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ON THE GRÜSS INEQUALITY FOR UNITAL 2–POSITIVE LINEAR MAPS

SRIRAM BALASUBRAMANIAN

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Abstract. In a recent work, Moslehian and Rajić have shown that the Grüss inequality holds for unital n-positive linear maps $\phi : \mathscr{A} \to B(H)$, where $\mathscr{A}$ is a unital C^*-algebra and H is a Hilbert space, if $n \geqslant 3$. They also demonstrate that the inequality fails to hold, in general, if $n = 1$ and question whether the inequality holds if $n = 2$. In this article, we provide an affirmative answer to this question.

1. Introduction

A classical theorem of Grüss (see [4]) states that if f and g are bounded real valued integrable functions on $[a,b]$ and $m_1 \leqslant f(x) \leqslant M_1$ and $m_2 \leqslant g(x) \leqslant M_2$ for all $x \in [a,b]$, then

$$\left| \frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2}\left(\int_a^b f(x)dx\right)\left(\int_a^b g(x)dx\right)\right| \leqslant \frac{1}{4}\alpha\beta,$$

where $\alpha = (M_1 - m_1)$ and $\beta = (M_2 - m_2)$.

A generalized operator version of the Grüss inequality was given by Renaud in [8], where he proved the following result.

THEOREM 1. *Let $A,B \in B(H)$ and suppose that their numerical ranges are contained in disks of radii R and S respectively. If $T \in B(H)$ is a positive operator with Tr(T) = 1, where Tr stands for the trace, then*

$$|Tr(TAB) - Tr(TA)Tr(TB)| \leqslant 4RS.$$

If A,B are normal, then the constant 4 on the right hand side can be replaced by 1.

Among other operator versions of the Grüss inequality, of particular interest to us are those of Perić and Rajić (see [7]), where they prove the Grüss inequality for completely bounded maps, and Moslehian and Rajić (see [5]), where they prove the Grüss inequality for n-positive unital linear maps, for $n \geqslant 3$. In [5], the authors show that the inequality fails to hold in general, if $n = 1$ and question whether it holds for the case $n = 2$. The main result of this article gives an affirmative answer to this question.

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Before we state the main result, we shall introduce some notation and definitions. Throughout this article, $\mathscr{A}$ will denote a unital C^*-algebra, $M_n(\mathscr{A})$ the C^*-algebra of $n\times n$ matrices over $\mathscr{A}$, H and K complex Hilbert spaces and $B(H)$ the C^*-algebra of bounded operators on H. The notations e, 1 will denote the unit elements in $\mathscr{A}$ and $B(H)$ respectively and $\phi : \mathscr{A} \to B(H)$, a unital linear map, i.e. a linear map such that $\phi(e) = 1$. The map ϕ is said to be positive if $\phi(a)$ is positive in $B(H)$ for all positive $a \in \mathscr{A}$. For more details, see [6]. It is easy to see that the map $\phi_n : M_n(\mathscr{A}) \to M_n(B(H))$ defined by $\phi_n((a_{ij})) = (\phi(a_{ij}))$ is unital and linear for each $n \in \mathbb{N}$. The map ϕ is said to be n-positive if ϕ_n is a positive map, completely positive if ϕ is n-positive for all $n \in \mathbb{N}$ and completely bounded if $\sup_{n\in\mathbb{N}} \|\phi_n\| < \infty$.

The main result of this article is the following.

THEOREM 2. *Let $\mathscr{A}$ be a C^*-algebra with unit e. If $\phi : \mathscr{A} \to B(H)$ is a unital 2-positive linear map, then*

$$\|\phi(ab) - \phi(a)\phi(b)\| \leqslant \left(\inf_{\lambda\in\mathbb{C}} \|a - \lambda e\|\right)\left(\inf_{\mu\in\mathbb{C}} \|b - \mu e\|\right). \tag{1}$$

for all $a, b \in \mathscr{A}$.

To prove Theorem 2, we use the well-known theorems of Stinespring, Russo-Dye, Fuglede-Putnam, and the result due to Choi (see Lemma 3).

2. Preliminaries

In this section we include some lemmas which will be used in the sequel. Observe that if $\mathscr{A}$ and $\mathscr{B}$ are unital C^*-algebras and $\gamma : \mathscr{A} \to \mathscr{B}$ is a unital n-positive linear map, then it is m-positive for all $m = 1, 2, \ldots, n$. In particular γ is positive. It is well known that positive maps are *-preserving. i.e. $\gamma(a^*) = \gamma(a)^*$ for all $a \in \mathscr{A}$. Moreover $\|\gamma\| = 1$.

LEMMA 1. *If $P, Q, R \in B(H)$, then $A = \begin{pmatrix} P & R \\ R^* & Q \end{pmatrix} \succeq 0$ in $M_2(B(H))$ if and only if $P, Q \succeq 0$ and $|\langle Rx, y\rangle|^2 \leqslant \langle Px, x\rangle\langle Qy, y\rangle$, for all $x, y \in H$. Moreover, if $A \succeq 0$, then $\|R\|^2 \leqslant \|P\|\|Q\|$.*

LEMMA 2. *Let $A = \begin{pmatrix} T & S \\ S^* & R \end{pmatrix} \in B(H \oplus K)$. If $R \in B(K)$ be invertible, then the following statements are equivalent.*

(i) $A \succeq 0$

(ii) $T, R \succeq 0$ *and* $T \succeq SR^{-1}S^*$.

The above two lemmas are well known. Their proofs can be found in [1].

LEMMA 3. (Choi) *Let $\mathscr{U}$ and $\mathscr{V}$ be C^*-algebras and $\phi : \mathscr{U} \to \mathscr{V}$ be a positive linear map. If $x, y \in \mathscr{U}$ and* $\begin{pmatrix} x & y \\ y^* & x \end{pmatrix} \succeq 0$, *then* $\begin{pmatrix} \phi(x) & \phi(y) \\ \phi(y^*) & \phi(x) \end{pmatrix} \succeq 0.$

For a proof of Lemma 3, please see Corollary 4.4 of [3].

PROPOSITION 1. *If $\mathscr{B}$ is a unital C^*-algebra and $\phi : \mathscr{B} \to B(H)$ is a unital 2-positive linear map, then*

$$\|\phi(ab) - \phi(a)\phi(b)\|^2 \leqslant \|\phi(aa^*) - \phi(a)\phi(a)^*\| \|\phi(b^*b) - \phi(b)^*\phi(b)\|, \tag{2}$$

for all unitaries $a, b \in \mathscr{B}$.

Proof. Since ϕ is positive, recall that $\phi(x^*) = \phi(x)^*$ for all $x \in \mathscr{B}$. Let $a, b \in \mathscr{B}$ be unitary. Consider the matrix

$$A = \left(\begin{array}{ccc|c} a^*a & a^*b & a^* & a^*(a^*b) \\ b^*a & b^*b & b^* & b^*(a^*b) \\ a & b & a^*a & a^*b \\ \hline (b^*a)a & (b^*a)b & b^*a & b^*b \end{array}\right).$$

Since a, b are unitaries, it follows that $R = b^*b = e$ and

$$T = \begin{pmatrix} a^*a & a^*b & a^* \\ b^*a & b^*b & b^* \\ a & b & a^*a \end{pmatrix} = \begin{pmatrix} a^*(a^*b) \\ b^*(a^*b) \\ a^*b \end{pmatrix} \big((b^*a)a \ (b^*a)b \ b^*a\big) = SS^* = SR^{-1}S^*.$$

Thus Lemma 2 implies that $A \succeq 0$. This is equivalent to

$$\left(\begin{array}{cc|cc} a^*a & a^*b & a^* & a^*(a^*b) \\ b^*a & b^*b & b^* & b^*(a^*b) \\ \hline a & b & a^*a & a^*b \\ (b^*a)a & (b^*a)b & b^*a & b^*b \end{array}\right) \succeq 0. \tag{3}$$

By Lemma 3 applied to the unital positive map ϕ_2 and the 2×2 block matrix in equation (3), it follows that

$$\begin{pmatrix} \phi(a^*a) & \phi(a^*b) & \phi(a)^* & \phi(a^*(a^*b)) \\ \phi(b^*a) & \phi(b^*b) & \phi(b)^* & \phi(b^*(a^*b)) \\ \phi(a) & \phi(b) & \phi(a^*a) & \phi(a^*b) \\ \phi((b^*a)a) & \phi((b^*a)b) & \phi(b^*a) & \phi(b^*b) \end{pmatrix} \succeq 0. \tag{4}$$

This in turn implies that

$$\begin{pmatrix} \phi(a^*a) & \phi(a^*b) & \phi(a)^* \\ \phi(b^*a) & \phi(b^*b) & \phi(b)^* \\ \phi(a) & \phi(b) & \phi(a^*a) \end{pmatrix} \succeq 0. \tag{5}$$

By Lemma 2 and the fact that $\phi(a^*a) = \phi(e) = 1$, equation (5) is equivalent to

$$\begin{pmatrix} \phi(a^*a) & \phi(a^*b) \\ \phi(b^*a) & \phi(b^*b) \end{pmatrix} - \begin{pmatrix} \phi(a)^* \\ \phi(b)^* \end{pmatrix} \begin{pmatrix} \phi(a) & \phi(b) \end{pmatrix} \succeq 0, \tag{6}$$

i.e.

$$\begin{pmatrix} \phi(a^*a) - \phi(a)^*\phi(a) & \phi(a^*b) - \phi(a)^*\phi(b) \\ \phi(b^*a) - \phi(b)^*\phi(a) & \phi(b^*b) - \phi(b)^*\phi(b) \end{pmatrix} \succeq 0. \tag{7}$$

An application of Lemma 1 to the operator matrix in equation (7) yields

$$\|\phi(a^*b) - \phi(a)^*\phi(b)\|^2 \leqslant \|\phi(a^*a) - \phi(a)^*\phi(a)\| \|\phi(b^*b) - \phi(b)^*\phi(b)\| \tag{8}$$

for all unitaries $a, b \in \mathscr{B}$. Replacing a by a^* in (8) completes the proof. □

The following three theorems are well known.

THEOREM 3. (Fuglede-Putnam) *Let $\mathscr{A}$ be a C^*-algebra. If $x, y \in \mathscr{A}$ are such that x is normal and $xy = yx$, then $x^*y = yx^*$.*

For more on the Fuglede-Putnam theorem, please see [2].

THEOREM 4. (Stinespring's Dilation Theorem) *If $\mathscr{B}$ is a unital C^*-algebra and $\phi : \mathscr{B} \to B(H)$ is a unital completely positive map, then there exist a Hilbert space K, an isometry $v : H \to K$ and a unital *-homomorphism $\pi : \mathscr{B} \to B(K)$ such that $\phi(x) = v^*\pi(x)v$ for all $x \in \mathscr{B}$.*

For a proof of Stinespring's dilation theorem, please see [6].

THEOREM 5. (Russo-Dye) *Let $\mathscr{A}$ be a unital C^*-algebra. If $a \in \mathscr{A}$ is such that $\|a\| < 1$, then a is a convex combination of unitary elements in $\mathscr{A}$.*

For a proof and more on the Russo-Dye theorem, please see [2].

3. The Proof

This section contains the proof of our main result, i.e. Theorem 2. The following theorem and corollary lead us to it.

THEOREM 6. *If a, b are commuting normal elements in the unital C^*-algebra $\mathscr{A}$ and $\phi : \mathscr{A} \to B(H)$ is a unital positive linear map, then*

$$\|\phi(ab) - \phi(a)\phi(b)\| \leqslant \left(\inf_{\lambda \in \mathbb{C}} \|a - \lambda e\|\right) \left(\inf_{\mu \in \mathbb{C}} \|b - \mu e\|\right), \tag{9}$$

i.e. the Grüss inequality holds for such $a, b \in \mathscr{A}$.

Proof. The proof is adapted from [7]. Let $\lambda,\mu \in \mathbb{C}$. Since a,b are commuting normal elements in the C^*-algebra $\mathscr{A}$, it follows from Theorem 3 that the C^*-subalgebra of $\mathscr{A}$, say $\mathscr{B}$, generated by a,b and e is commutative. Since the restricted map $\phi : \mathscr{B} \to B(H)$ is positive and $\mathscr{B}$ is commutative, it follows that $\phi : \mathscr{B} \to B(H)$ is in fact completely positive (see e.g. [6]). By Theorem 4, it follows that there exist a Hilbert space K, an isometry $v : H \to K$ and a unital *-homomorphism $\pi : \mathscr{B} \to B(K)$ such that $\phi(x) = v^*\pi(x)v$ for all $x \in \mathscr{B}$. Since π is a unital *-homomorphism, it is completely positive and hence is a complete contraction. In particular $\|\pi\| \leqslant 1$. It follows that

$$\begin{aligned}
\|\phi(ab)-\phi(a)\phi(b)\| &= \|\phi((a-\lambda e)(b-\mu e))-\phi(a-\lambda e)\phi(b-\mu e)\| \\
&= \|v^*\pi((a-\lambda e)(b-\mu e))v - v^*\pi(a-\lambda e)vv^*\pi(b-\mu e)v\| \\
&= \|v^*\pi(a-\lambda e)\pi(b-\mu e)v - v^*\pi(a-\lambda e)vv^*\pi(b-\mu e)v\| \\
&= \|v^*\pi(a-\lambda e)(1-vv^*)\pi(b-\mu e)v\| \\
&\leqslant \|a-\lambda e\|\|1-vv^*\|\|b-\mu e\| \\
&\leqslant \|a-\lambda e\|\|b-\mu e\|.
\end{aligned}$$

The proof is complete by taking infimums on the above inequality first with respect to λ and then with respect to μ. □

REMARK 1. It is easy to see that if $\mathscr{A}$ is commutative or ϕ is completely positive, in the statement of Theorem 6, then the entire proof of Theorem 6 goes through with $\mathscr{B}$ replaced by $\mathscr{A}$, for arbitrary a and b, i.e. the Grüss inequality (9) holds if $\mathscr{A}$ is commutative or ϕ is completely positive.

COROLLARY 1. *If ϕ and a are as in Theorem 6, then*

$$\|\phi(aa^*)-\phi(a)\phi(a)^*\| \leqslant \left(\inf_{\lambda\in\mathbb{C}}\|a-\lambda e\|\right)^2.$$

Proof. The proof follows by taking $b = a^*$ in Theorem 6. □

Proof of Theorem 2. Recall a, b, $\mathscr{A}$, H and ϕ from the statement of Theorem 2. Let $\varepsilon > 0$. By Theorem 5, there exist unitary elements $u_1,\dots,u_k$ and $v_1,\dots,v_\ell$ in $\mathscr{A}$ such that $\frac{a}{(\|a\|+\varepsilon)} = \sum_{i=1}^k \alpha_i u_i$ and $\frac{b}{(\|b\|+\varepsilon)} = \sum_{j=1}^\ell \beta_j v_j$, where $\alpha_i,\beta_j \geqslant 0$ and $\sum_{i=1}^k \alpha_i = \sum_{j=1}^\ell \beta_j = 1$. It follows from Proposition 1 and Corollary 1 that

$$\begin{aligned}
&\frac{1}{(\|a\|+\varepsilon)}\frac{1}{(\|b\|+\varepsilon)}\|\phi(ab)-\phi(a)\phi(b)\| \\
&\qquad = \left\|\phi\left(\left(\sum_{i=1}^k \alpha_i u_i\right)\left(\sum_{j=1}^\ell \beta_j v_j\right)\right) - \phi\left(\sum_{i=1}^k \alpha_i u_i\right)\phi\left(\sum_{j=1}^\ell \beta_j v_j\right)\right\| \\
&\qquad \leqslant \sum_{i=1}^k\sum_{j=1}^\ell \alpha_i\beta_j\|\phi(u_iv_j)-\phi(u_i)\phi(v_j)\| \qquad (10)
\end{aligned}$$

$$\begin{aligned}
&\leqslant \sum_{i=1}^{k}\sum_{j=1}^{\ell}\alpha_i\beta_j\|\phi(u_iu_i^*)-\phi(u_i)\phi(u_i)^*\|^{\frac{1}{2}}\|\phi(v_j^*v_j)-\phi(v_j)^*\phi(v_j)\|^{\frac{1}{2}} \qquad (11)\\
&\leqslant \sum_{i=1}^{k}\sum_{j=1}^{\ell}\alpha_i\beta_j\left(\inf_{\lambda\in\mathbb{C}}\|u_i-\lambda e\|\right)\left(\inf_{\mu\in\mathbb{C}}\|v_j-\mu e\|\right)\\
&\leqslant \sum_{i=1}^{k}\sum_{j=1}^{\ell}\alpha_i\beta_j\|u_i\|\|v_j\|\\
&= \sum_{i=1}^{k}\sum_{j=1}^{\ell}\alpha_i\beta_j\\
&= \left(\sum_{i=1}^{k}\alpha_i\right)\left(\sum_{j=1}^{\ell}\beta_j\right)\\
&= 1.
\end{aligned}$$

Letting $\varepsilon \to 0$ above yields,

$$\|\phi(ab)-\phi(a)\phi(b)\| \leqslant \|a\|\|b\|. \tag{12}$$

Let $\lambda,\mu \in \mathbb{C}$ be arbitrary. It follows from equation (12) that

$$\begin{aligned}
\|\phi(ab)-\phi(a)\phi(b)\| &= \|\phi((a-\lambda e)(b-\mu e))-\phi(a-\lambda e)\phi(b-\mu e)\|\\
&\leqslant \|(a-\lambda e)\|\|(b-\mu e)\|.
\end{aligned}$$

Taking infimums in the above inequality, first with respect to λ and then with respect to μ completes the proof. □

The Grüss inequality fails, in general, when ϕ in Theorem 2 is assumed only to be positive, i.e. when $n=1$, as the following example shows. We point out that [5] contains an example of such a map $\phi: M_2(\mathbb{C}) \to M_2(\mathbb{C})$.

EXAMPLE. Let $k \geqslant 2$, $\beta = \{e_1,e_2,\dots,e_k\}$ be an orthonormal set in H, $E = \operatorname{span}(\beta)$, and $\theta: M_k(\mathbb{C}) \to M_k(\mathbb{C})$ denote the transpose map. It is well known that θ is a unital positive linear map, which is not 2-positive (see [10]). Define $\phi: M_k(\mathbb{C}) \to B(H)$ by $\phi(a) = \begin{pmatrix}\theta(a) & \mathbf{0}\\ \mathbf{0} & \mathbf{1}\end{pmatrix}$. The block structure is with respect to the orthogonal decomposition $E\oplus E^{\perp}$ of H. Here $\mathbf{1}$ denotes the identity operator and $\mathbf{0}$ denotes the zero operator. It is easy to see that ϕ is a unital positive linear map which is not 2-positive. Let $a = \begin{pmatrix}1 & 3\\ 3 & 3\end{pmatrix}\oplus \mathbf{0}_{k-2} \in M_k(\mathbb{C})$ and $b = \begin{pmatrix}1 & 0\\ 0 & 3\end{pmatrix}\oplus \mathbf{0}_{k-2} \in M_k(\mathbb{C})$. A simple computation shows that the eigenvalues of a belong to $\{0, 2\pm\sqrt{10}\}$ and those of b belong to $\{0,1,3\}$. Since a and b are normal, it follows from [9] that,

$$\inf_{\lambda\in\mathbb{C}}\|a-\lambda e\| = \sqrt{10} \quad \text{and} \quad \inf_{\mu\in\mathbb{C}}\|b-\mu e\| = \frac{3}{2}. \tag{13}$$

Moreover

$$\begin{aligned}\phi(ab)-\phi(a)\phi(b) &= \left(\left(\begin{pmatrix}1 & 3\\ 9 & 9\end{pmatrix}\oplus \mathbf{0}_{k-2}\right)\oplus \mathbf{1}\right)-\left(\left(\begin{pmatrix}1 & 9\\ 3 & 9\end{pmatrix}\oplus \mathbf{0}_{k-2}\right)\oplus \mathbf{1}\right)\\ &= \left(\left(\begin{pmatrix}0 & -6\\ 6 & 0\end{pmatrix}\oplus \mathbf{0}_{k-2}\right)\oplus \mathbf{0}\right).\end{aligned}$$

Thus,

$$\|\phi(ab)-\phi(a)\phi(b)\| = 6 > \sqrt{10}\cdot\frac{3}{2} = \left(\inf_{\lambda\in\mathbb{C}}\|a-\lambda e\|\right)\left(\inf_{\mu\in\mathbb{C}}\|b-\mu e\|\right).$$

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Sriram Balasubramanian
Department of Mathematics
IIT Madras, Chennai – 600036, India
e-mail: bsriram@iitm.ac.in, bsriram80@yahoo.co.in