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NON–SELF–ADJOINT FOURTH–ORDER DISSIPATIVE OPERATORS AND THE COMPLETENESS OF THEIR EIGENFUNCTIONS

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Abstract. A class of non-self-adjoint fourth order differential operators with general separated boundary conditions in Weyl's limit circle case is studied. The dissipation property of the considered operators in $L^2[a,b)$ is proven by analysis and by using the characteristic determinant, the completeness of the system of eigenfunctions and associated functions of these dissipative operators also be proven.

1. Introduction

Non-self-adjoint spectral problems have more and more applications. For example, interesting non-classical wavelets can be obtained from eigenfunctions and associated functions for non-self-adjoint spectral problems. Thus, such problems are receiving more and more attention, especially the discreteness of the spectrum and the completeness of eigenfunctions.

The non-self-adjointness of spectral problems can be caused by one or more of the following factors: the non-linear dependence of the problems on the spectral parameter, the non-symmetry of the differential expressions used, and the non-self-adjointness of the boundary conditions (BCs) involved.

Non-self-adjoint spectral problems associated with differential operators having only a discrete spectrum and depending polynomially on the spectral parameter have been considered by Gohberg and Krein [1] and by Keldysh [2]. They studied the spectrum and principal functions of such problems and showed the completeness of the principal functions in the corresponding Hilbert function spaces.

Non-self-adjoint Sturm-Liouville operators generated in $L_2(\mathbb{R}_+)$, by differential expression

$$l_0(y) = -y'' + q(x)y,\ x \in \mathbb{R}_+ = [0, +\infty)$$

together with J-self-adjoint BCs, where $q(x)$ is a complex valued function, have been investigated in [3, 4, 5].

Non-self-adjoint differential operators generated by symmetric differential expressions together with non-self-adjoint BCs have been investigated in [6, 7, 8, 9, 10, 11,

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12, 13, 14]. The determinant of perturbation connected with the dissipative operator L generated in $L^2[a,b)$ by the Sturm-Liouville differential expression in Weyl's limit circle case has been studied by Bairamov and Ugurlu in [10], they using the Livšic theorem, investigated the problem of completeness of the system of eigenfunctions and associated functions of L.

There are also some results for fourth order dissipative operators [11, 12, 13, 14]. However, these studies only restricted into some special boundary conditions. For second order Sturm-Liouville differential expression, Z. Wang and H. Wu in [7] give all the non-self-adjoint boundary conditions which generate the operators dissipative. And in [15, 16] the authors show that the fourth order boundary conditions are classified into three types: separated, mixed and coupled, and the canonical forms for self-adjoint BCs of each type are given.

Based on [15, 16] and following the ideas of [7] and [12], in this paper, a class of non-self-adjoint fourth order differential operators in Weyl's limit circle case with general separated BCs is investigated. The results here are more general than previous known results, and the process is much complicated.

This paper is organized as follows. In Section 2, we introduce our notation and recall some basic results. The dissipation of the fourth order operator is proved in Section 3. In Section 4, we review the characteristic function and the characteristic determinant. The completeness of eigenfunctions and associated functions is studied in Section 5.

2. Preliminaries

Consider the fourth order differential expression

$$l(y) = y^{(4)} + q(x)y, \quad on \quad x \in I = [a,b), \tag{2.1}$$

where $-\infty < a < b \leqslant +\infty$, $q(x)$ is a real-valued function on I and $q(x) \in L^1_{loc}(I)$. Suppose that the endpoint a is regular, the endpoint b is singular, and the Weyl's limit-circle case holds for the differential expression $l(y)$. There are several discussions about the Weyl's limit-circle theory or applications [6, 17, 18, 19, 20, 21].

Let

$$\Omega = \{y \in L^2(I) : y, y', y'', y''' \in AC_{loc}(I), l(y) \in L^2(I)\}.$$

For all $y,z \in \Omega$, we set

$$[y,z]_x = -y\overline{z'''} + y'\overline{z''} - y''\overline{z'} + y'''\overline{z} = -R_{\overline{z}}(x)Q(x)C_y(x), \;\; x \in I,$$

where the bar over a function denotes complex conjugate, and

$$Q(x) = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \;\; R_z(x) = (z(x), z'(x), z''(x), z'''(x)), \;\; C_{\overline{z}}(x) = R_z^*(x),$$

and $R_z^*(x)$ is complex conjugate transpose of $R_z(x)$.

Let $l(y) = \lambda y$, and we consider the boundary value problem consisting of the differential equation

$$y^{(4)} + q(x)y = \lambda y \quad \text{on} \quad x \in I, \tag{2.2}$$

and the BC :

$$l_1(y) = \gamma_1 y(a) + \gamma_2 y'(a) - y'''(a) = 0, \tag{2.3}$$

$$l_2(y) = \gamma_2 y(a) + \gamma_3 y'(a) + y''(a) = 0, \tag{2.4}$$

$$l_3(y) = \gamma_5 [y, z_1]_b + \gamma_4 [y, z_2]_b - [y, z_4]_b = 0, \tag{2.5}$$

$$l_4(y) = \gamma_4 [y, z_1]_b + \gamma_6 [y, z_2]_b + [y, z_3]_b = 0, \tag{2.6}$$

where λ is a complex parameter, γ_i $(i = 1,2,3,4)$ are real numbers with $\gamma_1\gamma_3 - \gamma_2^2 > 0$ (hence $\gamma_1\gamma_3 \neq 0$), γ_i $(i = 5,6)$ are complex numbers with $\Im(\gamma_5 + \gamma_6) \geqslant 0$ and $4\Im\gamma_5\Im\gamma_6 \geqslant |\frac{\gamma_2}{\gamma_3}\gamma_5 - \frac{\gamma_2}{\gamma_1}\overline{\gamma_6}|^2$ and $z_i(x) \in L^2(I)$ $(i = 1,2,3,4)$, which are given later, are the linearly independent solutions of equation $l(y) = 0$ on interval I.

In $L^2(I)$, define the operator L as $Ly = l(y)$ on $D(L)$, where

$$D(L) = \{y \in \Omega |\ l_i(y) = 0, i = 1,2,3,4\}.$$

Denote by $\phi_i(x,\lambda)$ $(i = 1,2,3,4)$ the solutions of Eq. (2.2), satisfying the initial conditions

$$(C_{\phi_1}(a,\lambda), C_{\phi_2}(a,\lambda), C_{\phi_3}(a,\lambda), C_{\phi_4}(a,\lambda)) = \begin{pmatrix} 1 & -\frac{\gamma_2}{\gamma_1} & 0 & \frac{1}{\gamma_1} \\ -\frac{\gamma_2}{\gamma_3} & 1 & -\frac{1}{\gamma_3} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.7}$$

Since Weyl's limit-circle case holds for the differential expression $l(y)$ on I, the solutions $\phi_i(x,\lambda)$ $(i = 1,2,3,4)$ belong to $L^2(I)$. Let $z_i(x) = \phi_i(x,0)$ $(i = 1,2,3,4)$, so $z_i(x)$ $(i = 1,2,3,4)$ are the solutions of the equation $l(y) = 0$ $(x \in I)$, satisfying the initial conditions

$$(C_{z_1}(a), C_{z_2}(a), C_{z_3}(a), C_{z_4}(a)) = \begin{pmatrix} 1 & -\frac{\gamma_2}{\gamma_1} & 0 & \frac{1}{\gamma_1} \\ -\frac{\gamma_2}{\gamma_3} & 1 & -\frac{1}{\gamma_3} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.8}$$

and $z_i(x) \in L^2(I)$ $(i = 1,2,3,4)$; moreover $z_i(x) \in \Omega$ $(i = 1,2,3,4)$. Consequently, for each $y \in \Omega$, the values $[y, z_i]_b$ $(i = 1,2,3,4)$ exist and are finite.

Let $\Phi(x)$ be the Wronskian matrix of the solutions $z_i(x)$ $(i = 1,2,3,4)$ in I, i.e.

$$\Phi(x) = (C_{z_1}(x), C_{z_2}(x), C_{z_3}(x), C_{z_4}(x)).$$

From

$$[z_i, z_j]_x = -R_{\overline{z}_j}(x)Q(x)C_{z_i}(x) \ \ (1 \leqslant i,j \leqslant 4),$$

we have

$$([z_i, z_j]_x)^T = -\Phi^*(x)Q(x)\Phi(x) \ \ (1 \leqslant i,j \leqslant 4).$$

By the nonzero constant of the Wronskian of the solutions $z_1(x), z_2(x), z_3(x), z_4(x)$ in I, and from (2.8) one obtain

$$([z_i, z_j]_x)^T = -\Phi^*(x)Q(x)\Phi(x) = J \quad (1 \leqslant i, j \leqslant 4), \tag{2.9}$$

where

$$J = -\Phi^*(x)Q(x)\Phi(x) = \begin{pmatrix} 0 & 0 & \frac{\gamma_2}{\gamma_3} & 1 \\ 0 & 0 & -1 & -\frac{\gamma_2}{\gamma_1} \\ -\frac{\gamma_2}{\gamma_3} & 1 & 0 & 0 \\ -1 & \frac{\gamma_2}{\gamma_1} & 0 & 0 \end{pmatrix},$$

and

$$J^{-1} = \frac{1}{1-\frac{\gamma_2^2}{\gamma_1\gamma_3}} \begin{pmatrix} 0 & 0 & \frac{\gamma_2}{\gamma_1} & -1 \\ 0 & 0 & 1 & -\frac{\gamma_2}{\gamma_3} \\ -\frac{\gamma_2}{\gamma_1} & -1 & 0 & 0 \\ 1 & \frac{\gamma_2}{\gamma_3} & 0 & 0 \end{pmatrix}.$$

By deduction above the following lemmas follows.

LEMMA 1. *(see [17], page 261.)*

$$Q(x) = -(\Phi^*(x))^{-1} J \Phi^{-1}(x), \quad x \in I.$$

LEMMA 2. *For arbitrary $y \in D(L)$*

$$([y, z_1]_x, [y, z_2]_x, [y, z_3]_x, [y, z_4]_x)^T = J\Phi^{-1}(x)C_y(x), \quad x \in I.$$

COROLLARY 1. *For arbitrary $y_1, y_2, y_3, y_4 \in D(L)$, let $Y(x) = (C_{y_1}(x), C_{y_2}(x), C_{y_3}(x), C_{y_4}(x))$ be the Wronskian matrix of y_1, y_2, y_3, y_4, then*

$$J\Phi^{-1}(x)Y(x) = \begin{pmatrix} [y_1,z_1]_x & [y_2,z_1]_x & [y_3,z_1]_x & [y_4,z_1]_x \\ [y_1,z_2]_x & [y_2,z_2]_x & [y_3,z_2]_x & [y_4,z_2]_x \\ [y_1,z_3]_x & [y_2,z_3]_x & [y_3,z_3]_x & [y_4,z_3]_x \\ [y_1,z_4]_x & [y_2,z_4]_x & [y_3,z_4]_x & [y_4,z_4]_x \end{pmatrix}, \quad x \in I.$$

LEMMA 3. *For arbitrary $y, z \in D(L)$*

$$\begin{aligned} [y,z]_x = \frac{1}{1-\frac{\gamma_2^2}{\gamma_1\gamma_3}} \Big(& -\frac{\gamma_2}{\gamma_1}\overline{[z,z_1]_x}[y,z_3]_x + \overline{[z,z_1]_x}[y,z_4]_x - \overline{[z,z_2]_x}[y,z_3]_x + \frac{\gamma_2}{\gamma_3}\overline{[z,z_2]_x}[y,z_4]_x \\ & + \frac{\gamma_2}{\gamma_1}\overline{[z,z_3]_x}[y,z_1]_x + \overline{[z,z_3]_x}[y,z_2]_x - \overline{[z,z_4]_x}[y,z_1]_x - \frac{\gamma_2}{\gamma_3}\overline{[z,z_4]_x}[y,z_2]_x \Big), \quad x \in I. \end{aligned}$$

Proof. By Lemma 1 and Lemma 2, it follows

$$
\begin{aligned}
[y,z]_x &= -R_{\bar z}(x)Q(x)C_y(x) = R_{\bar z}(x)(\Phi^*(x))^{-1}J\Phi^{-1}(x)C_y(x)\\
&= (\Phi^{-1}(x)C_z(x))^* J(\Phi^{-1}(x)C_y(x))\\
&= \left(J^{-1}\begin{pmatrix}[z,z_1]_x\\ [z,z_2]_x\\ [z,z_3]_x\\ [z,z_4]_x\end{pmatrix}\right)^* J\left(J^{-1}\begin{pmatrix}[y,z_1]_x\\ [y,z_2]_x\\ [y,z_3]_x\\ [y,z_4]_x\end{pmatrix}\right)\\
&= \left(\frac{1}{1-\frac{\gamma_2^2}{\gamma_1\gamma_3}}\right)^2\left(\begin{pmatrix}0&0&\frac{\gamma_2}{\gamma_1}&-1\\ 0&0&1&-\frac{\gamma_2}{\gamma_3}\\ -\frac{\gamma_2}{\gamma_1}&-1&0&0\\ 1&\frac{\gamma_2}{\gamma_3}&0&0\end{pmatrix}\begin{pmatrix}[z,z_1]_x\\ [z,z_2]_x\\ [z,z_3]_x\\ [z,z_4]_x\end{pmatrix}\right)^*
\end{aligned}
$$

$$
\begin{pmatrix}0&0&\frac{\gamma_2}{\gamma_3}&1\\ 0&0&-1&-\frac{\gamma_2}{\gamma_1}\\ -\frac{\gamma_2}{\gamma_3}&1&0&0\\ -1&\frac{\gamma_2}{\gamma_1}&0&0\end{pmatrix}\left(\begin{pmatrix}0&0&\frac{\gamma_2}{\gamma_1}&-1\\ 0&0&1&-\frac{\gamma_2}{\gamma_3}\\ -\frac{\gamma_2}{\gamma_1}&-1&0&0\\ 1&\frac{\gamma_2}{\gamma_3}&0&0\end{pmatrix}\begin{pmatrix}[y,z_1]_x\\ [y,z_2]_x\\ [y,z_3]_x\\ [y,z_4]_x\end{pmatrix}\right)
$$

$$
\begin{aligned}
&= \frac{1}{1-\frac{\gamma_2^2}{\gamma_1\gamma_3}}\left(-\overline{[z,z_1]}_x,\ -\overline{[z,z_2]}_x,\ -\overline{[z,z_3]}_x,\ -\overline{[z,z_4]}_x\right)\begin{pmatrix}\frac{\gamma_2}{\gamma_1}[y,z_3]_x-[y,z_4]_x\\ [y,z_3]_x-\frac{\gamma_2}{\gamma_3}[y,z_4]_x\\ -\frac{\gamma_2}{\gamma_1}[y,z_1]_x-[y,z_2]_x\\ [y,z_1]_x+\frac{\gamma_2}{\gamma_3}[y,z_2]_x\end{pmatrix}\\
&= \frac{1}{1-\frac{\gamma_2^2}{\gamma_1\gamma_3}}\Big(-\frac{\gamma_2}{\gamma_1}\overline{[z,z_1]}_x[y,z_3]_x+\overline{[z,z_1]}_x[y,z_4]_x-\overline{[z,z_2]}_x[y,z_3]_x+\frac{\gamma_2}{\gamma_3}\overline{[z,z_2]}_x[y,z_4]_x\\
&\quad+\frac{\gamma_2}{\gamma_1}\overline{[z,z_3]}_x[y,z_1]_x+\overline{[z,z_3]}_x[y,z_2]_x-\overline{[z,z_4]}_x[y,z_1]_x-\frac{\gamma_2}{\gamma_3}\overline{[z,z_4]}_x[y,z_2]_x\Big). \qquad \square
\end{aligned}
$$

3. Dissipative operators

At first, we give the definition of dissipative operator.

DEFINITION 1. (see [1]) A linear operator L, acting in the Hilbert space $L^2(I)$ and having domain $D(L)$, is said to be dissipative if $\Im(Lf,f) \geqslant 0$, $\forall f \in D(L)$.

For a bounded operator L (defined on Hilbert space $L^2(I)$), we can always introduce two operators

$$L_{\Im} = \frac{L-L^*}{2i}, \quad L_{\Re} = \frac{L+L^*}{2i}.$$

The condition of dissipation is equivalent to the condition that its imaginary component $L_{\Im}$ is nonnegative. The operators $L_{\Re}$ and $L_{\Im}$ are self-adjoint operators, and $L = L_{\Re} + iL_{\Im}$, $L^* = L_{\Re} - iL_{\Im}$.

Now set

$$\begin{cases} v_3(x) = \gamma_5 z_1(x) + \gamma_4 z_2(x) - z_4(x), \\ v_4(x) = \gamma_4 z_1(x) + \gamma_6 z_2(x) + z_3(x), \end{cases} x \in L^2(I). \tag{3.1}$$

It is clear that the solutions $v_3(x)$, $v_4(x)$ satisfy the boundary conditions (2.5) and (2.6), $z_1(x)$, $z_2(x)$ satisfy the boundary conditions (2.3) and (2.4).

REMARK 1. $v_3(x)$ does not satisfy the boundary condition (2.6), but satisfy (2.5); $v_4(x)$ does not satisfy the boundary condition (2.5), but satisfy (2.6); $z_1(x)$ and $z_2(x)$ satisfy the boundary conditions (2.3) and (2.4).

Setting

$$\theta = det(C_{z_1}(x), C_{z_2}(x), C_{v_3}(x), C_{v_4}(x)), \quad x \in I.$$

Then we have the following property of the operator L defined in this paper, which is also a general property of dissipative operators.

LEMMA 4. *Zero is not an eigenvalue of L; i.e. $kerL = \{0\}$.*

Proof. Let $y \in D(L)$ and $Ly = 0$, then $y^{(4)} + q(x)y = 0$ and the function y satisfies the boundary conditions (2.3)–(2.6). Therefore there exist constants c_1, c_2, c_3, c_4 such that

$$y = c_1 z_1(x) + c_2 z_2(x) + c_3 v_3(x) + c_4 v_4(x), \quad x \in I.$$

The substitution y in the conditions (2.3)–(2.6) and the function $\theta \neq 0$, we find $c_i = 0$ $(i = 1,2,3,4)$. $\square$

THEOREM 1. *The operator L is dissipative in $L^2(I)$, i.e.*

$$\Im(Ly, y) \geqslant 0, \quad \forall y \in D(L).$$

Proof. For each $y \in D(L)$, from the Green's formula, it follows

$$(Ly, y) - (y, Ly) = [y, y]_b - [y, y]_a. \tag{3.2}$$

Since $y \in D(L)$ by the boundary conditions (2.3) and (2.4), one get

$$[y, y]_a = -y\overline{y'''}(a) + y'\overline{y''}(a) - y''\overline{y'}(a) + y'''\overline{y}(a) = 0. \tag{3.3}$$

Using the boundary conditions (2.5), (2.6) and by Lemma 3, it can be obtained that

$$\begin{aligned}
[y, y]_b = {} & \frac{1}{1 - \frac{\gamma_2^2}{\gamma_1 \gamma_3}} \Big(-\frac{\gamma_2}{\gamma_1} \overline{[y, z_1]}_b [y, z_3]_b + \overline{[y, z_1]}_b [y, z_4]_b - \overline{[y, z_2]}_b [y, z_3]_b + \frac{\gamma_2}{\gamma_3} \overline{[y, z_2]}_b [y, z_4]_b \\
& + \frac{\gamma_2}{\gamma_1} \overline{[y, z_3]}_b [y, z_1]_b + \overline{[y, z_3]}_b [y, z_2]_b - \overline{[y, z_4]}_b [y, z_1]_b - \frac{\gamma_2}{\gamma_3} \overline{[y, z_4]}_b [y, z_2]_b \Big) \\
= {} & \frac{1}{1 - \frac{\gamma_2^2}{\gamma_1 \gamma_3}} \Big((\gamma_5 - \overline{\gamma_5}) [y, z_1]_b \overline{[y, z_1]}_b + (\gamma_6 - \overline{\gamma_6}) [y, z_2]_b \overline{[y, z_2]}_b \\
& + \left(\frac{\gamma_2}{\gamma_1} \gamma_6 - \frac{\gamma_2}{\gamma_3} \overline{\gamma_5} \right) \overline{[y, z_1]}_b [y, z_2]_b + \left(\frac{\gamma_2}{\gamma_3} \gamma_5 - \frac{\gamma_2}{\gamma_1} \overline{\gamma_6} \right) [y, z_1]_b \overline{[y, z_2]}_b \Big).
\end{aligned} \tag{3.4}$$

Now inserting (3.3), (3.4) into (3.2), ones have

$$2i\Im(Ly,y) = (Ly,y)-(y,Ly) = \frac{1}{1-\frac{\gamma_2^2}{\gamma_1\gamma_3}}(\overline{[y,z_1]_b},\overline{[y,z_2]_b},\overline{[y,z_3]_b},\overline{[y,z_4]_b})$$

$$\begin{pmatrix} \gamma_5-\overline{\gamma_5} & \frac{\gamma_2}{\gamma_3}\gamma_5-\frac{\gamma_2}{\gamma_1}\overline{\gamma_6} & 0 & 0 \\ \frac{\gamma_2}{\gamma_1}\gamma_6-\frac{\gamma_2}{\gamma_3}\overline{\gamma_5} & \gamma_6-\overline{\gamma_6} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} [y,z_1]_b \\ [y,z_2]_b \\ [y,z_3]_b \\ [y,z_4]_b \end{pmatrix}, \tag{3.5}$$

and hence

$$2\Im(Ly,y) = \frac{1}{1-\frac{\gamma_2^2}{\gamma_1\gamma_3}}(\overline{[y,z_1]_b},\overline{[y,z_2]_b},\overline{[y,z_3]_b},\overline{[y,z_4]_b}) \begin{pmatrix} r & c & 0 & 0 \\ \overline{c} & s & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} [y,z_1]_b \\ [y,z_2]_b \\ [y,z_3]_b \\ [y,z_4]_b \end{pmatrix}, \tag{3.6}$$

where

$$r = 2\Im\gamma_5, \quad s = 2\Im\gamma_6, \quad c = i(\frac{\gamma_2}{\gamma_3}\gamma_5-\frac{\gamma_2}{\gamma_1}\overline{\gamma_6}). \tag{3.7}$$

Note that the 4 by 4 matrix in (3.6) is Hermitian. The eigenvalues of the Hermitian matrix are

$$0 \text{ and } \frac{r+s\pm\sqrt{(r-s)^2+4|c|^2}}{2},$$

and they are all non-negative if and only if

$$r+s \geqslant 0, \quad rs \geqslant |c|^2.$$

Since $\Im(\gamma_5+\gamma_6)\geqslant 0$, $4\Im\gamma_5\Im\gamma_6 \geqslant |\frac{\gamma_2}{\gamma_3}\gamma_5-\frac{\gamma_2}{\gamma_1}\overline{\gamma_6}|^2$ and $\gamma_1\gamma_3-\gamma_2^2>0$, it has

$$\Im(Ly,y)\geqslant 0,$$

i.e. L is dissipative in $L^2(I)$. □

THEOREM 2. *Let the notations of (3.7) hold. And if $r>0$, $s>0$ and $rs>|c|^2$, then the operator L has no real eigenvalue.*

Proof. Let λ_0 be a real eigenvalue of L and let $\phi_0(x)=\phi(x,\lambda_0)\neq 0$ be the corresponding eigenfunction, since

$$\Im(L\phi_0,\phi_0) = \Im(\lambda_0\|\phi_0\|^2) = 0,$$

from (3.6), it follows

$$\Im(L\phi_0,\phi_0) = \frac{1}{2(1-\frac{\gamma_2^2}{\gamma_1\gamma_3})}(\overline{[\phi_0,z_1]_b},\overline{[\phi_0,z_2]_b}) \begin{pmatrix} r & c \\ \overline{c} & s \end{pmatrix} \begin{pmatrix} [\phi_0,z_1]_b \\ [\phi_0,z_2]_b \end{pmatrix} = 0, \tag{3.8}$$

since $r>0,\ s>0$ and $rs>|c|^2$, the matrix $\begin{pmatrix} r & c \\ \overline{c} & s \end{pmatrix}$ is positive definite. So it can be funded that $[\phi_0,z_1]=0$ and $[\phi_0,z_2]=0$. By the boundary conditions (2.5) and (2.6), one has that $[\phi_0,z_3]=0$ and $[\phi_0,z_4]=0$. Let $\tau_0(x)=\tau(x,\lambda_0)$, $\eta_0(x)=\eta(x,\lambda_0)$, $\delta_0(x)=\delta(x,\lambda_0)$ and $\phi_0(x)=\phi(x,\lambda_0)$ be the independent solutions of $l(y)=\lambda_0 y$. By using Corollary 1, then

$$\begin{pmatrix} [\phi_0,z_1]_b & [\tau_0,z_1]_b & [\eta_0,z_1]_b & [\delta_0,z_1]_b \\ [\phi_0,z_2]_b & [\tau_0,z_2]_b & [\eta_0,z_2]_b & [\delta_0,z_2]_b \\ [\phi_0,z_3]_b & [\tau_0,z_3]_b & [\eta_0,z_3]_b & [\delta_0,z_3]_b \\ [\phi_0,z_4]_b & [\tau_0,z_4]_b & [\eta_0,z_4]_b & [\delta_0,z_4]_b \end{pmatrix} = J\Phi^{-1}(b)(C_{\phi_0}(b),C_{\tau_0}(b),C_{\eta_0}(b),C_{\delta_0}(b)).$$

It is evident that the determinant of the left hand side is equal to zero, the value of the Wronskian of the solutions $\tau(x,\lambda_0)$, $\eta(x,\lambda_0)$, $\delta(x,\lambda_0)$ and $\phi(x,\lambda_0)$ is not equal to zero, so the determinant of the right hand side is not equal to zero. This is a contradiction, so the theorem is proven. $\square$

4. Characteristic function and characteristic determinant

In this section, to prepare the operator L for the completeness, we review Green's function and use it to study the inverse of L.

The element $y\in D(L)$, $y\neq 0$, is called a root vector of the operator L corresponding to the eigenvalue λ_0, if all powers of L are defined on this element and $(L-\lambda_0 I)^n y=0$ for some integer $n>0$. The set of all root vectors of L corresponding to the same eigenvalue λ_0 with the vector $y\neq 0$ forms a linear set N_{λ_0} and is called the root lineal. The dimension of the lineal set N_{λ_0} is called the algebraic multiplicity of the eigenvalue λ_0. Consequently the completeness of the system of all eigenvectors and associated vectors of L is equivalent to the completeness of the system of all root vectors of this operator.

Denote the class of all nuclear and the Hilbert-Schmidt operators in $L^2(I)$ by σ_1 and σ_2, respectively. Let $\{\mu_j(L)\}_{j=1}^{\nu(L)}$ be a sequence of all nonzero eigenvalues of $L\in\sigma_p$, $p=1,2$ arranged by considering algebraic multiplicity and with decreasing modulus, where $\nu(L)(\leqslant\infty)$ is a sum of algebraic multiplicities of all nonzero eigenvalues of L. If $L\in\sigma_1$, then $\sum_{j=1}^{\nu(L)}\mu_j(L)$ is called the trace of L and is denoted by trL.

DEFINITION 2. (see [1]) Let g be an entire function. If for each $\varepsilon>0$ there exists a finite constant $C_\varepsilon>0$, such that $|g(\lambda)|\leqslant C_\varepsilon e^{\varepsilon|\lambda|}$, $\lambda\in\mathbb{C}$, then g is called an entire function with growth of order $\leqslant 1$ and minimal type.

For each $\lambda\in\mathbb{C}$, the functions $\phi_1(x,\lambda)$, $\phi_2(x,\lambda)$, $\phi_3(x,\lambda)$, $\phi_4(x,\lambda)$ form a fundamental system of solutions of (2.2), and hence determine the eigenvalues of L.

For all $x\in[a,b)$, setting

$$\psi_{ij}(x,\lambda)=[\phi_i(\cdot,\lambda),z_j(\cdot)]_x,\ \ (i,j=1,2,3,4), \tag{4.1}$$

and

$$\psi_{ij}(\lambda) = [\phi_i(x,\lambda), z_j(x)]_b, \ \ (i,j = 1,2,3,4), \tag{4.2}$$

so $\psi_{ij}(\lambda) = \psi_{ij}(b,\lambda)$ and it is evident that

$$\sigma_d(L) = \{\lambda : \lambda \in \mathbb{C},\ \widetilde{a_i}(\lambda) = 0,\ \widetilde{b_i}(\lambda) = 0,\ i = 1,2,3,4\},$$

where $\sigma_d(L)$ denotes the set of all eigenvalues of L and

$$\begin{cases} \widetilde{a_i}(\lambda) = \gamma_5\psi_{i1}(\lambda) + \gamma_4\psi_{i2}(\lambda) - \psi_{i4}(\lambda), \\ \widetilde{b_i}(\lambda) = \gamma_4\psi_{i1}(\lambda) + \gamma_6\psi_{i2}(\lambda) + \psi_{i3}(\lambda), \end{cases} \ (i = 1,2,3,4).$$

LEMMA 5. *The functions* $\psi_{ij}(\lambda)$ $(i,j = 1,2,3,4)$ *are the entire functions of* λ *with growth order* $\leqslant 1$ *and of minimal type.*

Proof. By (4.1), one has

$$\psi_{b_1,4j}(\lambda) = [\phi_4(x,\lambda), z_j(x)]_{b_1} \ \ (j = 1,2,3,4),$$

where $a \leqslant b_1 < b$, since for arbitrary fixed b_1, the functions $\phi_4(b_1,\lambda)$, $\phi_4'(b_1,\lambda)$, $\phi_4''(b_1,\lambda)$, $\phi_4'''(b_1,\lambda)$ are the entire functions of λ of order $\frac{1}{2}$, consequently, the functions $\psi_{b_1,4j}(\lambda)$ $(j = 1,2,3,4)$ have the same property. Now we prove that the entire functions $\psi_{b_1,4j}(\lambda)$ converge to $\psi_{4j}(\lambda)$ as $b_1 \to b$, uniformly in λ in each compact set of the complex plane $\mathbb{C}$.

Let $y = y(x)$ be the solution of Eq. (2.2), then by Lemma 2 one has

$$\begin{aligned} y(x) = & \frac{1}{1 - \frac{\gamma_2^2}{\gamma_1\gamma_3}} \Big([y,z_1]_x\Big(-\frac{\gamma_2}{\gamma_1}z_3(x) + z_4(x)\Big) - [y,z_2]_x\Big(z_3(x) - \frac{\gamma_2}{\gamma_3}z_3(x)\Big) \\ & + [y,z_3]_x\Big(z_2(x) + \frac{\gamma_2}{\gamma_1}z_1(x)\Big) - [y,z_4]_x\Big(z_1(x) + \frac{\gamma_2}{\gamma_3}z_2(x)\Big)\Big), \ \ x \in I. \end{aligned} \tag{4.3}$$

Let

$$f_j(x,\lambda) = [y,z_j]_x, \ \ (j = 1,2,3,4), \ \ x \in I.$$

Then following Green's formula, one has that $f_j(x,\lambda) = [y,z_j]_x$ satisfy a system of the first order differential equations

$$\frac{\partial}{\partial x} f_j(x,\lambda) = \lambda y(x,\lambda) z_j(x) \ \ (j = 1,2,3,4), \ \ x \in I.$$

Using (4.3) one obtain

$$\frac{\partial}{\partial x} f(x,\lambda) = \lambda G(x) f(x,\lambda), \ \ x \in I, \tag{4.4}$$

where

$$f(x,\lambda) = (f_1(x,\lambda), f_2(x,\lambda), f_3(x,\lambda), f_4(x,\lambda))^T,$$

$$G(x)=\begin{pmatrix} z_1(-\frac{\gamma_2}{\gamma_1}z_3+z_4)(x) & -z_1(z_3-\frac{\gamma_2}{\gamma_3}z_4)(x) & z_1(z_2+\frac{\gamma_2}{\gamma_1}z_1)(x) & -z_1(z_1+\frac{\gamma_2}{\gamma_3}z_2)(x)\\ z_2(-\frac{\gamma_2}{\gamma_1}z_3+z_4)(x) & -z_2(z_3-\frac{\gamma_2}{\gamma_3}z_4)(x) & z_2(z_2+\frac{\gamma_2}{\gamma_1}z_1)(x) & -z_2(z_1+\frac{\gamma_2}{\gamma_3}z_2)(x)\\ z_3(-\frac{\gamma_2}{\gamma_1}z_3+z_4)(x) & -z_3(z_3-\frac{\gamma_2}{\gamma_3}z_4)(x) & z_3(z_2+\frac{\gamma_2}{\gamma_1}z_1)(x) & -z_3(z_1+\frac{\gamma_2}{\gamma_3}z_2)(x)\\ z_4(-\frac{\gamma_2}{\gamma_1}z_3+z_4)(x) & -z_4(z_3-\frac{\gamma_2}{\gamma_3}z_4)(x) & z_4(z_2+\frac{\gamma_2}{\gamma_1}z_1)(x) & -z_4(z_1+\frac{\gamma_2}{\gamma_3}z_2)(x) \end{pmatrix},$$

and the elements of $G(x)$ are in $L^1(I)$. For $\omega=(\omega_1,\ \omega_2,\ \omega_3,\ \omega_4)^T$ put $\|\omega\|=|\omega_1|+|\omega_2|+|\omega_3|+|\omega_4|$, and the norm of a square 4×4 matrix will be denoted by $\|\cdot\|$, the inclusion $\|G(x)\|\in L^1(I)$ holds.

If $y(x,\lambda)=\phi_4(x,\lambda)$, then the system (4.4) is equivalent to the integral equation

$$f(x,\lambda)=f(b_1,\lambda)+\lambda\int_{b_1}^{x}G(t)f(t,\lambda)\mathrm{d}t,\quad x\in I, \tag{4.5}$$

where

$$f(b_1,\lambda)=\begin{pmatrix}\psi_{b_1,41}(\lambda)\\ \psi_{b_1,42}(\lambda)\\ \psi_{b_1,43}(\lambda)\\ \psi_{b_1,44}(\lambda)\end{pmatrix},\quad f(b,\lambda)=\begin{pmatrix}\psi_{41}(\lambda)\\ \psi_{42}(\lambda)\\ \psi_{43}(\lambda)\\ \psi_{44}(\lambda)\end{pmatrix}.$$

Using Gronwall's inequality from (4.5), one finds that

$$\|f(x,\lambda)\|\leqslant\|f(b_1,\lambda)\|\exp(|\lambda|\int_{b_1}^{x}\|G(t)\|\mathrm{d}t),\ \ x\in I.$$

Hence

$$\|f(b,\lambda)-f(b_1,\lambda)\|\leqslant|\lambda|(\int_{b_1}^{b}\|G(t)\|\mathrm{d}t)\exp(|\lambda|\int_{a}^{b}\|G(t)\|\mathrm{d}t), \tag{4.6}$$

$$\|f(b,\lambda)\|\leqslant\|f(b_1,\lambda)\|\exp(|\lambda|\int_{b_1}^{b}\|G(t)\|\mathrm{d}t). \tag{4.7}$$

It follows from (4.6) that $\psi_{b_1,4j}(\lambda)$ converges to $\psi_{4j}(\lambda)$ as $b_1\to b$, uniformly in λ in a compact set. Consequently $\psi_{4j}(\lambda)$ $(j=1,2,3,4)$ are the entire functions of λ. Hence $\psi_{4j}(\lambda)$ are of not higher than first order. Since, for arbitrary fixed b_1, the functions $\psi_{b_1,4j}(\lambda)$ $(j=1,2,3,4)$ are the entire functions of λ of order $\frac{1}{2}$, from (4.7) we obtain that the entire functions $\psi_{4j}(\lambda)$ $(j=1,2,3,4)$ are of growth and minimal type.

Similarly, It can be provn that $\psi_{ij}(\lambda)$ $(i=1,2,3;\ j=1,2,3,4)$ are the entire functions of λ with order $\leqslant 1$, and are of growth and minimal type. Hence the proof is completed. $\square$

Rewrite the boundary conditions (2.3)–(2.6) in matrix form. i.e.

$$\begin{pmatrix}\gamma_1&\gamma_2&0&-1\\ \gamma_2&\gamma_3&1&0\\ 0&0&0&0\\ 0&0&0&0\end{pmatrix}\begin{pmatrix}y(a)\\ y'(a)\\ y''(a)\\ y'''(a)\end{pmatrix}+\begin{pmatrix}0&0&0&0\\ 0&0&0&0\\ \gamma_5&\gamma_4&0&-1\\ \gamma_4&\gamma_6&1&0\end{pmatrix}\begin{pmatrix}[y,z_1]_b\\ [y,z_2]_b\\ [y,z_3]_b\\ [y,z_4]_b\end{pmatrix}=0,$$

and set

$$A = \begin{pmatrix} \gamma_1 & \gamma_2 & 0 & -1 \\ \gamma_2 & \gamma_3 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \gamma_5 & \gamma_4 & 0 & -1 \\ \gamma_4 & \gamma_6 & 1 & 0 \end{pmatrix},$$

then one has the following well known function

$$\Delta(\lambda) = \det(A\Phi(a,\lambda) + B\Phi(b,\lambda)),$$

where

$$\Phi(a,\lambda) = (C_{\phi_1}(a,\lambda), C_{\phi_2}(a,\lambda), C_{\phi_3}(a,\lambda), C_{\phi_4}(a,\lambda)),$$
$$\Phi(b,\lambda) = ([\phi_i(\cdot,\lambda), z_j(\cdot)]_b)^T,$$

By direct calculation

$$\Delta(\lambda) = \frac{(\gamma_1\gamma_3 - \gamma_2^2)^2}{\gamma_1\gamma_3} \det\begin{pmatrix} \widetilde{a}_3(\lambda) & \widetilde{a}_4(\lambda) \\ \widetilde{b}_3(\lambda) & \widetilde{b}_4(\lambda) \end{pmatrix} + \frac{\gamma_2(\gamma_1\gamma_3 - \gamma_2^2)}{\gamma_1\gamma_3} \det\begin{pmatrix} \widetilde{a}_2(\lambda) & \widetilde{a}_3(\lambda) \\ \widetilde{b}_2(\lambda) & \widetilde{b}_3(\lambda) \end{pmatrix}$$
$$+ \frac{\gamma_2(\gamma_1\gamma_3 - \gamma_2^2)}{\gamma_1\gamma_3} \det\begin{pmatrix} \widetilde{a}_1(\lambda) & \widetilde{a}_4(\lambda) \\ \widetilde{b}_1(\lambda) & \widetilde{b}_4(\lambda) \end{pmatrix} - \frac{\gamma_2^2}{\gamma_1\gamma_3} \det\begin{pmatrix} \widetilde{a}_1(\lambda) & \widetilde{a}_2(\lambda) \\ \widetilde{b}_1(\lambda) & \widetilde{b}_2(\lambda) \end{pmatrix}, \tag{4.8}$$

where

$$\widetilde{a}_i(\lambda) = \gamma_5\psi_{i1}(\lambda) + \gamma_4\psi_{i2}(\lambda) - \psi_{i4}(\lambda),$$
$$\widetilde{b}_i(\lambda) = \gamma_4\psi_{i1}(\lambda) + \gamma_6\psi_{i2}(\lambda) + \psi_{i3}(\lambda),$$
$$(i, j = 1,2,3,4).$$

By $\gamma_1\gamma_3 \neq 0$, $\gamma_1\gamma_3 - \gamma_2^2 > 0$ and Lemma 5, $\Delta(\lambda)$ is an entire function of λ.

The function $\Delta(\lambda)$ is called the characteristic function of L, The analytic multiplicity of an eigenvalue λ_0 is the order of λ_0 as a zero of $\Delta(\lambda)$; it is known that the algebraic multiplicity of any eigenvalue of L is equal to the analytic multiplicity of the eigenvalue (see [6]) concerning $\Delta(\lambda)$, then the following direct consequence of Lemma 5 which is well known can be obtained.

LEMMA 6. (see [12]) *A complex number is an eigenvalue of L if and only if it is a zero of the entire function $\Delta(\lambda)$.*

LEMMA 7. (see [22]) *The entire function $\Delta(\lambda)$ is also of growth order $\leqslant 1$ and minimal type: for any $\varepsilon > 0$, there exists a finite constant C_ε such that*

$$|\Delta(\lambda)| \leqslant C_\varepsilon e^{\varepsilon|\lambda|}, \quad \forall\lambda \in \mathbb{C}, \tag{4.9}$$

and hence

$$\limsup_{|\lambda|\to\infty} \frac{\ln|\Delta(\lambda)|}{|\lambda|} \leqslant 0. \tag{4.10}$$

One can deduce the following properties of the zeros of $\Delta(\lambda)$.

LEMMA 8. (see [22]) *Denote by λ_j a sequence of all zeros of $\Delta(\lambda)$ counting analytic multiplicity, then:*

(1) the limit

$$\lim_{r\to\infty}\sum_{|\lambda_j|\leqslant r}\frac{1}{\lambda_j}$$

exists and is finite;

(2) the number $n(r)$ of zeros λ_j lying in the circle $|\lambda_j|\leqslant r$ has a limit

$$\lim_{r\to\infty}\frac{n(r)}{r}=0;$$

(3) when $\Delta(0)\neq 0$, then

$$\Delta(\lambda)=\Delta(0)\lim_{r\to\infty}\Pi_{|\lambda_j|\leqslant r}\Big(1-\frac{\lambda}{\lambda_j}\Big),\quad \forall\lambda\in\mathbb{C}.$$

It is possible that $\Delta\equiv 0$, i.e., every complex number is an eigenvalue of L. However, since $\Delta(\lambda)$ is an entire function of λ and is not a constant or by Lemma 4, this dose not happen when L is dissipative, i.e., it has the following result.

LEMMA 9. (see [7]) *If L is dissipative, then the eigenvalues of L form a discrete subset of $\mathbb{C}$.*

By Lemma 4, we know that zero is not an eigenvalue of L (i.e., $kerL=\{0\}$). Thus, the inverse operator L^{-1} of L exists. To find an explicit formula for L^{-1}, we first calculate the Green's function.

For $y\in D(L)$, the equation $Ly=-f(x)$ is equivalent to the inhomogeneous differential equation

$$l(y)=-f(x),\quad x\in I=[a,b). \tag{4.11}$$

One can represent the general solution of homogeneous differential equation $l(y)=0$ in the form

$$y(x)=c_1z_1(x)+c_2z_2(x)+c_3v_3(x)+c_4v_4(x),\quad x\in I=[a,b), \tag{4.12}$$

where $c_i\ (i=1,2,3,4)$ are arbitrary constants.

By applying the standard method of variation of constants, one shall search the general solution of the inhomogeneous differential equation (4.11) in the form

$$y(x)=C_1(x)z_1(x)+C_2(x)z_2(x)+C_3(x)v_3(x)+C_4(x)v_4(x),\quad x\in I=[a,b), \tag{4.13}$$

where the functions $C_i(x)\ (i=1,2,3,4)$ satisfy the linear system of equations

$$\begin{cases} C_1'(x)z_1(x)+C_2'(x)z_2(x)+C_3'(x)v_3(x)+C_4'(x)v_4(x)=0,\\ C_1'(x)z_1'(x)+C_2'(x)z_2'(x)+C_3'(x)v_3'(x)+C_4'(x)v_4'(x)=0,\\ C_1'(x)z_1''(x)+C_2'(x)z_2''(x)+C_3'(x)v_3''(x)+C_4'(x)v_4''(x)=0,\\ C_1'(x)z_1'''(x)+C_2'(x)z_2'''(x)+C_3'(x)v_3'''(x)+C_4'(x)v_4'''(x)=f(x),\end{cases} \tag{4.14}$$

Calculated properly, one obtain that

$$y(x) = \int_a^b \widetilde{K}(x,t)f(t)\mathrm{d}t + c_1 z_1(x) + c_2 z_2(x) + c_3 v_3(x) + c_4 v_4(x),\ x \in I. \tag{4.15}$$

where $c_i\ (i = 1,2,3,4)$ are arbitrary constants and

$$\widetilde{K}(x,t) = \begin{cases} \frac{Z(x,t)}{\theta}, & a \leqslant t \leqslant x \leqslant b, \\ 0, & a \leqslant x \leqslant t < b, \end{cases}$$

where

$$\theta = det(C_{z_1}(x), C_{z_2}(x), C_{v_3}(x), C_{v_4}(x)),\ \ x \in I,$$

$$Z(x,t) = \begin{vmatrix} z_1(t) & z_2(t) & v_3(t) & v_4(t) \\ z_1'(t) & z_2'(t) & v_3'(t) & v_4'(t) \\ z_1''(t) & z_2''(t) & v_3''(t) & v_4''(t) \\ z_1'''(x) & z_2'''(x) & v_3'''(x) & v_4'''(x) \end{vmatrix}.$$

Substituting (4.15) into (2.3)–(2.6), it has $c_i = 0\ (i = 1,2,3,4)$.

Then

$$y(x) = \int_a^b \widetilde{K}(x,t)f(t)\mathrm{d}t,\ \ x \in I.$$

Set

$$G(x,t) = \begin{cases} \widetilde{K}(x,t), & a \leqslant t \leqslant x \leqslant b, \\ 0, & a \leqslant x \leqslant t < b. \end{cases}$$

Then

$$y(x) = \int_a^b G(x,t)f(t)\mathrm{d}t.$$

Let K denote the integral operator defined by the formula

$$Kf = \int_a^b G(x,t)f(t)\mathrm{d}t,\ \ \forall f \in L^2(I). \tag{4.16}$$

Since $z_1(x),\ z_2(x),\ v_3(x),\ v_4(x) \in L^2(I)$, one has $K \in \sigma_2$. It is evident that $K = L^{-1}$. Consequently the root lineal of the operators L and K coincides, therefore, the completeness in $L^2(I)$ of the system of all eigenvectors and associated vectors of L is equivalent to the completeness of those for K. Since the algebraic multiplicity of nonzero eigenvalues of a compact operator is finite, each eigenvector of L may have only a finite number of linear independent associated vectors.

5. Completeness of eigenfunctions

In this section, by using characteristic determinant, the completeness of the system of eigenfunctions and associated functions of A (let A denote the linear non-self-adjoint operator in the Hilbert space $L^2(I)$ with the domain $D(A)$) is considered. In order to investigate this problem, the following basics and lemmas are needed.

The determinant

$$\det(I-\mu A)=\prod_{j=1}^{\nu(A)}[1-\mu\mu_j(A]),\ \ A\in\sigma_1,$$

is called the characteristic determinant of A and is denoted by $D_A(\mu)$. The characteristic determinant $D_A(\mu)$ is an entire function of μ, since for any $A\in\sigma_1$,

$$\sum_{j=1}^{\nu(A)}|\mu_j|<\infty.$$

For any $A\in\sigma_2$, the regularized characteristic determinant is defined by

$$\widetilde{D}_A(\mu)=\prod_{j=1}^{\nu(A)}[1-\mu\mu_j(A)]e^{\mu\mu_j(A)}. \tag{5.1}$$

If the operator $I-\mu A$ has a bounded inverse defined on the whole space $L^2(I)$, then the complex number μ is called an F-regular point (regular in the sense of Fredholm) for A.

Let A and B be linear bounded operators in $L^2(I)$ and $A-B\in\sigma_1$. If the point μ is an F-regular point of B, then

$$(I-\mu A)(I-\mu B)^{-1}=I-\mu(A-B)(I-\mu B)^{-1},$$

where $\mu(A-B)(I-\mu B)^{-1}\in\sigma_1$. Consequently, the determinant

$$D_{A/B}(\mu)=\det[(I-\mu A)(I-\mu B)^{-1}]$$

is meaningful and is called the determinant of perturbation of the operator B by the operator $K=A-B$.

The following two theorems are well-known.

THEOREM 3. (see [1]) *If $A,\ B\in\sigma_2$, $A-B\in\sigma_1$ and μ is an F-regular point of B, then*

$$D_{A/B}(\mu)=\frac{\widetilde{D}_A(\mu)}{\widetilde{D}_B(\mu)}e^{\mu tr(B-A)}.$$

THEOREM 4. (see [1]) *Let A and B be bounded dissipative operators (in particular, one of them or both may be self-adjoint) and $A-B\in\sigma_1$, then for any β_0 $(0<\beta_0<\frac{\pi}{2})$ the relation.*

$$\lim_{\delta\to\infty}\left[\frac{1}{\delta}|D_{A/B}(\delta e^{i\beta})|\right]=0$$

holds uniformly with respect to β in the sector

$$\left\{\lambda:\lambda=\delta e^{i\beta},0<\delta<\infty,\left|\frac{\pi}{2}-\beta\right|<\beta_0\right\}.$$

THEOREM 5. (Livšic theorem) (see [1]) *Let A be compact dissipative operator and $A_{\Im} \in \sigma_1$. In order that the system of all root vectors of A be complete, it is necessary and sufficient that*

$$\sum_{j=1}^{\nu(A)} \Im \mu_j(A) = trA_{\Im}. \tag{5.2}$$

Now return to the integral operator K defined by (4.16), the inverse of A. Set $K = K_1 + iK_2$ with $K_1 = K_{\Re}$ and $K_2 = K_{\Im}$. By the discussion above, K and K_1 are the Hilbert-Schmidt operators, and K_1 is a self-adjoint Hilbert-Schmidt operator in $L^2(I)$ and K_2 is the self-adjoint with a range space of dimension two. It is easy to verify that K_1 is the inverse of A_1, i.e. $A_1^{-1} = K_1$. Let $T = -K$ and $T = T_1 + iT_2$, where $T_1 = -K_1$, $T_2 = -K_2$.

Denote by λ_j and γ_j the eigenvalues of the operators A and A_1, respectively. Then the eigenvalues of T are $-\frac{1}{\lambda_j}$ and the eigenvalues of T_1 are $-\frac{1}{\gamma_k}$. Since A_1 is a self-adjoint operator, therefore $\Im \gamma_k = 0$ for all k.

THEOREM 6.

$$\sum_j \Im(-\frac{1}{\lambda_j}) = trT_2.$$

Proof. Using Lemma 3 for $A = T_1$ and $B = T$ one obtain

$$D_{T_1/T}(\mu) = \frac{\widetilde{D}_{T_1}(\mu)}{\widetilde{D}_T(\mu)} e^{\mu tr(T-T_1)} = \frac{\widetilde{D}_{T_1}(\mu)}{\widetilde{D}_T(\mu)} e^{i\mu trT_2}. \tag{5.3}$$

By (5.1) one has that

$$\widetilde{D}_T(\mu) = \prod_j \left(1 + \frac{\mu}{\lambda_j}\right) e^{-\frac{\mu}{\lambda_j}}, \quad \widetilde{D}_{T_1}(\mu) = \prod_j \left(1 + \frac{\mu}{\gamma_j}\right) e^{-\frac{\mu}{\gamma_j}}. \tag{5.4}$$

Set $\gamma_5 = \Re\gamma_5 + i\Im\gamma_5$, $\gamma_6 = \Re\gamma_6 + i\Im\gamma_6$, $\widetilde{a_i'}(\mu) = \Re\gamma_5 \psi_{i1}(\mu) + \gamma_4 \psi_{i2}(\mu) - \psi_{i4}(\mu)$, $\widetilde{b_i'}(\mu) = \gamma_4 \psi_{i1}(\mu) + \Re\gamma_6 \psi_{i2}(\mu) + \psi_{i3}(\mu)$, so by (4), one has

$$\begin{aligned}
\Delta(\mu) = & \frac{(\gamma_1\gamma_3 - \gamma_2^2)^2}{\gamma_1\gamma_3} \left[\det \begin{pmatrix} \widetilde{a_3'}(\mu) & \widetilde{a_4'}(\mu) \\ \widetilde{b_3'}(\mu) & \widetilde{b_4'}(\mu) \end{pmatrix} - \Im\gamma_5 \Im\gamma_6 \det \begin{pmatrix} \psi_{31}(\mu) & \psi_{41}(\mu) \\ \psi_{32}(\mu) & \psi_{42}(\mu) \end{pmatrix} \right] \\
& + \frac{\gamma_2(\gamma_1\gamma_3 - \gamma_2^2)}{\gamma_1\gamma_3} \left[\det \begin{pmatrix} \widetilde{a_2'}(\mu) & \widetilde{a_3'}(\mu) \\ \widetilde{b_2'}(\mu) & \widetilde{b_3'}(\mu) \end{pmatrix} - \Im\gamma_5 \Im\gamma_6 \det \begin{pmatrix} \psi_{21}(\mu) & \psi_{31}(\mu) \\ \psi_{22}(\mu) & \psi_{32}(\mu) \end{pmatrix} \right] \\
& + \frac{\gamma_2(\gamma_1\gamma_3 - \gamma_2^2)}{\gamma_1\gamma_3} \left[\det \begin{pmatrix} \widetilde{a_1'}(\mu) & \widetilde{a_4'}(\mu) \\ \widetilde{b_1'}(\mu) & \widetilde{b_4'}(\mu) \end{pmatrix} - \Im\gamma_5 \Im\gamma_6 \det \begin{pmatrix} \psi_{11}(\mu) & \psi_{41}(\mu) \\ \psi_{12}(\mu) & \psi_{42}(\mu) \end{pmatrix} \right] \\
& - \frac{\gamma_2^2}{\gamma_1\gamma_3} [\det \begin{pmatrix} \widetilde{a_1'}(\mu) & \widetilde{a_2'}(\mu) \\ \widetilde{b_1'}(\mu) & \widetilde{b_2'}(\mu) \end{pmatrix} - \Im\gamma_5 \Im\gamma_6 \det \begin{pmatrix} \psi_{11}(\mu) & \psi_{21}(\mu) \\ \psi_{12}(\mu) & \psi_{22}(\mu) \end{pmatrix}].
\end{aligned}$$

it has $\Delta(0) \neq 0$, and

$$\Delta_1(\mu) = \frac{(\gamma_1\gamma_3 - \gamma_2^2)^2}{\gamma_1\gamma_3}\det\begin{pmatrix} \widetilde{a_3'}(\mu) & \widetilde{a_4'}(\mu) \\ \widetilde{b_3'}(\mu) & \widetilde{b_4'}(\mu) \end{pmatrix} + \frac{\gamma_2(\gamma_1\gamma_3 - \gamma_2^2)}{\gamma_1\gamma_3}\det\begin{pmatrix} \widetilde{a_2'}(\mu) & \widetilde{a_3'}(\mu) \\ \widetilde{b_2'}(\mu) & \widetilde{b_3'}(\mu) \end{pmatrix}$$
$$+ \frac{\gamma_2(\gamma_1\gamma_3 - \gamma_2^2)}{\gamma_1\gamma_3}\det\begin{pmatrix} \widetilde{a_1'}(\mu) & \widetilde{a_4'}(\mu) \\ \widetilde{b_1'}(\mu) & \widetilde{b_4'}(\mu) \end{pmatrix} - \frac{\gamma_2^2}{\gamma_1\gamma_3}\det\begin{pmatrix} \widetilde{a_1'}(\mu) & \widetilde{a_2'}(\mu) \\ \widetilde{b_1'}(\mu) & \widetilde{b_2'}(\mu) \end{pmatrix}.$$

By Lemma 8, it has

$$D_{-T}(\mu) = \frac{\triangle(\mu)}{\triangle(0)} = \prod_j \Big(1 - \frac{\mu}{\lambda_j}\Big), \ \ D_{-T_1}(\mu) = \frac{\triangle_1(\mu)}{\triangle_1(0)} = \prod_j \Big(1 - \frac{\mu}{\gamma_j}\Big).$$

Therefore

$$\widetilde{D}_T(\mu) = D_{-T}(-\mu)e^{-\mu\Sigma_j \frac{1}{\lambda_j}}, \ \ \widetilde{D}_{T_1}(\mu) = D_{-T_1}(-\mu)e^{-\mu\Sigma_j \frac{1}{\gamma_j}}, \tag{5.5}$$

and hence

$$\begin{aligned} D_{T_1/T}(\mu) &= \frac{D_{-T_1}(-\mu)e^{-\mu\Sigma_j \frac{1}{\gamma_j}}}{D_{-T}(-\mu)e^{-\mu\Sigma_j \frac{1}{\lambda_j}}} e^{i\mu trT_2} \\ &= \frac{D_{-T_1}(-\mu)}{D_{-T}(-\mu)} \exp\Big(\mu\sum_j \frac{1}{\lambda_j} - \mu\sum_j \frac{1}{\gamma_j} + i\mu trT_2\Big), \quad (\gamma_j \in \mathbb{R}). \end{aligned} \tag{5.6}$$

Note that $\Im\lambda_j \geqslant 0$ for each j since A is dissipative, so by taking $\mu = it$ $(0 < t < \infty)$ in (5.6), then get

$$\frac{1}{t}\ln|D_{T_1/T}(it)| = \frac{1}{t}\ln\Big|\prod_j\Big(1 + \frac{it}{\gamma_j}\Big)\Big| - \frac{1}{t}\ln\Big|\prod_j\Big(1 + \frac{it}{\lambda_j}\Big)\Big| - \sum_j \Im\frac{1}{\lambda_j} - trT_2. \tag{5.7}$$

By virtue of Theorem 4 and (4.10), one has that

$$\lim_{t\to\infty}\frac{1}{t}\ln|D_{T_1/T}(it)| = 0 \tag{5.8}$$

and

$$\limsup_{t\to\infty}\frac{1}{t}\ln\Big|\prod_j\Big(1 + \frac{it}{\gamma_j}\Big)\Big| \leqslant 0, \ \ \limsup_{t\to\infty}\frac{1}{t}\ln\Big|\prod_j\Big(1 + \frac{it}{\lambda_j}\Big)\Big| \leqslant 0. \tag{5.9}$$

On the other hand, for $t > 0$, it has the following estimates: for any $t > 0$ and each j,

$$\Big|1 + \frac{it}{\lambda_j}\Big|^2 = 1 + 2t\frac{\Im\lambda_j}{|\lambda_j|^2} + \frac{t^2}{|\lambda_j|^2} \geqslant 1, \ \ \Big|1 + \frac{it}{\gamma_j}\Big|^2 = 1 + \frac{t^2}{\gamma_j^2} \geqslant 1, \tag{5.10}$$

which imply that

$$\frac{1}{t}\ln\Big|\prod_j\Big(1 + \frac{it}{\gamma_j}\Big)\Big| \geqslant 0, \ \ \frac{1}{t}\ln\Big|\prod_j\Big(1 + \frac{it}{\lambda_j}\Big)\Big| \geqslant 0. \tag{5.11}$$

From (5.9) and (5.11) one deduces that

$$\limsup_{t\to\infty}\frac{1}{t}\ln\left|\prod_j\left(1+\frac{it}{\gamma_j}\right)\right|=0,\ \ \limsup_{t\to\infty}\frac{1}{t}\ln\left|\prod_j\left(1+\frac{it}{\lambda_j}\right)\right|=0. \tag{5.12}$$

Now, taking the limit $t\to\infty$, in (5.7) and making use of (5.8) and (5.12), one get that

$$\sum_j \Im\left(-\frac{1}{\lambda_j}\right)=trT_2. \quad \square \tag{5.13}$$

Therefore, by Livšic's theorem, the system of eigenfunctions and associated functions of $-K$ is complete in $L^2(I)$, and hence the same is true for A.

As a direct consequence of Theorem 6, one has the following fact.

COROLLARY 2. *The dissipative operator A has infinitely many eigenvalues.*

Proof. Since each lineal of A is finite dimensional, the completeness in $L^2(I)$ of the system of eigenfunctions and associated functions of A implies that A has infinitely many eigenvalues. $\square$

THEOREM 7. *The system of all root vectors of the dissipative operator T (also of K)is complete in $L^2(I)$.*

Since the completeness in $L^2(I)$ of the system of all eigenvectors and associated vectors of A(also L) in $L^2(I)$ is equivalent to the completeness of those for K, from Theorem 6 one obtain

THEOREM 8. *The system of all eigenvectors and associated vectors of L is complete in $L^2(I)$.*

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