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SPECTRAL PROPERTIES OF k–QUASI–$*$–n–PARANORMAL OPERATORS

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Abstract. For positive integers n and k, an operator T is said to be k-quasi-$*$-n-paranormal if $||T^{1+n+k}x||^{\frac{1}{1+n}}||T^kx||^{\frac{n}{1+n}} \geqslant ||T^*T^kx||$ for all $x \in H$, which is a generalization of $*$-paranormal operator. In this paper, we prove that the spectrum is continuous on the class of all k-quasi-$*$-n-paranormal operators. Let λ be an isolated point of $\sigma(T)$ and E be the Riesz idempotent with respect to λ. We also prove that (1) if $\lambda \neq 0$, then E is self-adjoint and $R(E) = N(T-\lambda) = N(T-\lambda)^*$. (2) if $\lambda = 0$, then $R(E) = N(T^{k+1})$.

1. Introduction

Let $B(H)$ denote the C^*-algebra of all bounded linear operators on an infinite dimensional separable Hilbert space H. In paper [18] authors introduced the class of k-quasi-$*$-n-paranormal operators defined as follows:

DEFINITION 1.1. For positive integers n and k, an operator T is said to be k-quasi-$*$-n-paranormal if $||T^{1+n+k}x||^{\frac{1}{1+n}}||T^kx||^{\frac{n}{1+n}} \geqslant ||T^*T^kx||$ for all $x \in H$.

A k-quasi-$*$-n-paranormal operator for positive integers n and k is an extension of $*$-n-paranormal operator, i.e., $||T^{1+n}x||^{\frac{1}{1+n}} \geqslant ||T^*x||$ for unit vector x and k-quasi-$*$-paranormal operator, i.e., $||T^{k+2}x||||T^kx|| \geqslant ||T^*T^kx||^2$ for all $x \in H$. A $*$-1-paranormal operator is called a $*$-paranormal operator and a 1-quasi-$*$-paranormal operator is called a quasi-$*$-paranormal operator. It is known that quasi-$*$-paranormal operator is normaloid [10], i.e., $||T^n|| = ||T||^n$, for $n \in \mathbb{N}$ (equivalently, $||T|| = r(T)$, the spectral radius of T). $*$-paranormal operator and quasi-$*$-paranormal operator have been studied by many authors and it is known that they have many interesting properties similar to those of hyponormal operators (see [5, 9, 13]).

It is clear that

$$*\text{-}n\text{-paranormal} \Rightarrow \text{quasi-}*\text{-}n\text{-paranormal} \Rightarrow \text{normaloid}$$

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and

$$\text{quasi-}*\text{-}n\text{-paranormal} \Rightarrow k\text{-quasi-}*\text{-}n\text{-paranormal} \\ \Rightarrow (k+1)\text{-quasi-}*\text{-}n\text{-paranormal}.$$

EXAMPLE 1.1. Given a bounded sequence of positive numbers $\alpha : \alpha_1, \alpha_2, \alpha_3, \ldots$ (called weights), the unilateral weighted shift W_α associated with α is the operator on l_2 defined by $W_\alpha e_n = \alpha_n e_{n+1}$ for all $n \geqslant 1$, where $\{e_n\}_{n=1}^{\infty}$ is the canonical orthogonal basis for l_2. Straightforward calculations show that W_α is a k-quasi-$*$-n-paranormal operator if and only if

$$W_\alpha = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ \alpha_1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & \alpha_2 & 0 & 0 & 0 & \cdots \\ 0 & 0 & \alpha_3 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \alpha_4 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix},$$

where

$$(\alpha_{i+1+n}\alpha_{i+n}\cdots\alpha_{i+2}\alpha_{i+1})^{\frac{1}{1+n}} \geqslant \alpha_i \quad (i = k, k+1, k+2, \cdots).$$

Now it is natural to ask whether k-quasi-$*$-n-paranormal operators are normaloid or not. For $k > 1$, an answer has been given: there exists a nilpotent operator which is a k-quasi-$*$-n-paranormal operator. But it need not be normaloid.

In section 2, we give a necessary and sufficient condition for T to be a k-quasi-$*$-n-paranormal operator. In section 3, we prove that the spectrum is continuous on the class of all k-quasi-$*$-n-paranormal operators. In section 4, we study the Riesz idempotent of k-quasi-$*$-n-paranormal operators.

2. k-quasi-$*$-n-paranormal operators

In the sequel, we shall write $N(T)$ and $R(T)$ for the null space and range space of T, respectively.

LEMMA 2.1. [18] *T is a k-quasi-$*$-n-paranormal operator if and only if*

$$T^{*1+n+k}T^{1+n+k} - (n+1)\lambda^n T^{*k}TT^*T^k + n\lambda^{1+n}T^{*k}T^k \geqslant 0 \ \text{ for any } \lambda > 0.$$

THEOREM 2.2. *If T does not have a dense range, then the following statements are equivalent:*

(1) *T is a k-quasi-$*$-n-paranormal operator;*

(2) $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ *on $H = \overline{R(T^k)} \oplus N(T^{*k})$, where $T_1^{*1+n}T_1^{1+n} - (1+n)\lambda^n(T_1T_1^* + T_2T_2^*) + n\lambda^{1+n} \geqslant 0$ for all $\lambda > 0$ and $T_3^k = 0$. Furthermore, $\sigma(T) = \sigma(T_1) \cup \{0\}$.*

Proof. (1) $\Rightarrow$ (2) Consider the matrix representation of T with respect to the decomposition $H = \overline{R(T^k)} \oplus N(T^{*k})$:

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}.$$

Let P be the projection onto $\overline{R(T^k)}$. Since T is a k-quasi-$*$-n-paranormal operator, we have

$$P(T^{*1+n}T^{1+n} - (1+n)\lambda^n TT^* + n\lambda^{1+n})P \geqslant 0 \text{ for all } \lambda > 0.$$

Therefore

$$T_1^{*1+n}T_1^{1+n} - (1+n)\lambda^n(T_1T_1^* + T_2T_2^*) + n\lambda^{1+n} \geqslant 0 \text{ for all } \lambda > 0.$$

On the other hand, for any $x = (x_1, x_2) \in H$, we have

$$(T_3^k x_2, x_2) = (T^k(I-P)x, (I-P)x) = ((I-P)x, T^{*k}(I-P)x) = 0,$$

which implies $T_3^k = 0$.

Since $\sigma(T) \cup M = \sigma(T_1) \cup \sigma(T_3)$, where M is the union of the holes in $\sigma(T)$ which happens to be subset of $\sigma(T_1) \cap \sigma(T_3)$ by Corollary 7 of [8], and $\sigma(T_1) \cap \sigma(T_3)$ has no interior point and T_3 is nilpotent, we have $\sigma(T) = \sigma(T_1) \cup \{0\}$.

(2) $\Rightarrow$ (1) Suppose that $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $H = \overline{R(T^k)} \oplus N(T^{*k})$, where $T_1^{*1+n}T_1^{1+n} - (1+n)\lambda^n(T_1T_1^* + T_2T_2^*) + n\lambda^{1+n} \geqslant 0$ for all $\lambda > 0$ and $T_3^k = 0$. Since

$$T^k = \begin{pmatrix} T_1^k & \sum\limits_{j=0}^{k-1} T_1^j T_2 T_3^{k-1-j} \\ 0 & 0 \end{pmatrix},$$

we have

$$\begin{aligned}
&T^{*k}(T^{*1+n}T^{1+n} - (1+n)\lambda^n TT^* + n\lambda^{1+n})T^k \\
&= \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}^{*k} \\
&\quad \times \left(\begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}^{*1+n} \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}^{1+n} - (1+n)\lambda^n \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}^{*} + n\lambda^{1+n} \right) \\
&\quad \times \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}^{k} \\
&= \begin{pmatrix} T_1^{*k} & 0 \\ (\sum\limits_{j=0}^{k-1} T_1^j T_2 T_3^{k-1-j})^* & 0 \end{pmatrix} \begin{pmatrix} D & C \\ C^* & F \end{pmatrix} \begin{pmatrix} T_1^k & \sum\limits_{j=0}^{k-1} T_1^j T_2 T_3^{k-1-j} \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} T_1^{*k} D T_1^k & T_1^{*k} D \sum\limits_{j=0}^{k-1} T_1^j T_2 T_3^{k-1-j} \\ (\sum\limits_{j=0}^{k-1} T_1^j T_2 T_3^{k-1-j})^* D T_1^k & (\sum\limits_{j=0}^{k-1} T_1^j T_2 T_3^{k-1-j})^* D \sum\limits_{j=0}^{k-1} T_1^j T_2 T_3^{k-1-j} \end{pmatrix}
\end{aligned}$$

where $D = T_1^{*1+n}T_1^{1+n} - (1+n)\lambda^n(T_1T_1^* + T_2T_2^*) + n\lambda^{1+n}$,

$$F = (\sum_{j=0}^{n} T_1^jT_2T_3^{n-j})^* \sum_{j=0}^{n} T_1^jT_2T_3^{n-j} + T_3^{*1+n}T_3^{1+n} - (1+n)\lambda^nT_3T_3^* + n\lambda^{1+n},$$

$C = T_1^{*1+n} \sum_{j=0}^{n} T_1^jT_2T_3^{n-j} - (1+n)\lambda^nT_2T_3^*$.

Let $\lambda > 0$ be arbitrary and $v = x \oplus y$ be a vector in $H = \overline{R(T^k)} \oplus N(T^{*k})$, where $x \in \overline{R(T^k)}$ and $y \in N(T^{*k})$. Then

$$\begin{aligned}
&(T^{*k}(T^{*1+n}T^{1+n} - (1+n)\lambda^nTT^* + n\lambda^{1+n})T^kv, v)\\
&= (T_1^{*k}DT_1^kx, x) + (T_1^{*k}Dy, x) + ((\sum_{j=0}^{k-1} T_1^jT_2T_3^{k-1-j})^*DT_1^kx, y)\\
&\quad + ((\sum_{j=0}^{k-1} T_1^jT_2T_3^{k-1-j})^*D\sum_{j=0}^{k-1} T_1^jT_2T_3^{k-1-j}y, y)\\
&= (D(T_1^kx + \sum_{j=0}^{k-1} T_1^jT_2T_3^{k-1-j}y), (T_1^kx + \sum_{j=0}^{k-1} T_1^jT_2T_3^{k-1-j}y)).
\end{aligned}$$

Since

$$D = T_1^{*1+n}T_1^{1+n} - (1+n)\lambda^n(T_1T_1^* + T_2T_2^*) + n\lambda^{1+n} \geqslant 0 \text{ for all } \lambda > 0,$$

$$(T^{*k}(T^{*1+n}T^{1+n} - (1+n)\lambda^nTT^* + n\lambda^{1+n})T^kv, v) \geqslant 0 \text{ for all } v \in H,$$

hence

$$T^{*k}(T^{*1+n}T^{1+n} - (1+n)\lambda^nTT^* + n\lambda^{1+n})T^k \geqslant 0 \text{ for all } \lambda > 0.$$

Thus T is a k-quasi-$*$-n-paranormal operator. □

COROLLARY 2.3. *If T is a k-quasi-$*$-n-paranormal operator and $R(T^k)$ is not dense, then*

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \quad on \quad H = \overline{R(T^k)} \oplus N(T^{*k}),$$

where T_1 is a $*$-n-paranormal operator on $\overline{R(T^k)}$ and $T_3^k = 0$.

COROLLARY 2.4. *If T is a k-quasi-$*$-n-paranormal operator and $R(T^k)$ is dense, then T is a $*$-n-paranormal operator.*

COROLLARY 2.5. *If T is a k-quasi-$*$-n-paranormal operator, $0 \neq \lambda \in \sigma_p(T)$ and T is of the form $T = \begin{pmatrix} \lambda & A \\ 0 & B \end{pmatrix}$ on $H = N(T-\lambda) \oplus N(T-\lambda)^\perp$, then $A = 0$.*

Proof. Let E be the orthogonal projection onto $N(T-\lambda)$. Without loss of generality, assume that $\lambda = 1$. Since T is a k-quasi-$*$-n-paranormal operator, T satisfies

$$T^{*k}(T^{*1+n}T^{1+n} - (1+n)\lambda^n TT^* + n\lambda^{1+n})T^k \geqslant 0 \text{ for all } \lambda > 0.$$

Taking $\lambda = 1$, we have

$$E(T^{*1+n+k}T^{1+n+k} - (1+n)T^{*k}TT^*T^k + nT^{*k}T^k)E \geqslant 0.$$

We remark

$$E(T^{*1+n+k}T^{1+n+k})E = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$E(T^{*k}TT^*T^k)E = \begin{pmatrix} 1+AA^* & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$ET^{*k}T^kE = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus

$$\begin{pmatrix} -(n+1)AA^* & 0 \\ 0 & 0 \end{pmatrix} \geqslant 0.$$

Hence $A = 0$. □

COROLLARY 2.6. *If T is a k-quasi-$*$-n-paranormal operator and $\lambda \neq 0$, then $Tx = \lambda x$ implies $T^*x = \bar{\lambda}x$.*

Proof. It follows from Corollary 2.5. □

The following example provides an operator T which is a k-quasi-$*$-n-paranormal operator, however, the relation $N(T) \subseteq N(T^*)$ does not hold.

EXAMPLE 2.7. [13] Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ be operators on $\mathbb{R}^2$, and let $H_n = \mathbb{R}^2$ for all positive integers n. Consider the operator $T_{A,B}$ on $\oplus_{n=1}^{+\infty} H_n$ defined by

$$T_{A,B} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ A & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & B & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & B & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & B & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & B & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then $T_{A,B}$ is a quasi-$*$-paranormal operator, hence $T_{A,B}$ is a k-quasi-$*$-paranormal operator, however for the vector $x = (0,0,1,-1,0,0,\cdots)$, $T_{A,B}(x) = 0$, but $T^*_{A,B}(x) \neq 0$. Therefore, the relation $N(T_{A,B}) \subseteq N(T^*_{A,B})$ does not always hold.

3. The spectral continuity of k-quasi-$*$-n-paranormal operators

For every $T \in B(H)$, $\sigma(T)$ is a compact subset of $\mathbb{C}$. The function σ viewed as a function from $B(H)$ into the set of all compact subsets of $\mathbb{C}$, equipped with the Hausdorff metric, is well known to be upper semi-continuous, but fails to be continuous in general. Conway and Morrel [2] have carried out a detailed study of spectral continuity in $B(H)$. Recently, it has been proved that the spectrum is continuous in the set of normal operators and hyponormal operators in [7]. And this result has been extended to quasihyponormal operators by Djordjević in [3], to p-hyponormal operators by Hwang and Lee in [11], and to (p,k)-quasihyponormal, M-hyponormal, $*$-paranormal and paranormal operators by Duggal, Jeon and Kim in [4]. In this section we extend this result to k-quasi-$*$-n-paranormal operators.

LEMMA 3.1. *Let T be a k-quasi-$*$-n-paranormal operator. Then the following assertions hold:*

(1) *If T is quasinilpotent, then $T^{k+1} = 0$.*

(2) *For every non-zero $\lambda \in \sigma_p(T)$, the matrix representation of T with respect to the decomposition $H = N(T-\lambda) \oplus (N(T-\lambda))^{\perp}$ is: $T = \begin{pmatrix} \lambda & 0 \\ 0 & B \end{pmatrix}$ for some operator B satisfying $\lambda \notin \sigma_p(B)$ and $\sigma(T) = \{\lambda\} \cup \sigma(B)$.*

Proof. (1) Suppose T is a k-quasi-$*$-n-paranormal operator. If $R(T^k)$ is dense, then T is a $*$-n-paranormal operator, which leads to that T is normaloid, hence $T = 0$. If $R(T^k)$ is not dense, then

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \quad \text{on} \quad H = \overline{R(T^k)} \oplus N(T^{*k})$$

where T_1 is a $*$-n-paranormal operator, $T_3^k = 0$ and $\sigma(T) = \sigma(T_1) \cup \{0\}$ by Theorem 2.2. Since $\sigma(T_1) = \{0\}$, $T_1 = 0$. Thus

$$T^{k+1} = \begin{pmatrix} 0 & T_2 \\ 0 & T_3 \end{pmatrix}^{k+1} = \begin{pmatrix} 0 & T_2T_3^k \\ 0 & T_3^{k+1} \end{pmatrix} = 0.$$

(2) If $\lambda \neq 0$ and $\lambda \in \sigma_p(T)$, we have that $N(T-\lambda)$ reduces T by Corollary 2.5. So we have that $T = \begin{pmatrix} \lambda & 0 \\ 0 & B \end{pmatrix}$ on $H = N(T-\lambda) \oplus (N(T-\lambda))^{\perp}$ for some operator B satisfying $\lambda \notin \sigma_p(B)$ and $\sigma(T) = \{\lambda\} \cup \sigma(B)$. $\square$

LEMMA 3.2. [1] *Let H be a complex Hilbert space. Then there exists a Hilbert space K such that $H \subset K$ and a map $\varphi : B(H) \to B(K)$ such that*

(1) *φ is a faithful $*$-representation of the algebra $B(H)$ on K;*

(2) *$\varphi(A) \geqslant 0$ for any $A \geqslant 0$ in $B(H)$;*

(3) *$\sigma_a(T) = \sigma_a(\varphi(T)) = \sigma_p(\varphi(T))$ for any $T \in B(H)$.*

THEOREM 3.3. *The spectrum σ is continuous on the set of k-quasi-$*$-n-paranormal operators.*

Proof. Suppose T is a k-quasi-$*$-n-paranormal operator. Let $\varphi\colon B(H) \to B(K)$ be Berberian's faithful $*$-representation of Lemma 3.2. In the following, we shall show that $\varphi(T)$ is also a k-quasi-$*$-n-paranormal operator. In fact, since T is a k-quasi-$*$-n-paranormal operator, we have

$$T^{*k}(T^{*1+n}T^{1+n} - (1+n)\lambda^n TT^* + n\lambda^{1+n})T^k \geqslant 0 \text{ for all } \lambda > 0.$$

Hence we have

$$\begin{aligned}
&\varphi(T)^{*k}(\varphi(T)^{*1+n}\varphi(T)^{1+n} - (1+n)\lambda^n\varphi(T)\varphi(T)^* + n\lambda^{1+n})\varphi(T)^k \\
&= \varphi(T^{*k}(T^{*1+n}T^{1+n} - (1+n)\lambda^n TT^* + n\lambda^{1+n})T^k) \text{ by Lemma 3.2} \\
&\geqslant 0 \text{ by Lemma 3.2,}
\end{aligned}$$

so $\varphi(T)$ is also a k-quasi-$*$-n-paranormal operator. By Lemma 3.1, we have T belongs to the set $C(i)$ (see definition in [4]). Therefore, we have that the spectrum σ is continuous on the set of k-quasi-$*$-n-paranormal operators by [4, Theorem 1.1]. □

4. Isolated point of spectrum of k-quasi-$*$-n-paranormal operator

Let λ be an isolated point of the spectrum of T. Then the Riesz idempotent E of T with respect to λ is defined by

$$E = \frac{1}{2\pi i}\int_{\partial D}(\mu - T)^{-1}d\mu,$$

where D is a closed disk centered at λ which contains no other points of the spectrum of T. Stampfli [14] showed that if T is hyponormal, then E is self-adjoint and $R(E) = N(T-\lambda) = N(T-\lambda)^*$. Recently, Tanahashi and Uchiyama [16] obtained Stampfli's result for $*$-paranormal operators. We shall show that for every k-quasi-$*$-n-paranormal operator T and each isolated point λ of $\sigma(T)$, the Riesz idempotent E satisfies that

$$\begin{aligned}
&R(E) = N(T^{k+1}) \text{ if } \lambda = 0, \\
&R(E) = N(T-\lambda) = N(T-\lambda)^* \text{ and } E \text{ is self-adjoint if } \lambda \neq 0.
\end{aligned}$$

LEMMA 4.1. *Let T be a $*$-n-paranormal operator, λ be an isolated point of $\sigma(T)$ and E be the Riesz idempotent with respect to λ. Then $R(E) = N(T-\lambda)$.*

Proof. Recall from [12, Proposition 4.8] that if T is a $*$-n-paranormal operator, then T is a $(n+1)$-paranormal operator, i.e., $||T^{2+n}x||^{\frac{1}{2+n}} \geqslant ||Tx||$ for unit vector x. [16, Theorem 2] implies $R(E) = N(T-\lambda)$. □

THEOREM 4.2. *Let T be a k-quasi-$*$-n-paranormal operator, λ be an isolated point of $\sigma(T)$ and E be the Riesz idempotent with respect to λ. Then the following assertions hold.*

(1) *if $\lambda \neq 0$, then E is self-adjoint and $R(E) = N(T-\lambda) = N(T-\lambda)^*$.*

(2) *if $\lambda = 0$, then $R(E) = N(T^{k+1})$.*

Proof. If $\lambda \neq 0$, assume that $R(T^k)$ is dense. Then T is a $*$-n-paranormal operator and $R(E) = N(T-\lambda)$ by Lemma 4.1. So we may assume that T^k does not have dense range. Then by Theorem 2.2 the operator T can be decomposed as follows:

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \text{ on } H = \overline{R(T^k)} \oplus N(T^{*k}),$$

where T_1 is a $*$-n-paranormal operator on $\overline{R(T^k)}$. Now if λ is a non-zero isolated point of $\sigma(T)$, then $\lambda \in \text{iso}\,\sigma(T_1)$ because $\sigma(T) = \sigma(T_1) \cup \{0\}$. Therefore λ is a simple pole of the resolvent of T_1 and the $*$-n-paranormal operator T_1 can be written as follows:

$$T_1 = \begin{pmatrix} T_{11} & 0 \\ 0 & T_{12} \end{pmatrix} \text{ on } \overline{R(T^k)} = N(T_1-\lambda) \oplus \overline{R(T_1-\lambda)},$$

where $\sigma(T_{11}) = \{\lambda\}$. Therefore

$$T-\lambda = \begin{pmatrix} 0 & 0 & T_{21} \\ 0 & T_{12}-\lambda & T_{22} \\ 0 & 0 & T_3-\lambda \end{pmatrix} = \begin{pmatrix} 0 & G \\ 0 & M \end{pmatrix}$$

on

$$H = N(T_1-\lambda) \oplus \overline{R(T_1-\lambda)} \oplus N(T^{*k}),$$

where

$$M = \begin{pmatrix} T_{12}-\lambda & T_{22} \\ 0 & T_3-\lambda \end{pmatrix}.$$

Now, we claim that M is an invertible operator on $\overline{R(T_1-\lambda)} \oplus N(T^{*k})$. First we verify that $T_{12}-\lambda$ is invertible. If not, then λ will be an isolated point in $\sigma(T_{12})$. Since T_{12} is $*$-n-paranormal, λ is an eigenvalue of T_{12} and thus $T_{12}x = \lambda x$ for some non-zero vector x in $\overline{R(T_1-\lambda)}$. On the other hand, $T_1 x = T_{12}x$ implying x is in $N(T_1-\lambda)$. Hence x must be a zero vector. This contradiction shows that $T_{12}-\lambda$ is invertible. Since $T_3-\lambda$ is also invertible, it follows that M is invertible. It is easy to show that $R(E) = N(T-\lambda)$.

We prove $N(T-\lambda) = N(T-\lambda)^*$. Since $N(T-\lambda) \subseteq N(T-\lambda)^*$ by Corollary 2.6, it suffices to show that $N(T-\lambda)^* \subseteq N(T-\lambda)$. Let $x = \binom{x_1}{x_2} \in N(T-\lambda)^*$. Then

$$\begin{aligned} 0 = (T-\lambda)^* x &= \begin{pmatrix} (T_1-\lambda)^* & 0 \\ T_2^* & (T_3-\lambda)^* \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= \begin{pmatrix} (T_1-\lambda)^* x_1 \\ T_2^* x_1 + (T_3-\lambda)^* x_2 \end{pmatrix}. \end{aligned}$$

Hence $x_1 \in N(T_1-\lambda)^* = N(T_1-\lambda)$. Since

$$(T-\lambda)\begin{pmatrix} x_1 \\ 0 \end{pmatrix} = \begin{pmatrix} T_1-\lambda & T_2 \\ 0 & T_3-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ 0 \end{pmatrix} = \begin{pmatrix} (T_1-\lambda)x_1 \\ 0 \end{pmatrix} = 0,$$

we have

$$0 = (T-\lambda)^*\begin{pmatrix} x_1 \\ 0 \end{pmatrix} = \begin{pmatrix} (T_1-\lambda)^* x_1 \\ T_2^* x_1 \end{pmatrix}$$

by Corollary 2.6, hence $T_2^* x_1 = 0$. This implies $(T_3-\lambda)^* x_2 = 0$ and $x_2 = 0$ because T_3 is nilpotent. Thus

$$x = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \in N(T_1-\lambda) \oplus \{0\} = N(T-\lambda).$$

Next, we show that E is self-adjoint. Since E is the Riesz idempotent of T with respect to λ and T is a k-quasi-$*$-n-paranormal operator, $R(E) = N(T-\lambda)$ and $N(E) = R(T-\lambda)$. Since $N(T-\lambda) \subseteq N(T-\lambda)^*$ by Corollary 2.6, then $N(T-\lambda)$ and $R(T-\lambda)$ are orthogonal. Hence $R(E)^\perp = N(E)$, and so E is self-adjoint.

If $\lambda = 0$, $\sigma(T|_{R(E)}) = 0$, then $(T|_{R(E)})^{k+1} = 0$ by Lemma 3.1, hence $R(E) = N(T^{k+1})$. $\square$

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