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ARROWHEAD OPERATORS ON A HILBERT SPACE

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Abstract. The arrowhead matrices define a class of one-term Sylvester matrix (OTSM) operators on a finite-dimensional Hilbert space through an elementary UDL factorization. It enables us to consider the infinite invertible arrowhead matrices UDL factored properly for introducing, under suitable conditions, the arrowhead operators and their associated class of OTSM operators on an infinite-dimensional Hilbert space. Properties regarding convergence, inertia, inverses, and spectra are also considered.

1. Introduction

The arrowhead matrices of finite order are of interest in the symmetric eigenvalue problem [14, 18], and some literature is dedicated to compute their eigenpairs; see e.g. [15]. In addition, there exists a current and increasing use of the arrowhead matrices of large order, with applications in networks, wireless communication, and the world wide web; see [12] and the references therein. Infinite arrowhead matrices arise also in physical applications [3, 7], although a further analysis remains. For details on infinite matrices see [6, 17]. Furthermore, it is of interest the introduction of explicit inertia criteria with respect the half planes, based on the well-known theory [4, 5, 13, 16]. In particular, for linear operators on Hilbert space with potential applications.

An arrowhead matrix can be defined as a particular diagonal plus rank-two matrix. There are two similar kinds, up-arrowhead matrices and down-arrowhead matrices. In the finite case, we obtain one from the other with an elementary similarity transformation. The up-arrowhead matrices can be defined adequately in the infinite case. Although asymptotic analysis for large order arrowhead matrices is possible, an infinite down-arrowhead matrix cannot be defined. Hence, we manage finite arrowhead matrices of the form

$$A=\left(\begin{array}{c|cccc} b_0 & c_1 & c_2 & \cdots & c_{n-1} \\ \hline a_1 & b_1 & 0 & \ddots & \vdots \\ a_2 & 0 & b_2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ a_{n-1} & \cdots & 0 & 0 & b_{n-1} \end{array}\right)=\left(\begin{array}{c|c} b_0 & c' \\ \hline a & D_b \end{array}\right), \tag{1}$$

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with $D_b = \operatorname{diag}(b_1, b_2, \ldots, b_{n-1})$, and c' denotes the transpose of c.

An $n \times n$ matrix A is said to be arrowhead without loss of generality if A is of the form (1) and bordered irreducible, i.e. D_b is nonsingular, with distinct nonzero diagonal entries, and $a_i \neq 0$, $c_i \neq 0$, for $i = 1, 2, \ldots, n-1$. Bordered irreducible arrowhead matrices ordered adequately have been studied largely in the available literature. The ordering is by convenience of the proposed methods, but in general it is not necessary; see e.g. [12, 15] and the references therein. In fact the symmetric eigenvalue problem [14, 18] has been focused on bordered irreducible arrowhead matrices.

The aim is to begin with a class of OTSM operators [10] on a finite-dimensional Hilbert space defined through a direct triangular factorization $A = UDL$ of the arrowhead matrices (1). The matrices U and L are upper and lower triangular, respectively, with unit diagonal. The matrix $D = \operatorname{diag}(u, D_b)$ is diagonal, with $u \neq 0$ to be defined. Well-known and new inertial and spectral criteria for these operators are given. In order to introduce the arrowhead operators and their related class of OTSM operators on an infinite-dimensional Hilbert space, we consider infinite invertible arrowhead matrices UDL factored properly so that associativity of multiplication of the involved matrices succeeds.

The operators defined from finite arrowhead matrices have nice properties. Most of these properties are preserved even for the arrowhead operators and their related OTSM operators on an infinite-dimensional Hilbert space. We shall consider some of their properties, e.g. Cauchy's interlacing property for eigenvalues, inverse decomposition, and inertia.

The outline is as follows, in Section 2 a UDL factorization for finite arrowhead matrices is handled, focusing on the nonsingular ones. Their related OTSM operators are defined and some results regarding inversion, inertial, and spectral properties are detailed. Further analysis on the hermitian case is done. The infinite invertible arrowhead matrices UDL factored properly are then introduced in Section 3. The arrowhead operators and their associated OTSM operators on an infinite-dimensional Hilbert space are introduced in Section 4. Properties regarding convergence, inertia, and spectra are also considered.

2. Finite arrowhead matrices and OTSM operators

Every arrowhead matrix $A \in \mathbb{C}^{n\times n}$ has a triangular factorization $A = UDL$. For nonsingular arrowhead matrices (1) such a factorization is unique, with D nonsingular.

LEMMA 1. *Given a nonsingular arrowhead matrix $A \in \mathbb{C}^{n\times n}$ as in (1):*

a). *A has a unique UDL triangular factorization of the form,*

$$A = \left(\begin{array}{c|c} 1 & (D_b^{-1}c)' \\ \hline 0 & I_{n-1} \end{array}\right) \left(\begin{array}{c|c} u & 0 \\ \hline 0 & D_b \end{array}\right) \left(\begin{array}{c|c} 1 & 0 \\ \hline D_b^{-1}a & I_{n-1} \end{array}\right), \tag{2}$$

with $u = b_0 - \sum_{i=1}^{n-1} \frac{a_i c_i}{b_i} \neq 0$, and I_{n-1} the identity matrix of order $n-1$.

b). *A^{-1} has a unique $L^{-1}D^{-1}U^{-1}$ triangular factorization of the form,*

$$A^{-1} = \left(\begin{array}{c|c} 1 & 0 \\ \hline -D_b^{-1}a & I_{n-1} \end{array}\right)\left(\begin{array}{c|c} \frac{1}{u} & 0 \\ \hline 0 & D_b^{-1} \end{array}\right)\left(\begin{array}{c|c} 1 & -\left(D_b^{-1}c\right)' \\ \hline 0 & I_{n-1} \end{array}\right). \tag{3}$$

c). *A^{-1} is decomposable as a diagonal plus a rank-one matrix,*

$$A^{-1} = diag\left(0, D_b^{-1}\right) + \frac{1}{u}\begin{pmatrix} 1 \\ -D_b^{-1}a \end{pmatrix}\left(1 \; -\left(D_b^{-1}c\right)'\right). \tag{4}$$

Proof. a). Straightforwardly from (1) and (2), with A and D_b nonsingular matrices, $A = UDL$ if and only if $u = b_0 - \sum_{i=1}^{n-1} \frac{a_i c_i}{b_i} \neq 0$.

The proofs for b). and c). are also trivial. $\square$

From Lemma 1, the UDL factorization of the nonsingular arrowhead matrices defines a particular class of OTSM operators [10] on the set of all $n \times n$ diagonal nonsingular matrices $\mathrm{Diag}\left(\mathbb{C}^{n\times n}\right)$. It is of interest for linking inertial and spectral properties of nonsingular arrowhead matrices with that of diagonal matrices belong to $\mathrm{Diag}\left(\mathbb{C}^{n\times n}\right)$.

DEFINITION 1. The class of the matrix operators

$$\mathscr{T} : \mathrm{Diag}\left(\mathbb{C}^{n\times n}\right) \to \mathbb{C}^{n\times n}; \quad T\left(U,L\right)\left[D\right] := UDL = A,$$

is a class of OTSM operators defined from the UDL factorization (2) of the nonsingular arrowhead matrix A. The argument matrix D, and the matrices L and U are as given in Lemma 1.

Notice as the definition of the operator $T \in \mathscr{T}$ is dual because it also defines an arrowhead operator on $T : \mathbb{C}^n \to \mathbb{C}^n$. Therefore in the finite-dimensional case, we can study such operators from the associated nonsingular arrowhead matrix A as usual.

PROPOSITION 1. *Given a nonsingular arrowhead matrix $A \in \mathbb{C}^{n\times n}$:*

a). *Its right eigenvectors are of the form $x_{\lambda_i} = \xi\left(1 \; -(D_{b-\lambda_i}^{-1}a)'\right)'$, with $\xi \neq 0$ an arbitrary scalar, and their distinct nonzero eigenvalues $\{\lambda_i\}$, $i = 1,2,\ldots,k \leqslant n$, are the roots of the complex-valued function*

$$f(\lambda) : \mathscr{D} \subset \mathbb{C} \to \mathbb{C}\,; \quad f(\lambda) = b_0 - \lambda - \sum_{i=1}^{n-1} \frac{c_i a_i}{b_i - \lambda}. \tag{5}$$

b). *A is simple if and only if $y'_{\lambda_i} x_{\lambda_i} \neq 0$, for $i = 1,2,\ldots,n$, where the vector $y'_{\lambda_i} = \eta\left(1 \; -(D_{b-\lambda_i}^{-1}c)'\right)$, with $\eta \neq 0$ arbitrary, is a left eigenvector of A associated to the simple nonzero eigenvalue λ_i. Therefore, its spectral decomposition is*

$$A = \sum_{i=1}^{n} \lambda_i x_{\lambda_i} y'_{\lambda_i} = \sum_{i=1}^{n} \lambda_i \begin{pmatrix} 1 \\ -D_{b-\lambda_i}^{-1}a \end{pmatrix}\left(\eta_i \; -\eta_i\left(D_{b-\lambda_i}^{-1}c\right)'\right), \tag{6}$$

with the scalars η_i chosen so that $y'_{\lambda_i} x_{\lambda_j} = \delta_{ij}$, Kronecker's delta, for $i,j = 1,2,\ldots,n$.

Proof. a). An eigenvector x_λ of A belongs to the nontrivial kernel of $A - \lambda I_n$. Since A is a nonsingular arrowhead matrix (1), bordered and irreducible, the form of the eigenvectors and the function $f(\lambda)$, related with the well-known secular equation, are obtained readily.

The eigenvalues of A satisfy $\lambda_i \neq b_j$, $j = 1,\dots,n-1$, so that $f(\lambda)$ from (5) has not indeterminacies. Notice the trivial kernels of $A - b_j I_n$, with independence of the value b_0. The special case $\lambda_i = b_0 \neq 0$, and $b_0 \neq b_j$, implies $u = b_0 \neq 0$, and $x_{\lambda_i} = \xi \left(1 - (D^{-1}_{b-b_0} a)'\right)'$.

The function $f(\lambda) = P(\lambda)/\prod_{i=1}^{n-1}(b_i - \lambda)$ has the same roots than the characteristic polynomial $P(\lambda)$ of A. Thus if $x_{\lambda_i} = \xi \left(1 - (D^{-1}_{b-\lambda_i} a)'\right)'$, with $\lambda_i \neq 0$ a root of $f(\lambda)$, then x_{λ_i} is an eigenvector associated to λ_i.

b). Although $P(\lambda)$ and $f(\lambda)$ have the same roots, as a rule, their derivatives $P'(\lambda)$ and $f'(\lambda)$ have distinct roots. However, if the root λ_i is not simple, λ_i is also a root of $P'(\lambda)$ and $f'(\lambda)$, i.e. $P'(\lambda_i) = f'(\lambda_i) = 0$. Hence, the derivative $f'(\lambda)$ evaluated at $\lambda = \lambda_i$, a possible eigenvalue of A with algebraic multiplicity greater than 1, is null. The eigenvalues of A are into the domain of analyticity of $f(\lambda)$, where its derivative is well defined,

$$f'(\lambda) = -1 - \sum_{i=1}^{n-1} \frac{c_i a_i}{(b_i - \lambda)^2} = 0, \text{ for } \lambda = \lambda_i, \text{ if and only if } y'_{\lambda_i} x_{\lambda_i} = 0,$$

the vector $y'_{\lambda_i} = \eta \left(1 - (D^{-1}_{b-\lambda_i} c)'\right)$ is a left eigenvector related with λ_i. Notice y_{λ_i} belonging to the kernel of $A' - \lambda_i I_n$.

Under the given assumptions A is simple, i.e. with n distinct nonzero eigenvalues, and hence semisimple (similar to a diagonal matrix). The spectral decomposition (6) of A is straightforward from the spectral theorem for finite simple matrices; see e.g. page 154 (Theorem 3) from [13]. □

2.1. Hermitian arrowhead matrices

The hermitian arrowhead matrices, satisfying $A^* = A$, with A^* the conjugate transpose of A, are of main use in applications. Since the hermitian arrowhead matrices also satisfy factorization (2), their related OTSM operators are $T(L)[D] := L^* D L = A$.

PROPOSITION 2. *Given a nonsingular hermitian arrowhead matrix $A \in \mathbb{C}^{n\times n}$, where $\mathbb{C}^n$ is the unitary space with inner product $\langle \cdot, \cdot \rangle$, and the diagonal entries of D_b ordered without loss of generality in decreasing order:*

a). *A is simple and between two nearer singularities of $f|_{\mathbb{R}}(\lambda)$, the restriction of $f(\lambda)$ to $\mathbb{R}$, there is one and only one real eigenvalue.*

b). *In addition, assume that $b_0 > b_1$. There exist real numbers θ_i $(i = 1,2,\dots,n)$, so that the eigenvalues of A satisfy:*

$$\lambda_i = b_{i-1} + \theta_i \|a\|, \textit{ with } |\theta_i| \leqslant 1, \textit{ and } \sum_{j=2}^{n} \theta_j = -\theta_1.$$

Proof. a). The ordering is obtained using an elementary similarity transformation $A = P\tilde{A}P^{-1}$ on a nonsingular hermitian arrowhead matrix $\tilde{A}$. The eigenvalues of A are real and nonzero. Also, $\lambda \neq b_j$, for $j = 1,\ldots,n-1$. Therefore, we can use the rational function $f|_{\mathbb{R}}(\lambda)$, the restriction of $f(\lambda)$ to $\mathbb{R}$, by observing its real range,

$$f|_{\mathbb{R}}(\lambda) : \mathscr{D}' \subset \mathbb{R} \to \mathbb{R}\,; \quad f|_{\mathbb{R}}(\lambda) = b_0 - \lambda - \sum_{i=1}^{n-1} \frac{|a_i|^2}{b_i - \lambda}.$$

Thus into the domain of analyticity of $f|_{\mathbb{R}}(\lambda)$, its derivative satisfies

$$f|'_{\mathbb{R}}(\lambda) = -1 - \sum_{i=1}^{n-1} \frac{|a_i|^2}{(b_i - \lambda)^2} < 0.$$

From Proposition 1, A is simple. The b_i, $i = 1,\ldots,n-1$, are singularities of $f|_{\mathbb{R}}(\lambda)$. Since $f|'_{\mathbb{R}}(\lambda) < 0$, taking arbitrary b_{k-1} and b_k, $k = 2,\ldots,n$, into the open interval (b_{k-1}, b_k) there is one and only one root of $f|_{\mathbb{R}}(\lambda)$.

b). The hermitian arrowhead matrix is decomposed by convenience in the form

$$A = D_{b_0} + F_2 = \operatorname{diag}(b_0, D_b) + \left(\begin{array}{c|c} 0 & a^* \\ \hline a & O_{n-1} \end{array}\right), \tag{7}$$

with $b_j \in \mathbb{R},\ j = 0,1,\ldots,n-1$. The matrix O_{n-1} is the zero matrix of order $n-1$. The nonzero eigenvalues of F_2 are simples, $+\|a\|$, $-\|a\|$. Since the matrices involved are hermitian and $b_0 > b_1 > \ldots > b_{n-1}$, using Courant-Fischer's theorem for the characterization of the eigenvalues $\{\lambda_i\}$ of A, with decreasing ordering, we obtain for $i = 1,2,\ldots,n$,

$$b_{i-1} - \|a\| \leqslant \lambda_i \leqslant b_{i-1} + \|a\| \Rightarrow \lambda_i = b_{i-1} + \theta_i \|a\|,$$

with $|\theta_i| \leqslant 1$. Since $\operatorname{trace}\left(A - D_{b_0}\right) = 0$, we have $\sum_{i=1}^{n} \theta_i = 0$.

As a simple consequence of Proposition 2 a)., Cauchy's interlacing property for the eigenvalues of A also holds,

$$\lambda_1 > b_1 > \lambda_2 > b_2 > \ldots > b_{n-2} > \lambda_{n-1} > b_{n-1} > \lambda_n.$$

Therefore, $-1 \leqslant \theta_i < 0$, $i = 2,3,\ldots,n$, $0 < \theta_1 \leqslant 1$, and $\sum_{i=2}^{n} \theta_i = -\theta_1$. $\square$

Analogous results as Proposition 2 b). can be obtained for distinct values of b_0 and orderings of the diagonal entries of D_b.

2.2. Explicit inertia criteria for arrowhead matrices

For every nonsingular matrix A, the Main Inertia Theorem [16] assures the existence of a hermitian matrix H with the same inertia than A, with respect the half planes. The interesting case for arrowhead matrices is when $H = D$, from (2), the matrix argument of $T(U,L)[D]$. Obviously, if an OTSM operator preserves inertia, then it preserves stability. If A is hermitian, it is well known that A and D have the same inertia.

COROLLARY 1. *Given a nonsingular arrowhead matrix* $A = UDL \in \mathbb{C}^{n\times n}$, *factored as in* (2), *where* $\mathbb{C}^n$ *is the unitary space. Assuming for the OTSM operator* $T(U,L)[D]$, *a real diagonal matrix argument* $D = diag(u, D_b)$:

a). *If* A *is hermitian, then the OTSM operator* $T(L)[D]$ *preserves inertia.*

b). *If* A *is not hermitian and the following condition on the matrix entries is satisfied,*

$$uRe(b_0) - \frac{1}{4}\sum_{i=1}^{n-1} \frac{|ua_i + b_i\bar{c}_i|^2}{b_i^2} > 0, \tag{8}$$

with $Re(b_0)$ *the real part of* b_0, *then the OTSM operator* $T(U,L)[D]$ *preserves inertia,* $In(A) = In(D)$.

Proof. a). By the given assumptions D is trivially hermitian. Since $A = L^*DL$, the matrices A and D are congruent and Sylvester's law of inertia holds; see page 187 (Theorem 3) from [13]. Thus if D is stable, negative definite, $\mathrm{In}(D) = (0,n,0) = \mathrm{In}(A)$, and the matrix A is also stable.

b). If D is solution of Lyapunov's equation $AD + DA^* = W$, with W positive definite and invertible, the Main Inertia Theorem [16] assures the same inertia $(\pi, \nu, 0)$ for the matrices A and D,

$$W = AD + DA^* = \left(\begin{array}{c|c} 2u\mathrm{Re}(b_0) & (ua + D_b\bar{c})^* \\ ua + D_b\bar{c} & 2D_b^2 \end{array}\right), \tag{9}$$

where $\bar{c}$ denotes the conjugate of c. The matrix W is arrowhead and hermitian, hence its eigenvalues are real and Cauchy's interlacing property holds. Thus for Proposition 2, W is positive and invertible if and only if the argument matrix $D_W = \mathrm{diag}\left(u_W, D_b^2\right)$ of the triangular factorization (2) of W is positive. Since the remaining diagonal entries of D_W are positive, we only may check the condition $u_W > 0$, it gives condition (8). $\square$

3. Infinite arrowhead matrices with a UDL factorization

Given an infinite-dimensional matrix $A = (\alpha_{ik})$, the matrix $B = (\beta_{ik})$ is a classical inverse of A if both matrices A and B satisfy $AB = BA = I$, with I the identity matrix. The matrix A may have (or may not have) classical inverse. Alternatively, A may have two classical inverses, and then infinitely many classical inverses; see e.g. [6, 17, 1].

We are interested in the infinite invertible arrowhead matrices (1) defined properly, with the sequences a and c belonging to ℓ_2, the space of the complex square summable sequences, and D_b an infinite invertible diagonal matrix. Recall that the arrowhead matrices have been defined here as bordered and irreducible, i.e. a_i, b_i, and c_i, are nonzero, $i = 1, 2, \ldots$.

No every infinite-dimensional arrowhead matrix has a UDL factorization (2) preserving associativity of multiplication of the matrices involved [6]. In order to extend the inverse decompositions (3) and (4) to the infinite-dimensional case, the UDL factorization for the infinite arrowhead matrices becomes essential. Further conditions on the entries of A may be introduced.

LEMMA 2. *Given an infinite arrowhead matrix A of the form* (1)*, with a, $D_b^{-1}a = a/b$, c, and $D_b^{-1}c = c/b$, sequences belonging to $\mathbb{C}^\infty$:*

a). *The matrix A can be factored in the form* (2) *preserving associativity of matrix product, with $u = b_0 - \sum_{k=1}^{\infty} \frac{a_k c_k}{b_k} \in \mathbb{C}$.*

b). *If A is invertible, its classical inverse A^{-1} can be factored in the form* (3)*, well defined for matrix product, and*

$$u = b_0 - \sum_{k=1}^{\infty} \frac{a_k c_k}{b_k} \neq 0. \tag{10}$$

c). *If A is invertible, its classical inverse can be decomposed as a determined diagonal plus rank-one matrix as given in* (4).

Proof. a). The result is checked directly using the assumptions on the involved sequences and the associativity property of the product of the infinite triangular matrices and the infinite diagonal matrices involved in factorization (2).

b). If a matrix A with a UDL factorization (2) is also invertible, then $u \neq 0$, and such a triangular factorization is unique. Using associativity of multiplication of the infinite matrices involved in (3),

$$AA^{-1} = (UD)(LL^{-1})(D^{-1}U^{-1}) = U(DD^{-1})U^{-1} = UU^{-1} = I.$$

In a similar way, $A^{-1}A = I$. Therefore, A^{-1} is a classical inverse of A. The uniqueness of the factorization for the classical inverse A^{-1} is a consequence of the uniqueness of the inverses of the diagonal and the unit triangular matrices involved in the UDL factorization (2) of A.

c). Taking the product $(L^{-1}D^{-1})U^{-1} = L^{-1}(D^{-1}U^{-1})$, a decomposition for the classical inverse, analogous that of the finite case, follows. $\square$

EXAMPLE 1. First, we illustrate with an infinite arrowhead matrix A_1 with $b_0 = 1$, $D_b = \mathrm{diag}\left(\frac{2}{3}, \frac{3}{4}, \ldots, \frac{n+1}{n+2}, \ldots\right)$, and the sequences a and c belonging to $\mathbb{C}^\infty$, $a_j = 1/(j+2)$, $c_j = 1/(j+1)$,

$$A_1 = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots \\ \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 & \cdots \\ \frac{1}{4} & 0 & \frac{3}{4} & 0 & 0 & \cdots \\ \frac{1}{5} & 0 & 0 & \frac{4}{5} & 0 & \cdots \\ \frac{1}{6} & 0 & 0 & 0 & \frac{5}{6} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Since the entries of $D_b^{-1} = \mathrm{diag}\left(\frac{3}{2}, \frac{4}{3}, \ldots, \frac{n+2}{n+1}, \ldots\right)$ are uniformly bounded, and $D_b^{-1}a = \left(\frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{(n+1)}, \ldots\right)'$, $D_b^{-1}c = \left(\frac{3}{4}, \frac{4}{9}, \ldots, \frac{n+2}{(n+1)^2}, \ldots\right)'$, belonging to $\mathbb{C}^\infty$, A_1 can be

UDL factored properly. The entry u from the factorization (2) is obtained using the series of Basel's problem,

$$u = 1 - \sum_{k=1}^{\infty} \frac{1}{(k+1)^2} = 2 - \frac{\pi^2}{6} > 0.$$

Hence by (10), A_1 is invertible. The classical inverse A_1^{-1} decomposed as in (4) is obtained straightforwardly.

EXAMPLE 2. We manage now a symmetric arrowhead matrix A_2 with its main diagonal densely defined in the closed interval $\left[\frac{1}{2}, \frac{3}{2}\right]$. It is provided by reiterating a bisection method on such an interval. Thus, for $j = 1, 2, \ldots$, the $b_j = \frac{2^m + 2k + 1}{2^{m+1}}$, with $j = 2^m + k$, $2^m \leqslant j < 2^{m+1}$, and $0 \leqslant k < 2^m$. Taking by convenience $b_0 = 3/2$, the entries on the diagonal of A_2 are distinct, and the diagonal is dense in $\left[\frac{1}{2}, \frac{3}{2}\right]$. Also, $a = c \in \mathbb{C}^\infty$, with $a_j = 1/(j+1)$,

$$A_2 = \begin{pmatrix} \frac{3}{2} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots \\ \frac{1}{2} & 1 & 0 & 0 & 0 & \cdots \\ \frac{1}{3} & 0 & \frac{3}{4} & 0 & 0 & \cdots \\ \frac{1}{4} & 0 & 0 & \frac{5}{4} & 0 & \cdots \\ \frac{1}{5} & 0 & 0 & 0 & \frac{5}{8} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Since $D_b^{-1} a = \left(\frac{1}{2}, \frac{4}{9}, \ldots, \frac{a_n}{b_n}, \ldots\right)' \in \mathbb{C}^\infty$, A_2 can be UDL factored properly. Notice the uniformly bounded entries of $D_b^{-1} = \operatorname{diag}\left(1, \frac{4}{3}, \ldots, \frac{1}{b_n}, \ldots\right)$, i.e. $2/3 < 1/b_j < 2$, and $a \in \mathbb{C}^\infty$. The condition (10) ($u \neq 0$) is also satisfied,

$$u = \frac{3}{2} - \sum_{k=1}^{\infty} \frac{1}{b_k} \frac{1}{(k+1)^2}, \text{ but } \frac{2}{3}\left(\frac{\pi^2-6}{6}\right) < \sum_{k=1}^{\infty} \frac{1}{b_k} \frac{1}{(k+1)^2} < 2\left(\frac{\pi^2-6}{6}\right).$$

Therefore, $0 < \frac{7}{2} - \frac{\pi^2}{3} < u < \frac{13}{6} - \frac{\pi^2}{9}$, and A_2 is invertible. The classical inverse A_2^{-1} decomposed as in (4) is

$$A_2^{-1} = \operatorname{diag}\left(0, D_b^{-1}\right) + \frac{1}{u} x_0 x_0', \text{ with } \quad x_0 = \left(1 \; - \left(D_b^{-1} a\right)'\right)' \in \mathbb{C}^\infty.$$

4. The arrowhead operators and their related OTSM operators.

Let $\ell_2 = \ell_2(\mathbb{N})$ be the infinite-dimensional Hilbert space of the complex square summable sequences $x = (\xi_1, \xi_2, \ldots)' = (\xi_i)$, with the usual inner product $\langle \cdot, \cdot \rangle$. We consider the operators defined from arrowhead matrices. In particular, we ask about (maximal) bounded linear operators $T_A = T : \ell_2 \to \ell_2$, $T \in \mathrm{B}(\ell_2)$, defined from the infinite invertible arrowhead matrices (1), not necessarily factored as UDL, given in Section 3,

$$y = (\eta_i) = Tx \in \ell_2, \qquad \text{so that,} \qquad \eta_j = \sum_{k=1}^{\infty} \alpha_{jk}\xi_k, \quad j = 1,2,\dots$$

As usual, the arrowhead matrices are referred to the standard orthonormal basis $\{e_i\}$ of ℓ_2, unless otherwise stated.

Some restrictions may be introduced. As observed in Section 3, it is necessary that the sequences $a = (a_i)$ and $c = (c_i)$ from (1) be square summable sequences. In addition, for infinite arrowhead matrices under the conditions given in Lemma 2, the sequences $a/b = (a_i/b_i)$ and $c/b = (c_i/b_i)$ should be sequences belonging to ℓ_2.

For $T \in \mathrm{B}(\ell_2)$, defined from an arrowhead matrix A factored as in (2), $\|T\| \leqslant K_U \cdot K_D \cdot K_L = K \in \mathbb{R}$, where $K_D = \sup\{|u|, |b_i|, \mid i \in \mathbb{Z}^+\} < \infty$, and the first row of U and the first column of L belonging to ℓ_2. Therefore, $\|Tx\| = \|y\| \leqslant K\|x\|$, and the operator T is defined on the whole of ℓ_2. Here $\|T\|$ is the norm of the operator T, $\|T\| = \sup\left\{\frac{\|Tx\|}{\|x\|} \mid x \in \mathscr{D}(T)\ , x \neq 0_{\ell_2}\right\}$, where 0_{ℓ_2} is the trivial sequence, and $\|x\| = \sqrt{\langle x, x\rangle}$.

Related with arrowhead operators, we can define a class of OTSM operators on the set of invertible diagonal operator on ℓ_2, $\mathrm{Diag}(\ell_2) \subset \mathrm{B}(\ell_2)$.

DEFINITION 2. The class of OTSM operators

$$\mathscr{T} : \mathrm{Diag}(\ell_2) \to \mathrm{B}(\ell_2); \quad T = T(U,L)[D] := UDL,$$

is the class related with the operators arising from arrowhead matrices UDL factored properly as in (2).

4.1. Compact arrowhead operators

To handle with the arrowhead operators, we recover the infinite arrowhead matrix A decomposed by convenience in the form

$$A = (\alpha_{ik}) = D_{b_0} + F_2 = \mathrm{diag}(b_0, (b_i)) + \left(\begin{array}{c|c} 0 & c' \\ \hline a & O \end{array}\right), \tag{11}$$

where the sequence $b = (b_i)$ is not necessarily neither square summable nor convergent. The matrices D_{b_0}, O, and F_2 are the matrix representations of a diagonal operator, the zero operator, and a finite rank-two operator, respectively. We can introduce a wide class of arrowhead operators on ℓ_2 defined from matrices of the form (11), not only those defined from the UDL factorization (2). However, such UDL factorization is compulsory for managing their related OTSM operators.

A sufficient condition for defining an operator $T \in \mathrm{B}(\ell_2)$ from a matrix A is $\sum_i \sum_j |\alpha_{ij}|^2 < \infty$; see e.g. [8, 11]. Applying it to an arrowhead operator, we state the following, PROPOSITION 3. *Given a linear operator T defined from an infinite arrowhead matrix $A = (\alpha_{ik})$ decomposed as in (11), with (a_i), (b_i), (c_i), sequences belong to ℓ_2:*

a). *The arrowhead operator T is bounded, $T \in B(\ell_2)$.*

b). *If the sequence $a/b = (a_i/b_i) \in \ell_2$, and the condition (10) is satisfied, then $T \in B(\ell_2)$ is invertible, but its related OTSM operator cannot be well defined.*

c). *If in addition to the assumptions given in b)., the sequence $c/b = (c_i/b_i) \in \ell_2$, so that the matrix A, which induces T, can be UDL factored properly. Then, its related OTSM operator can be well defined, and the inverse operator T^{-1} is defined from a matrix decomposed as in (4).*

Proof. a). Trivially, $\sum_i \sum_j |\alpha_{ij}|^2 = |b_0|^2 + \sum_{j=1}^{\infty} \left(|a_j|^2 + |b_j|^2 + |c_j|^2\right) < \infty$, and the arrowhead matrix A from (11) defines a bounded linear operator T.

b). Since $A = D_{b_0} + F_2$ is decomposed in the form (11) and $(a_i/b_i) \in \ell_2$, the next decomposition is well defined,

$$A = \operatorname{diag}(1,(b_i))\left[D_{b_0}^{\diamond} + F_2^{\diamond}\right] = \operatorname{diag}(1,(b_i))\left[\left(\begin{array}{c|c} b_0 & 0'_{\ell_2} \\ \hline 0_{\ell_2} & I \end{array}\right) + \left(\begin{array}{c|c} 0 & c' \\ \hline D_b^{-1}a & O \end{array}\right)\right].$$

Since the operator $\operatorname{diag}(1,(b_i))$ is diagonal invertible, it is sufficient to check the invertibility of the operator defined from $\left[D_{b_0}^{\diamond} + F_2^{\diamond}\right]$ by finding its kernel, $\operatorname{Ker}(\left[D_{b_0}^{\diamond} + F_2^{\diamond}\right])$, where I is the matrix representation of the identity operator,

$$\operatorname{Ker}(\left[D_{b_0}^{\diamond} + F_2^{\diamond}\right]) = \left\{x \in \ell_2 \mid \left[D_{b_0}^{\diamond} + F_2^{\diamond}\right]x = \left(\begin{array}{c|c} b_0 & c' \\ \hline D_b^{-1}a & I \end{array}\right)x = 0_{\ell_2}\right\}.$$

Hence, $b_0\xi_1 + \langle(\xi_j)_{j=2}^{\infty}, \bar{c}\rangle = 0$, and $\xi_1 D_b^{-1}a + (\xi_j)_{j=2}^{\infty} = 0_{\ell_2}$. We obtain $(\xi_j)_{j=2}^{\infty} = -\xi_1 D_b^{-1}a = -\xi_1 a/b$, and $b_0\xi_1 - \xi_1\langle a/b, \bar{c}\rangle = 0$.

A nontrivial sequence x belongs to the kernel of $\left[D_{b_0}^{\diamond} + F_2^{\diamond}\right]$ if and only if the condition $\langle a/b, \bar{c}\rangle = b_0$ is satisfied. Such a x is of the form $x = \xi(1 - a/b) \in \ell_2$, with $\xi = \xi_1$ an arbitrary scalar. However, such a condition is in contradiction with the assumption (10) unless $\xi = 0$, i.e. $x = 0_{\ell_2}$.

The factorization $A = \operatorname{diag}(1,(b_i))\left[D_{b_0}^{\diamond} + F_2^{\diamond}\right]$ allows us to consider the inverse operator T^{-1}. Although $\left[D_{b_0}^{\diamond} + F_2^{\diamond}\right]$ is not an arrowhead matrix as defined here, it can be UDL factored trivially. Thus $A = D_1 UDL$, where $D_1 = \operatorname{diag}(1,(b_i))$. Hence $A^{-1} = (L^{-1}D^{-1}U^{-1})D_1^{-1}$ is a factorization of the matrix that induces the inverse operator T^{-1}. There is not difficulty with the matrix product $A^{-1}A = I$. However, taking the matrix product AA^{-1},

$$\begin{aligned} AA^{-1} &= (D_1UD)(LL^{-1})(D^{-1}U^{-1})D_1^{-1} = (D_1U(DD^{-1})U^{-1})D_1^{-1} \\ &= D_1(UU^{-1})D_1^{-1} = D_1D_1^{-1} = I. \end{aligned}$$

The product $U^{-1}D_1^{-1}$ should be avoided because it cannot be well defined. Since the difficulties with the matrix U, the UDL factorization of A becomes unsuitable. Its related OTSM operator cannot be well defined. Although T^{-1} exists, it cannot be well defined from the decompositions (3) and (4).

c). If in addition to the assumptions given in b)., the sequence $\left(D_b^{-1}c\right) = c/b \in \ell_2$, then the matrix A from (11) can be UDL factored properly, and its related OTSM operator can be well defined. Also, the matrix decompositions (3) and (4) define the operator T^{-1}. $\square$

LEMMA 3. *An arrowhead operator $T \in B(\ell_2)$, defined from matrix* (11) *is compact if and only if* $\lim_{j\to\infty} b_j = 0$.

Proof. Since $T = D_{b_0} + F_2$, with F_2 a finite rank-two operator, T is compact if and only if D_{b_0} is compact. But the diagonal operator D_{b_0}, with $b = (b_i) \in \mathbb{C}^\infty$ having nonzero components, is compact if and only if $\lim_{j\to\infty} b_j = 0$; see e.g. [2, 11].

Another simple but constructive proof exploits the uniform convergence of the sequence $\left(T_{(n)}\right)$ of (compact) finite rank operators defined, from the nth principal sections of A, on ℓ_2. Indeed $\|T_{(n)} - T\| \to 0$, taking into account that a and c belong to ℓ_2 and the matrix representation of $T_{(n)} - T$,

$$A_{(n)} - A = \left(\begin{array}{c|c} O_n & c^{(n)\prime} \\ \hline a^{(n)} & D_b^{(n)} \end{array}\right) \underset{n\to\infty}{\longrightarrow} O \quad \Leftrightarrow \quad \lim_{j\to\infty} |b_j| = 0,$$

where $a^{(n)}$, $c^{(n)}$, and $D_b^{(n)}$, are a, c, and D_b, shifted $n-1$ positions forward. $\square$

LEMMA 4. *Given an arrowhead operator $T \in B(\ell_2)$:*

a). *If T satisfies the assumptions of Proposition* 3 *b). or c)., then T is an invertible compact operator and its inverse operator T^{-1} is unbounded.*

b). *If T satisfies the assumptions of Proposition* 3 *c)., then its related OTSM operator $T(U,L)[D]$ preserves compactness.*

Proof. a). It is not difficult to see that the operator T defined in Proposition 3 satisfies also the conditions of Lemma 3. Thus $0 \in \sigma(T)$, but by Proposition 3, the value 0 is not an eigenvalue, and T is invertible.

The inverse of an invertible compact operator is necessarily unbounded. It is checked easily using the matrix representations of T^{-1} in Proposition 3 b). or c)., and observing that a positive real number K, satisfying $\|T^{-1}y\| \leqslant K\|y\|$, for all $y \in \ell_2$, does not exist. Thus T^{-1} is unbounded.

b). Since D is assumed compact, and L and U operators belonging to $\mathrm{B}(\ell_2)$, the operator $T = T(U,L)[D] = UDL \in \mathrm{B}(\ell_2)$, is compact. $\square$

A compact operator on an infinite Hilbert space may have (or may not have) spectral values distinct than 0, but for Riesz-Schauder's theorem, if they exist must be eigenvalues [2, 9, 11]. For bounded (not necessarily compact) arrowhead operators, conditions on the existence of their eigenvalues agree with the finite-dimensional case.

PROPOSITION 4. *Given an invertible arrowhead operator $T \in B(\ell_2)$, defined from a matrix of the form* (11)*, with a, c, and $D_b^{-1}a$, square summable sequences satisfying condition* (10)*. The operator T has an eigenpair of the form $\{\lambda , x_\lambda\}$, with $x_\lambda = \xi\left(1-(a/(b-\lambda))'\right)'$, and $\xi \neq 0$ an arbitrary scalar, if and only if $\lambda \neq 0$ is a root of the complex-valued function*

$$f(\lambda):\mathscr{D}\subset\mathbb{C}\to\mathbb{C};\quad f(\lambda)=(b_0-\lambda)-\langle(a/(b-\lambda)),\bar{c}\rangle.$$

Proof. The resolvent R_λ does not exist if and only if $T-\lambda I$ has a nontrivial kernel. In addition, it is not difficult to prove that $\lambda \neq b_j$, $j=1,2,\ldots$; see Proposition 1 for the finite case. Therefore, $D_{b-\lambda}^{-1}$ exists and reasoning as in the proof of Proposition 3 for invertible compact arrowhead operators, it is sufficient to check that a nontrivial sequence $x_\lambda = (\xi_i) \in \ell_2$ satisfies

$$\left(\begin{array}{c|c} b_0-\lambda & c' \\ \hline D_{b-\lambda}^{-1}a & I \end{array}\right) x_\lambda = 0_{\ell_2}, \quad \text{with} \quad D_{b-\lambda}=\operatorname{diag}((b_i-\lambda)).$$

We obtain $(\xi_j)_{j=2}^{\infty} = -\xi_1 D_{b-\lambda}^{-1}a = -\xi_1(a/(b-\lambda))$. Also, the following condition should be satisfied $(b_0-\lambda)\xi_1 - \xi_1\langle(a/(b-\lambda)),\bar{c}\rangle = 0$. Thus T has eigenpairs of the form $\{\lambda , x_\lambda\} = \left\{\lambda , \xi\left(1-(a/(b-\lambda))'\right)'\right\}$, where $\xi \neq 0$ is an arbitrary scalar, if and only if λ is a root of $f(\lambda)$. From condition (10), $f(0) = b_0 - \langle a/b, \bar{c}\rangle = u \neq 0$, and 0 is not an eigenvalue of T. $\square$

Analogous to the well-known Equation (5) for the finite case, the eigenvalues of bounded invertible arrowhead operators under the assumptions of Proposition 4 are the roots of the complex-valued function $f(\lambda)$ defined on the domain $\mathscr{D} = \mathbb{D}\setminus\{b\}\subset\mathbb{C}$. Here $\mathbb{D} = \mathbb{D}(0,r)$ can be any open disc with center 0 and radius $r > \|T\|$.

$$f(\lambda)=\begin{cases} b_0-\langle a/b,\bar{c}\rangle = u \neq 0, & \text{if } \lambda = 0; \\ b_0-\lambda-\langle(a/(b-\lambda)),\bar{c}\rangle, & \text{if } \lambda \neq 0, \lambda \neq b_k. \end{cases} \tag{12}$$

For invertible compact arrowhead operators, $f(\lambda)$ is meromorphic on $\mathscr{D}_0 = \mathbb{D}\setminus\{0,b\}\subset\mathbb{C}$. The simple poles of $f(\lambda)$ accumulate only in $0 \notin \mathscr{D}_0$. It is reasonable to argue that, under additional conditions on the sequences a, b, and c, there exist infinitely many countable roots of $f(\lambda)$ accumulate only in 0. For example, it happens if the invertible compact arrowhead operator is also self-adjoint. Indeed, it is not difficult to observe that $f(\lambda)$ can be restricted to $f|_{\mathbb{R}}(\lambda)\colon \mathscr{D}'\subset\mathbb{R}\to\mathbb{R}$, and $f|'_{\mathbb{R}}(\lambda) < 0$ in $\mathscr{D}'_0 = \mathscr{D}'\setminus\{0\}\subset\mathbb{R}$. Between two nearer singularities exists one and only one root of $f|_{\mathbb{R}}(\lambda)$. The eigenvalues are a countable set accumulate in the value 0, and Cauchy's interlacing property for the eigenvalues also holds.

4.2. Nonsingular arrowhead operators

An operator $T \in \mathrm{B}(\ell_2)$ is said to be nonsingular if T^{-1} exists and is bounded, $T^{-1} \in \mathrm{B}(\ell_2)$; see e.g. [9]. Therefore, nonsingular arrowhead operators defined, on the whole of ℓ_2, from a matrix of the form (11) are bounded invertible (not compact) arrowhead operators. Thus T is bijective, for all $x \in \ell_2$; $k\|x\| \leqslant \|Tx\| \leqslant K\|x\|$, with k and K positive real numbers, and its inverse operator T^{-1} is also nonsingular.

PROPOSITION 5. *Given a linear operator $T \in B(\ell_2)$ defined from an infinite arrowhead matrix A (11), with a, c, and $a/b = D_b^{-1}a$, nontrivial square summable sequences satisfying condition (10):*

a). *The arrowhead operator T is nonsingular if and only if the entries of D_b are uniformly bounded and $\lim_{j\to\infty}|b_j| \neq 0$.*

b). *If in addition the nontrivial sequence $c/b = D_b^{-1}c \in \ell_2$, so that the matrix A, which induces T, has an appropriate UDL factorization, then the inverse (nonsingular) operator $T^{-1} \in B(\ell_2)$ is defined from a matrix decomposable as in (3) and (4). Its related OTSM operator $T(U,L)[D]$ is well defined and preserves nonsingularity.*

Proof. a). Notice that $0 < |u| < \infty$. Also, we can use the factorization given in the proof of Proposition 3, for the matrix $A = \operatorname{diag}(1,(b_i))\left[D_{b_0}^{\diamond} + F_2^{\diamond}\right] = D_1UDL$. Therefore, $A^{-1} = (L^{-1}D^{-1}U^{-1})D_1^{-1}$. Here the operators defined for U, D, and L are trivially nonsingular operators. Thus T is nonsingular if and only if the diagonal operator defined from $D_1 = \operatorname{diag}(1,(b_i))$ is nonsingular. But recalling that D_b has distinct nonzero entries on the diagonal, such operator is nonsingular if and only if the entries of D_b are uniformly bounded and $\lim_{j\to\infty}|b_j| \neq 0$.

b). The nonsingularity of T^{-1} follows from a). The proof for the decomposition of A^{-1} is analogous to the proof of Proposition 3 c). Since the nonsingularity of the operators U, D, and L, its related OTSM operator $T(U,L)[D]$ is well defined, and the nonsingularity of D is preserved. □

Proposition 5 jointly with the proof of Lemma 3 establish that, for a nonsingular arrowhead operator T, the sequence $\left(T_{(n)}\right)$ of the finite rank operators does not converge uniformly to T. However, it is not difficult to check the strong convergence of $\left(T_{(n)}\right)$.

We prove that the sequence $\left(T_{(n)}^{-1}\right)$ of finite rank operators $T_{(n)}^{-1} \in \mathrm{B}(\ell_2)$, defined from the inverses of the nth principal sections of the matrix A, strongly converges to T^{-1}, so that $T^{-1} \in \mathrm{B}(\ell_2)$, but $\left(T_{(n)}^{-1}\right)$ is not uniformly convergent. For simplicity, the arrowhead operators are defined from matrices UDL factored properly.

THEOREM 6. *Given a nonsingular arrowhead operator $T \in B(\ell_2)$, satisfying all the assumptions of Proposition 5, and the sequence $\left(T_{(n)}^{-1}\right)$ of finite rank operators on ℓ_2, defined from the inverses of the principal finite sections of the matrix A that induces T. The sequence $\left(T_{(n)}^{-1}\right) \xrightarrow{s} T^{-1}$ (strongly).*

Proof. Since the $T_{(n)}^{-1}$ are compact, these are bounded operators. Also by Proposition 5, the entries of D_b are uniformly bounded and $\lim_{j\to\infty} b_j \neq 0$. Thus for all $x \in \ell_2$, the sequence $\left(\|T_{(n)}^{-1}x\|\right) \leqslant k_x$. Then, from Banach-Steinhaus's (uniform boundedness) theorem, the sequence $\left(\|T_{(n)}^{-1}\|\right) \leqslant k$ is bounded on ℓ_2. Besides, for all $x \in \ell_2$, the sequence $\left(y_{(n)}\right) = \left(T_{(n)}^{-1}x\right)$ is a (fundamental) Cauchy sequence. These results characterize the strong convergence of $\left(T_{(n)}^{-1}\right)$, i.e. $\left(T_{(n)}^{-1}\right) \xrightarrow{s} T^{-1}$; see e.g. [11].

We give another equivalent and constructive proof using the matrix representation (4) for both $T_{(n)}^{-1}$ and T^{-1}. The matrix representation for the difference operator $T_{(n)}^{-1} - T^{-1}$ is

$$\begin{aligned}
A_{(n)}^{-1} - A^{-1} &= \operatorname{diag}\left(O_n, -\left(D_b^{(n)}\right)^{-1}\right) - \frac{1}{u}\begin{pmatrix} 0_n \\ -\left(D_b^{(n)}\right)^{-1} a^{(n)} \end{pmatrix}\left(0_n - \left(\left(D_b^{(n)}\right)^{-1} c^{(n)}\right)'\right) \\
&\quad + \left(\frac{u-u_n}{u_n u}\right)\begin{pmatrix} 1 \\ -D_{(n-1)b}^{-1} a_{(n-1)} \\ 0_{\ell_2} \end{pmatrix}\left(1 - \left(D_{(n-1)b}^{-1} c_{(n-1)}\right)' 0'_{\ell_2}\right) \\
&= \operatorname{diag}\left(O_n, -\left(D_b^{(n)}\right)^{-1}\right) - \frac{1}{u} x_a^{(n)} \left(x_c^{(n)}\right)' + \left(\frac{u-u_n}{u_n u}\right) x_{a(n-1)} \left(x_{c(n-1)}\right)'.
\end{aligned}$$

We have denoted $u_n \neq 0$ as the value of u for the matrix $A_{(n)}^{-1}$, the vectors $D_{(n-1)b}^{-1} a_{(n-1)} = \left(\frac{a_1}{b_1}, \ldots, \frac{a_{n-1}}{b_{n-1}}\right)'$, $D_{(n-1)b}^{-1} c_{(n-1)} = \left(\frac{c_1}{b_1}, \ldots, \frac{c_{n-1}}{b_{n-1}}\right)'$, and 0_n is the zero vector of $\mathbb{C}^n$. Since Proposition 5 is assumed,

$$\frac{1}{u} x_a^{(n)} \left(x_c^{(n)}\right)' \underset{n\to\infty}{\longrightarrow} O = 0 \cdot x_{a(\infty)} \left(x_{c(\infty)}\right)' \underset{n\to\infty}{\longleftarrow} \left(\frac{u-u_n}{u_n u}\right) x_{a(n-1)} \left(x_{c(n-1)}\right)'.$$

We obtain,

$$\lim_n \left(A_{(n)}^{-1} - A^{-1}\right) = \lim_n \operatorname{diag}\left(O_n, -\left(D_b^{(n)}\right)^{-1}\right) \neq O,$$

because the entries of D_b are uniformly bounded and $\lim_{n\to\infty} 1/b_n \neq 0$. Therefore, $\lim_{n\to\infty} \|T_{(n)}^{-1} - T^{-1}\| > 0$, and the sequence $\left(T_{(n)}^{-1}\right)$ is not uniformly convergent to T^{-1}. However, the nonsingularity of the operator defined from D_b is used for observing the strong convergence of $\left(T_{(n)}^{-1}\right)$,

$$\lim_n \left(A_{(n)}^{-1} x - A^{-1} x\right) = \lim_n \operatorname{diag}\left(O_n, -\left(D_b^{(n)}\right)^{-1}\right) x = 0_{\ell_2}.$$

Therefore, $\forall x \in \ell_2 : \lim_{n\to\infty} \|T_{(n)}^{-1} x - T^{-1} x\| = 0$, and the sequence $\left(T_{(n)}^{-1}\right)$ converges strongly to T^{-1}. $\square$

4.3. Explicit inertia criteria for invertible arrowhead operators $T \in \mathbf{B}(\ell_2)$

Inertial properties with respect the half planes of invertible arrowhead operators $T \in \mathrm{B}(\ell_2)$ can be connected with that of invertible diagonal operators $D \in \mathrm{Diag}\left(\hat{\ell}_2\right)$, with $\hat{\ell}_2$ the Hilbert space of the real square summable sequences. Analogous to the finite-dimensional case, the connection is through $\mathscr{T}(U,L)[D]$, the class of OTSM operators.

COROLLARY 2. *Given an invertible arrowhead operator $T \in B(\ell_2)$, defined from an arrowhead matrix under the assumptions of Lemma 2. Assuming its related OTSM operator well defined, $T(U,L)[D]\colon Diag\left(\hat{\ell}_2\right) \to B(\ell_2)$, with real matrix argument $D = diag(u, D_b)$:*

a). *If A is hermitian, then the OTSM operator $T(L)[D]$ preserves inertia.*

b). *If A is not hermitian and the following condition on the matrix entries is satisfied,*

$$uRe(b_0) - \tfrac{1}{4}\|\left(u\tfrac{a}{b} + \bar{c}\right)\|^2 = uRe(b_0) - \tfrac{1}{4}\sum_{i=1}^{\infty}\tfrac{|ua_i + b_i\bar{c}_i|^2}{b_i^2} > 0, \tag{13}$$

with $Re(b_0)$ the real part of b_0, then the OTSM operator $T(U,L)[D]$ preserves inertia.

Proof. a). It is an immediate consequence of the infinite-dimensional version of Sylvester's law of inertia; see Corollary 5 from [4].

b). We use the infinite arrowhead matrix $W = 2\mathrm{Re}(\mathrm{AD}) = AD + DA^*$, of the form (9). If condition (13) holds, then its related arrowhead operator W is positive and invertible, and the result follows from [5], Theorem 5. □

4.4. Examples of simple arrowhead operators

An arrowhead operator T is said to be simple if T has countable distintc eigenvalues. From Proposition 4, an invertible self-adjoint arrowhead operator $T \in \mathrm{B}(\ell_2)$ is simple. It is illustrated with the maximal operator T_2 defined from the arrowhead matrix A_2 from Section 3.

EXAMPLE 3. The self-adjoint operator $T_2 \in \mathrm{B}(\ell_2)$ is also nonsingular. The existence of infinitely many real and simple eigenvalues and its corresponding (total) orthonormal set of eigenvectors $\left\{x_{\lambda_i}\right\}$, and the fact that the extremal eigenvalues are $\lambda_{\max} > \frac{3}{2}$ and $\lambda_{\min} < \frac{1}{2}$, follow from Proposition 4. Rayleigh's quotients are $m = \lambda_{\min} = \inf\{\langle Tx, x\rangle | x \in \ell_2, \|x\| = 1\}$, $M = \lambda_{\max} = \sup\{\langle Tx, x\rangle | x \in \ell_2, \|x\| = 1\}$.

Since T_2 satisfies the assumptions of Proposition 5 b)., its related OTSM operator $T_2(L)[D]$ is well defined. By Corollary 2 a)., $T_2(L)[D]$ preserves inertia, $\sigma(T_2) \subset (0,\infty)$, and $0 < m < \frac{1}{2}$. The spectrum of T_2 is

$$\sigma(T_2) = \{m \cup \left[\tfrac{1}{2}, \tfrac{3}{2}\right] \cup M\} \subset (0,\infty).$$

Every $\alpha \in \left[\frac{1}{2}, \frac{3}{2}\right]$ belongs to $\sigma(T_2)$. Indeed, α is either an eigenvalue or belongs to the continuous spectrum. Furthermore since T_2 is nonsingular, the spectrum of T_2^{-1} is simply

$$\sigma\left(T_2^{-1}\right) = \left\{\tfrac{1}{M} \cup \left[\tfrac{2}{3}, 2\right] \cup \tfrac{1}{m}\right\} \subset (0, \infty).$$

As a rule, if an arrowhead operator is simple, it is not necessarily self-adjoint, or normal.

EXAMPLE 4. The maximal operator $T_1 \in \mathrm{B}(\ell_2)$, defined from the arrowhead matrix A_1 given in Section 3, is nonsingular. Although T_1 is not self-adjoint, the existence of infinitely many real and simple eigenvalues and their corresponding eigenvectors $\left\{x_{\lambda_i}\right\}$ also follows from Proposition 4. Notice $f(\lambda)$ restricted to $f|_{\mathbb{R}}(\lambda)\colon \mathscr{D}' \subset \mathbb{R} \to \mathbb{R}$, and $f|'_{\mathbb{R}}(\lambda) < 0$. The interlacing property for the eigenvalues also holds.

$$0 < \lambda_{\min} = \lambda_1 < b_1 < \lambda_2 < b_2 < \ldots < b_j < \lambda_{j+1} < b_{j+1} < \ldots.. \left(j \in \mathbb{Z}^+\right).$$

Thus T_1 has a unique eigenvalue $\lambda_{\max} > 1$. Since $f|_{\mathbb{R}}(0) = f(0) = u > 0$, $\lambda_{\min}$ is the unique eigenvalue of T_1 lying in the interval $\left(0, \frac{2}{3}\right)$.

Alternatively, T_1 satisfies also Proposition 5 b). and $T_1(U, L)[D]$, its associate OTSM operator, is well defined. Since condition (13) is

$$\left(2 - \frac{\pi^2}{6}\right) - \frac{1}{4}\left(3 - \frac{\pi^2}{6}\right)^2 \sum_{k=1}^{\infty} \frac{1}{(k+1)^2} > 0,$$

from Corollary 2 b)., $T_1(L, U)[D]$ preserves inertia, and $\sigma(T_1) \subset (0, \infty)$.

The only accumulation point of eigenvalues of T_1 is 1, from below. We have also the additional $\lambda_{\max} > 1$. The spectrum is $\sigma(T_1) = \{\lambda_j, \lambda_{\max}\} \cup \{1\}$. The spectrum of the inverse operator T_1^{-1} is easily obtained from that of T_1. For example, the only accumulation point of eigenvalues of T_1^{-1} is 1, from above, and $1/\lambda_{\max}$ is its unique eigenvalue less than 1.

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