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BACKWARD ERRORS AND PSEUDOSPECTRA FOR STRUCTURED NONLINEAR EIGENVALUE PROBLEMS

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Abstract. Minimal norm structured perturbations (backward errors) are constructed such that an approximate eigenpair of a nonlinear eigenvalue problem is an exact eigenpair of an appropriately perturbed problem. Structured and unstructured backward errors are compared. These results extend previous results for (structured) matrix polynomials to more general functions.

1. Introduction

In this paper we consider the perturbation analysis for the *nonlinear eigenvalue problem* (in homogeneous form) of computing points on the *Riemann sphere* $\{(c,s) \in \mathbb{C}^2,\ |c|^2+|s|^2=1\}$ and vectors $x \in \mathbb{C}^n$ such that the nonlinear system of equations

$$\left(\sum_{j=0}^{m} M_j f_j(c,s)\right) x = 0 \tag{1}$$

holds, where we assume that the scalar valued functions $f_0(c,s), f_1(c,s), \ldots, f_m(c,s)$ and the coefficient matrices $M_0, M_1, M_2, \ldots, M_m \in \mathbb{C}^{n,n}$ are given data.

Simultaneously, we study the problem in non-homogeneous form to determine complex numbers λ and vectors $x \in \mathbb{C}^n$ such that

$$\left(\sum_{j=0}^{m} M_j f_j(\lambda)\right) x = 0. \tag{2}$$

We call pairs (c,s) satisfying (1) or λ satisfying (2) *eigenvalues* and the associated vectors x *eigenvectors*. If the functions f_i are all rational in c,s then by introducing $\lambda = \frac{c}{s}$ and scaling with appropriate powers of c and s, then these two versions of the problem can be easily transformed into each other, but this transformation to nonhomogeneous form is not always possible, see problem (9).

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For convenience we often write problems (1) and (2) in product notation $(M \otimes f(c,s))x = 0$, $(M \otimes f(\lambda))x = 0$, respectively, with tuples of matrices $M := (M_0, M_1, \ldots, M_m) \in (\mathbb{C}^{n\times n})^{m+1}$ and tuples $f(c,s) := (f_0(c,s), f_1(c,s), \ldots, f_m(c,s))$, or $f(\lambda) := (f_0(\lambda), \ldots, f_m(\lambda))$, respectively.

Backward error analysis, see e.g. [20], studies the question to compute, for an approximate eigenvalue (γ,μ) (or λ) and associated approximate eigenvector x, the smallest (in some appropriate measure) perturbation tuple of coefficient matrices $\Delta M := (\Delta M_0, \Delta M_1, \ldots, \Delta M_m) \in (\mathbb{C}^{n\times n})^{m+1}$ and perturbation vectors $\Delta f = (\delta f_0, \delta f_1, \ldots, \delta f_m) \in \mathbb{C}^{m+1}$ such that the perturbed nonlinear equations

$$((M+\Delta M)\otimes(f+\Delta f)(\gamma,\mu))x = 0,\ ((M+\Delta M)\otimes(f+\Delta f)(\lambda))x = 0, \tag{3}$$

hold, respectively.

Classical backward error analysis would construct the perturbations both for the nonlinear functions $(f_j+\delta f_j)(c,s)$ $((f_j+\delta f_j)(\lambda))$ and the coefficients $M_j + \Delta M_j$, $j = 0,\ldots,m$. This problem is, however, extremely difficult for general sets of functions f_j. Instead, in this paper we assume that the perturbations in the functions f_j are known (so that for simplicity we may assume that there are no perturbations in these functions at all) and we consider only perturbations in the coefficient matrices M_j. This is a reasonable assumption in many applications, since the f_j are typically elementary scalar functions and thus the perturbation analysis is well understood. However, if there are errors in the functions f_j, then these would have to be incorporated in the analysis.

Under this assumption, for a given eigenvalue (γ,μ) or λ, we can express the function values $f_j(\gamma,\mu)$ or $f_j(\lambda)$, respectively, in a uniform way as scalars ψ_j, where we have the following choices that incorporate all cases that are considered in this paper.

1. The choice $\psi_j = \lambda^j$, $j = 0,1,\ldots,m$ encodes the nonhomogeneous polynomial case $f_j(\lambda) = \lambda^j$.
2. The choice $\psi_j = \gamma^j\mu^{m-j}$, $j = 0,1,\ldots,m$ encodes the homogeneous polynomial case.
3. The choice $\psi_j = f_j(\lambda)$, $j = 0,1,\ldots,m$ encodes the nonhomogeneous case with general functions f_j.
4. The choice $\psi_j = f_j(\gamma,\mu)$, $j = 0,1,\ldots,m$ encodes the homogeneous case with general functions f_j.

The backward error then is the smallest perturbation ΔM (in some measure) such that

$$((M+\Delta M)\otimes\psi)x = \left(\sum_{j=0}^{m}(M_j+\Delta M_j)\psi_j\right)x = 0. \tag{4}$$

Nonlinear eigenvalue problems of the described form arise in many applications, see e.g., [34, 40] for surveys with a large number of applications and [9] for a collection of benchmark examples.

A nonrational eigenvalue problem of the form

$$\left(K + \imath\sqrt{\kappa^2 - \kappa_c^2}\,D - \kappa^2 M\right)x = 0$$

has been studied in [43]. Here κ is an unknown, κ_c is a fixed reference frequency, and K, M, D are large and sparse symmetric stiffness, mass and damping matrices, respectively. This problem can be turned into a polynomial eigenvalue problem by introducing $\lambda = \sqrt{\kappa^2 - \kappa_c^2}$.

In [12] a rational eigenvalue problem arising in the numerical solution of a fluid-structure interaction is introduced. It has the form

$$\left(\frac{\lambda^2}{a^2}M + K + \frac{\lambda^2}{\lambda\beta + \alpha}D\right)x = 0, \tag{5}$$

where a is the speed of sound in the given material, and α, β are positive constants. The matrices M, K are large sparse symmetric positive definite mass and stiffness matrices, respectively, and the symmetric positive semidefinite matrix D describes the effect of an absorbing wall. Clearing out the denominator in (5) leads to a cubic eigenvalue problem

$$(\lambda^3\beta M + \lambda^2(\alpha M + a^2 A) + \lambda(a^2\beta K) + a^2\alpha K)x = 0. \tag{6}$$

Similar rational eigenvalue problems arise in the finite element simulation of mechanical problems, see e.g., [33, 36, 42].

For (6) in the polynomial setting (and also in many other cases), to deal with infinite eigenvalues arising from a singular leading coefficient, it is convenient to use the homogeneous framework, setting $\lambda = \frac{c}{s}$, and studying

$$(s^3\beta M + s^2c(\alpha M + a^2 A) + sc^2(a^2\beta K) + c^3a^2\alpha K)x = 0, \tag{7}$$

which can be written as $M \otimes f(c,s) = M_0 f_0(c,s) + M_1 f_1(c,s) + M_2 f_2(c,s) + M_3 f_3(c,s)$, where $f_i = c^i s^{m-i}$, $i = 0,1,2,3$, $m = 3$, $M_0 = \beta M$, $M_1 = (\alpha M + a^2 A)$, $M_2 = (a^2\beta K)$, $M_3 = a^2\alpha K$. This representation may still not cure all the difficulties with infinite eigenvalues, e.g., the homogeneous version of (5)

$$(M \otimes f(c,s))\,x := \left(\frac{s^2}{c^2a^2}M + K + \frac{s^2}{c(s\beta + c\alpha)}A\right)x = 0, \tag{8}$$

still is problematic if $c = 0$. We often prefer nevertheless to work in the homogeneous framework also for this particular problem.

Once a nonlinear eigenvalue problem of the form (3) has been converted into a polynomial eigenvalue problem, it can subsequently be turned into a linear eigenvalue problem by one of the usual linearization approaches, see e.g., [16, 31, 32]. This approach of turning a rational problem into a larger linear problem is successful in many practical applications, see e.g., [24, 25, 29]. However, the size of the problem may substantially increase and, moreover, typically extra eigenvalues are introduced which have to be recognized and removed from the computed spectrum.

EXAMPLE 1.1. Consider the symmetric rational eigenvalue problem

$$R(\lambda)x := \begin{bmatrix} \lambda-\alpha+\frac{1}{\lambda-1} & 1 \\ 1 & 0 \end{bmatrix} x = 0,$$

which has no finite eigenvalues, since $\det R(\lambda) = -1$. Scaling the problem by $\lambda - 1$, the rational eigenvalue problem becomes a symmetric polynomial eigenvalue problem

$$P(\lambda)x = \begin{bmatrix} (\lambda-1)(\lambda-\alpha)+1 & \lambda-1 \\ (\lambda-1) & 0 \end{bmatrix} x = 0,$$

which has a double eigenvalue at ∞ and also a double eigenvalue at 1.

Considering a symmetric linearization [22, 31] of the polynomial problem one obtains

$$L(\lambda)z = \left(\lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} -(\alpha+1) & 1 & \alpha & -1 \\ 1 & 0 & -1 & 0 \\ \alpha & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \right) z = 0.$$

Analyzing $L(\lambda)$ for different α, one sees that it has a Jordan block of size 2 at ∞ and two Jordan blocks of size 1 for the eigenvalue $\lambda = \alpha$.

Due to the Jordan block at ∞ this problem is very sensitive to perturbations. If, e.g., we perturb the problem to

$$P_\varepsilon(\lambda)x = \left(\begin{bmatrix} (\lambda-1)(\lambda-\alpha)+1 & \lambda-1 \\ \lambda-1 & \varepsilon \end{bmatrix} \right) x = 0,$$

then the problem has two finite eigenvalues as roots of $(\lambda^2-(\alpha+1)\lambda+1)\varepsilon-(\lambda-1)^2$ and by appropriate choices of ε and α large perturbations may arise.

There also exist practical problems where a nonlinear eigenvalue problem cannot be turned into a polynomial eigenvalue problem. Consider, e.g., the non-rational eigenvalue problem of the form

$$\left(\lambda M_0 + M_1 + M_2 e^{-\tau\lambda}\right) x = 0, \tag{9}$$

where the M_i are real matrix coefficients, and τ is a real parameter. Such problems arise in the stability analysis of single delay differential equations [14, 18, 26, 35], where τ describes the delay time.

Example 1.1 shows some of the difficulties that may arise in eigenvalue problems of the form (3) and it shows the need for a careful perturbation analysis on the original data. This analysis is still mainly open and even for polynomial problems the (structured) perturbation theory and the computation of (structured) backward errors is only very recent, see e.g., [1, 3, 5, 7, 8, 10, 11, 21, 22, 23, 30, 38, 39].

There is very little literature that deals with the perturbation analysis of rational or more general nonlinear eigenvalue problems, see e.g., [12, 15, 37], but in these articles usually only problems without infinite eigenvalues are considered. But, as we will see below, it turns out that for the discussed class of backward errors, the theory developed in [7, 8] for the polynomial case can be easily extended.

The paper is organized as follows. In Section 2, we introduce the notation and recall some of the techniques for polynomial eigenvalue problems from [7, 8]. In Sections 3 and 4 we then construct structured backward errors for complex symmetric/skew-symmetric and Hermitian/skew-Hermitian problems, respectively, and compare these to the corresponding unstructured backward errors.

2. Notation and preliminaries

For a nonnegative vector $w = [w_1, w_2, \ldots, w_n]^T \in \mathbb{R}^n$ and a vector $x \in \mathbb{C}^n$, we introduce the weighted (semi-)norm $\|x\|_{w,2} := \|[w_1x_1, w_2x_2, \ldots, w_nx_n]^T\|_2$, where $\|\ \|_2$ denotes the classical Euclidean norm in $\mathbb{C}^n$. If w is strictly positive, then this is a norm, and if w has zero components then it is a semi-norm. For a nonnegative vector $w \in \mathbb{R}^n$, we define the componentwise inverse via $w^{-1} := [w_1^{-1}, w_2^{-1}, \ldots, w_n^{-1}]^T$, where we use the convention that $w_i^{-1} = 0$ if $w_i = 0$. By $\sigma_{\max}(A)$ and $\sigma_{\min}(A)$, we denote the largest and smallest singular values of a matrix A, respectively. The identity matrix is denoted by I, A^T stands for transpose and A^H for the conjugate transpose of a matrix $A \in \mathbb{C}^{n,n}$. For $x \in \mathbb{C}^n$ with $x^Hx = 1$, we frequently use the projector $P_x := I - xx^H$ onto the orthogonal complement of the space spanned by x.

We will construct structured and unstructured backward errors both in spectral and Frobenius norm on $\mathbb{C}^{n,n}$, which are defined by

$$\|A\|_2 := \max_{\|x\|=1} \|Ax\|_2, \ \|A\|_F := (\mathrm{trace} A^H A)^{1/2},$$

respectively, and we sometimes use $\|A\|_q$, where $q \in \{2, F\}$. For $z \in \mathbb{C}$, define $\mathrm{sign}(z) := \dfrac{\overline{z}}{|z|^2}$ if $z \neq 0$, and $\mathrm{sign}(z) := 0$ if $z = 0$.

The vector space of all tuples $M = (M_0, M_1, M_2, \ldots, M_m)$ with coefficients in $M_i \in \mathbb{C}^{n,n}$, is denoted by $\mathscr{M}_{m+1}(\mathbb{C}^{n,n})$. With a nonnegative weight vector $w \in \mathbb{R}^{m+1}$, it can be equipped with a weighted norm/seminorm $|||.|||_{w,q}$ given by

$$|||M|||_{w,q} := \|(M_0, \ldots, M_m)\|_{w,q} = (w_0^2\|M_1\|_q^2 + \ldots + w_m^2\|M_m\|_q^2)^{1/2},$$

for $q \in \{2, F\}$, respectively. For convenience, if $w := [1, 1, \ldots, 1]^T$ then we leave off the subscript w.

In the following, we consider matrix functions of the form $M \otimes f$ as in (1) or (2). Such a matrix function is called *regular* if $\det(M \otimes f) \neq 0$ for some eigenvalue (γ, μ) (λ), otherwise it is called *singular*. Let (γ, μ) (λ) be an approximation to an eigenvalue with corresponding approximate right eigenvector $x \neq 0$. Then for the tuple $\psi =: (\psi_0, \psi_1, \ldots, \psi_m)$ constructed as in (4), we construct *Frobenius and spectral norm backward errors*

$$\eta_{w,q}(\psi, x, M) := \inf\{|||\Delta M|||_{w,q}, \ ((M + \Delta M) \otimes \psi)x = 0\}.$$

If the problem has coefficients M_i that are structured in a subset $\mathbf{S} \subset \mathscr{M}_m(\mathbb{C}^{n,n})$, then we *construct structured backward errors*

$$\eta^{\mathbf{S}}_{w,q}(\psi,x,M) := \inf\{|||\Delta M|||_{w,q},\ \Delta M \in \mathbf{S},\ ((M+\Delta M)\otimes\psi)x = 0\}.$$

Such backward errors were introduced for matrix polynomials in [19, 38], but here we follow [3, 4, 5].

In order to compute the backward errors, we will need the partial gradient $\nabla_i \|z\|_{w,2}$ of a map $\mathbb{C}^{m+1} \to \mathbb{R},\ z \mapsto \|z\|_{w,2}$ which is just the gradient of the map $\mathbb{C} \to \mathbb{R},\ z_i \mapsto \|[z_0,\dots,z_m]^T\|_{w,2}$ which fixes the variables $z_0,\dots,z_{i-1},z_{i+1},\dots,z_m$ as constants. The gradient of the map $\mathbb{C}^{m+1} \to \mathbb{R},\ z \mapsto \|z\|_{w,2}$, is then defined as

$$\nabla(\|z\|_{w,2}) = [\nabla_0\|z\|_{w,2}, \nabla_1\|z\|_{w,2},\dots,\nabla_m\|z\|_{w,2}]^T.$$

With these definitions we have the following proposition, see [4, 8].

PROPOSITION 2.1. *Consider the map* $H_{w,2} : \mathbb{C}^{m+1}\setminus\{0\} \to \mathbb{R}$ *given by* $H_{w,2}(z) := \|[z_0,\dots,z_m]^T\|_{w,2}$. *Then* $H_{w,2}$ *is differentiable on* $\mathbb{C}^m$ *and*

$$\nabla_j H_{w,2}(z) = \frac{w_j^2 z_j}{H_{w,2}(z)}, \quad j = 0,1,\dots,m.$$

Furthermore,

$$\sum_{j=0}^m z_j \frac{\overline{\nabla_j H_{w,2}(z)}}{H_{w,2}(z)} = 1, \quad \sum_{j=0}^m w_j^{-2}|\nabla_j H_{w,2}(z)|^2 = 1.$$

In order to simplify the presentation, in the following we use the abbreviations

$$z_{M_j} := \frac{\nabla_j H_{w,2}}{H_{w,2}}\Big|_z, \quad j = 0,\dots,m, \tag{10}$$

where $z := [z_0,z_1,\dots,z_m]^T$.

We will construct backward errors for the following structured nonlinear eigenvalue problems, which extend the polynomial classes that were introduced in nonhomogeneous form in [31]. Defining $M^T := (M_0^T, M_1^T,\dots,M_m^T)$ and $M^H := (M_0^H, M_1^H, \dots, M_m^H)$, we say that problem (4) is complex *symmetric/skew-symmetric* if $(M^T\otimes\psi) = \pm(M\otimes\psi)$, and *Hermitian/skew-Hermitian* if $M^H\otimes\psi = \pm(M\otimes\overline{\psi})$.

For symmetric/skew-symmetric problems of the form (4), if $x \in \mathbb{C}^n$ is a right eigenvector of (1) or (2) corresponding to an eigenvalue (γ,μ) (λ), then $\bar{x}$ is a left eigenvector. For Hermitian/skew Hermitian eigenvalue problems, if $x \in \mathbb{C}^n$ and $y \in \mathbb{C}^n$ are right and left eigenvectors corresponding to an eigenvalue (γ,μ) (λ) of (1) or (2), then y and x are right and left eigenvector corresponding to the eigenvalue $(\overline{\gamma},\overline{\mu})$ $(\overline{\lambda})$.

For a given approximate eigenvalue (γ,μ) (λ), we can determine the smallest norm perturbation ΔM to the tuple M that makes this an eigenvalue, and when this is known we can determine a concrete perturbation with this norm and a given right eigenvector x. This follows from the following proposition.

PROPOSITION 2.2. *Let $M \in \mathscr{M}_{m+1}(\mathbb{C}^{n,n})$ and a set of functions f_j, $j = 0,\ldots,m$ be given. For an approximate eigenvalue (γ,μ) (λ), consider the tuple $\psi = (\psi_0,\ldots,\psi_m)$ as in (4). Set*

$$H_{w,2}(\psi) = \|[w_0\psi_1,\ldots,w_m\psi_m]^T\|_2, \tag{11}$$

where we assume that not all weights w_i are zero. Then the backward error satisfies

$$\eta_{w,2}(\psi,x,M) = \min_{\|x\|=1} \frac{\|(M\otimes\psi)x\|}{H_{w^{-1},2}(\psi)}.$$

Proof. The tuple that defines the backward error satisfies $((M+\Delta M)\otimes\psi)x = 0$ for some normalized vector x, which implies that $(M\otimes\psi)x = -(\Delta M\otimes\psi)x$. Hence we have that

$$\|(M\otimes\psi)x\| \leqslant |||\Delta M|||_{w,2} H_{w^{-1},2}(\psi),$$

where $w^{-1} = (w_0^{-1}, w_1^{-1},\ldots,w_n^{-1})$, with $w_i^{-1} = 0$ if $w_i = 0$. The perturbation ΔM_i depends on w_i and hence if $w_i = 0$, then there is no perturbation in the ith coefficient matrix of $M\otimes\psi$, i.e., $\Delta M_i = 0$. If at least one w_i is nonzero and $(M\otimes\psi)x + (\Delta M\otimes\psi)x = 0$, then

$$\frac{\|(M\otimes\psi)x\|}{H_{w^{-1},2}(\psi)} \leqslant |||\Delta M|||_{w,2}$$

i.e., we have

$$\eta_{w,2}(\psi,x,M) \geqslant \frac{\|(M\otimes\psi)x\|}{H_{w^{-1},2}(\psi)}.$$

To show that equality can be achieved, consider any normalized vector x, a normalized vector y with $y^H x = 1$, as well as the rank one matrix $(M\otimes\psi)xy^H$, and choose for $i = 0,\ldots,m$,

$$\Delta M_i = \frac{-w_i^{-2}\operatorname{sign}\psi_i|\psi_i|}{H_{w^{-1},2}(\psi)^2}(M\otimes\psi)xy^H.$$

Then we have that $(M\otimes\psi + \Delta M\otimes\psi)x = 0$, which implies that

$$\frac{\|(M\otimes\psi)xy^H x\|}{H_{w^{-1},2}(\psi)} = \frac{\|(M\otimes\psi)x\|}{H_{w^{-1},2}(\psi)}.$$

Minimizing over all possible normalized vectors x then gives the desired inequality. $\square$

We will also make use of the following completion result which is a direct corollary of Theorem 1.2 in [13].

PROPOSITION 2.3.

1. *Let* $A = \pm A^T, C = \pm B^T \in \mathbb{C}^{n,n}$ *and* $\chi := \sigma_{\max}\left(\begin{bmatrix} A \\ B \end{bmatrix}\right)$. *Then there exists a symmetric/skew-symmetric matrix* $X \in \mathbb{C}^{n,n}$ *such that* $\sigma_{max}\left(\begin{bmatrix} A & \pm B^T \\ B & X \end{bmatrix}\right) = \chi$, *and* X *has the form*

$$X := -K\overline{A}K^T + \chi(I - KK^H)^{1/2}Z(I - \overline{K}K^T)^{1/2},$$

where $K := B(\chi^2 I - \overline{A}A)^{-1/2}$ *and* $Z = \pm Z^T \in \mathbb{C}^{n,n}$ *is an arbitrary matrix such that* $\|Z\|_2 \leqslant 1$.

2. *For* $A = \pm A^H, B = \pm B^H$, *set* $\chi := \left\| \begin{bmatrix} A \\ B \end{bmatrix} \right\|_2$. *Then there exists an Hermitian/skew-Hermitian matrix* D, *respectively, such that* $\left\| \begin{bmatrix} A & \pm B^H \\ B & D \end{bmatrix} \right\|_2 = \chi$ *and* D *is of the form* $D := -KAK^H + \chi(I - KK^H)^{1/2}Z(I - KK^H)^{1/2}$, *where* $K := B(\chi^2 I - A^2)^{-1/2}$ *and* $Z = \pm Z^H$ *is an arbitrary matrix such that* $\|Z\|_2 \leqslant 1$.

3. Backward errors for symmetric/skew-symmetric nonlinear eigenvalue problems

In this section we will construct backward error formulas for non-homogeneous/homogeneous symmetric/skew-symmetric nonlinear eigenvalue problems.

THEOREM 3.1. *Consider a regular symmetric/skew-symmetric nonlinear matrix equation of the form* (4) *with* $M \in \mathscr{M}_{m+1}(\mathbb{C}^{n,n})$, *and let* $x \in \mathbb{C}^n$ *with* $x^H x = 1$. *For an approximate eigenvalue* (γ, μ) (λ), *construct the tuple* $\psi := (\psi_0, \psi_1, \ldots, \psi_m)$ *as in* (4), *and let* $k := -(M \otimes \psi)x$. *Introduce the perturbation matrices*

$$\Delta M_j := \begin{cases} -\overline{x}x^T M_j x x^H + \overline{z_{M_j}} \left[\overline{x}k^T + kx^H - 2(x^T k)\overline{x}x^H\right] & \text{if } M_j = M_j^T, \\ \overline{z_{M_j}} \left[kx^H - \overline{x}k^T\right] & \text{if } M_j = -M_j^T, \end{cases}$$

for $j = 0, \ldots, m$, *where* z_{M_j} *is as in* (10). *Then* ΔM *has the desired symmetry structure and satisfies* $((M \otimes \psi) + (\Delta M \otimes \psi))x = 0$, *i.e.,* (γ, μ) (λ) *is an exact eigenvalue and*

$$(M + \Delta M \otimes \psi))x = 0.$$

Proof. The proof is a slight modification of the proof for the polynomial case in [7]. In the symmetric case we have for all $j = 0, \ldots, m$ that $\Delta M_j = \Delta M_j^T$. Hence ΔM is symmetric, and we have that

$$
\begin{aligned}
((M+\Delta M)\otimes\psi)x &= \left(\sum_{j=0}^{m}(M_j+\Delta M_j)\psi_j\right)x \\
&= \sum_{j=0}^{m}\psi_j\left[M_jx-\overline{x}x^TM_jx+\overline{z_{M_j}}\left[\overline{x}k^Tx+k-2(x^Tk)\overline{x}\right]\right] \\
&= (I-\overline{x}x^T)(\sum_{j=0}^{m}M_j\psi_j)x+\left[\overline{x}k^Tx+k-2(x^Tk)\overline{x}\right]\sum_{j=0}^{m}\overline{z_{M_j}}\psi_j.
\end{aligned}
$$

By Proposition 2.1, we have $\sum_{j=0}^{m}\overline{z_{M_j}}\psi_j=1$, and also $k^Tx=x^Tk$. Hence

$$
\begin{aligned}
((M+\Delta M)\otimes\psi))x &= -(I-\overline{x}x^T)k+\overline{x}k^Tx+k-2(x^Tk)\overline{x} \\
&= -k+\overline{x}(x^Tk)+\overline{x}(k^Tx)+k-2(x^Tk)\overline{x}=0.
\end{aligned}
$$

The proof for the skew-symmetric case follows analogously. $\square$

Using Theorem 3.1, we then obtain the following backward errors for complex symmetric nonlinear eigenvalue problems.

THEOREM 3.2. *Let $M\in\mathscr{M}_{m+1}(\mathbb{C}^{n,n})$ be as in (4) with complex symmetric coefficients, and let $x\in\mathbb{C}^n$ be such that $x^Hx=1$. Let (γ,μ) (λ) be an approximate eigenvalue, let the associated tuple $\psi:=(\psi_0,\psi_1,\ldots,\psi_m)$ be as in (4), and let $k:=-(M\otimes\psi)x$.*

i) The structured backward error with respect to the Frobenius norm is then given by

$$
\eta^{\mathbf{S}}_{w,F}(\psi,x,M)=\frac{\sqrt{2\|k\|_2^2-|x^Tk|^2}}{H_{w^{-1},2}(\psi)}
$$

and there exists a unique complex symmetric ΔM with coefficients

$$
\Delta M_j=\overline{z_{M_j}}\left[\overline{x}k^T+kx^H-(x^Tk)\overline{x}x^H\right],\quad j=0,\ldots,m,
$$

such that the structured backward error satisfies $\eta^{\mathbf{S}}_{w,F}(\psi,x,M)=|||\Delta M|||_{w,F}$ and $\overline{x}$, x are left and right eigenvectors corresponding to the eigenvalue of $(M+\Delta M)\otimes\psi$, respectively.

ii) The structured backward error with respect to the spectral norm is given by

$$
\eta^{\mathbf{S}}_{w,2}(\psi,x,M)=\frac{\|k\|_2}{H_{w^{-1},2}(\psi)}
$$

and there exists a complex symmetric ΔM with coefficients

$$
\Delta M_j:=\overline{z_{M_j}}\left[\overline{x}k^T+kx^H-(k^Tx)\overline{x}x^H-\frac{\overline{x^Tk}(I-\overline{x}x^T)kk^T(I-xx^H)}{\|k\|_2^2-|x^Tk|^2}\right]
$$

such that $|||\Delta M|||_{w,2} = \eta^{\mathbf{S}}_{w,2}(\psi,x,M)$ *and* $((M+\Delta M)\otimes\psi)x = 0$*, where* $\|k\|_2^2 \neq |x^T k|^2$*. Then* (γ,μ)*, (* λ *) is an exact eigenvalue satisfying* $((M+\Delta M)\otimes\psi)x = 0$.

Proof. The proof is a slight modification of the proof for the polynomial case [7]. By Theorem 3.1 we have that $((M+\Delta M)\otimes\psi)x = 0$ and hence $k = (\Delta M\otimes\psi)x$. Now we construct a unitary matrix U which has x as its first column, i.e., $U = [x, U_1] \in \mathbb{C}^{n\times n}$ and let $\widetilde{\Delta M_j} := U^T \Delta M_j U = \begin{bmatrix} d_{j,j} & d_j^T \\ d_j & D_{j,j}\end{bmatrix}$, where $D_{j,j} = D_{j,j}^T \in \mathbb{C}^{(n-1)\times(n-1)}$. Then

$$\overline{U}(\widetilde{\Delta M}\otimes\psi)U^H = \overline{U}U^T(\Delta M\otimes\psi)U^H U = \widetilde{\Delta M}\otimes\psi,$$

and hence

$$\overline{U}(\widetilde{\Delta M}\otimes\psi)U^H x = (\widetilde{\Delta M}\otimes\psi)x = k,$$

which implies that

$$(\widetilde{\Delta M}\otimes\psi)U^H x = U^T k = \begin{bmatrix} x^T k \\ U_1^T k\end{bmatrix}.$$

Therefore, we get that

$$\begin{bmatrix} \sum_{j=0}^m \psi_j d_{j,j} \\ \sum_{j=0}^m \psi_j d_j \end{bmatrix} = \begin{bmatrix} \sum_{j=0}^m w_j d_{j,j}\psi_j w_j^{-1} \\ \sum_{j=0}^m w_j \psi_j d_j w_j^{-1}\end{bmatrix} = \begin{bmatrix} x^T k \\ U_1^T k\end{bmatrix}.$$

To minimize the norm of the perturbation, we use the same procedure as in the polynomial case [7] and solve this system for the parameters $d_{j,j}, d_j$ in a least squares sense which, together with Proposition 2.1, yields

$$d_{j,j} = \overline{z_{M_j}} x^T k, \quad d_j = \overline{z_{M_j}} U_1^T k, \quad j = 0,1,\ldots,m,$$

and, since $U_1 U_1^H = (I - xx^H)$, we have

$$\begin{aligned}\Delta M_j &= \overline{U}\widetilde{\Delta M}U^H = \overline{x} d_{j,j} x^H + \overline{U_1} d_j x^H + \overline{x} d_j^T U_1^H + \overline{U_1} D_{j,j} U_1^H \\ &= \overline{z_{M_j}}[(\overline{x}x^T k x^H) + \overline{U_1}U_1^T k x^H + \overline{x}k^T U_1 U_1^H)] + \overline{U_1} D_{j,j} U_1^H \\ &= \overline{z_{M_j}}\left[k x^H + \overline{x}k^T - (k^T x)\overline{x}x^H\right] + \overline{U_1} D_{j,j} U_1^H. \end{aligned} \tag{12}$$

Further, we note that if $w_j = 0$, then by our definition $w_j^{-1} = 0$, and hence there will be no perturbation to the coefficient matrix M_j, i.e., $\Delta M_j = 0$.

In Frobenius norm, the unique minimal perturbation is obtained by taking $D_{j,j} = 0$ and hence we get

$$\begin{aligned}\|\Delta M_j\|_F^2 &= |d_{j,j}|^2 + 2\|d_j\|_2^2 = |z_{M_j}|^2(|x^T k|^2 + 2\|U_1^T k\|_2^2) \\ &= |\nabla_j H_{w^{-1},2}(\psi)|^2 \frac{2\|k\|_2^2 - |x^T k|^2}{H_{w^{-1},2}(\psi)^2},\end{aligned}$$

since $\|U^T k\|_2^2 = |x^T k|^2 + \|U_1^T k\|_2^2$. By Proposition 2.1, we have that

$$\sum_{j=0}^{m} w_j^2 |\nabla_j H_{w^{-1},2}(\psi)|^2 = 1, \quad \text{and hence,} \quad |||\Delta M|||_{w,F} = \sqrt{\frac{2\|k\|_2^2 - |x^T k|^2}{H_{w^{-1},2}(\psi)^2}}.$$

As $k^T x$ is a scalar constant, it follows that all ΔM_j and thus also ΔM are symmetric and

$$\begin{aligned}((M+\Delta M)\otimes\psi)x &= \sum_{j=0}^{m} \psi_j (M_j + \Delta M_j)x = -k + (\sum_{j=0}^{m} \psi_j \Delta M_j)x \\ &= -k + \sum_{j=0}^{m} f_j \overline{z_{M_j}} [kx^H + \bar{x}k^T - \bar{x}k^T x x^H]x = 0,\end{aligned}$$

where we have again used Proposition 2.1, i.e., $\sum_{j=0}^{m} \psi_j \overline{z_{M_j}} = 1$.

For the spectral norm we can apply Proposition 2.3 to (12) and get

$$\begin{aligned}D_{j,j} = &-\frac{\overline{z_{M_j}}}{P^2}\left[\overline{x^T k}(U_1^T k)(U_1^T k)^T\right] \\ &+\chi\left[I - \frac{(U_1^T k)(U_1^T k)^H}{P^2}\right]^{1/2} Z \left[I - \frac{\overline{U_1^T k}(U_1^T k)^T}{P^2}\right]^{1/2},\end{aligned}$$

where $Z = Z^T$ and $\|Z\|_2 \leqslant 1$, $P^2 = \|k\|_2^2 - |x^T k|^2$, $\chi := \sqrt{\|d_{j,j}\|^2 + \|d_j\|_2^2}$. With the special choice $Z = 0$ we get $D_{j,j} = -\frac{\overline{z_{M_j}}}{P^2}\left[\overline{x^T k}(U_1^T k)(U_1^T k)^T\right]$, where $P \neq 0$ and

$$\overline{U_1} D_{j,j} U_1^H = -\frac{\overline{z_{M_j}}}{P^2}\overline{x^T k}\,\overline{U_1} U_1^T k k^T U_1 U_1^H = -\frac{\overline{z_{M_j}}}{P^2}\overline{x^T k}(I - \bar{x}x^T)kk^T(I - xx^H).$$

Hence,

$$\Delta M_j = \overline{z_{M_j}}\left[kx^H + \bar{x}k^T - \bar{x}(k^T x)x^H\right] - \frac{\overline{z_{M_j}}}{P^2}\overline{x^T k}(I - \bar{x}x^T)kk^T(I - xx^H),$$

ΔM is symmetric, and $((M\otimes\psi) + (\Delta M\otimes\psi))x = 0$. With

$$\chi := \sigma_{\max}\left(\begin{bmatrix} d_{j,j} \\ d_j \end{bmatrix}\right) = |z_{M_j}|\sqrt{|x^T k|^2 + \|U_1^T k\|^2}\frac{|\nabla_j H_{w^{-1},2}(\psi)|}{H_{w^{-1},2}(\psi)}\|k\|_2,$$

then by Proposition 2.3 we have $\chi = \|\Delta M_j\|_2$, and again by Proposition 2.1,

$$\eta_{w,2}^{\mathbf{S}}(\psi, x, M) = |||\Delta M|||_{w,2} = \frac{\|k\|_2}{H_{w^{-1},2}(\psi)}. \quad \square$$

As a corollary we have the following relations between structured and unstructured backward errors.

COROLLARY 3.3. *Consider a regular problem* (1) *with symmetric* $M \in \mathscr{M}_{m+1}(\mathbb{C}^{n\times n})$ *and let* $x \in \mathbb{C}^n$ *be such that* $x^H x = 1$. *For a given approximate eigenvalue* (γ,μ) *(* λ *) define* $\psi = (\psi_0, \psi_1, \ldots, \psi_m)$ *as in (4) and let* $k := -(M \otimes \psi)x$. *Then*

$$\eta^{\mathbf{S}}_{w,F}(\psi,x,M) \leqslant \sqrt{2}\eta_{w,2}(\psi,x,M),$$
$$\eta^{\mathbf{S}}_{w,2}(\psi,x,M) = \eta_{w,2}(\psi,x,M).$$

We obtain an analogous result in the case of real problems and real perturbations which we omit here for brevity, we just mention that we need that the function evaluations f_j yield real values to obtain a real backward error. Defining ψ as in (4), in this case the minimal perturbation has the form $\sum_{j=0}^{m} \psi_j \Delta M_j$, with coefficients

$$\Delta M_j = z_{M_j}\left[xk^T + kx^T - (x^T k)xx^T\right], \quad j = 1,2,\ldots,m.$$

The same technique of proof also applies in the complex skew-symmetric case. We state the results here for completeness.

THEOREM 3.4. *Consider problem* (1) *with complex-symmetric* $M \in \mathscr{M}_{m+1}(\mathbb{C}^{n,n})$, *and let* $x \in \mathbb{C}^n$ *be such that* $x^H x = 1$. *For an approximate eigenvalue* (γ,μ), *(* λ *) define* $\psi = (\psi_0, \ldots, \psi_m)$ *as in (4) and let* $k := -(M \otimes \psi)x$. *Introducing* $\Delta M \otimes \psi = \sum_{j=1}^{m} \psi_j \Delta M_j$ *with coefficient matrices*

$$\Delta M_j := -\overline{z_{M_j}}\left[\overline{x}k^T - kx^H\right], \quad j = 0,1,2,\ldots,m,$$

we have that ΔM *is complex skew-symmetric, and* (γ,μ) *is an exact eigenvalue with eigenvector* x *of the perturbed problem* $((M+\Delta M)\otimes\psi)x = 0$.

THEOREM 3.5. *Consider problem* (1) *with complex-symmetric* $M \in \mathscr{M}_{m+1}(\mathbb{C}^{n,n})$, *and let* $x \in \mathbb{C}^n$ *be such that* $x^H x = 1$. *For an approximate eigenvalue* (γ,μ), *(* λ *) define* $\psi = (\psi_0, \ldots, \psi_m)$ *as in* (4) *and let* $k := -(M \otimes \psi)x$. *Then the structured backward errors with respect to the Frobenius norm and spectral norm are given by*

$$\eta^{\mathbf{S}}_{w,F}(\psi,x,M) = \frac{\sqrt{2\|k\|_2^2}}{H_{w^{-1},2}(\psi)},$$
$$\eta^{\mathbf{S}}_{w,2}(\psi,x,M) = \frac{\|k\|_2}{H_{w^{-1},2}(\psi)},$$

respectively.

The relation between structured and unstructured backward errors is then clearly the same as in the symmetric case for the spectral norm.

In this section we have shown that the backward error results for symmetric and skew-symmetric matrix functions carry over from the polynomial case to (4) with very little modifications.

4. Backward errors for Hermitian/skew-Hermitian nonlinear eigenvalue problems

In this section we present the results for the Hermitian and skew-Hermitian case.

THEOREM 4.1. *Consider problem* (1) *with Hermitian or skew-Hermitian* $M \in \mathscr{M}_{m+1}(\mathbb{C}^{n,n})$, *and let* $x \in \mathbb{C}^n$ *be such that* $x^H x = 1$. *For an approximate eigenvalue* (γ,μ) (λ), *define* $\psi = (\psi_0,\ldots,\psi_m)$ *as in* (4) *and let* $k := -(M \otimes \psi)x$. *Introducing* $\Delta M \otimes \psi = \sum_{j=1}^{m} \psi_j \Delta M_j$ *with coefficient matrices*

$$\Delta M_j := \begin{cases} -xx^H M_j xx^H + \left[z_{M_j} xk^H P_x + \overline{z_{M_j}} P_x kx^H\right], & \text{if } M_j = M_j^H, \\ -xx^H M_j xx^H - \left[z_{M_j} xk^H P_x - \overline{z_{M_j}} P_x kx^H\right], & \text{if } M_j = -M_j^H, \end{cases}$$

we have that ΔM *is Hermitian or skew-Hermitian, respectively, and* (γ,μ) (λ) *is an exact eigenvalue with eigenvector* x *satisfying* $((M+\Delta M)\otimes\psi)x = 0$.

Proof. The proof follows in the same way as for the symmetric/skew symmetric problems. □

For the construction of the backward errors we introduce

$$T := \begin{bmatrix} \Re(\psi_0)w_0^{-1} & \ldots & \Re(\psi_m)w_m^{-1} \\ \Im(\psi_0)w_0^{-1} & \ldots & \Im(\psi_m)w_m^{-1} \end{bmatrix},$$

and set

$$t = [t_0,\ldots,t_m]^T := T^+ \begin{bmatrix} \Re(x^H k) \\ \Im(x^H k) \end{bmatrix}, \tag{13}$$

where T^+ denotes the Moore-Penrose inverse of T, see [17]. Denoting by e_j the j-th unit vector, we have the following structured backward errors.

THEOREM 4.2. *Consider problem* (1) *with Hermitian* $M \in \mathscr{M}_{m+1}(\mathbb{C}^{n,n})$, *and let* $x \in \mathbb{C}^n$ *be such that* $x^H x = 1$. *For an approximate eigenvalue* (γ,μ), (λ) *define* $\psi = (\psi_0,\ldots,\psi_m)$ *as in* (4) *and let* $k := -(M\otimes\psi)x$.

i) The structured backward error in Frobenius norm is given by

$$\eta^{\mathbf{S}}_{w,F}(\psi,x,M) = \frac{\sqrt{2\|k\|_2^2 - |x^H k|^2}}{H_{w^{-1},2}(\psi)},$$

with

$$\Delta M_j = z_{M_j}\left[kx^H + xk^H - (k^H x)xx^H\right] + U_1 \Delta D_{j,j} U_1^H.$$

If all ψ_j, $j = 0,\ldots,m$ *are real then*

$$\eta^{\mathbf{S}}_{w,F}(\psi,x,M) = \sqrt{\sum_{j=0}^{m} \frac{\|e_j t\|_2^2}{w_j^2} + 2\frac{\|k\|_2^2 - |x^H k|^2}{H_{w^{-1},2}(\psi)^2}},$$

with

$$\Delta M_j = w_j^{-1} x e_j^T t x^H + \overline{z_{M_j}} P_x k x^H + z_{M_j} x k^H P_x + U_1 \Delta D_{j,j} U_1^H.$$

ii) The structured backward error in spectral norm is given by

$$\eta^{\mathbf{S}}_{w,2}(\psi,x,M) = \frac{\|k\|_2}{H_{w^{-1},2}(\psi)},$$

with

$$\Delta E_j = \Delta M_j - \frac{z_{M_j} x^H k P_x k k^H P_x}{P^2},$$

if all ψ_j, $j = 1,\ldots,m$ *are real, where* ΔM_j *is as in part i), and*

$$\eta^{\mathbf{S}}_{w,2}(\psi,x,M) = \sqrt{\sum_{j=0}^{m} \frac{\|e_j t\|_2^2}{w_j^2} + \frac{\|k\|_2^2 - |x^H k|^2}{H_{w^{-1},2}(\psi)^2}},$$

with

$$\Delta E_j = \Delta M_j - \frac{w_j^{-1} e_j^T t P_x k k^H P_x}{\|k\|_2^2 - |x^H k|^2}, \quad \text{and} \quad \|k\|_2^2 \neq |x^H k|^2,$$

otherwise, where ΔM_j *is as in* (i).

Proof. The proof follows as in the proof of Theorem 3.3 in [8]. □

The result for the skew-Hermitian case is obtained analogously, just replacing M_j by $\imath M_j$, we omit the result here.

The relationship between the structured and unstructured backward errors for the Hermitian/skew-Hermitian case is the same as in the symmetric/skew-symmetric case. Here we state the results for the Hermitian case, the analogous results for the skew-Hermitian case are again omitted.

COROLLARY 4.3. *Consider problem* (1) *with Hermitian or skew-Hermitian* $M \in \mathscr{M}_{m+1}(\mathbb{C}^{n,n})$*, and let* $x \in \mathbb{C}^n$ *be such that* $x^H x = 1$*. For an approximate eigenvalue* (γ,μ) *(*λ*), define* $\psi = (\psi_0,\ldots,\psi_m)$ *as in* (4) *and let* $k := -(M \otimes \psi)x$*. Then we have the following relations between the structured and unstructured backward errors.*

i) $\eta^{\mathbf{S}}_{w,2}(\psi,x,M) = \eta_{w,2}(\psi,x,M)$, if $\psi_j \in \mathbb{R}$ for $0 \leqslant j \leqslant m$,

ii) $\eta^{\mathbf{S}}_{w,2}(\psi,x,M) = \eta_{w,2}(\psi,x,M)$, if $\psi_j \in \imath\mathbb{R}$ for $0 \leqslant j \leqslant m$,

iii) $\eta^{\mathbf{S}}_{w,2}(\psi,x,M) \leqslant \eta_{w,2}(\psi,x,M)$, otherwise, if $H_{w^{-1},2}(\psi)\|T^+\| \leqslant 1$,

iv) $\eta^{\mathbf{S}}_{w,F}(\psi,x,M) \leqslant \sqrt{2}\eta_{w,2}(\psi,x,M)$, if $\psi_j \in \mathbb{R}$ for $0 \leqslant j \leqslant m$,

v) $\eta^{\mathbf{S}}_{w,F}(\psi,x,M) \leqslant \sqrt{2}\eta_{w,2}(\psi,x,M)$, otherwise, if $H_{w^{-1},2}(\psi)\|T^+\| \leqslant \sqrt{2}$.

Proof. By Theorem 4.2, we have

$$
\begin{aligned}
\eta^{\mathbf{S}}_{w,F}(\psi,x,M) &= \sqrt{\sum_{j=0}^{m}\left|\frac{t_j}{w_j}\right|^2 + 2\frac{\|k\|_2^2-|x^Hk|^2}{H_{w^{-1},2}(\psi)^2}} \\
&\leqslant \sqrt{\|T^+\||x^Hk|^2 - 2\frac{|x^Hk|^2}{H_{w^{-1},2}(\psi)^2} + 2\frac{\|k\|^2}{H_{w^{-1},2}(\psi)^2}} \\
&= \sqrt{|x^Hk|^2\left[\|T^+\|^2 - \frac{2}{H_{w^{-1},2}(\psi)^2}\right] + 2\frac{\|k\|^2}{H_{w^{-1},2}(\psi)^2}} \\
&\leqslant \sqrt{2}\eta_{w,2}(\psi,x,M)
\end{aligned}
$$

if $H_{w^{-1},2}(\psi)\|T^+\| \leqslant \sqrt{2}$, where using (13) we have $\sum_{j=0}^{m}|t_j|^2 = \|t\|^2 = |x^Hk|^2\|T^+\|^2$. The other results follow from Theorem 4.2. $\square$

We illustrate the results with some examples.

EXAMPLE 4.4. Consider the delay differential equation $\dot{x}(t)+M_1x(t)+M_0x(t-\tau)=0$, where

$$
M_0 = -\begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}, \quad M_1 = -\begin{bmatrix} -149 & -50i & -154i+1 \\ 50i & 7 & 4+i \\ 154i+1 & 4-i & 1 \end{bmatrix}, \quad x = \begin{bmatrix} \frac{i}{\sqrt{3}} \\ -\frac{i}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix},
$$

and the delay is $\tau = 1000$. For given λ and ψ as in (4) the associated eigenvalue problem is $M\otimes\psi = \gamma I + M_1 + M_0e^{-\lambda\tau}$ with $\psi = (\psi_0,\psi_1,\psi_2) = (-e^{-\lambda\tau},-1,\lambda)$. The coefficient matrices $M_2 = I$, M_1, M_0 are Hermitian and if λ is such that $\psi_j \in \mathbb{R}$, for $j = 0,\ldots,m$, then we obtain the following backward errors.

λ	$\eta_2(\psi,x,M)$	$\eta_2^{\mathbf{S}}(\psi,x,M)$	$\eta_F^{\mathbf{S}}(\psi,x,M)$
0.3	0.3703	0.3703	0.5104
3	0.9000	0.9000	0.9074
10	0.9780	0.9780	0.9787
10^{-5}	140.5133	140.5133	168.3081

If $\lambda \in \mathbb{C}$ such that $\psi_j \in \mathbb{C}$ for $j = 0,\ldots,m$, then we obtain

λ	$\eta_2(\psi,x,M)$	$\eta_2^{\mathbf{S}}(\psi,x,M)$	$\eta_F^{\mathbf{S}}(\psi,x,M)$
$20+3i$	0.9903	1.0158	1.0158
$3+5i$	0.9725	1.0176	1.0176
$0.3+10i$	0.9953	1.0163	1.0163
$3i$	60.279	112.5229	112.5229

EXAMPLE 4.5. Consider the rational eigenvalue problem in homogeneous form

$$\left(\mu M_0 + \gamma M_1 + \frac{\gamma\mu}{\mu - 10\gamma} M_2\right) x = 0,$$

with Hermitian coefficients

$$M_0 = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}, \quad M_1 = \begin{bmatrix} -149 & -50i & -154i+1 \\ 50i & 7 & 4+i \\ 154i+1 & 4-i & 1 \end{bmatrix},$$

$$M_2 = \begin{bmatrix} 1 & 1+i & 2i \\ 1-i & 2 & 3i \\ -2i & -3i & 2 \end{bmatrix}, \quad \text{and} \quad x = \begin{bmatrix} \frac{i}{\sqrt{3}} \\ -\frac{i}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}.$$

If (γ, μ) is such that $\psi_j \in \mathbb{R}$ for $j = 0, \ldots, m$, then we obtain the backward errors

(γ,μ)	$\eta_2(\psi,x,M)$	$\eta_2^{\mathbf{S}}(\psi,x,M)$	$\eta_F^{\mathbf{S}}(\psi,x,M)$
$(2,3)$	110.3015	110.3015	132.1514
$(0,2)$	0.4076	0.4076	0.5483
$(4,3)$	159.6050	159.6050	191.2182
$(4,0)$	200.0725	200.0725	239.7038

If (γ, μ) is such that $\psi_j \in \mathbb{C}$ for $j = 0, \ldots, m$, then we obtain the backward errors

(γ,μ)	$\eta_2(\psi,x,M)$	$\eta_2^{\mathbf{S}}(\psi,x,M)$	$\eta_F^{\mathbf{S}}(\psi,x,M)$
$(0.1+0.2i, -0.3+0.9i)$	50.4250	149.6831	149.6831
$(2-3i, -4+3i)$	116.9143	168.9691	168.9691
$(-2-5i, 3+7i)$	115.3996	168.5062	168.5062

We see that in the complex case the structured and unstructured backward errors may differ substantially.

REMARK 4.6. Backward errors are closely related to pseudospectra which have been studied in detail for matrices, matrix pencils, and matrix polynomials, see e.g., [2, 6, 23, 27, 28, 39, 41]. Thus, the presented results on structured backward errors can be immediately transferred to corresponding results for structured pseudospectra. For brevity we do not present these results here.

5. Conclusion

We have extended the construction of structured backward errors from polynomial eigenvalue problems to nonlinear eigenvalue problems that are linear in the matrix coefficients and have derived a systematic framework for the construction of appropriately structured backward errors for the classes of complex symmetric, complex skew-symmetric, Hermitian, and skew-Hermitian problems. The resulting minimal perturbation is unique in the case of Frobenius norm and there are infinitely many solutions for the case of spectral norm. The results show no real surprise, the relation between structured and unstructured backward errors is similar as in the polynomial case.

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