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ON FUNCTIONAL IDENTITIES OF DEGREE 2 AND CENTRALIZING MAPS IN TRIANGULAR RINGS

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Abstract. Let R be a triangular ring with center $Z(R)$. Let $F_1,F_2,G_1,G_2 : R \to R$ be maps such that

$$F_1(x)y+F_2(y)x+xG_2(y)+yG_1(x) \in Z(R)$$

for all $x,y \in R$. The aim of the paper is to give a solution of this functional identity in certain triangular rings. As applications, centralizing additive maps of certain triangular rings are determined.

1. Introduction

Let R be an associative ring with center $Z(R)$. The functional identities of degree 2 are

$$F_1(x)y+F_2(y)x+xG_2(y)+yG_1(x) = 0 \tag{1}$$

for all $x,y \in R$ and a slightly more general one,

$$F_1(x)y+F_2(y)x+xG_2(y)+yG_1(x) \in Z(R) \tag{2}$$

for all $x,y \in R$. In 1995, Brešar [3] investigated both (1) and (2) in prime rings, which initiated the theory of functional identities (see the book [5] for a full account on this theory).

In 2013, Eremita [9] gave a solution of (1) in certain triangular rings. As applications he discussed commuting additive maps and generalized inner biderivations of certain triangular rings. It should be mentioned that (1) was considered in the context of nest algebras by Zhang, Feng, Li and Wu in [14]. In 2015, Eremita [10] gave a solution of (1) in a much wider class of triangular rings. Recently, the author [12] gave a solution of (1) in arbitrary triangular rings. As applications commuting additive maps and generalized inner biderivations of arbitrary triangular rings are determined.

In the present paper, we shall give a solution of (2) in certain triangular rings. As consequences, the solution of (2) in upper triangular matrix rings and nest algebras are determined. As an application of our main result, centralizing additive maps of certain triangular rings are determined.

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2. The main result

A unital ring R with a nontrivial idempotent e is a triangular ring, if eRf is a faithful (eRe, fRf)-bimodule and $fRe = 0$, where $f = 1 - e$. Each triangular ring R has the so-called Peirce decomposition:

$$R = eRe \oplus eRf \oplus fRf.$$

The result [6, Proposion 3] tells us that the center of R is

$$Z(R) = \{a + b \in eRe + fRf \mid am = mb \text{ for all } m \in eRf\}.$$

Furthermore, there exists a unique ring isomorphism $\tau : Z(R)e \to Z(R)f$ such that $am = m\tau(a)$ for all $m \in eRf$ and for any $a \in Z(R)e$.

We set $[x,y] = xy - yx$ for $x, y \in R$. Let S be a subset of R. we set

$$Z_2(S) = \{a \in S \mid [[a,x],x] = 0 \text{ for all } x \in S\}.$$

We begin with the following technical result, which will be used in the proof of the main result.

LEMMA 2.1. *Let A be a unital ring such that*

(i) $Z_2(A) = Z(A)$*;*

(ii) *A does not contain nonzero central ideals.*

Suppose that there exist $a, b \in A$ such that

$$ax + xb \in Z(A) \tag{3}$$

for all $x \in A$. Then $a = -b \in Z(A)$.

Proof. Letting $x = 1$ in (3) we get that $a + b \in Z(A)$. Set $c = a + b$. It follows from (3) that

$$[a,x] + xc \in Z(A)$$

for all $x \in A$, which yields that $[[a,x],x] = 0$ for all $x \in A$. Hence, $a \in Z_2(A) = Z(A)$ and hence $b \in Z(A)$. Consequently, $(a+b)A \subseteq Z(A)$. Since A does not contain nonzero central ideals, we get that $a + b = 0$ and then $a = -b \in Z(A)$. □

A map $F : R \to R$ is said to be additive modulo $Z(R)$ if

$$F(x+y) - F(x) - F(y) \in Z(R)$$

for all $x, y \in R$. For each map $F : R \to R$ and each positive integer n we define a map $\delta_{n,F} : R^n \to R$ by

$$\delta_{n,F}(x_1, \dots, x_n) = F(x_1 + \cdots + x_n) - F(x_1) - \cdots - F(x_n).$$

Obviously, $\delta_{n,F}(R^n) \subseteq Z(R)$ if F is additive modulo $Z(R)$ (see [9, Lemma 3.1]).

Using the same arguments as in [9] we are ready to give the main result of the paper.

THEOREM 2.1. *Let R be a triangular ring such that*

$$Z(eRe) = Z(R)e \quad \text{and} \quad Z(fRf) = Z(R)f.$$

Assume that one of the following conditions is satisfied:

(i) $Z_2(eRe) = Z(eRe)$ *and* eRe *does not contain nonzero central ideals;*

(ii) $Z_2(fRf) = Z(fRf)$ *and* fRf *does not contain nonzero central ideals.*

Suppose that $F_1, F_2, G_1, G_2 : R \to R$ *are arbitrary maps such that*

$$F_1(x)y + F_2(y)x + xG_2(y) + yG_1(x) \in Z(R)$$

for all $x, y \in R$*. Then there exist* $p_1, p_2, q_1, q_2, r_1, r_2 \in R$ *and maps* $\alpha_1, \alpha_2 : R \to Z(R)$ *such that* $p_1 + p_2 = r_1 + r_2 \in Z(R)$*,* $p_i[x,y] - [x,y]r_i \in Z(R)$*,* $i = 1,2$*, and*

$$\begin{aligned}
F_1(x) &= xq_1 - p_1x + \alpha_1(x) \\
F_2(x) &= xq_2 - p_2x + \alpha_2(x) \\
G_1(x) &= xr_2 - q_2x - \alpha_1(x) \\
G_2(x) &= xr_1 - q_1x - \alpha_2(x)
\end{aligned}$$

for all $x, y \in R$*.*

Proof. Let

$$H(x,y) = F_1(x)y + F_2(y)x + xG_2(y) + yG_1(x)$$

for all $x, y \in R$. We claim that F_1, F_2, G_1, G_2 are additive modulo $Z(R)$. Since $H(x_1 + x_2, y) - H(x_1, y) - H(x_2, y) \in Z(R)$ for all $x_1, x_2, y \in R$ we see that

$$\delta_{2,F_1}(x_1,x_2)y + y\delta_{2,G_1}(x_1,x_2) \in Z(R)$$

for all $x_1, x_2, y \in R$. In view of [8, Lemma 3.1] or [9, Lemma 3.1] we obtain

$$\delta_{2,F_1}(x_1,x_2) = -\delta_{2,G_1}(x_1,x_2) \in Z(R)$$

for all $x_1, x_2 \in R$. Thus, both F_1 and G_1 are additive modulo $Z(R)$. Similarly, we obtain that both F_2 and G_2 are additive modulo $Z(R)$.

Since $H(x,1) \in Z(R)$ and $H(1,x) \in Z(R)$ we get

$$\begin{aligned}
F_1(x) + G_1(x) + F_2(1)x + xG_2(1) &\in Z(R) \\
F_2(x) + G_2(x) + F_1(1)x + xG_1(1) &\in Z(R)
\end{aligned} \tag{4}$$

for all $x \in R$. From $eH(x,f)f = 0$ it follows that

$$eF_1(x)f + eF_2(f)xf + exG_2(f)f = 0$$

and so

$$eF_1(x)f = -eF_2(f)xf - exG_2(f)f \tag{5}$$

for all $x \in R$. Analogously, we get

$$\begin{aligned} eF_2(x)f &= -eF_1(f)xf - exG_1(f)f \\ eG_1(x)f &= -eF_2(e)xf - exG_2(e)f \\ eG_2(x)f &= -eF_1(e)xf - exG_1(e)f \end{aligned} \tag{6}$$

for all $x \in R$. Since $H(exe, fyf) \in Z(R)$ we get

$$F_1(exe)fyf + F_2(fyf)exe + exeG_2(fyf) + fyfG_1(exe) \in Z(R)$$

for all $x, y \in R$. That is,

$$\begin{aligned} eF_1(exf)fyf + fF_1(exe)fyf + eF_2(fyf)exe + exeG_2(fyf)e \\ + exeG_2(fyf)f + fyfG_1(exe)f \in Z(R) \end{aligned}$$

for all $x, y \in R$. Then [6, Proposition 3] tells us that

$$eF_1(exf)fyf + exeG_2(fyf)f = 0$$

for all $x \in R$ and so

$$fF_1(exe)fyf + eF_2(fyf)exe + exeG_2(fyf)e + fyfG_1(exe)f \in Z(R)$$

for all $x \in R$. This implies that

$$(eF_2(fyf)exe + exeG_2(fyf)e) + (fF_1(exe)fyf + fyfG_1(exe)f) \in Z(R) \tag{7}$$

for all $x, y \in R$. Without loss of generality we may assume that the assumption (i) is satisfied. It follows from (7) that

$$eF_2(fyf)exe + exeG_2(fyf)e \in Z(eRe)$$

for all $x, y \in R$. By Lemma 2.1 we get

$$eF_2(fyf)e = -eG_2(fyf)e \in Z(eRe) \tag{8}$$

for all $y \in R$. Using (8) togather with (7) we get

$$fF_1(exe)fyf + fyfG_1(exe)f = 0$$

for all $x, y \in R$. Letting $y = f$ in the last relation we get that $fF_1(exe)f = -fG_1(exe)f$ and then

$$fF_1(exe)f = -fG_1(exe)f \in Z(fRf) \tag{9}$$

for all $x \in R$. In an analogous manner, we get

$$\begin{aligned} eF_1(fxf)e &= -eG_1(fxf)e \in Z(eRe) \\ fF_2(exe)f &= -fG_2(exe)f \in Z(fRf) \\ eF_2(fxf)e &= -eG_2(fxf)e \in Z(eRe) \end{aligned} \tag{10}$$

for all $x \in R$. From $H(exe, eyf) \in Z(R)$ it follows that

$$\begin{aligned} eF_1(exe)eyf + eF_2(eyf)exe + exeG_2(eyf)e + exeG_2(eyf)f \\ + eyfG_1(exe)f \in Z(R) \end{aligned} \tag{11}$$

for all $x, y \in R$. On the one hand, we get from (11) that

$$eF_1(exe)eyf + exeG_2(eyf)f + eyfG_1(exe)f = 0$$

for all $x, y \in R$. Consequently, using (6) we get

$$eF_1(exe)eyf - exeF_1(e)eyf - exeyfG_1(e)f + eyfG_1(exe)f = 0$$

for all $x, y \in R$. Using (9), it follows that

$$\left(eF_1(exe)e - exeF_1(e) - exe\tau^{-1}(fG_1(e)f) + \tau^{-1}(fG_1(exe)f)\right) eyf = 0$$

for all $x, y \in R$. Since eRf is faithful as a left eRe-module we obtain

$$eF_1(exe)e = exe\left(F_1(e)e + \tau^{-1}(fG_1(e)f)\right) - \tau^{-1}(fG_1(exe)f) \tag{12}$$

for all $x \in R$. Analogously, we obtain

$$eF_2(exe)e = exe\left(F_2(e)e + \tau^{-1}(fG_2(e)f)\right) - \tau^{-1}(fG_2(exe)f)$$

for all $x \in R$. On the other hand, we get from (11) that

$$eF_2(eyf)exe + exeG_2(eyf)e = 0$$

for all $x \in R$. This yields that

$$eF_2(eyf)e = -eG_2(eyf)e \in Z(eRe)$$

for all $y \in R$. Analogously, we obtain

$$\begin{aligned} eF_1(eyf)e &= -eG_1(eyf)e \in Z(eRe) \\ fF_1(eyf)f &= -fG_1(eyf)f \in Z(fRf) \\ fF_2(eyf)f &= -fG_2(eyf)f \in Z(fRf) \end{aligned} \tag{13}$$

for all $x \in R$. Since $H(exf, eyf) = 0$ we have

$$F_1(exf)eyf + F_2(eyf)exf + exfG_2(eyf) + eyfG_1(exf) = 0$$

and hence, using (13), we get

$$\left(eF_1(exf)e - \tau^{-1}(fF_1(exf)f)\right) eyf + \left(eF_2(eyf)e - \tau^{-1}(fF_2(eyf)f)\right) exf = 0$$

for all $x, y \in R$. It follows from [9, Lemma 2.1] that

$$\begin{aligned} eF_1(exf)e &= \tau^{-1}(fF_1(exf)f) \\ eF_2(exf)e &= \tau^{-1}(fF_2(exf)f) \end{aligned} \tag{14}$$

for all $x \in R$. Since $eH(fxf, eyf)f = 0$ we have

$$eF_1(fxf)eyf + eF_2(eyf)fxf + eyfG_1(fxf)f = 0$$

for all $x, y \in R$. Using (6) togather with (10), it follows that

$$eyf\,(\tau(eF_1(fxf)e) - (\tau(eF_1(f)e) + fG_1(f)f)fxf + fG_1(fxf)f) = 0$$

for all $x \in R$, $y \in R$. Since eRf is faithful as a right fRf-module we get

$$fG_1(fxf)f = (\tau(eF_1(f)e) + fG_1(f)f)fxf - \tau(eF_1(fxf)e) \tag{15}$$

for all $x \in R$. Analogously,

$$fG_2(fxf)f = (\tau(eF_2(f)e) + fG_2(f)f)fxf - \tau(eF_2(fxf)e)$$

for all $x \in R$. Note that

$$fF_1(fxf)f + fF_2(f)fxf + fxfG_2(f)f + fG_1(fxf)f = H(fxf, f)f$$

and hence, using (15), we obtain

$$\begin{aligned} fF_1(fxf)f = &-(fF_2(f)f + fG_1(f)f + \tau(eF_1(f)e))fxf - fxfG_2(f)f \\ &+ \tau(eF_1(fxf)e) + H(fxf, f)f \end{aligned} \tag{16}$$

for all $x \in R$.

We are now ready to describe F_1. Using (5), (9), (12) and (16) we get

$$\begin{aligned} F_1(x) &= eF_1(x)e + eF_1(x)f + fF_1(x)f \\ &= eF_1(exe)e + eF_1(exf)e + eF_1(fxf)e + eF_1(x)f + fF_1(exe)f \\ &\quad + fF_1(exf)f + fF_1(fxf)f + \delta_{3,F_1}(exe, exf, fxf) \\ &= exe(F_1(e)e + \tau^{-1}(fG_1(e)f)) - \tau^{-1}(fG_1(exe)f) + eF_1(exf)e \\ &\quad + eF_1(fxf)e - eF_2(f)xf - exG_2(f)f - fG_1(exe)f + fF_1(exf)f \\ &\quad - (fF_2(f)f + fG_1(f)f + \tau(eF_1(f)e))fxf - fxfG_2(f)f \\ &\quad + \tau(eF_1(fxf)e) + H(fxf, f)f + \delta_{3,F_1}(exe, exf, fxf) \end{aligned}$$

for all $x \in R$. Define a map α_1 by

$$\begin{aligned} \alpha_1(x) = &(eF_1(exf)e + fF_1(exf)f) + (eF_1(fxf)e + \tau(eF_1(fxf)e)) \\ &- (\tau^{-1}(fG_1(exe)f) + fG_1(exe)f) + \delta_{3,F_1}(exe, exf, fxf) \end{aligned}$$

for all $x \in R$. According to (9), (10), and (14) we see that $\alpha_1(R) \subseteq Z(R)$. Thus,

$$\begin{aligned} F_1(x) = &exe(F_1(e)e + \tau^{-1}(fG_1(e)f)) - eF_2(f)xf - exG_2(f)f \\ &- (fF_2(f)f + fG_1(f)f + \tau(eF_1(f)e))fxf - fxfG_2(f)f \\ &+ H(fxf, f)f + \alpha_1(x) \end{aligned} \tag{17}$$

for all $x \in R$. Note that

$$eF_2(f)exf = eF_2(f)ex - xeF_2(f)e,$$

since $eF_2(f)e \in Z(eRe)$. We can now rewrite (17) as

$$\begin{aligned} F_1(x) = x\big(&F_1(e)e + \tau^{-1}(fG_1(e)f) - G_2(f)f + eF_2(f)e\big) \\ &- (F_2(f)f + fG_1(f)f + \tau(eF_1(f)e) + eF_2(f)e)x \\ &+ H(fxf,f)f + \alpha_1(x) \end{aligned}$$

for all $x \in R$. Moreover, since

$$\begin{aligned} -G_2(f)f + eF_2(f)e &= -G_2(f)f - eG_2(f)e = -G_2(f), \\ F_2(f)f + eF_2(f)e &= F_2(f) \end{aligned}$$

Thus, we can rewrite (17) as

$$\begin{aligned} F_1(x) = x\big(&eF_1(e)e - G_2(f) + \tau^{-1}(fG_1(e)f)\big) \\ &- (fG_1(f)f + F_2(f) + \tau(eF_1(f)e))x + H(fxf,f)f + \alpha_1(x) \end{aligned} \tag{18}$$

for all $x \in R$. In an analogous manner we get

$$\begin{aligned} F_2(x) = x(&eF_2(e)e - G_1(f) + \tau^{-1}(fG_2(e)f)) \\ &- (fG_2(f)f + F_1(f) + \tau(eF_2(f)e))x + H(f,fxf)f + \alpha_2(x) \end{aligned} \tag{19}$$

for all $x \in R$, where

$$\begin{aligned} \alpha_2(x) = (&eF_2(exf)e + fF_2(exf)f) + (eF_2(fxf)e + \tau(eF_2(fxf)e)) \\ &- (\tau^{-1}(fG_2(exe)f) + fG_2(exe)f) + \delta_{3,F_2}(exe,exf,fxf) \in Z(R). \end{aligned}$$

Now, using (4) we have

$$\begin{aligned} G_1(x) &= -F_1(x) - F_2(1)x - xG_2(1) \\ &= -x(eF_1(e)e + G_2(1) - G_2(f) + \tau^{-1}(fG_1(e)f)) \\ &\quad + (fG_1(f)f - F_2(1) + F_2(f) + \tau(eF_1(f)e))x \\ &\quad - H(fxf,f)f - \alpha_1(x) \end{aligned} \tag{20}$$

and

$$\begin{aligned} G_2(x) &= -F_2(x) - F_1(1)x - xG_1(1) \\ &= -x(eF_2(e)e + G_1(1) - G_1(f) + \tau^{-1}(fG_2(e)f)) \\ &\quad + (fG_2(f)f - F_1(1) + F_1(f) + \tau(eF_2(f)e))x \\ &\quad - H(f,fxf)f - \alpha_2(x). \end{aligned} \tag{21}$$

Set

$$
\begin{aligned}
q_1 &= eF_1(e)e - G_2(f) + \tau^{-1}(fG_1(e)f),\\
q_2 &= eF_2(e)e - G_1(f) + \tau^{-1}(fG_2(e)f),\\
\lambda &= (eF_2(f)e + \tau(eF_2(f)e)) + (\tau^{-1}(fG_1(e)f) + fG_1(e)f),\\
\mu &= (eF_1(f)e + \tau(eF_1(f)e)) + (\tau^{-1}(fG_2(e)f) + fG_2(e)f).
\end{aligned}
$$

Note that $\lambda, \mu \in Z(R)$. Using (6) and (10), we see that

$$
\begin{aligned}
\lambda - q_1 &= eF_2(f)e + \tau(eF_2(f)e) + fG_1(e)f - F_1(e)e + G_2(f)\\
&= \tau(eF_2(f)e) + fG_1(e)f - F_1(e)e + eG_2(f)f + fG_2(f)f\\
&= \tau(eF_2(f)e) + fG_1(e)f - F_1(e)e - F_1(e)f - fG_1(e)f + fG_2(f)f\\
&= fG_2(f)f + \tau(eF_2(f)e) - F_1(e).
\end{aligned}
$$

Similarly, using (6) and (10), we obtain

$$
\begin{aligned}
\mu - q_2 &= eF_1(f)e + \tau(eF_1(f)e) + fG_2(e)f - F_2(e)e + G_1(f)\\
&= \tau(eF_1(f)e) + fG_2(e)f - F_2(e)e + eG_1(f)f + fG_1(f)f\\
&= \tau(eF_1(f)e) + fG_2(e)f - F_2(e)e - F_2(e)f - fG_2(e)f + fG_1(f)f\\
&= fG_1(f)f + \tau(eF_1(f)e) - F_2(e).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
fG_1(f)f + F_2(f) + \tau(eF_1(f)e) &= \mu - q_2 + F_2(e) + F_2(f),\\
fG_2(f)f + F_1(f) + \tau(eF_2(f)e) &= \lambda - q_1 + F_1(e) + F_1(f),\\
eF_1(e)e + G_2(1) - G_2(f) + \tau^{-1}(fG_1(e)f) &= G_2(1) + q_1,\\
eF_2(e)e + G_1(1) - G_1(f) + \tau^{-1}(fG_2(e)f) &= G_1(1) + q_2.
\end{aligned}
$$

Consequently, (18)-(21) can be rewritten as

$$
\begin{aligned}
F_1(x) &= xq_1 - (\mu - q_2 + F_2(e) + F_2(f))x + H(fxf, f)f + \alpha_1(x)\\
F_2(x) &= xq_2 - (\lambda - q_1 + F_1(e) + F_1(f))x + H(f, fxf)f + \alpha_2(x)\\
G_1(x) &= -x(G_2(1) + q_1 - \mu + \delta_{2,F_2}(e,f)) - q_2x - H(fxf, f)f - \alpha_1(x)\\
G_2(x) &= -x(G_1(1) + q_2 - \lambda + \delta_{2,F_1}(e,f)) - q_1x - H(f, fxf)f - \alpha_2(x)
\end{aligned}
$$

for all $x \in R$. Setting

$$
\begin{aligned}
p_1 &= \mu - q_2 + F_2(e) + F_2(f),\\
p_2 &= \lambda - q_1 + F_1(e) + F_1(f),\\
r_2 &= -G_2(1) - q_1 + \mu - \delta_{2,F_2}(e,f),\\
r_1 &= -G_1(1) - q_2 + \lambda - \delta_{2,F_1}(e,f).
\end{aligned}
$$

and

$$
\tau_1(x) = H(fxf, f)f \quad \text{and} \quad \tau_2(x) = H(f, fxf)f
$$

for all $x \in R$. Now, (2) yields

$$p_1xy + p_2yx - xyr_1 - yxr_2 + \tau_1(x)y + \tau_2(y)x - x\tau_2(y) - y\tau_1(x) \in Z(R) \tag{22}$$

for all $x, y \in R$. It suffices to prove that $p_1 + p_2 = r_1 + r_2 \in Z(R)$, $\tau_1 = \tau_2 = 0$, and

$$p_i[x,y] - [x,y]r_i \in Z(R)$$

for all $x, y \in R$, $i = 1, 2$. For every $r \in R$, letting $x = frf$ and $y = f$ in (22) we obtain

$$(p_1 + p_2)frf - frf(r_1 + r_2) \in Z(R)$$

for all $r \in R$. This implies that $e(p_1 + p_2)f = 0$ and

$$f(p_1 + p_2)frf - frf(r_1 + r_2) = 0$$

for all $r \in R$. Hence, $f(p_1 + p_2)f = f(r_1 + r_2)f \in Z(fRf)$. Similarly, letting $x = ere$ and $y = e$ in (22) we obtain

$$(p_1 + p_2)ere - ere(r_1 + r_2) \in Z(R)$$

for all $r \in R$. This implies that $e(p_1 + p_2)e = e(r_1 + r_2)e \in Z(eRe)$ and $e(r_1 + r_2)f = 0$. Hence, $p_1 + p_2 = r_1 + r_2$.

Letting $x = e$ and $y = erf$ in (22) we get

$$p_1erf - erfr_1 \in Z(R)$$

for all $r \in R$. This implies that

$$ep_1erf - erfr_1f = 0$$

for all $r \in R$. That is, $ep_1 \cdot erf = erf \cdot fr_1f \in Z(R)$. [6, Proposion 3] tells us that $ep_1e + fr_1f \in Z(R)$. Similarly, letting $x = erf$ and $y = e$ we obtain $ep_2e + fr_2f \in Z(R)$. Thus, we have

$$e(p_1 + p_2)e + f(r_1 + r_2)f \in Z(R).$$

Since $p_1 + p_2 = r_1 + r_2$ and $e(p_1 + p_2)f = 0$, we see that

$$p_1 + p_2 = e(p_1 + p_2)e + f(p_1 + p_2)f \in Z(R).$$

Replacing x, y by exf and fyf in (22) respectively, we see that

$$ep_1exfyf - exfyfr_1f - exf\tau_2(y) \in Z(R) \tag{23}$$

for all $x, y \in R$. Since $ep_1e + fr_1f \in Z(R)$ we get from (23) that $exf\tau_2(y) = 0$ for all $x, y \in R$. Since eRf is faithful as a right fRf-module we obtain $\tau_2 = 0$. Similarly, replacing x and y by fxf and eyf in (22) respectively, we obtain $\tau_1 = 0$.

Now, (22) yields

$$p_1xy + p_2yx - xyr_1 - yxr_2 \in Z(R) \tag{24}$$

for all $x,y\in R$. Let $\gamma=p_1+p_2$. Then (24) implies

$$\begin{aligned}p_1xy+p_2yx-xyr_1-yxr_2&=(\gamma-p_2)xy+p_2yx-xyr_1-yx(\gamma-r_1)\\&=(\gamma-p_2)xy+(p_2-\gamma)yx-xyr_1+yxr_1\\&=p_1[x,y]-[x,y]r_1\in Z(R)\end{aligned}$$

and similarly $p_2[x,y]-[x,y]r_2\in Z(R)$ for all $x,y\in R$. □

LEMMA 2.2. *Let R be a triangular ring. Then R does not contain nonzero central ideals and $Z_2(R)=Z(R)$.*

Proof. In view of [1, Lemma 2.6] we see that R does not contain nonzero central ideals. Let $a\in Z_2(R)$. We have

$$[[a,x],x]=0 \tag{25}$$

for all $x\in R$. Letting $x=f$ in (25) we get $eaf=0$. Thus, $a=eae+faf$. Letting $x=erf+f$ in (25) we get

$$eaerf-erfaf=0$$

for all $r\in R$. This implies that $eae+faf\in Z(R)$. Hence, $a\in Z(R)$. This implies that $Z_2(R)\subseteq Z(R)$. The inclusion $Z(R)\subseteq Z_2(R)$ is trivial. Therefore, $Z_2(R)=Z(R)$. □

As a consequence of Theorem 2.1 we obtain the following result.

COROLLARY 2.1. *Let S be a unital ring and let $n\geqslant 3$. Let $T_n(S)$ be the ring of all $n\times n$ upper triangular matrices over S. Suppose that $F_1,F_2,G_1,G_2:T_n(S)\to T_n(S)$ are arbitrary maps such that*

$$F_1(x)y+F_2(y)x+xG_2(y)+yG_1(x)\in Z(S)\cdot I$$

for all $x,y\in T_n(S)$. Then there exist $p_1,p_2,q_1,q_2,r_1,r_2\in T_n(S)$ and maps $\alpha_1,\alpha_2: T_n(S)\to Z(S)\cdot I$ such that $p_1+p_2=r_1+r_2\in Z(S)\cdot I$, $p_i[x,y]-[x,y]r_i\in Z(S)\cdot I$, $i=1,2$, and

$$\begin{aligned}F_1(x)&=xq_1-p_1x+\alpha_1(x)\\F_2(x)&=xq_2-p_2x+\alpha_2(x)\\G_1(x)&=xr_2-q_2x-\alpha_1(x)\\G_2(x)&=xr_1-q_1x-\alpha_2(x)\end{aligned}$$

for all $x,y\in T_n(S)$.

Proof. Let $e=e_{11}$ and $f=I-e_{11}$. It is easy to check that $T_n(S)$ is a triangular ring. Clearly, $Z(T_n(S))e=Z(eT_n(S)e)$ and $Z(T_n(S))f=Z(fT_n(S)f)$. Note that $fT_n(S)f\cong T_{n-1}(S)$ is a triangular ring. By Lemma 2.2 we see that $fT_n(S)f$ does not contain nonzero central ideals and $Z_2(fT_n(S)f)=Z(fT_n(S)f)$. Thus, all assumptions of Theorem 2.1 are met. Then the conclusion follows from Theorem 2.1. □

COROLLARY 2.2. *Let S be a unital noncommutative prime ring. Suppose that $F_1, F_2, G_1, G_2 : T_2(S) \to T_2(S)$ are arbitrary maps such that*

$$F_1(x)y + F_2(y)x + xG_2(y) + yG_1(x) \in Z(S) \cdot I$$

for all $x, y \in T_2(S)$. Then there exist $p_1, p_2, q_1, q_2, r_1, r_2 \in T_2(S)$ and maps $\alpha_1, \alpha_2 : T_2(S) \to Z(S) \cdot I$ such that $p_1 + p_2 = r_1 + r_2 \in Z(S) \cdot I$, $p_i[x,y] - [x,y]r_i \in Z(S) \cdot I$, $i = 1, 2$, and

$$\begin{aligned} F_1(x) &= xq_1 - p_1x + \alpha_1(x) \\ F_2(x) &= xq_2 - p_2x + \alpha_2(x) \\ G_1(x) &= xr_2 - q_2x - \alpha_1(x) \\ G_2(x) &= xr_1 - q_1x - \alpha_2(x) \end{aligned}$$

for all $x, y \in T_2(S)$.

Proof. Let $e = e_{11}$ and $f = e_{22}$. Note that $fT_2(S)f \cong Sf$ is a noncommutative prime ring. In view of a well-known result of Posner [11, Theorem 2] we get that $fT_2(S)f$ does not contain nonzero central ideals and $Z_2(fT_2(S)f) = Z(fT_2(S)f)$. Thus, all assumptions of Theorem 2.1 are met. Then the conclusion follows from Theorem 2.1. $\square$

A *nest* $\mathscr{N}$ is a totally ordered set of closed subspaces of a Hilbert space H such that $\{0\}$, $H \in \mathscr{N}$, and $\mathscr{N}$ is closed under the taking of arbitrary intersections and closed linear spans of its elements. The *nest algebra* associated to $\mathscr{N}$ is the set

$$\mathscr{T}(\mathscr{N}) = \{T \in \mathscr{B}(H) \mid TN \subseteq N \quad \text{for all } N \in \mathscr{N}\}.$$

A nest algebra $\mathscr{T}(\mathscr{N})$ is called *trivial* if $\mathscr{N} = \{0, H\}$. A nontrivial nest algebra can be viewed as a triangular algebra. Namely, if $N \in \mathscr{N} \setminus \{0, H\}$ and E is the orthonormal projection onto N, then $\mathscr{N}_1 = E(\mathscr{N})$ and $\mathscr{N}_2 = (1-E)(\mathscr{N})$ are nests of N and $N^{\perp}$, respectively. Moreover, $\mathscr{T}(\mathscr{N}_1) = ET(\mathscr{N})E$, $\mathscr{T}(\mathscr{N}_2) = (1-E)\mathscr{T}(\mathscr{N})(1-E)$ are nest algebras and

$$\mathscr{T}(\mathscr{N}) = \begin{pmatrix} \mathscr{T}(\mathscr{N}_1) & E\mathscr{T}(\mathscr{N})(1-E) \\ & \mathscr{T}(\mathscr{N}_2) \end{pmatrix}.$$

We refer the reader to [7] for the general theory of nest algebras. Note that $Z(\mathscr{T}(\mathscr{N})) = \mathbb{C}1$ [7, Corollary 19.5].

As a consequence of Theorem 2.1 we have

COROLLARY 2.3. *Let $\mathscr{N}$ be a nest of a complex Hilbert space H with $\dim_{\mathbb{C}} H > 2$. Let $\mathscr{T}(\mathscr{N})$ be a nest algebra. Suppose that $F_1, F_2, G_1, G_2 : \mathscr{T}(\mathscr{N}) \to \mathscr{T}(\mathscr{N})$ are arbitrary maps such that*

$$F_1(x)y + F_2(y)x + xG_2(y) + yG_1(x) \in \mathbb{C}1$$

for all $x,y \in \mathscr{T}(\mathscr{N})$. *Then there exist* $p_1,p_2,q_1,q_2,r_1,r_2 \in \mathscr{T}(\mathscr{N})$ *and maps* $\alpha_1,\alpha_2 : \mathscr{T}(\mathscr{N}) \to \mathbb{C}1$ *such that* $p_1+p_2 = r_1+r_2 \in \mathbb{C}1$, $p_i[x,y]-[x,y]r_i \in \mathbb{C}1$, $i=1,2$, *and*

$$F_1(x) = xq_1 - p_1x + \alpha_1(x)$$
$$F_2(x) = xq_2 - p_2x + \alpha_2(x)$$
$$G_1(x) = xr_2 - q_2x - \alpha_1(x)$$
$$G_2(x) = xr_1 - q_1x - \alpha_2(x)$$

for all $x,y \in \mathscr{T}(\mathscr{N})$.

Proof. If $\mathscr{N}$ is a trivial nest, then $\mathscr{T}(\mathscr{N}) = \mathscr{B}(H)$ is a prime ring and hence the conclusion follows from [3, Theorem 4.8]. Thus, we may assume that $\mathscr{N}$ is a nontrivial nest. Since $\dim_{\mathbb{C}} H > 2$ it follows that either $\dim_{\mathbb{C}} N > 1$ or $\dim_{\mathbb{C}} N^{\perp} > 1$. If $\dim_{\mathbb{C}} N > 1$, then either $\mathscr{T}(\mathscr{N}_1) = \mathscr{B}(N)$ is a noncommutative prime ring or $\mathscr{T}(\mathscr{N}_1)$ is a triangular ring. Similarly, if $\dim_{\mathbb{C}} N^{\perp} > 1$, then either $\mathscr{T}(\mathscr{N}_2) = \mathscr{B}(N^{\perp})$ is a noncommutative prime ring or $\mathscr{T}(\mathscr{N}_2)$ is a triangular ring. In view of a well-known result of Posner [11, Theorem 2] and Lemma 2.2 we see that all assumptions of Theorem 2.1 are met. Then the conclusion follows from Theorem 2.1. $\square$

3. Centralizing additive maps

Let R be a ring. Recall that an additive map $F : R \to R$ is said to be *commuting* if $[F(x),x] = 0$ for all $x \in R$. An additive map $F : R \to R$ is said to be *centralizing* if $[F(x),x] \in Z(R)$. In 1993, Brešar [2] determined commuting and centralizing additive maps on prime rings. Commuting maps appear in many areas and have been studied intensively (see the survey article [4]).

In 2001, Cheung [6] described the form of commuting linear maps for a certain class of triangular algebras. Later, a similar result [1, Remark 2.8] was obtained for another class of triangular algebras. In 2013, Eremita [9] described the form of commuting additive maps for yet another class of triangular rings. In 2015, Eremita [10] described the form of commuting additive maps for a much wider class of triangular rings. Recently, the author [12] described the form of commuting additive maps for arbitrary triangular rings.

In 2012, Du and Wang [8] described the form of k-commuting linear maps for a certain class of triangular algebras. In particular, they obtained a description of centralizing linear maps for a certain class of triangular algebras (see [8, Theorem 1.1]). Recently, the author obtained a description of centralizing linear maps for a more general class of triangular algebras (see [13, Proposition 2.1]).

Using Theorem 2.1 we obtain a description of centralizing additive maps for a certain class of triangular rings.

THEOREM 3.1. *Let* R *be a triangular ring such that*

$$Z(eRe) = Z(R)e \quad \textit{and} \quad Z(fRf) = Z(R)f.$$

Assume that one of the following conditions is satisfied:

(i) $Z_2(eRe) = Z(R)e$ *and* eRe *does not contain nonzero central ideals;*

(ii) $Z_2(fRf) = Z(fRf)$ *and* fRf *does not contain nonzero central ideals.*

Suppose that a map $F : R \to R$ *is additive modulo* $Z(R)$. *If* F *is centralizing, then there exist* $\lambda \in Z(R)$ *and a map* $\tau : R \to Z(R)$ *such that*

$$F(x) = \lambda x + \tau(x)$$

for all $x \in R$.

Proof. Since F is additive modulo $Z(R)$, the linearization of $[F(x), x] \in Z(A)$ for all $x \in R$, gives

$$F(x)y + F(y)x - xF(y) - yF(x) \in Z(R)$$

for all $x, y \in R$. Now, Theorem 2.1 implies that

$$\begin{aligned} F(x) &= xq - px + \alpha_1(x) \\ -F(x) &= xr - qx - \alpha_2(x) \end{aligned} \tag{26}$$

for some $p, q, r \in R$ and maps $\alpha_1, \alpha_2 : R \to Z(R)$. It follows from (26) that

$$x(q + r) - (p + q)x \in Z(R)$$

for all $x \in R$. In view of [8, Lemma 3.1] or [9, Lemma 3.1] we get that $q + r = p + q \in Z(R)$. So, $r = p$. Setting $c = p + q$ we get from (26) that

$$F(x) = qx + xq - cx + \alpha_1(x)$$

for all $x \in R$. Since $[F(x), x] \in Z(R)$ for all $x \in R$ we get

$$[qx + xq, x] \in Z(R) \tag{27}$$

for all $x \in R$. Letting $x = e$ in (27) we obtain that $eqf = 0$. Thus,

$$q = eqe + fqf.$$

For every $r \in R$, letting $x = e + erf$ in (27) we get

$$\begin{aligned} [qx + xq, x] &= [(eqe + fqf)(e + erf) + (e + erf)(eqe + fqf), e + erf] \\ &= [2eqe + eqerf + erfqf, e + erf] \\ &= -eqerf - erfqf + 2eqerf \\ &= eqerf - erfqf \in Z(R) \end{aligned}$$

for all $r \in R$. This yields that $eqerf - erfqf = 0$ and so $eqe \cdot erf = erf \cdot fqf$ for all $r \in R$. Hence,

$$q = eqe \oplus fqf \in Z(R).$$

Consequently, $p \in Z(R)$. Setting $\lambda = q - p$ and $\tau = \alpha_1$ we conclude that $F(x) = \lambda x + \tau(x)$ for all $x \in R$. $\square$

As consequences of Theorem 3.1 we have

COROLLARY 3.1. *Let S be a unital ring and let $n \geqslant 3$. Suppose that a map $F : T_n(S) \to T_n(S)$ is additive modulo $Z(S) \cdot I$. If F is centralizing, then there exist $\lambda \in Z(S) \cdot I$ and a map $\tau : T_n(S) \to Z(S) \cdot I$ such that*

$$F(x) = \lambda x + \tau(x)$$

for all $x \in T_n(S)$.

COROLLARY 3.2. *Let S be a unital noncommutative prime ring. Suppose that a map $F : T_2(S) \to T_2(S)$ is additive modulo $Z(S) \cdot I$. If F is centralizing, then there exist $\lambda \in Z(S) \cdot I$ and a map $\tau : T_2(S) \to Z(S) \cdot I$ such that*

$$F(x) = \lambda x + \tau(x)$$

for all $x \in T_2(S)$.

COROLLARY 3.3. *Let $\mathscr{N}$ be a nest of a complex Hilbert space H with $\dim_{\mathbb{C}} H > 2$. Let $\mathscr{T}(\mathscr{N})$ be a nest algebra. Suppose that a map $F : \mathscr{T}(\mathscr{N}) \to \mathscr{T}(\mathscr{N})$ is additive modulo $\mathbb{C}1$. If F is centralizing, then there exist $\lambda \in \mathbb{C}1$ and a map $\tau : \mathscr{T}(\mathscr{N}) \to \mathbb{C}1$ such that*

$$F(x) = \lambda x + \tau(x)$$

for all $x \in \mathscr{T}(\mathscr{N})$.

Proof. If $\mathscr{N}$ is a trivial nest, then $\mathscr{T}(\mathscr{N}) = \mathscr{B}(H)$ is a prime ring and hence the conclusion follows from [2, Theorem A]. If $\mathscr{N}$ is a nontrivial nest, then the result follows from Theorem 3.1. □

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