t|||^{1/t}$ is harmonically log-convex, hence is harmonically convex, for any unitarily invariant norm $|||\ \ |||$.*

In particular, the function $f(p)=\|A\|_p, p\geqslant 1$ is harmonically log-convex for any $A\in\mathbb{M}_n$.

The following result follows immediately from Theorem 2.19 and Proposition 2.3.

COROLLARY 2.20. *Let* $A\in\mathbb{M}_n^+$. *Then the function* $f:(0,\infty)\to[0,\infty)$ *defined by*

$$f(t)=\left|\left|\left|A^{1/t}\right|\right|\right|^t$$

is log-convex, hence is convex.

2.4. Multiplicative Heinz-Type Means

Now we study the function $\nu\to|||A^\nu XB^{1-\nu}|||$ which will be the key to other results in this part of the paper.

PROPOSITION 2.21. *Let* $A,B\in\mathbb{M}_n^+$ *and* $X\in\mathbb{M}_n$. *Then the function*

$$f(\nu)=|||A^\nu XB^{1-\nu}|||$$

is log-convex on $[0,1]$, *hence is convex.*

Proof. Let $\nu_1,\nu_2,\ \nu\in[0,1]$. Apply Hölder's inequality (1.1) replacing X by $A^{\nu_2}XB^{1-\nu_1}$, A by $A^{\nu_1-\nu_2}$ and B by $B^{\nu_1-\nu_2}$ to get

$$\begin{aligned}
f(\nu\nu_1+(1-\nu)\nu_2)&=|||A^{\nu\nu_1+(1-\nu)\nu_2}XB^{1-\nu\nu_1-(1-\nu)\nu_2}|||\\
&=\left|\left|\left|\left(A^{\nu_1-\nu_2}\right)^\nu\left(A^{\nu_2}XB^{1-\nu_1}\right)\left(B^{\nu_1-\nu_2}\right)^{1-\nu}\right|\right|\right|\\
&\leqslant|||A^{\nu_1-\nu_2}\left(A^{\nu_2}XB^{1-\nu_1}\right)|||^\nu\ |||\left(A^{\nu_2}XB^{1-\nu_1}\right)B^{\nu_1-\nu_2}|||^{1-\nu}\\
&=|||A^{\nu_1}XB^{1-\nu_1}|||^\nu|||A^{\nu_2}XB^{1-\nu_2}|||^{1-\nu}\\
&=f^\nu(\nu_1)f^{1-\nu}(\nu_2).
\end{aligned}$$

This completes the proof for invertible matrices A and B. If A or B is not invertible, a standard limiting process yields the result for general matrices. □

By symmetry, we deduce that the function $f(\nu)=|||A^{1-\nu}XB^\nu|||$ is log-convex too.

Following the same computations, one can easily prove that the function

$$f(\nu)=|||A^\nu XB^\nu|||, 0\leqslant\nu\leqslant 1$$

is also log-convex. This gives a straightforward proof of Lemma 2, p. 150 of [2].

Since the product of two log-convex functions is log-convex, we have the following corollary.

COROLLARY 2.22. *Let* $A,B\in\mathbb{M}_n^+$ *and* $X\in\mathbb{M}_n$. *Then the function*

$$f(\nu)=|||A^\nu XB^{1-\nu}|||\ |||A^{1-\nu}XB^\nu|||$$

is log-convex, hence is convex.

Observe that when $f(\nu)=|||A^{\nu}XB^{1-\nu}|||$, the known Young's inequality $f(\nu)\leqslant f(1)^{\nu}f(0)^{1-\nu}$ follows from Theorem 2.9 by letting $n=0$. In fact as n increases inequality (2.1) becomes better as shown in Theorem 2.10.

For example, by letting $f(\nu)=|||A^{\nu}XB^{1-\nu}|||$ and $n=1$ in Theorem 2.9 we get the following refinement of Young's inequality (1.1).

COROLLARY 2.23. *Let $A,B\in\mathbb{M}_n^+$ and $X\in\mathbb{M}_n$. Then,*

$$|||A^{\nu}XB^{1-\nu}|||\leqslant\left\{\begin{array}{ll}|||XB|||^{1-2\nu}|||A^{1/2}XB^{1/2}|||^{2\nu}, & 0\leqslant\nu\leqslant\frac{1}{2}\\ |||A^{1/2}XB^{1/2}|||^{2-2\nu}|||AX|||^{2\nu-1}, & \frac{1}{2}\leqslant\nu\leqslant 1\end{array}\right\}\leqslant|||AX|||^{\nu}|||XB|||^{1-\nu}.$$

Then as suggested by Theorem 2.10, taking larger n implies better refinements.

Our next goal is to study the monotonicity of the function

$$f(\nu)=|||A^{\nu}XB^{1-\nu}|||\ |||A^{1-\nu}XB^{\nu}|||.$$

For this, we need the following lemma.

LEMMA 2.24. *Let $A,B\in\mathbb{M}_n^+$ and $X\in\mathbb{M}_n$. If $p\geqslant q\geqslant 0$, then the function*

$$g(r)=|||A^{p+r}XB^{q-r}|||\ |||A^{q-r}XB^{p+r}|||$$

is increasing on $[0,q]$.

Proof. Let $0\leqslant r_1\leqslant r_2\leqslant q$. Then

$$g(r_1)=|||A^{p+r_1}XB^{q-r_1}|||\ |||A^{q-r_1}XB^{p+r_1}|||.$$

Observe that because $p\geqslant q$ we have $p+r_1\geqslant q-r_1$, hence we may apply Inequality (1.3) twice with $p=p+r_1$, $q=q-r_1$ and $r=r_2-r_1$, noting that A has the bigger exponent $p+r_1$ in the first quantity and B has the bigger exponent $p+r_1$ in the second quantity. Then

$$\begin{aligned}g(r_1)&=|||A^{p+r_1}XB^{q-r_1}|||\ |||A^{q-r_1}XB^{p+r_1}|||\\&\leqslant\left(|||A^{p+r_2}XB^{q-r_2}|||^{\frac{p-q+r_1+r_2}{p-q+2r_2}}\,|||A^{q-r_2}XB^{p+r_2}|||^{\frac{r_2-r_1}{p-q+2r_2}}\right)\\&\quad\times\left(|||A^{p+r_2}XB^{q-r_2}|||^{\frac{r_2-r_1}{p-q+2r_2}}\,|||A^{q-r_2}XB^{p+r_2}|||^{\frac{p-q+r_1+r_2}{p-q+2r_2}}\right)\\&=|||A^{p+r_2}XB^{q-r_2}|||\ |||A^{q-r_2}XB^{p+r_2}|||\\&=g(r_2).\end{aligned}$$

This completes the proof. □

PROPOSITION 2.25. *Let $A,B\in\mathbb{M}_n^+$ and $X\in\mathbb{M}_n$. Then the function*

$$f(\nu)=|||A^{\nu}XB^{1-\nu}|||\ |||A^{1-\nu}XB^{\nu}|||$$

is decreasing on $[0,\frac{1}{2}]$ and increasing on $[\frac{1}{2},1]$.

Proof. If $0 \leqslant \nu \leqslant \frac{1}{2}$, then

$$f(\nu) = |||A^{\frac{1}{2}+(\frac{1}{2}-\nu)}XB^{\frac{1}{2}-(\frac{1}{2}-\nu)}|||\ |||A^{\frac{1}{2}-(\frac{1}{2}-\nu)}XB^{\frac{1}{2}+(\frac{1}{2}-\nu)}|||.$$

Write $p = q = \frac{1}{2}$, $\frac{1}{2} - \nu = r$. Then,

$$f(\nu) = |||A^{p+r}XB^{q-r}|||\ |||A^{q-r}XB^{p+r}|||, \quad 0 \leqslant r \leqslant \frac{1}{2}.$$

Consequently, f is increasing with r. Since $r = \frac{1}{2} - \nu$, f is decreasing with ν. This completes the proof for $[0, \frac{1}{2}]$.

For $\frac{1}{2} \leqslant \nu \leqslant 1$, observe that

$$f(\nu) = |||A^{\frac{1}{2}+(\nu-\frac{1}{2})}XB^{\frac{1}{2}-(\nu-\frac{1}{2})}|||\ |||A^{\frac{1}{2}-(\nu-\frac{1}{2})}XB^{\frac{1}{2}+(\nu-\frac{1}{2})}|||.$$

Following the same idea, we infer that f is increasing with $r := \nu - \frac{1}{2}$, hence so is with ν. This complete the proof of the proposition. □

Proposition 2.25 allows us to write the following multiplicative version of Heinz inequality (1.2).

COROLLARY 2.26. *Let $A, B \in \mathbb{M}_n^+$ and $X \in \mathbb{M}_n$. Then, for $\nu \in [0,1]$, we have*

$$|||A^{1/2}XB^{1/2}|||^2 \leqslant \|A^{\nu}XB^{1-\nu}|||\ |||A^{1-\nu}XB^{\nu}||| \leqslant |||AX|||\ |||XB|||.$$

Proof. Since $f(\nu) = |||A^{\nu}XB^{1-\nu}|||\ |||A^{1-\nu}XB^{\nu}|||$ is decreasing on $[0, \frac{1}{2}]$, increasing on $[\frac{1}{2}, 1]$ and is symmetric about $\nu = \frac{1}{2}$, we have

$$f(1/2) \leqslant f(\nu) \leqslant f(1) = f(0).$$

This implies the result. □

It should be noted that in [8], convexity of the function

$$\nu \to |||\ |A^{\nu}XB^{1-\nu}|^r\ |||\ |||\ |A^{1-\nu}XB^{\nu}|^r\ |||; \quad r > 0,$$

was proved and used to obtain some interesting inequalities.

We remark that the second inequality of Corollary 2.26 is trivial, however the first part of the inequality $|||A^{1/2}XB^{1/2}|||^2 \leqslant \|A^{\nu}XB^{1-\nu}|||\ |||A^{1-\nu}XB^{\nu}|||$ is not. At this point, it might be thought that $|||A^{1/2}XB^{1/2}||| \leqslant |||A^{\nu}XB^{1-\nu}|||$ and $|||A^{1/2}XB^{1/2}||| \leqslant |||A^{1-\nu}XB^{\nu}|||$. In fact this is not true. This can be seen by taking the numerical example: $A = 3, B = 5$ and $X = 1$. Then, for each $\nu \in (\frac{1}{2}, 1]$ we have $|||A^{1/2}XB^{1/2}||| > |||A^{\nu}XB^{1-\nu}|||$ but $|||A^{1/2}XB^{1/2}||| < |||A^{1-\nu}XB^{\nu}|||$.

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