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SEQUENCES OF k–REFLEXIVITY DEFECTS

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Abstract. Let $\mathscr{H}$ be a complex separable Hilbert space and let k be a positive integer. Given a sequence of nonnegative integers $r_1 \geqslant r_2 \geqslant \ldots \geqslant 0$ we show that there exists a subspace $\mathscr{S} \subseteq \mathscr{B}(\mathscr{H})$, such that $\mathrm{rd}_k(\mathscr{S}) = r_k$ for all $k \geqslant 1$.

1. Introduction

Let $(\mathscr{H}, \langle\cdot,\cdot\rangle)$ be a complex separable Hilbert space and let k be a positive integer. Let $\mathscr{B}(\mathscr{H})$, $\mathscr{C}_1(\mathscr{H})$, $\mathscr{F}(\mathscr{H})$ and $\mathscr{F}_k(\mathscr{H})$ denote the algebra of all bounded linear operators on $\mathscr{H}$, the ideal of trace class operators in $\mathscr{B}(\mathscr{H})$, the linear variety of finite rank operators on $\mathscr{H}$ and the subset of all operators in $\mathscr{B}(\mathscr{H})$ of rank k or less, respectively. Obviously, one has

$$\mathscr{F}_1(\mathscr{H}) \subseteq \mathscr{F}_2(\mathscr{H}) \subseteq \ldots \subseteq \mathscr{F}(\mathscr{H}) \subseteq \mathscr{C}_1(\mathscr{H}). \tag{1}$$

Given a subset $\mathscr{S} \subseteq \mathscr{B}(\mathscr{H})$ we define the *preannihilator* of $\mathscr{S}$ by $\mathscr{S}_\perp = \{T \in \mathscr{C}_1(\mathscr{H}) : \operatorname{tr}(ST) = 0 \text{ for all } S \in \mathscr{S}\}$. Similarly, the *annihilator* of a subset $\mathscr{T} \subseteq \mathscr{C}_1(\mathscr{H})$ is defined by $\mathscr{T}^\perp = \{A \in \mathscr{B}(\mathscr{H}) : \operatorname{tr}(AT) = 0 \text{ for all } T \in \mathscr{T}\}$. Let $\mathscr{S} \subseteq \mathscr{B}(\mathscr{H})$ be a linear subspace. The k*-reflexive cover* of the space $\mathscr{S}$ is defined by

$$\begin{aligned}\mathrm{Ref_k}\,\mathscr{S} = \{T \in \mathscr{B}(\mathscr{H}) :\ & \text{for every } \varepsilon > 0 \text{ and arbitrary } k\text{-tuple } x_1,\ldots,x_k \in \mathscr{H} \text{ exists} \\ & S \in \mathscr{S} \text{ such that } \|Tx_i - Sx_i\| < \varepsilon \text{ for all } 1 \leqslant i \leqslant k\}.\end{aligned}$$

A weakly* closed linear subspace $\mathscr{S} \subseteq \mathscr{B}(\mathscr{H})$ is said to be k*-reflexive* if $\mathrm{Ref_k}\,\mathscr{S} = \mathscr{S}$. The k*-reflexivity defect* of a subspace $\mathscr{S}$ is defined by $\mathrm{rd}_k\,\mathscr{S} = \dim\left(\mathrm{Ref_k}\,\mathscr{S}/\mathscr{S}\right)$. Thus, if the space $\mathscr{S}$ is finite dimensional, we have $\mathrm{rd}_k\,\mathscr{S} = \dim \mathrm{Ref_k}\,\mathscr{S} - \dim\mathscr{S}$. If $\mathscr{S} \subseteq \mathscr{B}(\mathscr{H})$ is a weakly* closed subspace, then by a result of Kraus and Larson [3, 4] one has

$$\mathrm{Ref_k}\,\mathscr{S} = (\mathscr{S}_\perp \cap \mathscr{F}_k(\mathscr{H}))^\perp. \tag{2}$$

Using (1) and (2) one gets

$$\mathscr{S} \subseteq \overline{\mathscr{S}}^{w^*} \subseteq \ldots \subseteq \mathrm{Ref_2}\,\mathscr{S} \subseteq \mathrm{Ref_1}\,\mathscr{S}, \tag{3}$$

where $\overline{\mathscr{S}}^{w^*}$ denotes the closure of $\mathscr{S}$ in weak* topology. Note that (3) implies $\mathrm{rd}_1\,\mathscr{S} \geqslant \mathrm{rd}_2\,\mathscr{S} \geqslant \ldots \geqslant 0$. In this paper we are dealing with the following question.

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QUESTION 1.1. Let $r_1 \geqslant r_2 \geqslant \ldots \geqslant 0$ be a sequence of nonnegative integers. Does there exist a subspace $\mathscr{S} \subseteq \mathscr{B}(\mathscr{H})$, such that $\mathrm{rd}_k\, \mathscr{S} = r_k$ for all $k \geqslant 1$?

In Section 2 we consider the case when the sequence $r_1 \geqslant r_2 \geqslant \ldots \geqslant 0$ has only finitely many nonzero elements. In this setting we construct a subspace $\mathscr{S}$ of $\mathbb{M}_N$, the algebra of N-by-N complex matrices, for an appropriate positive integer N such that $\mathrm{rd}_k\, \mathscr{S} = r_k$ for all $k \in \mathbb{N}$. In Section 3 we provide an affirmative answer to the Question 1.1 in the most general case.

2. Sequences with finitely many nonzero elements

Firstly observe that since every one dimensional subspace of $\mathscr{B}(\mathscr{H})$ is reflexive the case when $r_k = 0$ for every $k \in \mathbb{N}$ is not interesting. However, it is reasonable to consider the following special case of Question 1.1.

QUESTION 2.1. Let $r_1 \geqslant r_2 \geqslant \ldots \geqslant 0$ be a sequence of positive integers and assume that $r_k > 0$ and $r_j = 0$ for all $j > k$ where $k \geqslant 1$. Does there exist a subspace $\mathscr{S} \subseteq \mathscr{B}(\mathscr{H})$, such that $\mathrm{rd}_j\, \mathscr{S} = r_j$ for all $j \geqslant 1$?

In what follows we construct a finite dimensional subspace of operators that fulfills the conditions of Question 2.1. It is well known that a subspace $\mathscr{S} \subseteq \mathscr{B}(\mathscr{H})$ is k-reflexive iff the space $\mathscr{S}^{(k)} = \{S^{(k)} : S \in \mathscr{S}\}$ is reflexive. If $\mathscr{S}$ is a finite dimensional space, then [5, Theorem 2.6] yields that the k-reflexivity defect of $\mathscr{S}$ depends only on its finite rank elements. Hence, we can without any loss of generality assume that the underlying Hilbert space is finite dimensional. In this case, if $\mathscr{T} \subseteq \mathscr{B}(\mathscr{H})$ is a linear subspace, then $\mathscr{T} = (\mathscr{T}_\perp)^\perp$ obviously holds. We will construct a space $\mathscr{T}$ such that the annihilator $\mathscr{T}^\perp$ will satisfy the conditions of Question 2.1. First we consider a simple example of a sequence with only one nonzero element.

EXAMPLE 2.2. Let $r_1 = 1$ and $r_j = 0$ for all $j \geqslant 2$. Let us define the following linear subspaces of $\mathbb{M}_2$ and $\mathbb{M}_3$, respectively,

$$\mathscr{T}_1 = \left\{ \begin{pmatrix} t_1 & t_2 \\ 0 & -t_1 \end{pmatrix} : t_1, t_2 \in \mathbb{C} \right\} \quad \text{and} \quad \mathscr{T}_2 = \left\{ \begin{pmatrix} t_1 & t_2 & t_3 \\ 0 & -t_1 & t_4 \\ 0 & 0 & 0 \end{pmatrix} : t_1, \ldots, t_4 \in \mathbb{C} \right\}.$$

Now put $\mathscr{S}_1 = \mathscr{T}_1^\perp$ and $\mathscr{S}_2 = \mathscr{T}_2^\perp$, that is,

$$\mathscr{S}_1 = \left\{ \begin{pmatrix} s_1 & s_2 \\ 0 & s_1 \end{pmatrix} : s_1, s_2 \in \mathbb{C} \right\} \quad \text{and} \quad \mathscr{S}_2 = \left\{ \begin{pmatrix} s_1 & s_2 & s_3 \\ 0 & s_1 & s_4 \\ 0 & 0 & s_5 \end{pmatrix} : s_1, \ldots, s_5 \in \mathbb{C} \right\}.$$

Since $\mathrm{Ref}_j\, \mathscr{S}_1 = \left(\mathscr{T}_1 \cap \mathscr{F}_j(\mathbb{C}^2)\right)^\perp$ for every $j \in \mathbb{N}$ one has $\mathrm{rd}_1\, \mathscr{S}_1 = 1$ and $\mathrm{rd}_j\, \mathscr{S}_1 = 0$ for all $j \geqslant 2$. Similarly, $\mathrm{Ref}_j\, \mathscr{S}_2 = \left(\mathscr{T}_2 \cap \mathscr{F}_j(\mathbb{C}^3)\right)^\perp$ for every $j \in \mathbb{N}$, hence $\mathrm{rd}_1\, \mathscr{S}_2 = 1$ and $\mathrm{rd}_j\, \mathscr{S}_2 = 0$ holds for all $j \geqslant 2$.

The reader may notice that every element of the space $\mathscr{T}_1$ can be written as the sum of matrices that have no rank one summand in common. Namely, let $\{e_1, e_2\}$ denote the standard orthonormal basis in $\mathbb{C}^2$ and let $A = a(E_{11} - E_{22}) + bE_{12}$ $(a, b \in \mathbb{C})$ be an arbitrary element of $\mathscr{T}_1$. Here, $E_{ij} = e_i \otimes e_j$ denotes the matrix unit, i.e., a matrix with 1 in the i-th row and j-th column and zeros elsewhere. Obviously, matrices $E_{11} - E_{22}$ and E_{12} have no rank one summand in common. The same happens in the case of the subspace $\mathscr{T}_2$. This motivates the following construction.

PROPOSITION 2.3. *Let $r_1 \geqslant r_2 \geqslant \ldots \geqslant 0$ be a sequence of positive integers such that $r_k > 0$ and $r_j = 0$ for all $j > k$ where $k \geqslant 1$. Let*

$$N = \max\left\{r_k(k+1), (r_{j-1} - r_j + 1)j - 1 :\ 2 \leqslant j \leqslant k\right\}. \tag{4}$$

Then there exists a subspace $\mathscr{S} \subseteq \mathbb{M}_N$ such that $\mathrm{rd}_j\, \mathscr{S} = r_j$ *for all $j \in \mathbb{N}$.*

Proof. Due to the condition (4) one can define the following elements of $\mathbb{M}_N$,

$$C_{k+1,s} = \sum_{n=(s-1)(k+1)+1}^{s(k+1)} E_{nn} \qquad \text{for } 1 \leqslant s \leqslant r_k. \tag{5}$$

Let $2 \leqslant j \leqslant k$. If $r_{j-1} - r_j > 0$, then we define

$$C_{j,s} = \sum_{n=(s-1)j+1}^{sj} E_{n,n+j-1} \qquad \text{for } 1 \leqslant s \leqslant r_{j-1} - r_j. \tag{6}$$

Obviously, $C_{k+1,s}$ are rank $k+1$ matrices for $1 \leqslant s \leqslant r_k$ with nonzero entries strictly on the main diagonal. Similarly, for $2 \leqslant j \leqslant k$, $r_{j-1} - r_j > 0$ and $1 \leqslant s \leqslant r_{j-1} - r_j$ the matrices $C_{j,s}$ are of rank j and have nonzero entries strictly on the $(j-1)$-th upper diagonal. Now let $\mathscr{T}$ be the linear subspace of the algebra $\mathbb{M}_N$ spanned by the matrices of the form (5) and (6). Since these matrices are linearly independent it follows that $\dim \mathscr{T} = r_1$. If we put $\mathscr{S} = \mathscr{T}^{\perp}$, then $\dim \mathscr{S} = N^2 - r_1$. Since the subspace $\mathscr{T}$ has no elements of rank one we also get $\mathrm{Ref}_1\, \mathscr{S} = \mathbb{M}_N$, hence $\mathrm{rd}_1\, \mathscr{S} = r_1$. Let $2 \leqslant j \leqslant k$. For a subset $\mathscr{M}$ let $\mathrm{Lin}\,\mathscr{M}$ denote the linear span of $\mathscr{M}$. By the structure of the subspace $\mathscr{T} = \mathscr{S}_{\perp}$, we have $\dim(\mathrm{Lin}(\mathscr{S}_{\perp} \cap \mathscr{F}_j(\mathbb{C}^N))) = r_1 - r_j$. Thus, $\dim \mathrm{Ref}_j\, \mathscr{S} = N^2 + r_j - r_1$ and $\mathrm{rd}_j\, \mathscr{S} = r_j$. Now let $j \geqslant k+1$. Since $\mathscr{S}_{\perp} \cap \mathscr{F}_j(\mathbb{C}^N) = \mathscr{S}_{\perp}$ we have $\mathrm{Ref}_j\, \mathscr{S} = \mathscr{S}$ and $\mathrm{rd}_j\, \mathscr{S} = 0$ for every $j \geqslant k+1$, which completes the proof. $\square$

3. General case

Let $r_1 \geqslant r_2 \geqslant \ldots \geqslant 0$ be a sequence of nonnegative integers as in Question 1.1. Obviously, there exists $k \in \mathbb{N}$ such that $r_k = r_{k+1} = r_{k+n}$ for all $n \in \mathbb{N}$, i.e., the sequence is constant from the k-th term on. If $r_j \neq 0$ for $j \geqslant k$, then it is necessary for these conditions to hold that the underlying Hilbert space is infinite dimensional. In the following example we consider the simplest case of such a sequence, i.e., an infinite constant sequence.

EXAMPLE 3.1. Let r be a positive integer and let $r_j = r$ for all $j \in \mathbb{N}$. If there exists a space of operators satisfying the conditions of Question 1.1, then by (2) the preannihilator of this space cannot contain any operators of rank $j \geqslant 2$, thus its elements may be only rank one operators and trace class operators of infinite rank. However, since the space should not be j-reflexive for any $j \in \mathbb{N}$ its preannihilator has at least one element of infinite rank. For given vectors $x, y \in \mathscr{H}$ let $x \otimes y$ denote a bounded linear operator on $\mathscr{H}$ of rank one or less given by $z \mapsto \langle z, y \rangle x$. Let $\{\varphi_j\}_{j \in \mathbb{N}}$ denote the complete orthonormal system in $\mathscr{H}$. For $1 \leqslant j \leqslant r$ let us define

$$D_j = \sum_{n=1}^{\infty} \frac{1}{n^2} \varphi_{n+j-1} \otimes \varphi_n.$$

The reader can easily check that $D_1, \ldots, D_r$ are trace class operators of infinite rank. Let $\mathscr{T} = \operatorname{Lin}\{D_1, \ldots, D_r\}$, i.e., $\mathscr{T}$ is a r-dimensional subspace of $\mathscr{C}_1(\mathscr{H})$. If we define $\mathscr{S} = \mathscr{T}^{\perp}$, then $\mathscr{S}$ is a weak* closed linear subspace of $\mathscr{B}(\mathscr{H})$ and $\mathscr{S}_{\perp} = \mathscr{T}$ holds. It is easy to see that

$$\mathscr{S} = \left\{ T \in \mathscr{B}(\mathscr{H}) : \sum_{n=1}^{\infty} \frac{1}{n^2} \langle T\varphi_{n+j-1}, \varphi_n \rangle = 0 \text{ for } 1 \leqslant j \leqslant r \right\}.$$

For a normed space $\mathscr{X}$ let $\mathscr{X}^*$ denote the collection of all bounded linear functionals on $\mathscr{X}$. Using the formula in (2) we get $\operatorname{Ref}_j \mathscr{S} = (\mathscr{T} \cap \mathscr{F}_j(\mathscr{H}))^{\perp} = \mathscr{B}(\mathscr{H})$ for every $j \in \mathbb{N}$. It is well known that $\mathscr{B}(\mathscr{H})$ can be identified as the dual of the space $\mathscr{C}_1(\mathscr{H})$, therefore [1, Chapter III, Theorem 10.1] yields $\mathscr{B}(\mathscr{H})/\mathscr{S} = \mathscr{C}_1(\mathscr{H})^*/\mathscr{T}^{\perp} \cong \mathscr{T}^*$. Since $\mathscr{T}$ is a finite dimensional subspace of $\mathscr{C}_1(\mathscr{H})$ we have

$$\operatorname{rd}_j \mathscr{S} = \dim(\operatorname{Ref}_j \mathscr{S}/\mathscr{S}) = \dim \mathscr{T}^* = \dim \mathscr{T} = r$$

for all $j \in \mathbb{N}$. Hence, the space $\mathscr{S}$ satisfies the conditions of the Question 1.1 for the given constant sequence.

By using the construction in Example 3.1 together with the Proposition 2.3 we can now give an affirmative answer to the Question 1.1 in the most general case.

THEOREM 3.2. *Let $r_1 \geqslant r_2 \geqslant \ldots \geqslant 0$ be a sequence of nonnegative integers. Then there exist a complex Hilbert space $\mathscr{H}$ and a subspace $\mathscr{S} \subseteq \mathscr{B}(\mathscr{H})$, such that* $\operatorname{rd}_k \mathscr{S} = r_k$ *for all $k \geqslant 1$.*

Proof. Without any loss of generality we can assume that the given sequence is nonzero. If there exists a positive integer k such that $r_k > 0$ and $r_j = 0$ for all $j > k$, then Proposition 2.3 yields the existence of the appropriate subspace of complex square matrices. Let now $\mathscr{H}$ be a complex separable Hilbert space of infinite dimension. We can assume that there exists a positive integer k such that $r_j = r_k > 0$ for all $j \geqslant k$ and $r_{k-1} > r_k$. Let $2 \leqslant j \leqslant k$. If $r_{j-1} - r_j > 0$, then we define

$$C_{j,s} = \sum_{n=(s-1)j+1}^{sj} \varphi_n \otimes \varphi_{n+j-1} \qquad \text{for } 1 \leqslant s \leqslant r_{j-1} - r_j. \tag{7}$$

Obviously, $C_{j,1}, C_{j,2}, \dots, C_{j,r_{j-1}-r_j}$ are bounded linear operators on $\mathscr{H}$ of rank j for every $j \in \{2,\dots,k\}$. Next, let $1 \leqslant j \leqslant r_k$ and let

$$D_j = \sum_{n=1}^{\infty} \frac{1}{n^2}\, \varphi_{n+j-1} \otimes \varphi_n. \tag{8}$$

As it was pointed out in Example 3.1, the operators $D_1, D_2, \dots, D_{r_k}$ are trace class and of infinite rank. If we define

$$\mathscr{T} = \mathrm{Lin}\left\{D_1,\dots,D_{r_k}, C_{j,1},\dots,C_{j,r_{j-1}-r_j} : 2 \leqslant j \leqslant k\right\},$$

then $\mathscr{T}$ is a r_1-dimensional closed subspace of $\mathscr{C}_1(\mathscr{H})$ which has no elements of rank one. Moreover, for $2 \leqslant j \leqslant k$ we have $\dim(\mathrm{Lin}(\mathscr{T} \cap \mathscr{F}_j(\mathscr{H}))) = r_1 - r_j$. Now let $\mathscr{S} = \mathscr{T}^{\perp}$, hence $\mathscr{S}$ is a closed subspace of $\mathscr{B}(\mathscr{H})$ for which $\mathscr{S}_{\perp} = \mathscr{T}$ holds. By equation (2) we get $\mathrm{Ref}_1\, \mathscr{S} = (\mathscr{T} \cap \mathscr{F}_1(\mathscr{H}))^{\perp} = \mathscr{B}(\mathscr{H})$. As in Example 3.1, we identify $\mathscr{B}(\mathscr{H})$ with the dual of the space $\mathscr{C}_1(\mathscr{H})$ so that we get $\mathrm{Ref}_1\, \mathscr{S}/\mathscr{S} = \mathscr{C}_1(\mathscr{H})^*/\mathscr{T}^{\perp} \cong \mathscr{T}^*$ by [1, Chapter III, Theorem 10.1]. The latter yields $\mathrm{rd}_1\, \mathscr{S} = \dim \mathscr{T}^* = \dim \mathscr{T} = r_1$. For an arbitrary operator $T \in \mathscr{B}(\mathscr{H})$ let $t_{pq} = \langle T\varphi_p, \varphi_q\rangle$ for $p,q \in \mathbb{N}$. By this notation, let $(t_{pq})_{p,q=1}^{\infty}$ be a matrix representation of the operator T. It is not hard to see that

$$\begin{aligned}\mathscr{S} = \Bigg\{ (t_{pq})_{p,q=1}^{\infty} \in \mathscr{B}(\mathscr{H}) : \sum_{n=1}^{\infty} \frac{1}{n^2} t_{n+j-1,n} = 0 \text{ for } 1 \leqslant j \leqslant r_k, \sum_{n=(s-1)j+1}^{sj} t_{n,n+j-1} = 0 \\ \text{for } r_{j-1} - r_j > 0,\ 1 \leqslant s \leqslant r_{j-1} - r_j \text{ and } 2 \leqslant j \leqslant k \Bigg\}.\end{aligned} \tag{9}$$

Note that $\mathrm{Ref}_j\, \mathscr{S} = \left(\mathrm{Lin}\{C_{i,1}, C_{i,2}, \dots, C_{i,r_{i-1}-r_i} : 2 \leqslant i \leqslant j\}\right)^{\perp}$ holds for $2 \leqslant j \leqslant k$. Thus,

$$\begin{aligned}\mathrm{Ref}_j\, \mathscr{S} = \Bigg\{ (t_{pq})_{p,q=1}^{\infty} \in \mathscr{B}(\mathscr{H}) : \sum_{n=(s-1)i+1}^{si} t_{n,n+i-1} = 0 \text{ for } r_{i-1} - r_i > 0, \\ 1 \leqslant s \leqslant r_{i-1} - r_i \text{ and } 2 \leqslant i \leqslant j \Bigg\}.\end{aligned} \tag{10}$$

Combining (9) and (10) one then has $\mathrm{rd}_j\, \mathscr{S} = \dim(\mathrm{Ref}_j\, \mathscr{S}/\mathscr{S}) = r_j$ for $2 \leqslant j \leqslant k$. Moreover, since $\mathscr{T} \cap \mathscr{F}_j(\mathscr{H}) = \mathscr{T} \cap \mathscr{F}_k(\mathscr{H})$ holds for $j \geqslant k$ it follows $\mathrm{Ref}_j\, \mathscr{S} = \mathrm{Ref}_k\, \mathscr{S}$ and $\mathrm{rd}_j\, \mathscr{S} = r_k$ for all $j \geqslant k$. $\square$

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