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ON THE STRUCTURE OF SKEW SYMMETRIC OPERATORS

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Abstract. An operator T on a complex Hilbert space $\mathscr{H}$ is called skew symmetric if T can be represented as a skew symmetric matrix relative to some orthonormal basis for $\mathscr{H}$. We use multiplicity theory to characterize when there is an anti-conjugation commuting with a fixed positive operator, and give a description of such anti-conjugations. Based on these results, we provide a canonical model of skew symmetric operators in terms of multiplication operators on function spaces.

1. Introduction

Throughout this paper, we denote by $\mathscr{H}$ a complex separable Hilbert space endowed with the inner product $\langle\cdot,\cdot\rangle$, and by $\mathscr{B}(\mathscr{H})$ the algebra of all bounded linear operators on $\mathscr{H}$. Recall that a map C on $\mathscr{H}$ is called a *conjugation* if C is conjugate-linear, $C^{-1}=C$ and $\langle Cx,Cy\rangle=\langle y,x\rangle$ for all $x,y\in\mathscr{H}$. An operator $T\in\mathscr{B}(\mathscr{H})$ is said to be *skew symmetric* if $CTC=-T^*$ for some conjugation C on $\mathscr{H}$.

We remark that $T\in\mathscr{B}(\mathscr{H})$ is skew symmetric if and only if there exists an orthonormal basis (ONB, for short) $\{e_n\}$ of $\mathscr{H}$ such that $\langle Te_n,e_m\rangle=-\langle Te_m,e_n\rangle$ for all m,n; that is, T admits a skew symmetric matrix representation with respect to $\{e_n\}$ ([4, Lem. 1]). Thus skew symmetric operators can be viewed as an infinite dimensional analogue of skew symmetric matrices. The most obvious examples of skew symmetric operators on finite dimensional spaces are those Jordan blocks with even ranks (see [13, Ex. 1.7]).

Recently, there has been growing interest in skew symmetric operators, and some interesting results have been obtained [12, 13, 14, 15, 16, 17, 20]. In particular, skew symmetric normal operators, partial isometries, compact operators and weighted shifts are classified [13, 12, 17]. The reader is referred to [13] for more elementary properties of skew symmetric operators.

The primary motivation for the study of skew symmetric operators lies in its connections to complex symmetric operators, which have received much attention in the last decade [1, 3, 4, 5, 7, 8, 9, 10, 18, 19]. Recall that an operator $T\in\mathscr{B}(\mathscr{H})$ is said to be *complex symmetric* if $CTC=T^*$ for some conjugation C on $\mathscr{H}$. One can use complex symmetric operators to construct new skew symmetric operators ([13, Lem. 1.4]).

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In particular, if T is complex symmetric, then $T^*T - TT^*$ is skew symmetric. In view of the description of skew symmetric normal operators [13, Thm. 1.10], this provides a new approach to describing complex symmetric operators. In a recent paper [10], one can see such an application to Toeplitz operators.

Another motivation for the study of skew symmetric operators lies in the connection between skew symmetric operators and anti-automorphisms of singly generated C^*-algebras. Recall that an *anti-automorphism* of a C^*-algebra $\mathscr{A}$ is a vector space isomorphism $\varphi : \mathscr{A} \to \mathscr{A}$ with $\varphi(a^*) = \varphi(a)^*$ and $\varphi(ab) = \varphi(b)\varphi(a)$ for $a, b \in \mathscr{A}$. It is proved that each C^*-algebra generated by a skew symmetric operator admits an involutory anti-automorphism on it (see [16, Cor. 3.2]).

Recently, Li and Zhou [12] gave a description of the polar decomposition of a skew symmetric operator.

LEMMA 1.1. ([12], Lem. 2.3) *Let $T \in \mathscr{B}(\mathscr{H})$ be skew symmetric. Then $T = CK|T|$, where C is a conjugation on $\mathscr{H}$, and K is a partial anti-conjugation supported on* $\overline{\operatorname{ran}|T|}$ *which commutes with $|T|$.*

Recall that a map D on $\mathscr{H}$ is called an *anti-conjugation* if D is conjugate-linear, $C^{-1} = -C$ and $\langle Cx, Cy \rangle = \langle y, x \rangle$ for all $x, y \in \mathscr{H}$ (see [6]). For a subspace M of $\mathscr{H}$, a conjugate-linear map J on $\mathscr{H}$ is called a *partial anti-conjugation* supported on M if $\ker J = M^\perp$ reduces J and $J|_M$ is an anti-conjugation.

The decomposition asserted in Lemma 1.1 proves to be very useful in the study of skew symmetric operators (see [12, 17]). The main aim of this note is to complete this result. In other words, we shall exhibit the internal structure of skew symmetric operators by studying their polar decompositions.

First, we point out that the converse of Lemma 1.1 also holds.

THEOREM 1.2. *Let $T \in \mathscr{B}(\mathscr{H})$. Then T is skew symmetric if and only if $T = CK|T|$, where C is a conjugation on $\mathscr{H}$, and K is a partial anti-conjugation supported on* $\overline{\operatorname{ran}|T|}$ *which commutes with $|T|$.*

REMARK 1.3. (i) Let $T \in \mathscr{B}(\mathscr{H})$. By Theorem 1.2, if K is a partial anti-conjugation supported on $\overline{\operatorname{ran}|T|}$ commuting with $|T|$, then $CK|T|$ is skew symmetric for any conjugation C on $\mathscr{H}$. This provides a method to construct skew symmetric operators.

(ii) Garcia and Putinar [5, Thm. 2] obtained a refined polar decomposition theorem for complex symmetric operators, which asserts that an operator R is complex symmetric if and only if $R = CJ|R|$, where C is a conjugation, and J is a partial conjugation supported on $\overline{\operatorname{ran}|R|}$ commuting with $|R|$. Thus Theorem 1.2 is the skew symmetric version of their result.

We obtain the following result which completely characterizes when there exists an anti-conjugation J commuting with a fixed positive operator.

THEOREM 1.4. *Let $P \in \mathscr{B}(\mathscr{H})$ be positive. Then there exists an anti-conjugation J on $\mathscr{H}$ such that $JP = PJ$ if and only if $P \cong Q^{(2)}$ for some positive operator Q.*

Here and in what follows, $\cong$ denotes unitary equivalence and $Q^{(2)} = Q \oplus Q$. By the spectral multiplicity theory, each positive operator P is unitarily equivalent to a direct sum of some multiplication operators on function spaces. Based on this result, we give a description of anti-conjugations commuting with P (see Theorem 2.5). Moreover, we give a canonical model of skew symmetric operators. To state our main result, we need some extra notation.

In this note, we often write $[T_{i,j}]_{1\leqslant i,j\leqslant n}$ or $[T_{i,j}]_{n\times n}$ to denote the following operator matrix

$$\begin{bmatrix} T_{1,1} & T_{1,2} & \cdots & T_{1,n} \\ T_{2,1} & T_{2,2} & \cdots & T_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ T_{n,1} & T_{n,2} & \cdots & T_{n,n} \end{bmatrix},$$

where $n \in \mathbb{N} \cup \{\infty\}$. If d is a cardinal number and $\mathscr{K}$ is a Hilbert space, let $\mathscr{K}^{(d)}$ denote the direct sum of d copies of $\mathscr{K}$. For $A \in \mathscr{B}(\mathscr{K})$, let $A^{(d)}$ denote the direct sum of d copies of A.

Let μ be a finitely supported, positive Borel measure on $(0,+\infty)$. For $f \in L^2(\mu)$, define $(D_\mu f)(t) = \overline{f(t)}$. Then it is obvious that D_μ is a conjugation on $L^2(\mu)$. For $\varphi \in L^\infty(\mu)$, we let M_φ denote the "multiplication by φ" operator on $L^2(\mu)$. If n is even or $n=\infty$, we denote by $\Phi(\mu,n)$ the set of all conjugate-linear operators on $L^2(\mu)^{(n)}$ with the form

$$\left[M_{t\varphi_{i,j}} D_\mu\right]_{1\leqslant i,j\leqslant n}, \tag{1.1}$$

where $\varphi_{i,j} \in L^\infty(\mu)$, $\varphi_{i,j} = -\varphi_{j,i}$ for all $1 \leqslant i,j \leqslant n$ and $\left[\varphi_{i,j}(t)\right]_{1\leqslant i,j\leqslant n}$ is unitary almost everywhere with respect to μ. One can see that

$$\left[M_{t\varphi_{i,j}} D_\mu\right]_{1\leqslant i,j\leqslant n} = \left[M_{\varphi_{i,j}} D_\mu M_t\right]_{1\leqslant i,j\leqslant n} = \left[M_{\varphi_{i,j}} D_\mu\right]_{1\leqslant i,j\leqslant n} M_t^{(n)}. \tag{1.2}$$

The main result of this note is the following theorem which gives a canonical model of skew symmetric operators up to unitary equivalence.

THEOREM 1.5. *Let $T \in \mathscr{B}(\mathscr{H})$. Then T is skew symmetric if and only if there exist mutually singular, finitely supported measures $\mu_\infty, \mu_1, \mu_2, \cdots$ on $(0, \|T\|]$ (some of which may be absent) such that*

$$T \cong C\Big(\bigoplus_{0\leqslant n\leqslant\infty} T_n\Big),$$

where T_0 is the 0 operator on some Hilbert space $\mathscr{K}$ (which may be absent), $T_n \in \Phi(\mu_n, 2n)$ for $1 \leqslant n \leqslant \infty$ and C is a conjugation on

$$\mathscr{K} \oplus \Big(\bigoplus_{1\leqslant n\leqslant\infty} L^2(\mu_n)^{(2n)}\Big).$$

The proofs of Theorems 1.2, 1.4 and 1.5 will be provided in Section 2. In Section 3, we shall give several corollaries of our results.

2. Refined polar decomposition

This section is devoted to the analysis of the polar decomposition of a skew symmetric operator. We first give the proof of Theorem 1.2.

Proof of Theorem 1.2. By Lemma 1.1, it suffices to prove the sufficiency.

"$\Longleftarrow$". Assume that $T = CK|T|$, where C is a conjugation on $\mathscr{H}$, and K is a partial anti-conjugation supported on $\overline{\operatorname{ran}|T|}$ which commutes with $|T|$. Denote by P the orthogonal projection of $\mathscr{H}$ onto $\overline{\operatorname{ran}|T|}$. Then $P = -K^2$ and $PK = KP$.

We claim that $(CK)^* = -KC$. In fact, for $x, y \in \mathscr{H}$, we have

$$\begin{aligned}\langle (CK)^* x, y\rangle &= \langle x, CKy\rangle = \langle Ky, Cx\rangle = \langle Ky, PCx\rangle \\ &= -\langle Ky, K^2Cx\rangle = -\langle KPy, K^2Cx\rangle \\ &= -\langle KCx, Py\rangle = -\langle KCx, y\rangle.\end{aligned}$$

It follows that

$$CTC = K|T|C = |T|KC = -|T|(CK)^* = -T^*. \quad \square$$

LEMMA 2.1. *Let $T \in \mathscr{B}(\mathscr{H})$. If Q is a positive operator on a Hilbert space $\mathscr{K}$ and $T \cong Q^{(2)}$, then there exists an anti-conjugation on $\mathscr{H}$ commuting with T.*

Proof. It suffices to find an anti-conjugation J on $\mathscr{K}^{(2)}$ so that $JQ^{(2)} = Q^{(2)}J$.

Since Q is positive, Q is complex symmetric. Then we can choose a conjugation C on $\mathscr{K}$ so that $CQC = Q$. Set

$$J = \begin{bmatrix} 0 & -C \\ C & 0 \end{bmatrix} \begin{matrix} \mathscr{H} \\ \mathscr{H} \end{matrix}.$$

So J is conjugate-linear, isometric and $J^{-1} = -J$. Hence J is an anti-conjugation on $\mathscr{K}^{(2)}$. Now compte to see that

$$JQ^{(2)} = \begin{bmatrix} 0 & -C \\ C & 0 \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & Q \end{bmatrix} = \begin{bmatrix} 0 & -CQ \\ CQ & 0 \end{bmatrix} = \begin{bmatrix} 0 & -QC \\ QC & 0 \end{bmatrix} = Q^{(2)}J.$$

This completes the proof. $\square$

The following lemma is a linear algebra exercise. The reader is referred to [11, page 217] for a proof.

LEMMA 2.2. *Let $T = [t_{i,j}]_{1 \leqslant i,j \leqslant n}$ be a skew symmetric matrix, that is, $t_{i,j} = -t_{j,i}$ for $1 \leqslant i, j \leqslant n$. If T is a unitary matrix, then n is even.*

PROPOSITION 2.3. *Let μ be a finitely supported, positive Borel measure on $[0, +\infty)$ and $T = M_t^{(n)}$, where $n \in \mathbb{N} \cup \{\infty\}$. Denote by $\Psi(\mu, n)$ the set of all anti-conjugations on $L^2(\mu)^{(n)}$ commuting with T. Then* *(i)* $\Psi(\mu,n)\neq\emptyset$ *if and only if* n *is even or* $n=\infty$*;*

(ii) each anti-conjugation J *in* $\Psi(\mu,n)$ *has the form of*

$$\left[M_{\varphi_{i,j}}D_\mu\right]_{n\times n}, \tag{2.1}$$

where $\varphi_{i,j}\in L^\infty(\mu)$, $\varphi_{i,j}=-\varphi_{j,i}$ *for all* $1\leqslant i,j\leqslant n$ *and* $\left[\varphi_{i,j}(t)\right]_{n\times n}$ *is a unitary matrix for almost every* t *with respect to* μ*.*

Proof. By Lemma 2.1, the sufficiency of (i) is clear. Now we assume that J is an anti-conjugation on $L^2(\mu)^{(n)}$ commuting with T.

Set $D=D_\mu^{(n)}$ and $U=JD$. It is obvious that D is a conjugation on $L^2(\mu)^{(n)}$ and $U\in\mathscr{B}(L^2(\mu)^{(n)})$ is unitary. Noting that $M_tD_\mu=D_\mu M_t$, we have $M_t^{(n)}D=DM_t^{(n)}$. Then

$$UM_t^{(n)}=JDM_t^{(n)}=JM_t^{(n)}D=M_t^{(n)}JD=M_t^{(n)}U.$$

Assume that

$$U=\begin{array}{c}\begin{bmatrix}U_{1,1} & U_{1,2} & \cdots & U_{1,n}\\ U_{2,1} & U_{2,2} & \cdots & U_{2,n}\\ \vdots & \vdots & \ddots & \vdots\\ U_{n,1} & U_{n,2} & \cdots & U_{n,n}\end{bmatrix}\end{array}\begin{array}{c}L^2(\mu)\\ L^2(\mu)\\ \vdots\\ L^2(\mu)\end{array}.$$

It follows that $U_{i,j}M_t=M_tU_{i,j}$ for $1\leqslant i,j\leqslant n$. By [2, Prop. 10.18], there exists $\varphi_{i,j}\in L^\infty(\mu)$ such that $U_{i,j}=M_{\varphi_{i,j}}$ for $1\leqslant i,j\leqslant n$. Since U is unitary, one can check that $\left[\varphi_{i,j}(t)\right]_{n\times n}$ is a unitary matrix for almost every t with respect to μ. Moreover,

$$J=UD=\left[M_{\varphi_{i,j}}\right]_{n\times n}D=\left[M_{\varphi_{i,j}}D_\mu\right]_{n\times n}.$$

Since $DU^*=J^{-1}=-J=-UD$, we obtain

$$U=-DU^*D=-D\left[M_{\varphi_{i,j}}\right]_{n\times n}^*D=-\left[D_\mu M_{\overline{\varphi_{j,i}}}D_\mu\right]_{n\times n}.$$

Thus $M_{\varphi_{i,j}}=-D_\mu M_{\overline{\varphi_{j,i}}}D_\mu=-M_{\varphi_{j,i}}$ for $1\leqslant i,j\leqslant n$. That is, $\varphi_{i,j}=-\varphi_{j,i}$ for $1\leqslant i,j\leqslant n$. Noting that $\left[\varphi_{i,j}(t)\right]_{n\times n}$ is a unitary matrix almost everywhere with respect to μ, by Lemma 2.2, n is even or $n=\infty$. Hence we conclude the proof. $\square$

LEMMA 2.4. *Let* $A\in\mathscr{B}(\mathscr{H})$ *be positive and* $E(\cdot)$ *be the associated projection-valued spectral measure. If* J *is an anti-conjugation on* $\mathscr{H}$ *and* $JA=AJ$*, then* $JE(\sigma)=E(\sigma)J$ *for all Borel subset* σ *of* $\mathbb{C}$*.*

Proof. Since $JA=AJ$, for any polynomial $p(t)=\sum_{i=0}^k a_it^i$ with real coefficients a_i's, it is easy to see $Jp(A)=p(A)J$. Fix a Borel subset σ of $\mathbb{C}$. Noting that A is positive and $E(\cdot)$ is supported on a subset of $[0,\|A\|]$, there exist a sequence $\{p_n(t)\}_{n=1}^\infty$ of polynomials with real coefficients such that $p_n(A)\to E(\sigma)$ in the strong operator topology. It follows immediately that $JE(\sigma)=E(\sigma)J$. $\square$

Proof of Theorem 1.4. The sufficiency follows from Lemma 2.1. We need only prove the necessity.

"$\Longrightarrow$". Since P is positive, by the spectral theorem, there exist mutually singular, finitely supported, positive Borel measures $\mu_1, \mu_2, \cdots, \mu_m$ on $[0, \|P\|]$ $(1 \leqslant m \leqslant \infty)$ such that

$$P \cong \bigoplus_{1 \leqslant j \leqslant m} M_j^{(n_j)},$$

where $M_j \in \mathscr{B}(L^2(\mu_j))$ is the multiplication operator $f(t) \mapsto tf(t)$ and $1 \leqslant n_j \leqslant \infty$ for $1 \leqslant j \leqslant m$. It suffices to prove that n_j is even or $n_j = \infty$ for all $1 \leqslant j \leqslant m$.

By the hypothesis, there exists an anti-conjugation J on $\oplus_{1 \leqslant j \leqslant m} L^2(\mu_j)^{(n_j)}$ commuting with $\oplus_{1 \leqslant j \leqslant m} M_j^{(n_j)}$. Since μ_j's are mutually singular, it follows from Lemma 2.4 that $L^2(\mu_j)^{(n_j)}$ reduces J for each $1 \leqslant j \leqslant m$. For $1 \leqslant j \leqslant m$, set

$$J_j = J|_{L^2(\mu_j)^{(n_j)}}.$$

Then each J_i is an anti-conjugation on $L^2(\mu_j)^{(n_j)}$ commuting with $M_j^{(n_j)}$. By Proposition 2.3, n_j is even or $n_j = \infty$ for all $1 \leqslant j \leqslant m$. $\square$

Summarizing the results of Proposition 2.3 and Lemma 2.4, we have the following result.

THEOREM 2.5. *Let $\mu_1, \mu_2, \mu_3, \cdots$ be mutually singular, finitely supported, positive Borel measures on $[0, \delta]$, where $0 \leqslant \delta < \infty$. Suppose that*

$$P = \bigoplus_{j \geqslant 1} M_j^{(n_j)},$$

where $M_j \in \mathscr{B}(L^2(\mu_j))$ is the multiplication operator $f(t) \mapsto tf(t)$ and $1 \leqslant n_j \leqslant \infty$ for $j \geqslant 1$. If J is an anti-conjugation on $\oplus_{j \geqslant 1} L^2(\mu_j)^{(n_j)}$ commuting with P, then n_j is even or $n_j = \infty$ for all j, and $J = \oplus_{j \geqslant 1} J_j$, where each J_j is an anti-conjugation on $L^2(\mu_j)^{(n_j)}$ with the form of (2.1).

Now we are going to give the proof of Theorem 1.5.

Proof of Theorem 1.5. "$\Longleftarrow$". By Theorem 1.2, it suffices to prove that each T_n, $1 \leqslant n \leqslant \infty$, can be written as JP, where P is positive and J is an anti-conjugation commuting with P. Since $T_n \in \Phi(\mu_n, 2n)$, T_n has the form

$$\left[M_{t\varphi_{i,j}} D_\mu\right]_{2n \times 2n},$$

where $\varphi_{i,j} \in L^\infty(\mu_n)$, $\varphi_{i,j} = -\varphi_{j,i}$ for all $1 \leqslant i, j \leqslant 2n$ and $\left[\varphi_{i,j}(t)\right]_{2n \times 2n}$ is unitary almost everywhere with respect to μ_n. Set

$$J = \left[M_{\varphi_{i,j}} D_\mu\right]_{2n \times 2n}, \quad P = M_t^{(2n)}.$$

Then P is positive and, by Proposition 2.3, J is an anti-conjugation on $L^2(\mu_n)^{(2n)}$ commuting with P. By (1.2), we have $T_n = JP$. This proves the sufficiency.

"$\Longrightarrow$". By [2, Prop. 10.18], there exist mutually singular, finitely supported, positive Borel measures $\mu_1, \mu_2, \mu_3, \cdots, \mu_m$ on $(0, \|T\|]$, where $1 \leqslant m \leqslant \infty$, so that

$$|T| \cong T_0 \oplus \left(\bigoplus_{j=1}^{m} M_j^{(n_j)} \right) \triangleq T_0 \oplus P,$$

where T_0 is the 0 operator on $\ker T$, $M_j \in \mathscr{B}(L^2(\mu_j))$ is the multiplication operator $f(t) \mapsto tf(t)$ and $1 \leqslant n_j \leqslant \infty$ for $1 \leqslant j \leqslant m$. Obviously, we may directly assume that $n_i \neq n_j$ whenever $i \neq j$.

We can find an operator R, which is unitarily equivalent to T, so that $|R| = T_0 \oplus P$. Denote $\mathscr{K} = \ker T$. Then one can see that R acts on

$$\widetilde{\mathscr{H}} \triangleq \mathscr{K} \oplus \left(\bigoplus_{j=1}^{m} L^2(\mu_j)^{(n_j)} \right)$$

and $\ker R = \mathscr{K}$. Noting that R is skew symmetric, in view of Theorem 1.2, there exist a conjugation C on $\widetilde{\mathscr{H}}$ and a partial anti-conjugation K supported on $\oplus_{j=1}^m L^2(\mu_j)^{(n_j)}$ so that $K|R| = |R|K$ and $R = CK|R|$. Denote by K_0 the restriction of K to $\oplus_{j=1}^m L^2(\mu_j)^{(n_j)}$. Then K_0 is an anti-conjugation and $K_0 P = PK_0$. By Theorem 2.5, each n_j is even or $n_j = \infty$, and

$$K_0 = \oplus_{j=1}^m J_j,$$

where each J_j is an anti-conjugation on $L^2(\mu_j)^{(n_j)}$ with the form of (2.1). Then

$$K|R| = T_0 \oplus K_0 P = T_0 \oplus \left(\oplus_{j=1}^m J_j M_j^{(n_j)} \right).$$

Noting that each $J_j M_j^{(n_j)}$ has the form of (1.1), this proves the necessity. $\square$

3. Several corollaries

As applications of our results, we give in this section some corollaries.

The following result gives a characterization of the modulus of a skew symmetric operator.

COROLLARY 3.1. *Let $P \in \mathscr{B}(\mathscr{H})$ be positive and $M = \overline{\operatorname{ran} P}$. Then the following are equivalent:*

(i) there exists a skew symmetric operator $T \in \mathscr{B}(\mathscr{H})$ such that $|T| = P$;

(ii) there exists a skew symmetric normal operator $N \in \mathscr{B}(\mathscr{H})$ such that $|N| = P$;

(iii) there exists a partial anti-conjugation J supported on $(\ker P)^\perp$ so that $JP = PJ$;

(iv) $P|_M \cong Q^{(2)}$ for some positive operator Q.

Proof. The implication "(ii)$\Longrightarrow$(i)" is obvious, and the implications "(i)$\Longrightarrow$(iii)$\Longleftrightarrow$(iv)" follow from Theorems 1.2 and 1.4.

"(iv)$\Longrightarrow$(ii)". Assume that Q acts on $\mathscr{K}$. Then

$$P \cong \begin{bmatrix} 0 & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & Q \end{bmatrix} \begin{matrix} \ker P \\ \mathscr{K} \\ \mathscr{K} \end{matrix}.$$

Set

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -Q & 0 \\ 0 & 0 & Q \end{bmatrix} \begin{matrix} \ker P \\ \mathscr{K} \\ \mathscr{K} \end{matrix}.$$

Then $|A| = 0 \oplus Q \oplus Q \cong P$. In view of [13, Thm. 1.10], A is skew symmetric and normal. So there exists a skew symmetric, normal operator N such that $|N| = P$. This completes the proof. $\square$

By Corollary 3.1, the modulus of a skew symmetric operator can not have odd rank. Then the following corollary is clear.

COROLLARY 3.2. *A finite-rank skew symmetric operator must have even rank.*

On the other hand, it follows from Corollary 3.1 that the classical Volterra integration operator is not skew symmetric since each singular value of its modulus is of multiplicity one (see [5, Ex. 6]).

COROLLARY 3.3. *Let $T \in \mathscr{B}(\mathscr{H})$. Then there exists unitary $U \in \mathscr{B}(\mathscr{H})$ such that UT is skew symmetric if and only if $\dim\ker T = \dim\ker T^*$ and $|T|_{(\ker T)^\perp} \cong Q^{(2)}$ for some positive operator Q.*

Proof. "$\Longrightarrow$". Since UT is skew symmetric, we can choose a conjugation C such that $C(UT)C = -(UT)^*$. It follows that $\dim\ker T = \dim\ker T^*$. Noting that $|UT| = |T|$, it follows from Corollary 3.1 that $|T|_{(\ker T)^\perp} \cong Q^{(2)}$ for some positive operator Q.

"$\Longleftarrow$". Let $T = V|T|$ be the polar decomposition of T. Since $\dim\ker T = \dim\ker T^*$, we obtain $\dim\ker V = \dim\ker V^*$. Thus there is a unitary operator U_1 on $\mathscr{H}$ such that

$$U_1 T = U_1 V|T| = |T|.$$

From the proof of "(iv)$\Longrightarrow$ (ii)" in Corollary 3.1, one can find a unitary operator U_2 so that $U_2|T|$ is skew symmetric. Set $U = U_1U_2$. Then U satisfies all requirements. $\square$

We remark that the partial anti-conjugation in Theorem 1.2 can not be replaced by anti-conjugation (see Example 3.6). However, under a hypothesis of the dimension of null space, we obtain another decomposition result similar to Theorem 1.2.

COROLLARY 3.4. *Let $T \in \mathscr{B}(\mathscr{H})$. If $\dim\ker T$ is even or $\dim\ker T = \infty$, then T is skew symmetric if and only if T has the form $T = CJ|T|$, where C is a conjugation on $\mathscr{H}$ and J is an anti-conjugation on $\mathscr{H}$ commuting with $|T|$.*

By Theorem 1.4, the constraint in Corollary 3.4 on the dimension of $\ker T$ is necessary. To give the proof of Corollary 3.4, we first make some preparation.

Recall that a conjugate-linear map D on $\mathscr{H}$ is called an *anti-unitary operator* if C is invertible and $\langle Dx, Dy\rangle = \langle y, x\rangle$ for all $x, y \in \mathscr{H}$. Note that conjugations and anti-conjugations are anti-unitary operators.

LEMMA 3.5. *Let C, D be two anti-unitary operators on $\mathscr{H}$. Then $CD \in \mathscr{B}(\mathscr{H})$ is unitary and $(CD)^* = D^{-1}C^{-1}$.*

Proof. Since C, D are conjugate-linear, invertible and isometric, it follows that $CD \in \mathscr{B}(\mathscr{H})$ is unitary. For $x, y \in \mathscr{H}$, one can check that

$$\begin{aligned}\langle CDx, y\rangle &= \langle CDx, CC^{-1}y\rangle = \langle C^{-1}y, Dx\rangle \\ &= \langle DD^{-1}C^{-1}y, Dx\rangle = \langle x, D^{-1}C^{-1}y\rangle.\end{aligned}$$

This implies that $(CD)^* = D^{-1}C^{-1}$. $\square$

Proof of Corollary 3.4. The sufficiency follows from Theorem 1.2.

"$\Longrightarrow$". Since T is skew symmetric, we may assume that $CTC = -T^*$, where C is a conjugation on $\mathscr{H}$. By Lemma 1.1, there exists a partial anti-conjugation K supported on $(\ker|T|)^\perp$ such that $T = CK|T|$ and $K|T| = |T|K$. Then, with respect to the decomposition $\mathscr{H} = \ker|T| \oplus (\ker|T|)^\perp$, K can be written as

$$K = \begin{bmatrix} 0 & 0 \\ 0 & K_0 \end{bmatrix},$$

where $K_0 = K|_{(\ker|T|)^\perp}$.

Since $\dim\ker|T|$ is even or $\dim\ker|T| = \infty$, by [6, page 188], we can find an anti-conjugation K_1 on $\ker|T|$. Set

$$J = \begin{array}{c} \begin{bmatrix} K_1 & 0 \\ 0 & K_0 \end{bmatrix} \end{array} \begin{array}{l} \ker|T| \\ (\ker|T|)^\perp \end{array}.$$

Then J is an anti-conjugation on $\mathscr{H}$. Since $|T|$ admits the following matrix representation

$$|T| = \begin{array}{c} \begin{bmatrix} 0 & 0 \\ 0 & Q \end{bmatrix} \end{array} \begin{array}{l} \ker|T| \\ (\ker|T|)^\perp \end{array},$$

it follows that $K|T| = J|T| = |T|J$. Hence $T = CJ|T|$, which completes the proof. $\square$

The following example shows that the constraint in Corollary 3.4 on the dimension of $\ker T$ is necessary.

EXAMPLE 3.6. Let $\{e_n\}_{n=1}^\infty$ be an ONB of $\mathscr{H}$ and define $S \in \mathscr{B}(\mathscr{H})$ as

$$Se_n = e_{n+1}, \quad \forall n \geqslant 1.$$

Set $T = S^* \oplus S$. For $x \in \mathscr{H}$ with $x = \sum_{i=1}^{\infty} \alpha_i e_i$, we define $Ex = \sum_{i=1}^{\infty} (-1)^i \overline{\alpha_i} e_i$ One can verify that E is a conjugation on $\mathscr{H}$ and $ESE = -S$. So $ES^*E = -S^*$. Set

$$D = \begin{bmatrix} 0 & E \\ E & 0 \end{bmatrix} \begin{matrix} \mathscr{H} \\ \mathscr{H} \end{matrix}.$$

Then D is a conjugation on $\mathscr{H}^{(2)}$ and one can check that $DTD = -T^*$. So T is skew symmetric. A direct calculation shows that

$$\dim\ker|T| = \dim\ker T = \dim\ker S^* = 1.$$

By Theorem 1.4, there exists no anti-conjugation commuting with $|T|$.

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REFERENCES

[1] N. CHEVROT, E. FRICAIN, AND D. TIMOTIN, *The characteristic function of a complex symmetric contraction*, Proc. Amer. Math. Soc. **135** (2007), no. 9, 2877–2886 (electronic).
[2] J. B. CONWAY, *A course in functional analysis*, second ed., Graduate Texts in Mathematics, vol. 96, Springer-Verlag, New York, 1990.
[3] S. R. GARCIA AND D. E. POORE, *On the norm closure of the complex symmetric operators: compact operators and weighted shifts*, J. Funct. Anal. **264** (2013), no. 3, 691–712.
[4] S. R. GARCIA AND M. PUTINAR, *Complex symmetric operators and applications*, Trans. Amer. Math. Soc. **358** (2006), no. 3, 1285–1315 (electronic).
[5] S. R. GARCIA AND M. PUTINAR, *Complex symmetric operators and applications, II*, Trans. Amer. Math. Soc. **359** (2007), no. 8, 3913–3931 (electronic).
[6] S. R. GARCIA, J. E. TENER, *Unitary equivalence of a matrix to its transpose*, J. Operator Theory **68**(2012), no. 1, 179–203.
[7] S. R. GARCIA AND W. R. WOGEN, *Complex symmetric partial isometries*, J. Funct. Anal. **257** (2009), no. 4, 1251–1260.
[8] T. M. GILBREATH AND W. R. WOGEN, *Remarks on the structure of complex symmetric operators*, Integral Equations Operator Theory **59** (2007), no. 4, 585–590.
[9] K. GUO, Y. JI, AND S. ZHU, *A C^*-algebra approach to complex symmetric operators*, Trans. Amer. Math. Soc. **367** (2015), no. 10, 6903–6942.
[10] K. GUO AND S. ZHU, *A canonical decomposition of complex symmetric operators*, J. Operator Theory **72** (2014), no. 2, 529–547.
[11] R. A. HORN AND C. R. JOHNSON, *Matrix analysis*, Corrected reprint of the 1985 original, Cambridge University Press, Cambridge, 1990.
[12] C. G. LI AND T. T. ZHOU, *Skew symmetry of a class of operators*, Banach J. Math. Anal. **8** (2014), no. 1, 279–294.
[13] C. G. LI AND S. ZHU, *Skew symmetric normal operators*, Proc. Amer. Math. Soc. **141** (2013), no. 8, 2755–2762.
[14] S. M. ZAGORODNYUK, *On a J-polar decomposition of a bounded operator and matrices of J-symmetric and J-skew-symmetric operators*, Banach J. Math. Anal. **4** (2010), no. 2, 11–36.
[15] S. M. ZAGORODNYUK, *On the complex symmetric and skew-symmetric operators with a simple spectrum*, Symmetry, Integrability and Geometry: Methods and Applications **7** (2011), 1–9.
[16] S. ZHU, *Approximate unitary equivalence to skew symmetric operators*, Complex Anal. Oper. Theory **8** (2014), no. 7, 1565–1580.
[17] S. ZHU, *Skew symmetric weighted shifts*, Banach J. Math. Anal. **9** (2015), no. 1, 253–272.

[18] S. ZHU AND C. G. LI, *Complex symmetry of a dense class of operators*, Integral Equations Operator Theory **73** (2012), no. 2, 255–272.
[19] S. ZHU AND C. G. LI, *Complex symmetric weighted shifts*, Trans. Amer. Math. Soc. **365** (2013), no. 1, 511–530.
[20] S. ZHU AND J. ZHAO, *The Riesz decomposition theorem for skew symmetric operators*, J. Korean Math. Soc. **52** (2015), no. 2, 403–416.

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