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ALGORITHM TESTING FOR THE HYPERCYCLICITY OF FINITELY ABELIAN SUBGROUPS OF $GL(n,\mathbb{C})$

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Abstract. In this paper, we present an algorithm that tests the existence of dense orbits for finitely abelian subgroups of $GL(n,\mathbb{C})$. A test example is given.

1. Introduction

In [1], Ayadi and Marzougui have characterized abelian subgroups of $GL(n,\mathbb{C})$ which are hypercyclic (i.e. having a dense orbit). In this paper, we deal with the algorithmic aspect, we present an algorithm that tests the existence of dense orbits for any abelian finitely generated subgroup G of $GL(n,\mathbb{C})$.

The hypercyclicity condition presented in [1] is related to the density of an additive subgroup of $\mathbb{C}^n$. As a matter of fact, the authors [3] gave a simple criterion to test the density of discrete additive subgroups of $\mathbb{R}^n$ and $\mathbb{C}^n$. Our algorithm is actually based heavily on these two papers ([1], [3]). It determines, in the same handwork, explicitly the normal form of the group G (see definition below). For one matrix, the normal form is reduced to the Jordan canonical form and in this case, Weintranb [5] gave an algorithm.

On this matter, we can cite S. Goodwin [4] who gave an algorithm which tests the density of orbits for Borel subgroups.

To state our main results, we need to introduce the following notations and definitions:

Denote by $M_n(\mathbb{C})$ the set of complex square matrices of order $n \geqslant 1$, and $GL(n,\mathbb{C})$ the group of the invertible matrices of $M_n(\mathbb{C})$.

- The spectrum of a square matrix A, denoted by $\sigma(A)$ is the set of all eigenvalues of A.
- $\mathbb{T}_n(\mathbb{C})$ the set of all lower-triangular matrices over $\mathbb{C}$, of order n and with only one eigenvalue.
- $\mathbb{T}_n^*(\mathbb{C}) = \mathbb{T}_n(\mathbb{C}) \cap GL(n,\mathbb{C})$ (*i.e.* the subset of matrices of $\mathbb{T}_n(\mathbb{C})$ having a non zero eigenvalue), it is a subgroup of $GL(n,\mathbb{C})$.
- $\mathbb{D}_n(\mathbb{C})$ the set of diagonal matrix of $M_n(\mathbb{C})$.

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- $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$ and $\mathbb{N}_0 = \mathbb{N}\backslash\{0\}$.

Let $r \in \mathbb{N}_0$ and $\eta = (n_1, \dots, n_r) \in \mathbb{N}_0^r$ such that $\sum_{i=1}^{r} n_i = n$. Denote by:

- $\mathcal{K}_{\eta,r}(\mathbb{C}) = \{M = \mathrm{diag}(T_1, \dots, T_r) \in M_n(\mathbb{C}) : \ T_k \in \mathbb{T}_{n_k}(\mathbb{C}), \ k = 1, \dots, r\}$.
- $\mathcal{K}^*_{\eta,r}(\mathbb{C}) = \mathcal{K}_{\eta,r}(\mathbb{C}) \cap GL(n,\mathbb{C})$, it is a subgroup of $GL(n,\mathbb{C})$.
- v^T the transpose of a vector $v \in \mathbb{C}^n$.
- $\mathcal{E}_n = (e_1, \dots, e_n)$ the standard basis of $\mathbb{C}^n$.
- I_n the identity matrix on $\mathbb{C}^n$.

Denote by:

- $u_0 = [e_{1,1}, \dots, e_{r,1}]^T \in \mathbb{C}^n$, where $e_{k,1} = [1, 0, \dots, 0]^T \in \mathbb{C}^{n_k}$, $\quad 1 \leqslant k \leqslant r$.
- $e^{(k)} = [0_{\mathbb{C}^{n_1}}, \dots, 0_{\mathbb{C}^{n_{k-1}}}, e^T_{k,1}, 0_{\mathbb{C}^{n_{k+1}}}, \dots, 0_{\mathbb{C}^{n_r}}]^T$, $\quad 1 \leqslant k \leqslant r$.

In [1], the authors proved the following

PROPOSITION 1.1. ([1], Proposition 6.1.) *Let G be an abelian subgroup of $GL(n,\mathbb{C})$, then there exists $P \in GL(n,\mathbb{C})$ such that $\widetilde{G} = P^{-1}GP$ is a subgroup of $\mathcal{K}^*_{\eta,r}(\mathbb{C})$, for some $1 \leqslant r \leqslant n$ and $\eta \in \mathbb{N}_0^r$.*

We say that the group $\widetilde{G}$ is a normal form of G of length r.

THEOREM 1.2. ([1], Theorem 1.3) *Let G be an abelian subgroup of $GL(n,\mathbb{C})$ and $P \in GL(n,\mathbb{C})$ such that $P^{-1}GP \subset \mathcal{K}^*_{\eta,r}(\mathbb{C})$. Assume that G is generated by $A_1 = e^{B_1}, \dots, A_p = e^{B_p}$ with $B_1, \dots, B_p \in P\mathcal{K}^*_{\eta,r}(\mathbb{C})P^{-1}$. Then G is hypercyclic if and only if $\sum_{k=1}^{p} \mathbb{Z}B_k P u_0 + 2i\pi \sum_{k=1}^{r} \mathbb{Z}Pe^{(k)}$ is a dense additive subgroup of $\mathbb{C}^n$.*

COROLLARY 1.3. *Let G be an abelian subgroup of $GL(n,\mathbb{C})$, generated by $A_1, \dots, A_p$ and $P \in GL(n,\mathbb{C})$ such that $P^{-1}GP \subset \mathcal{K}^*_{\eta,r}(\mathbb{C})$. If $p + r \leqslant 2n$, G has no dense orbit.*

- A subset V of $\mathbb{C}^n$ is called G-invariant if for every $x \in V$, $Gx \subset V$.

2. Algorithm testing for the hypercyclicity of finitely abelian subgroups of $GL(n,\mathbb{C})$

2.1. Normal form of the group G

Let G be an abelian subgroup of $GL(n,\mathbb{C})$ generated by $A_1, \dots, A_p$.

2.1.1. Determination of generalized eigenspaces of G

The first part of the algorithm is to determine a matrix $P \in GL(n,\mathbb{C})$ such that $G' = P^{-1}GP$ is a subgroup of $\mathcal{K}^*_{\eta,r}(\mathbb{C})$ as given in proposition 1.1.

To do so, given the eigenvalues $\lambda_{k,1}, \dots, \lambda_{k,r_k}$ of A_k, $k = 1, \dots, p$, the algorithm determines the corresponding generalized eigenspaces:

$$E_{k,j} = \mathrm{Ker}(A_k - \lambda_{k,j} I_n)^{\alpha_{k,j}}, \quad j = 1, \dots, r_k, \ k = 1, \dots, p$$

where $\alpha_{k,j}$ is the multiplicity of $\lambda_{k,j}$ and r_k the number of distinct eigenvalues of A_k. After that, it determines all the intersections:

$$\bigcap_{k=1}^{p} E_{k,i_k}, \quad 1 \leqslant i_k \leqslant r_k, \text{ such that } \bigcap_{k=1}^{p} E_{k,i_k} \neq \{0\}.$$

Denote these spaces by $E_1, \ldots, E_r$, called the *generalized eigenspaces of G*.

PROPOSITION 2.1. *The spaces E_i defined as above verify:*

(i) $\bigoplus_{i=1}^{r} E_i = \mathbb{C}^n$ *and E_i are G-invariant.*

(ii) For every $M \in G$, and for every $1 \leqslant j \leqslant r$, the restriction $M_{|E_j}$ has only one eigenvalue.

Proof. *(i)* Since E_{1,i_1} is A_2-invariant for every $i_1 = 1, \ldots, r_1$, so we have

$$E_{1,i_1} = \bigoplus_{i_2=1}^{r_2} E_{1,i_1} \cap E_{2,i_2}$$

We now apply this argument again, with E_{1,i_1} replaced by $E_{1,i_1} \cap E_{2,i_2}$, to obtain

$$E_{1,i_1} \cap E_{2,i_2} = \bigoplus_{i_3=1}^{r_3} \left(E_{1,i_1} \cap E_{2,i_2} \cap E_{3,i_3}\right)$$

We continuous in this fashion obtaining

$$\begin{aligned}
\mathbb{C}^n &= \bigoplus_{i_1=1}^{r_1} E_{1,i_1} \\
&= \bigoplus_{i_1=1}^{r_1} \bigoplus_{i_2=1}^{r_2} (E_{1,i_1} \cap E_{2,i_2}) \\
&= \bigoplus_{i_1=1}^{r_1} \bigoplus_{i_2=1}^{r_2} \bigoplus_{i_3=1}^{r_3} \left(E_{1,i_1} \cap E_{2,i_2} \cap E_{3,i_3}\right) \\
&= \bigoplus_{i_1=1}^{r_1} \bigoplus_{i_2=1}^{r_2} \cdots \bigoplus_{i_p=1}^{r_p} \left(E_{1,i_1} \cap E_{2,i_2} \cap \ldots \cap E_{p,i_p}\right)
\end{aligned}$$

Finally, by ignoring those intersections which are equal to $\{0\}$, we obtain $\mathbb{C}^n = \bigoplus_{i=1}^{r} E_i$.

As for every $i = 1, \ldots, r$, $E_i = E_{1,i_1} \cap \ldots \cap E_{p,i_p}$ for some $1 \leqslant i_k \leqslant r_k$, $k = 1, \ldots, p$, then E_i is G-invariant as intersection of the G-invariant subspaces E_{k,i_k}.

(ii) Let $E_i = E_{1,i_1} \cap \ldots \cap E_{p,i_p}$ for every $i = 1, \ldots, r$. As $E_i \subset E_{k,i_k}$ then λ_{k,i_k} is the unique eigenvalue of $A_{k|E_i}$. Since the matrices $\left(A_{k|E_i}\right)_{1 \leqslant k \leqslant p}$ is pairwise commuting

for every $i = 1,\dots,r$ they are simultaneously trigonalized. It follows that for any $M = A_1^{n_1}A_2^{n_2}\dots A_p^{n_p} \in G$ with $n_1,n_2,\dots,n_p \in \mathbb{N}$,

$$M_{|E_i} = \left(A_{1|E_i}\right)^{n_1}\left(A_{2|E_i}\right)^{n_2}\dots\left(A_{p|E_i}\right)^{n_p}$$

and

$$\sigma(M_{|E_i}) \subset \prod_{k=1}^{p}\sigma\left((A_{k|E_i})^{n_k}\right) = \left\{\prod_{k=1}^{p}\lambda_{k,i_k}^{n_k}\right\}$$

Therefore $\sigma(M_{|E_i}) = \left\{\prod_{k=1}^{p}\lambda_{k,i_k}^{n_k}\right\}$. □

At this state, the algorithm determines the number r of generalized eigenspaces of G which corresponds to the number of blocs in the normal form of each matrix of G. If $p+r \leqslant 2n$ then there is no need to proceed further since by Corollary 1.3, G has no dense orbit.

The next step consists in finding a basis $\mathscr{C}_i$ for each space E_i and so by juxtaposing, a new basis $\mathscr{C} = (\mathscr{C}_1,\dots,\mathscr{C}_r)$ of $\mathbb{C}^n$. Denote by R the transition matrix from $\mathscr{E}_n$ to $\mathscr{C}$ and by $\widehat{A_k} = R^{-1}A_kR$, $k = 1,\dots,p$. Then $\widehat{A_k} = \mathrm{diag}(\widehat{A}_{k,1},\dots,\widehat{A}_{k,r})$. Actually, the set $\{A_1,\dots,A_p\}$ has been simultaneously block diagonalized.

A step further in order to simplify the structure of G, is to simultaneously trigonalize the set $\{\widehat{A}_{1,i},\dots,\widehat{A}_{p,i}\}$, $i = 1,\dots,r$. Since these matrices are pairwise commuting, so they have some common eigenvectors $(v_{r_1+1},\dots,v_{n_i})$. We complete these vectors to obtain a basis $\mathscr{R}_i = (w_1,\dots,w_{r_1},v_{r_1+1},\dots,v_{n_i})$ of E_i. Denote by $Q_{i,1}$ the transition matrix from the standard basis $\mathscr{E}_{n_i}$ of E_i to $\mathscr{R}_i$. Then, for every $k = 1,\dots,p$, we have

$$Q_{i,1}^{-1}\widehat{A}_{k,i}Q_{i,1} = \begin{bmatrix}\widehat{A}_{k,i}^{(1)} & 0\\ L_{k,i}^{(1)} & \mu_k I_{n_i-r_1}\end{bmatrix}$$

with $\widehat{A}_{k,i}^{(1)} \in GL(r_1,\mathbb{C})$ and $L_{k,i}^{(1)} \in M_{n_i-r_1,r_1}(\mathbb{C})$. Now, we consider the set of matrices $\left(\widehat{A}_{k,i}^{(1)}\right)_{1\leqslant k\leqslant p}$ which are also pairwise commuting. Therefore, we can apply the same type of reduction as before to obtain a transition matrix $\widehat{Q}_{i,2} \in GL(r_1,\mathbb{C})$ such that we get $\widehat{Q}_{i,2}^{-1}\widehat{A}_{k,i}^{(1)}\widehat{Q}_{i,2} = \begin{bmatrix}\widehat{A}_{k,i}^{(2)} & 0\\ L_{k,i}^{(2)} & \mu_k I_{r_1-r_2}\end{bmatrix}$ with $\widehat{A}_{k,i}^{(2)} \in GL(r_2,\mathbb{C})$ and $L_{k,i}^{(2)} \in M_{r_1-r_2,r_2}(\mathbb{C})$. Set $Q_{i,2} = Q_{i,1}\begin{bmatrix}\widehat{Q}_{i,2} & 0\\ 0 & I_{n_i-r_1}\end{bmatrix}$. Then

$$Q_{i,2}^{-1}\widehat{A}_{k,i}Q_{i,2} = \begin{bmatrix}\begin{bmatrix}\widehat{A}_{k,i}^{(2)} & 0\\ L_{k,i}^{(2)} & \mu_k I_{r_1-r_2}\end{bmatrix} & 0\\ L_{k,i}^{(1)}\widehat{Q}_{i,2} & \mu_k I_{n_i-r_1}\end{bmatrix}, \quad k = 1,\dots,p.$$

So, we continuous this process until we end up with a final basis of E_i (eventually a transition matrix called Q_i) so that $Q_i^{-1}\widehat{A}_{k,i}Q_i = \mathscr{T}_{k,i} \in \mathbb{T}_{n_i}^*(\mathbb{C})$, $k = 1,\dots,p$. Hence, if $Q = \mathrm{diag}(Q_1,\dots,Q_r)$ and $P = RQ$, then

$$\widetilde{A_k} := P^{-1}A_kP = Q^{-1}R^{-1}A_k(RQ) = Q^{-1}\widehat{A_k}Q = \mathrm{diag}(\mathscr{T}_{k,1},\dots,\mathscr{T}_{k,r})$$

where $\mathscr{T}_{k,i} \in \mathbb{T}^*_{n_i}(\mathbb{C})$.

2.2. Determination of matrices B_k

In this section, the algorithm shall construct matrices $B_1,\dots,B_p \in \mathscr{K}_{\eta,r}(\mathbb{C})$ satisfying $\widetilde{A}_k = e^{B_k}$, $k = 1,\dots,p$. Recall that $\widetilde{A}_k = P^{-1}A_kP = diag(\mathscr{T}_{k,1},\dots,\mathscr{T}_{k,r})$ where $\mathscr{T}_{k,i} \in \mathbb{T}^*_{n_i}(\mathbb{C})$. So it suffices to construct $T_{k,i} \in \mathbb{T}_{n_i}(\mathbb{C})$ so that $e^{T_{k,i}} = \mathscr{T}_{k,i}$ and then we take $B_k = \mathrm{diag}(T_{k,1},\dots,T_{k,r})$. So we need a method to construct for $T \in \mathbb{T}^*_m(\mathbb{C})$, $1 \leqslant m \leqslant n$, a matrix $N \in \mathbb{T}_m(\mathbb{C})$ such that $e^N = T$. For this, we use the following lemma:

LEMMA 2.2. ([1], Lemma 2.2) *If $N \in M_n(\mathbb{C})$ has only one eigenvalue such that $e^N \in \mathbb{T}^*_n(\mathbb{C})$ then $N \in \mathbb{T}_n(\mathbb{C})$.*

Let $J(\theta)$ denote the Jordan block in $\mathbb{T}_m(\mathbb{C})$ associated with θ (with lower-triangular form):

$$J(\theta) = \begin{bmatrix} \theta & & & & 0 \\ 1 & \ddots & & & \\ 0 & \ddots & \ddots & & \\ \vdots & \ddots & \ddots & \ddots & \\ 0 & \dots & 0 & 1 & \theta \end{bmatrix}$$

Then we have:

$$e^{J(\theta)} = e^{\theta} \begin{bmatrix} 1 & & & & & 0 \\ 1 & \ddots & & & & \\ \frac{1}{2} & \ddots & \ddots & & & \\ \vdots & \ddots & \ddots & \ddots & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \\ \frac{1}{(m-1)!} & \dots & \dots & \frac{1}{2} & 1 & 1 \end{bmatrix}$$

Since $\dim\left(\mathrm{Ker}(e^{J(\theta)} - e^{\theta}I_m)\right) = 1$, $J(e^{\theta})$ is the Jordan normal form of $e^{J(\theta)}$, so there is a matrix $U \in GL(m,\mathbb{C})$ such that:

$$U^{-1}e^{J(\theta)}U = J(e^{\theta}). \tag{2.1}$$

Let $T \in \mathbb{T}_m^*(\mathbb{C})$ and let $J = \mathrm{diag}(J_1(\lambda), \dots, J_s(\lambda)) \in \mathbb{T}_m(\mathbb{C})$, where

$$J_i(\lambda) = \begin{bmatrix} \lambda & & & & 0 \\ 1 & \ddots & & & \\ 0 & \ddots & \ddots & & \\ \vdots & \ddots & \ddots & \ddots & \\ 0 & \dots & 0 & 1 & \lambda \end{bmatrix} \in \mathbb{T}_{n_i}(\mathbb{C})$$

and $\sum_{i=1}^{s} n_i = m$, be the Jordan normal form of T. Since $\lambda \neq 0$, there exists $\mu \in \mathbb{C}$ such that $e^{\mu} = \lambda$. Applying equation 2.1 to each block of J, we obtain:

$$\begin{aligned} J &= \mathrm{diag}(J_1(\lambda), \dots, J_s(\lambda)) \\ &= \mathrm{diag}(J_1(e^{\mu}), \dots, J_s(e^{\mu}) \\ &= \mathrm{diag}(U_1^{-1} e^{J_1(\mu)} U_1, \dots, U_s^{-1} e^{J_s(\mu)} U_s) \\ &= U^{-1} e^{J'} U \end{aligned}$$

where $U = \mathrm{diag}(U_1, \dots, U_s)$ and $J' = \mathrm{diag}(J_1(\mu), \dots, J_s(\mu))$.

There exists $V \in GL(m, \mathbb{C})$ such that $V^{-1}TV = J$. Take $N = VU^{-1}J'UV^{-1}$, it follows that $e^N = T$. Since $e^N = T \in \mathbb{T}_m^*(\mathbb{C})$ and as N has only one eigenvalue, so by Lemma 2.2, $N \in \mathbb{T}_m(\mathbb{C})$.

2.3. Hypercyclicity of the group G

The last step of this algorithm, is to check the hypercyclicity of G using theorem 1.2, *i.e.* $H(G) := \sum_{k=1}^{p} \mathbb{Z}B_k u_0 + 2\pi i \sum_{k=1}^{r} \mathbb{Z}e^{(k)}$ is a dense additive subgroup of $\mathbb{C}^n$. To do so, we apply the algorithm given in [3] for the complex case. In order to make this article self contained, we briefly outline the different steps of this algorithm.

Let $q = p + r$. If $q \leqslant 2n$ or $\sum_{k=1}^{p} \mathbb{R}B_k u_0 + 2\pi i \sum_{k=1}^{r} \mathbb{R}e^{(k)} \neq \mathbb{C}^n$, then $H(G)$ is not dense in $\mathbb{C}^n$, otherwise, $q > 2n$ and $\sum_{k=1}^{p} \mathbb{R}B_k u_0 + 2\pi i \sum_{k=1}^{r} \mathbb{R}e^{(k)} = \mathbb{C}^n$. Let us write $H(G) = \sum_{k=1}^{q} \mathbb{Z}u_k$ where $(u_k = B_k u_0)_{k=1,\dots,p}$ and $\left(u_{p+k} = 2\pi i\, e^{(k)}\right)_{k=1,\dots,r}$. We can assume that $(u_1, \dots, u_{2n})$ is a $\mathbb{R}$-basis of $\mathbb{C}^n$.

Set $\widetilde{H}(G) = \sum_{k=1}^{q} \mathbb{Z}\tilde{u}_k$, where $\tilde{u}_k = [\Re(u_k), \Im(u_k)]^T$.

For every $k = 2n+1, \dots, q$, let $\alpha_{k,i}$ be the coordinates of $\tilde{u}_k$ in the basis $(\tilde{u}_1, \dots, \tilde{u}_{2n})$, *i.e.* $\tilde{u}_k = \sum_{i=1}^{2n} \alpha_{k,i} \tilde{u}_i$. Suppose that $1, \alpha_{k,i_1}, \dots, \alpha_{k,i_{r_k}}$ is the longest sequence extracted from the list $\{1, \alpha_{k,1}, \dots, \alpha_{k,2n}\}$ which contains 1 and such that its elements are independent over $\mathbb{Q}$. Then set $I_k := \{i_1, \dots, i_{r_k}\}$.

The next step is to write the scalars $\alpha_{k,j}$ for every $j \notin I_k$ as a function of 1 and the scalars $\{\alpha_{k,i}\ i \in I_k\}$, *i.e.*

$$\alpha_{k,j} = t_{k,j} + \sum_{i \in I_k} \gamma_{j,i}^{(k)} \alpha_{k,i}$$

where $\gamma_{j,i_1}^{(k)}, \dots, \gamma_{j,i_{r_k}}^{(k)}, t_{k,j} \in \mathbb{Q}$.

Moreover, we define the vectors $u'_{k,j},\ j \in I_k,\ k = 2n+1, \dots, q$ as

$$u'_{k,j} = q_k \tilde{u}_j + \sum_{i \notin I_k} m_{i,j}^{(k)} \tilde{u}_i$$

where $q_k \in \mathbb{N}^*$ and $m_{i,j}^{(k)} \in \mathbb{Z}$, are such that

$$\gamma_{i,j}^{(k)} = \frac{m_{i,j}^{(k)}}{q_k}$$

Finally, let $M_{\tilde{H}(G)}$ be the matrix of the coordinates of all the vectors $u'_{k,j}$.

Then by (Theorem 4.1, [3]) $H(G)$ is dense in $\mathbb{C}^n$ if and only if

$$\operatorname{rank}\left(M_{\tilde{H}(G)}\right) = 2n$$

3. The algorithm outline

1. Given the eigenvalues of $A_1, A_2, \dots, A_p$, determine the corresponding generalized eigenspaces $E_{k,j},\ j = 1, \dots, r_k,\ k = 1, \dots, p$.
2. Determine all the intersections $\bigcap_{k=1}^{p} E_{k,i_k} \neq \{0\},\ 1 \leqslant i_k \leqslant r_k$ and obtain the generalized eigenspaces $E_1, E_2, \dots, E_r$ of G.
3. If $p + r \leqslant 2n$ then G is not hypercyclic.
4. Otherwise, compute the normal form of G, *i.e.* determine the set $\{\widetilde{A}_1, \widetilde{A}_2, \dots, \widetilde{A}_p\}$.
5. Construct the matrices B_k such that $\widetilde{A}_k = e^{B_k},\ k = 1, \dots, p$.
6. If $\sum_{k=1}^{p} \mathbb{R} B_k u_0 + 2\pi i \sum_{k=1}^{r} \mathbb{R} e^{(k)} \neq \mathbb{C}^n$ then G is not hypercyclic.
7. Otherwise, consider the additive group $H(G) = \sum_{k=1}^{p} \mathbb{Z} B_k u_0 + 2\pi i \sum_{k=1}^{r} \mathbb{Z} e^{(k)}$ and determine $\widetilde{H}(G)$ and $M_{\tilde{H}(G)}$ as described in the last section.
8. G is hypercyclic if and only if $\operatorname{rank}\left(M_{\tilde{H}(G)}\right) = 2n$.

4. Example

Let G be the subgroup of $GL(3,\mathbb{C})$ generated by A_1,A_2, A_3, A_4, A_5 and A_6, where:

$$A_1 = \begin{bmatrix} e & 3-2e+i & -2+e-i \\ 0 & 2+i & -1-i \\ 0 & 1+i & -i \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 & -2+2e^{\sqrt{2}} & 1-e^{\sqrt{2}} \\ 0 & e^{\sqrt{2}} & 0 \\ 0 & 0 & e^{\sqrt{2}} \end{bmatrix}$$

$$A_3 = \begin{bmatrix} e^{\sqrt{3}} & -2e^{\sqrt{3}}+2e^{i} & e^{\sqrt{3}}-e^{i} \\ 0 & e^{i} & 0 \\ 0 & 0 & e^{i} \end{bmatrix}$$

$$A_4 = \begin{bmatrix} e^{i\sqrt{5}} & \sqrt{2}(\sqrt{2}+i)e-2e^{i\sqrt{5}} & e^{i\sqrt{5}}-(1+i\sqrt{2})e \\ 0 & (1+i\sqrt{2})e & -i\sqrt{2}e \\ 0 & i\sqrt{2}e & (1-i\sqrt{2})e \end{bmatrix}$$

$$A_5 = \begin{bmatrix} e & 2-2e+\sqrt{7}+i\sqrt{2} & e-1-\sqrt{7}-i\sqrt{2} \\ 0 & 1+\sqrt{7}+i\sqrt{2} & -\sqrt{7}-i\sqrt{2} \\ 0 & \sqrt{7}+i\sqrt{2} & 1-\sqrt{7}-i\sqrt{2} \end{bmatrix}$$

$$A_6 = \begin{bmatrix} 1 & i\sqrt{2} & -i\sqrt{2} \\ 0 & 1+i\sqrt{2} & -i\sqrt{2} \\ 0 & i\sqrt{2} & 1-i\sqrt{2} \end{bmatrix}$$

The spectrum $\sigma(A_k)$ of A_k are:

$$\begin{array}{ll} \sigma(A_1) = \{1,e\} & \sigma(A_2) = \{1,e^{\sqrt{2}}\} \\ \sigma(A_3) = \{e^{\sqrt{3}},e^{i}\} & \sigma(A_4) = \{e,e^{i\sqrt{5}}\} \\ \sigma(A_5) = \{1,e\} & \sigma(A_6) = \{1\} \end{array}$$

Here $r = 2$ which corresponds to two generalized eigenspaces E_1 and E_2 for G of dimension:

$$\dim(E_1) = 1, \qquad \dim(E_2) = 2$$

The normal form of G is given by:

$$\widetilde{A}_1 = \begin{bmatrix} e & 0 \\ 0 & \begin{bmatrix} 1 & 0 \\ \frac{1}{2}+\frac{1}{2}i & 1 \end{bmatrix} \end{bmatrix} \qquad \widetilde{A}_2 = \begin{bmatrix} 1 & 0 \\ 0 & \begin{bmatrix} e^{\sqrt{2}} & 0 \\ 0 & e^{\sqrt{2}} \end{bmatrix} \end{bmatrix}$$

$$\widetilde{A}_3 = \begin{bmatrix} e^{\sqrt{3}} & 0 \\ 0 & \begin{bmatrix} e^{i} & 0 \\ 0 & e^{i} \end{bmatrix} \end{bmatrix} \qquad \widetilde{A}_4 = \begin{bmatrix} e^{i\sqrt{5}} & 0 \\ 0 & \begin{bmatrix} e & 0 \\ \frac{\sqrt{2}}{2}e\,i & e \end{bmatrix} \end{bmatrix}$$

$$\widetilde{A}_5 = \begin{bmatrix} e & 0 \\ 0 & \begin{bmatrix} 1 & 0 \\ \frac{\sqrt{7}}{2}+i\frac{\sqrt{2}}{2} & 1 \end{bmatrix} \end{bmatrix} \qquad \widetilde{A}_6 = \begin{bmatrix} 1 & 0 \\ 0 & \begin{bmatrix} 1 & 0 \\ i\frac{\sqrt{2}}{2} & 1 \end{bmatrix} \end{bmatrix}$$

The matrices B_k such that $e^{B_k} = \tilde{A}_k$, $k = 1,\dots,6$ are given by:

$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & \begin{bmatrix} 0 & 0 \\ \frac{1}{2}+\frac{1}{2}i & 0 \end{bmatrix} \end{bmatrix} \qquad B_2 = \begin{bmatrix} 0 & 0 \\ 0 & \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \end{bmatrix}$$

$$B_3 = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} \end{bmatrix} \qquad B_4 = \begin{bmatrix} i\sqrt{5} & 0 \\ 0 & \begin{bmatrix} 1 & 0 \\ \frac{\sqrt{2}}{2}i & 1 \end{bmatrix} \end{bmatrix}$$

$$B_5 = \begin{bmatrix} 1 & 0 \\ 0 & \begin{bmatrix} 0 & 0 \\ \frac{\sqrt{7}}{2}+\frac{\sqrt{2}}{2}i & 0 \end{bmatrix} \end{bmatrix} \qquad B_6 = \begin{bmatrix} 0 & 0 \\ 0 & \begin{bmatrix} 0 & 0 \\ \frac{\sqrt{2}}{2}i & 0 \end{bmatrix} \end{bmatrix}$$

By Theorem 1.2, G is hypercyclic if and only if the complex additive group $H(G) = \sum_{k=1}^{6} \mathbb{Z}B_k u_0 + 2\pi i\,\mathbb{Z}e_1 + 2\pi i\,\mathbb{Z}e_2$ is dense, where $u_0 = [1,1,0]^T$.

We get

$$u_1 = \left[1,0,\tfrac{1}{2}+\tfrac{1}{2}i\right]^T \quad u_2 = \left[0,\sqrt{2},0\right]^T \quad u_3 = \left[\sqrt{3},i,0\right]^T \quad u_4 = \left[i\sqrt{5},1,\tfrac{\sqrt{2}}{2}i\right]^T$$

$$u_5 = \left[1,0,\tfrac{\sqrt{7}}{2}+\tfrac{\sqrt{2}}{2}i\right]^T \quad u_6 = \left[0,0,\tfrac{\sqrt{2}}{2}i\right]^T \quad u_7 = [2\pi i,0,0]^T \quad u_8 = [0,2\pi i,0]^T$$

Therefore

$$\tilde{u}_1 = \left[1,0,\tfrac{1}{2},0,0,\tfrac{1}{2}\right]^T \quad \tilde{u}_2 = \left[0,\sqrt{2},0,0,0,0\right]^T \quad \tilde{u}_3 = \left[\sqrt{3},0,0,0,1,0\right]^T$$

$$\tilde{u}_4 = \left[0,1,0,\sqrt{5},0,\tfrac{\sqrt{2}}{2}\right]^T \quad \tilde{u}_5 = \left[1,0,\tfrac{\sqrt{7}}{2},0,0,\tfrac{\sqrt{2}}{2}\right]^T \quad \tilde{u}_6 = \left[0,0,0,0,0,\tfrac{\sqrt{2}}{2}\right]^T$$

$$\tilde{u}_7 = [0,0,0,2\pi,0,0]^T \quad \tilde{u}_8 = [0,0,0,0,2\pi,0]^T$$

We have $\widetilde{H}(G) = \sum_{k=1}^{8} \mathbb{Z}\tilde{u}_k$.

The vectors $\tilde{u}_7$ and $\tilde{u}_8$ can be expressed in the basis $(\tilde{u}_1,\tilde{u}_2,\dots,\tilde{u}_6)$ as

$$\tilde{u}_7 = -\pi\frac{\sqrt{10}}{5}\tilde{u}_2 + 2\pi\frac{\sqrt{5}}{5}\tilde{u}_4 - 2\pi\frac{\sqrt{5}}{5}\tilde{u}_6$$

$$\tilde{u}_8 = -\pi\frac{7\sqrt{3}+\sqrt{21}}{3}\tilde{u}_1 + 2\pi\tilde{u}_3 + \pi\frac{\sqrt{3}+\sqrt{21}}{3}\tilde{u}_5 + \pi\frac{\sqrt{42}-2\sqrt{21}+7\sqrt{6}-2\sqrt{3}}{6}\tilde{u}_6$$

Now, we apply the algorithm given in [3] (see Theorem 4.1). We get the sets:

$I_7 = \{2,4\}$ and $I_8 = \{1,3,5,6\}$ obtained by using the fact that π is a transcendental number and that the set $\{\sqrt{n} :$ n is a squarefree number$\}$ is linearly independent over $\mathbb{Q}$ [2]. (Recall that an integer is squarefree if its prime factorization contains no prime more than once).

Now the vectors $u'_{k,j}$, $j \in I_k$, $k = 7,8$ are:

$$\begin{aligned} u'_{7,2} &= \tilde{u}_2 \\ u'_{7,4} &= \tilde{u}_4 - \tilde{u}_6 \\ u'_{8,1} &= \tilde{u}_1 \\ u'_{8,3} &= \tilde{u}_3 \\ u'_{8,5} &= \tilde{u}_5 \\ u'_{8,6} &= \tilde{u}_6 \end{aligned}$$

The matrix $M_{\widetilde{H}(G)}$ is given by:

$$M_{\widetilde{H}(G)} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Since $\operatorname{rank}(M_{\widetilde{H}(G)}) = 6$, we apply (Theorem 4.1, [3]) to get that $H(G)$ is dense in $\mathbb{R}^6$. We conclude by Theorem 1.2 that G is hypercyclic.

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