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# THE EXISTENCE AND THE REPRESENTATIONS FOR THE GROUP INVERSE OF BLOCK MATRICES UNDER SOME CONDITIONS

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*Abstract.* In this paper, we give the existence and the representations for the group inverse of block matrices $M = \begin{pmatrix} M_1 & A \\ B & D \end{pmatrix}$ having a sub-block $M_1$ which is a combination of sub-block $A$ and $B$, and give the existence and the representations for the group inverse of block matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ under some conditions. Finally, some numerical examples are given to illustrate our results.

## 1. Introduction

Let $\mathbb{C}^{n\times n}$ be the set of all $n\times n$ complex matrix. For a square matrix $A \in \mathbb{C}^{n\times n}$, the Drazin inverse of $A$ is the matrix $X$ satisfying

$$A^kXA = A^k \qquad XAX = X \qquad AX = XA \tag{1}$$

where $k = \mathrm{ind}(A)$ is the index of $A$, i.e., the smallest nonnegative integer $k$ satisfying $\mathrm{rank}\,(A^{k+1}) = \mathrm{rank}\,(A^k)$. In this case, when $\mathrm{ind}(A) = 1$, $X$ is called the group inverse of matrix $A$ and denoted by $X = A^{\#}$. If $X = A^{\#}$ exists, then it is unique.

Now we give some explanations on notations used in this paper. We denote the range, null space and rank of a matrix $A$ by $\mathrm{R}(A)$, $\mathrm{N}(A)$, $\mathrm{rank}\,(A)$, respectively. Let $A^{\pi} = I - AA^{\#}$. A matrix $A$ is regular if there exists a matrix $X$ satisfying $AXA = A$, then $X$ is called the $\{1\}$-inverse of matrix $A$. Let $\mathbb{K}^{m\times n}$ be the set of all $m\times n$ matrices over skew fields. Let $P_{L,M}$ be the transformation that carries any $x \in \mathbb{C}^{n\times n}$ into its projection on $L$ along $M$.

The group inverse of block matrices have numerous applications in many areas, such as singular differential equations, singular difference equations, Markov chains, iterative methods and so on [1–4]. So it is an important problem of investigating the existence and the representations for the group inverse of block matrices. In 1979, Campbell and Meyer proposed an open problem to find an explicit formula for Drazin (group) inverse of $2\times 2$ block matrices. Then authors studied the existence and representations for the group inverse of block matrices under different conditions. Until now, this problem has not been solved completely.

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Recently, more and more authors pay attention to them and a large number of works can be found in some literatures (Bu et al. 2009; Liu et al. 2012 and so on). For example, Ge et al. (2012) investigated the group inverse of $2\times 2$ block matrices over skew fields or rings and domains. In these papers [7, 8, 9, 14, 16], authors investigated the existence and representations for the group inverse of block matrices. The main forms including

(i) $\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}$ (see [8]);

(ii) $\begin{pmatrix} A & A \\ B & 0 \end{pmatrix}$ (see [14]);

(iii) $\begin{pmatrix} A^mB^n & A \\ B & 0 \end{pmatrix}$ where $m$ and $n$ are positive intrgers (see [16]);

(iv) $\begin{pmatrix} c_1A+c_2B & A \\ B & 0 \end{pmatrix}$ where non-zero element $c_1$, $c_2$ are in the center of $\mathbb{K}$ (see [16]);

(v) $\begin{pmatrix} AX+YB & A \\ B & 0 \end{pmatrix}$ where $AX=XA$, $A^{\#}$ exists and $X$ is invertible (see [7] and [9]).

Inspired by the above results, we mainly consider a class of block matrices with the form $\begin{pmatrix} M_1 & A \\ B & D \end{pmatrix}$. Obviously, this form include all cases in the above results (i)–(v).

In this paper, we give the necessary and sufficient conditions for the existence as well as the expressions of the group inverse for block matrices $\begin{pmatrix} M_1 & A \\ B & D \end{pmatrix}$ under some conditions, and give the existence and the representations for the group inverse of block matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ (there exists matrices $X$ and $Y$ satisfying $XA=C$, $AY=B$). Finally, some numerical examples are given to illustrate our results.

In order to prove the main results, we give some lemmas.

LEMMA 1.1. [1] *Let $A\in\mathbb{C}^{n\times n}$. Then $A^{\#}$ exists if and only if* $\operatorname{rank}(A)=\operatorname{rank}(A^2)$.

LEMMA 1.2. [7] *Let $A\in\mathbb{K}^{n\times m}$, $B\in\mathbb{K}^{m\times n}$,* $\operatorname{rank}(A)=\operatorname{rank}(B)=\operatorname{rank}(AB)=\operatorname{rank}(BA)$. *Then $(AB)^{\#}$ and $(BA)^{\#}$ exist, and the following equalities hold*

*(i) $(AB^{\#})A=A(BA)^{\#}$ and $(BA)^{\#}B=B(AB)^{\#}$;*

*(ii) $(AB)^{\#}=A[(BA)^{\#}]^2B$ and $(BA)^{\#}=B[(AB)^{\#}]^2A$;*

*(iii) $A(BA)^{\#}BA=(AB)^{\#}ABA=A$ and $B(AB)^{\#}AB=(BA)^{\#}BAB=B$;*

*(iv) $(BA)^{\#}BA=B(AB)^{\#}A$ and $(AB)^{\#}AB=A(BA)^{\#}B$.*

LEMMA 1.3. [3] *Suppose projectors $P_{L,M}$, where $L\oplus M=\mathbb{C}^n$. Then $P_{L,M}A=A$ if and only if* $\mathrm{R}(A)\subset L$ *and $AP_{L,M}=A$ if and only if* $\mathrm{N}(A)\supset M$.

LEMMA 1.4. [7] *Let $A\in\mathbb{K}^{n\times n}$, $A^{\#}$ exists. Then there exists matrices $X$ and $Y$ satisfying $A=YA^2=A^2X$. In this case, $A^{\#}=YAX=AX^2=Y^2A$.*

## 2. Main conclusions

THEOREM 2.1. *Let* $M = \begin{pmatrix} AX+YB+M_0 & A \\ B & D \end{pmatrix} \in \mathbb{C}^{n\times n}$, *where the matrix* $M_0$ *is arbitrary,* $A \in \mathbb{C}^{r\times(n-r)}$, $B \in \mathbb{C}^{(n-r)\times r}$, $D \in \mathbb{C}^{(n-r)\times(n-r)}$, $DX = B$, $\mathrm{N}(A) \supset \mathrm{N}(D)$, $\mathrm{R}(YB+M_0) \supset \mathrm{R}(A)$ *and* $(YB+M_0)^{\#}$ *exists. Then*

*(i)* $M^{\#}$ *exists if and only if* $\operatorname{rank}(D) = \operatorname{rank}(D^2)$.

*(ii) If* $M^{\#}$ *exists, then* $M^{\#} = \begin{pmatrix} (YB+M_0)^{\#} & -(YB+M_0)^{\#}AD^{\#} \\ -D^{\#}B(YB+M_0)^{\#} & D^{\#}B(YB+M_0)^{\#}AD^{\#}+D^{\#} \end{pmatrix}$.

*Proof.* (i) According to Lemma 1.3, we know there exists $D^{(1)}$ satisfying $AD^{(1)}D = A$. Then we have

$$M = \begin{pmatrix} I & AD^{(1)} \\ 0 & I \end{pmatrix} \begin{pmatrix} YB+M_0 & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} I & 0 \\ X & I \end{pmatrix}. \tag{2}$$

Similarly, we have

$$M^2 = \begin{pmatrix} I & AD^{(1)} \\ 0 & I \end{pmatrix} \begin{pmatrix} (YB+M_0)^2 & (YB+M_0)A \\ B(YB+M_0) & BA+D^2 \end{pmatrix} \begin{pmatrix} I & 0 \\ X & I \end{pmatrix}. \tag{3}$$

Note that $(YB+M_0)^{\#}$ exists, then

$$\begin{aligned} M^2 = &\begin{pmatrix} I & AD^{(1)} \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ B(YB+M_0)^{\#} & I \end{pmatrix} \begin{pmatrix} (YB+M_0)^2 & 0 \\ 0 & B(YB+M_0)^{\pi}A+D^2 \end{pmatrix} \\ &\times \begin{pmatrix} I & (YB+M_0)^{\#}A \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ X & I \end{pmatrix}. \end{aligned} \tag{4}$$

And since $\mathrm{R}(YB+M_0) \supset \mathrm{R}(A)$, then we get $(YB+M_0)(YB+M_0)^{(1)}A = A$ and

$$B(YB+M_0)^{\pi}A = B(A-(YB+M_0)(YB+M_0)^{\#}A) = 0. \tag{5}$$

Then

$$M^2 = \begin{pmatrix} I & AD^{(1)} \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ B(YB+M_0)^{\#} & I \end{pmatrix} \begin{pmatrix} (YB+M_0)^2 & 0 \\ 0 & D^2 \end{pmatrix} \begin{pmatrix} I & (YB+M_0)^{\#}A \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ X & I \end{pmatrix}. \tag{6}$$

Therefore, according to the equations (2) and (6), we have $\operatorname{rank}(M) = \operatorname{rank}(YB+M_0) + \operatorname{rank}(D)$ and $\operatorname{rank}(M^2) = \operatorname{rank}((YB+M_0)^2) + \operatorname{rank}(D^2)$.

Then we obtain the necessary and sufficient conditions for the existence of group inverse for block matrices $M$ is $\operatorname{rank}(D) = \operatorname{rank}(D^2)$.

(ii) Since $\mathrm{N}(D) \subset \mathrm{N}(A)$ and $\mathrm{R}(A) \subset \mathrm{R}(YB+M_0)$, so there exists matrices $D^{\#}$ and $(YB+M_0)^{\#}$ satisfying $AD^{\#}D = A$, $YB(YB)^{\#}A = A$.

Note that $M^{\#}$ exists, then $k$=ind($M$)=1. Let $S = YB + M_0$. Then

$$
\begin{aligned}
MM^{\#} &= \begin{pmatrix} AX+YB+M_0 & A \\ B & D \end{pmatrix} \begin{pmatrix} (YB+M_0)^{\#} & -(YB+M_0)^{\#}AD^{\#} \\ -D^{\#}B(YB+M_0)^{\#} & D^{\#}B(YB+M_0)^{\#}AD^{\#}+D^{\#} \end{pmatrix} \\
&= \begin{pmatrix} AXS^{\#}+SS^{\#}-AD^{\#}BS^{\#} & -AXS^{\#}AD^{\#}+SS^{\#}AD^{\#}+AD^{\#}BS^{\#}AD^{\#}+AD^{\#} \\ BS^{\#}-DD^{\#}BS^{\#} & -BS^{\#}AD^{\#}+DD^{\#}BS^{\#}AD^{\#}+DD^{\#} \end{pmatrix} \\
&= \begin{pmatrix} AXS^{\#}+SS^{\#}-AD^{\#}DXS^{\#} & -AXS^{\#}AD^{\#}+SS^{\#}AD^{\#}+AD^{\#}DXS^{\#}AD^{\#}+AD^{\#} \\ BS^{\#}-DD^{\#}DXS^{\#} & -BS^{\#}AD^{\#}+DD^{\#}DXS^{\#}AD^{\#}+DD^{\#} \end{pmatrix} \\
&= \begin{pmatrix} SS^{\#} & 0 \\ 0 & DD^{\#} \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
M^{\#}M &= \begin{pmatrix} (YB+M_0)^{\#} & -(YB+M_0)^{\#}AD^{\#} \\ -D^{\#}B(YB+M_0)^{\#} & D^{\#}B(YB+M_0)^{\#}AD^{\#}+D^{\#} \end{pmatrix} \begin{pmatrix} AX+YB+M_0 & A \\ B & D \end{pmatrix} \\
&= \begin{pmatrix} S^{\#}AX+S^{\#}S-S^{\#}AD^{\#}B & S^{\#}A-S^{\#}AD^{\#}D \\ -D^{\#}BS^{\#}AX-D^{\#}BS^{\#}S+D^{\#}BS^{\#}AD^{\#}B+D^{\#}B & -D^{\#}BS^{\#}A+D^{\#}BS^{\#}AD^{\#}D+D^{\#}D \end{pmatrix} \\
&= \begin{pmatrix} S^{\#}AX+S^{\#}S-S^{\#}AD^{\#}DX & 0 \\ -D^{\#}BS^{\#}AX-D^{\#}B+D^{\#}BS^{\#}AD^{\#}DX+D^{\#}B & D^{\#}D \end{pmatrix} \\
&= \begin{pmatrix} S^{\#}S & 0 \\ 0 & D^{\#}D \end{pmatrix} \\
&= MM^{\#}
\end{aligned}
$$

$$
\begin{aligned}
MM^{\#}M &= \begin{pmatrix} SS^{\#} & 0 \\ 0 & DD^{\#} \end{pmatrix} \begin{pmatrix} AX+YB+M_0 & A \\ B & D \end{pmatrix} \\
&= \begin{pmatrix} SS^{\#}AX+SS^{\#}S & SS^{\#}A \\ DD^{\#}B & DD^{\#}D \end{pmatrix} = \begin{pmatrix} SS^{\#}AX+S^{\#} & SS^{\#}A \\ DD^{\#}DX & DD^{\#}D \end{pmatrix} \\
&= \begin{pmatrix} AX+YB+M0 & A \\ B & D \end{pmatrix} = M
\end{aligned}
$$

$$
\begin{aligned}
M^{\#}MM^{\#} &= \begin{pmatrix} S^{\#}S & 0 \\ 0 & D^{\#}D \end{pmatrix} \begin{pmatrix} S^{\#} & -S^{\#}AD^{\#} \\ -D^{\#}BS^{\#} & D^{\#}BS^{\#}AD^{\#}+D^{\#} \end{pmatrix} \\
&= \begin{pmatrix} (YB+M_0)^{\#} & -(YB+M_0)^{\#}AD^{\#} \\ -D^{\#}B(YB+M_0)^{\#} & D^{\#}B(YB+M_0)^{\#}AD^{\#}+D^{\#} \end{pmatrix} = M^{\#}
\end{aligned}
$$

This completes the proof of Theorem 2.1. $\square$

COROLLARY 1. *Let* $M = \begin{pmatrix} AX+YB+M_0 & A \\ B & D \end{pmatrix}$, *where the matrix* $M_0$ *is arbitrary,* $A \in \mathbb{C}^{r\times(n-r)}$, $B \in \mathbb{C}^{(n-r)\times r}$, $D \in \mathbb{C}^{(n-r)\times(n-r)}$, $DX = B$, $\mathrm{N}(A) \supset \mathrm{N}(D)$, $S = YB + M_0$ *and* $S$ *is invertible. Then* *(i)* $M^\#$ *exists if and only if* $\operatorname{rank}(D) = \operatorname{rank}(D^2)$.

*(ii) If* $M^\#$ *exists, then* $M^\# = \begin{pmatrix} S^{-1} & -S^{-1}AD^\# \\ -D^\#BS^{-1} & D^\# + D^\#BS^{-1}AD^\# \end{pmatrix}$.

THEOREM 2.2. *Let* $M = \begin{pmatrix} AX+YB+M_0 & A \\ B & D \end{pmatrix} \in \mathbb{C}^{n\times n}$, *where the matrix* $M_0$ *is arbitrary,* $A \in \mathbb{C}^{r\times(n-r)}$, $B \in \mathbb{C}^{(n-r)\times r}$, $D \in \mathbb{C}^{(n-r)\times(n-r)}$, $YD = A$, $\mathrm{R}(B) \subset \mathrm{R}(D)$, $\mathrm{R}(AX+M_0) \supset \mathrm{R}(A)$ *and* $(AX+M_0)^\#$ *exists. Then*

*(i)* $M^\#$ *exists if and only if* $\operatorname{rank}(D) = \operatorname{rank}(D^2)$.

*(ii) If* $M^\#$ *exists, then* $M^\# = \begin{pmatrix} (AX+M_0)^\# & -(AX+M_0)^\#AD^\# \\ -D^\#B(AX+M_0)^\# & D^\#B(AX+M_0)^\#AD^\# + D^\# \end{pmatrix}$.

The proof is similar to Theorem 2.1.

COROLLARY 2. *Let* $M = \begin{pmatrix} AX+YB+M_0 & A \\ B & D \end{pmatrix} \in \mathbb{C}^{n\times n}$, *where the matrix* $M_0$ *is arbitrary,* $A \in \mathbb{C}^{r\times(n-r)}$, $B \in \mathbb{C}^{(n-r)\times r}$, $D \in \mathbb{C}^{(n-r)\times(n-r)}$, $DX = B$, $YD = A$, $\mathrm{R}(YDX+M_0) \supset \mathrm{R}(A)$ *and* $(YDX+M_0)^\#$ *exists. Then*

*(i)* $M^\#$ *exists if and only if* $\operatorname{rank}(D) = \operatorname{rank}(D^2)$.

*(ii) If* $M^\#$ *exists, then* $M^\# = \begin{pmatrix} (YDX+M_0)^\# & -(YDX+M_0)^\#AD^\# \\ -D^\#B(YDX+M_0)^\# & D^\#B(YDX+M_0)^\#AD^\# + D^\# \end{pmatrix}$.

THEOREM 2.3. *Let* $M = \begin{pmatrix} A & B \\ D & DX+YB+M_0 \end{pmatrix} \in \mathbb{C}^{n\times n}$, *where the matrix* $M_0$ *is arbitrary,* $A \in \mathbb{C}^{r\times r}$, $B \in \mathbb{C}^{r\times(n-r)}$, $D \in \mathbb{C}^{(n-r)\times r}$, $YA = D$, $\mathrm{R}(B) \subset \mathrm{R}(A)$, $\mathrm{N}(B) \supset \mathrm{N}(DX+M_0)$ *and* $(DX+M_0)^\#$ *exists. Then*

*(i)* $M^\#$ *exists if and only if* $\operatorname{rank}(A) = \operatorname{rank}(A^2)$.

*(ii) If* $M^\#$ *exists, then* $M^\# = \begin{pmatrix} A^\# + A^\#B(DX+M_0)^\#DA^\# & -A^\#B(DX+M_0)^\# \\ -(DX+M_0)^\#DA^\# & (DX+M_0)^\# \end{pmatrix}$.

*Proof.* (i) According to Lemma 1.3, we know there exists $A^{(1)}$ satisfying $AA^{(1)}B = B$. Then we have

$$M = \begin{pmatrix} I & 0 \\ Y & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & DX+M_0 \end{pmatrix} \begin{pmatrix} I & A^{(1)}B \\ 0 & I \end{pmatrix}. \tag{7}$$

Note that $(YB+M_0)^\#$ exists and $\mathrm{N}(B) \supset \mathrm{N}(DX+M_0)$ hold, similarly, we have

$$M^2 = \begin{pmatrix} I & 0 \\ Y & I \end{pmatrix} \begin{pmatrix} I & B(DX+M_0)^\# \\ 0 & I \end{pmatrix} \begin{pmatrix} A^2 & 0 \\ 0 & (DX+M_0)^2 \end{pmatrix} \begin{pmatrix} I & 0 \\ (DX+M_0)^\#D & I \end{pmatrix} \begin{pmatrix} I & A^{(1)}B \\ 0 & I \end{pmatrix}. \tag{8}$$

Therefore, according to the equations (7) and (8), we have $\operatorname{rank}(M) = \operatorname{rank}(DX+M_0) + \operatorname{rank}(A)$ and $\operatorname{rank}(M^2) = \operatorname{rank}((DX+M_0)^2) + \operatorname{rank}(A^2)$.

Then we obtain the necessary and sufficient condition for the existence of group inverse for matrix $M$ is $\operatorname{rank}(A) = \operatorname{rank}(A^2)$.

(ii) The verify is similar to Theorem 2.1.

THEOREM 2.4. *Let* $M=\begin{pmatrix} A & B \\ D & M_0+YB+DX \end{pmatrix}\in\mathbb{C}^{n\times n}$, *where the matrix* $M_0$ *is arbitrary,* $A\in\mathbb{C}^{r\times r}$, $B\in\mathbb{C}^{r\times(n-r)}$, $D\in\mathbb{C}^{(n-r)\times r}$, $AX=B$, $\mathrm{N}(D)\supset\mathrm{N}(A)$, $\mathrm{N}(B)\supset\mathrm{N}(YB+M_0)$ *and* $(YB+M_0)^{\#}$ *exists. Then*

*(i)* $M^{\#}$ *exists if and only if* $\operatorname{rank}(A)=\operatorname{rank}(A^2)$.

*(ii) If* $M^{\#}$ *exists, then* $M^{\#}=\begin{pmatrix} A^{\#}+A^{\#}B(YB+M_0)^{\#}DA^{\#} & -A^{\#}B(YB+M_0)^{\#} \\ -(YB+M_0)^{\#}DA^{\#} & (YB+M_0)^{\#} \end{pmatrix}$.

The proof is similar to Theorem 2.3.

COROLLARY 3. *Let* $M=\begin{pmatrix} A & B \\ D & YB+DX+M_0 \end{pmatrix}\in\mathbb{C}^{n\times n}$, *where the matrix* $M_0$ *is arbitrary,* $A\in\mathbb{C}^{r\times r}$, $B\in\mathbb{C}^{r\times(n-r)}$, $D\in\mathbb{C}^{(n-r)\times r}$, $YA=D$, $AX=B$, $\mathrm{N}(B)\supset\mathrm{N}(YAX+M_0)$ *and* $(YAX+M_0)^{\#}$ *exists. Then*

*(i)* $M^{\#}$ *exists if and only if* $\operatorname{rank}(A)=\operatorname{rank}(A^2)$.

*(ii) If* $M^{\#}$ *exists, then* $M^{\#}=\begin{pmatrix} A^{\#}+A^{\#}B(YAX+M_0)^{\#}DA^{\#} & -A^{\#}B(YAX+M_0)^{\#} \\ -(YAX+M_0)^{\#}DA^{\#} & (YAX+M_0)^{\#} \end{pmatrix}$.

COROLLARY 4. *Let* $M=\begin{pmatrix} A & B \\ D & DX+YB+M_0 \end{pmatrix}\in\mathbb{C}^{n\times n}$, *where the matrix* $M_0$ *is arbitrary,* $A\in\mathbb{C}^{r\times r}$, $B\in\mathbb{C}^{r\times(n-r)}$, $D\in\mathbb{C}^{(n-r)\times r}$, $YA=D$, $\mathrm{R}(A)\subset\mathrm{R}(D)$ *and* $S$ *is invertible, where* $S=M_0+DX$. *Then*

*(i)* $M^{\#}$ *exists if and only if* $\operatorname{rank}(A)=\operatorname{rank}(A^2)$.

*(ii) If* $M^{\#}$ *exists, then* $M^{\#}=\begin{pmatrix} A^{\#}+A^{\#}BS^{-1}DA^{\#} & -A^{\#}BS^{-1} \\ -S^{-1}DA^{\#} & S^{-1} \end{pmatrix}$.

Now we give the existence and the representations for the group inverse of block matrices $M=\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ (there exists matrices $X$ and $Y$ satisfying $XA=C$, $AY=B$).

THEOREM 2.5. *Let* $M=\begin{pmatrix} A & B \\ C & D \end{pmatrix}\in\mathbb{C}^{n\times n}$, *there exists matrices* $X$ *and* $Y$ *satisfying* $XA=C$, $AY=B$ *and* $S^{\#}$ *exists,* $S=D-XB$, $F=A^2+BS^{\pi}C$. *Then*

*(i)* $M^{\#}$ *exists if and only if* $\operatorname{rank}(A)=\operatorname{rank}(F)$.

*(ii) If* $M^{\#}$ *exists, then* $M^{\#}=\begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}$, *where*

$$
\begin{aligned}
M_1 &= AF^{(1)}(I+BS^{\#}X)AF^{(1)}A;\\
M_2 &= AF^{(1)}(I+BS^{\#}X)AF^{(1)}BS^{\pi}-AF^{(1)}BS^{\#};\\
M_3 &= S^{\pi}CF^{(1)}(I+BS^{\#}X)AF^{(1)}A-S^{\#}CF^{(1)}A;\\
M_4 &= (S^{\pi}CF^{(1)}(I+BS^{\#}X)A-S^{\#}C)F^{(1)}BS^{\pi}-S^{\pi}CF^{(1)}BS^{\#}+S^{\#}.
\end{aligned}
$$

*Proof.* We obtain the following by the elementary transformation of matrix $M$,

$$
M=\begin{pmatrix} I & 0 \\ X & I \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & S \end{pmatrix}\begin{pmatrix} I & Y \\ 0 & I \end{pmatrix}. \tag{9}
$$

Similarly, we have

$$M^2 = \begin{pmatrix} I & 0 \\ X & I \end{pmatrix} \begin{pmatrix} I & BS^\# \\ 0 & I \end{pmatrix} \begin{pmatrix} F & 0 \\ 0 & S^2 \end{pmatrix} \begin{pmatrix} I & 0 \\ S^\# C & I \end{pmatrix} \begin{pmatrix} I & Y \\ 0 & I \end{pmatrix}. \tag{10}$$

By the Lemma 1.1 and the two equations (9), (10), we obtain $M^\#$ exists if and only if $\operatorname{rank}(A) = \operatorname{rank}(F)$.

(ii) According to Lemma 1.4, we know there exists matrices $U$ and $V$ satisfying $M = M^2 U$, $M = VM^2$.

Let

$$U = \begin{pmatrix} I & -Y \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -S^\# C & I \end{pmatrix} \begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix} \begin{pmatrix} I & Y \\ 0 & I \end{pmatrix} \tag{11}$$

and

$$V = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \begin{pmatrix} V_1 & V_2 \\ V_3 & V_4 \end{pmatrix} \begin{pmatrix} I & -BS^\# \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -X & I \end{pmatrix}. \tag{12}$$

Where $U_1 = F^{(1)}A$, $U_2 = -F^{(1)}BS^\# S$, $U_3 = 0$, $U_4 = S^\#$, $V_1 = AF^{(1)}$, $V_2 = 0$, $V_3 = -SS^\# CF^{(1)}$, $V_4 = S^\#$.

So we calculate

$$\begin{aligned} M^\# &= VMU \\ &= \begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \begin{pmatrix} V_1 & V_2 \\ V_3 & V_4 \end{pmatrix} \begin{pmatrix} I & -BS^\# \\ 0 & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & S \end{pmatrix} \begin{pmatrix} I & 0 \\ -S^\# C & I \end{pmatrix} \begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix} \begin{pmatrix} I & Y \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \begin{pmatrix} V_1 & V_2 \\ V_3 & V_4 \end{pmatrix} \begin{pmatrix} A + BS^\# C & -BS^\# S \\ -SS^\# C & S \end{pmatrix} \begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix} \begin{pmatrix} I & Y \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} V_1 + XV_3 & V_2 + XV_4 \\ V_3 & V_4 \end{pmatrix} \begin{pmatrix} A + BS^\# C & -BS^\# S \\ -SS^\# C & S \end{pmatrix} \begin{pmatrix} U_1 & U_2 + U_1 Y \\ U_3 & U_4 + U_3 Y \end{pmatrix} \\ &= \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} \end{aligned}$$

where

$$\begin{aligned} M_1 &= AF^{(1)}(I + BS^\# X)AF^{(1)}A \\ M_2 &= AF^{(1)}(I + BS^\# X)AF^{(1)}BS^\pi - AF^{(1)}BS^\# \\ M_3 &= S^\pi CF^{(1)}(I + BS^\# X)AF^{(1)}A - S^\# CF^{(1)}A \\ M_4 &= (S^\pi CF^{(1)}(I + BS^\# X)A - S^\# C)F^{(1)}BS^\pi - S^\pi CF^{(1)}BS^\# + S^\#. \end{aligned}$$

This completes the proof of Theorem 2.5. $\square$

THEOREM 2.6. *Let* $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbb{C}^{n \times n}$, *there exists matrices* $X$ *and* $Y$ *satisfying* $XA = C$, $AY = B$ *and* $S^\#$ *exists,* $S = D - XB$, $F = A^2 + BS^\pi C$. *Then*

*(i)* $M^\#$ *exists if and only if* $\operatorname{rank}(A) = \operatorname{rank}(F)$.

*(ii) If $M^\#$ exists, then* $M^\# = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}$, *where*

$$\begin{aligned}
M_1 &= AF^{(1)}A(I+YS^\#C)F^{(1)}A; \\
M_2 &= AF^{(1)}A(I+YS^\#C)F^{(1)}BS^\pi - AF^{(1)}BS^\#; \\
M_3 &= S^\pi CF^{(1)}A(I+YS^\#C)F^{(1)}A - S^\# CF^{(1)}A; \\
M_4 &= (S^\pi CF^{(1)}A(I+YS^\#C) - S^\# C)F^{(1)}BS^\pi - S^\pi CF^{(1)}BS^\# + S^\#.
\end{aligned}$$

The proof is similar to Theorem 2.5.

REMARK. The results of Theorem 2.5 and Theorem 2.6 have two kinds of different forms for $M^\#$. It is easy to check the results are correct.

## 3. Numerical Example

EXAMPLE FOR THEOREM 2.1. Let

$$M_1 = \begin{pmatrix} 756 & 672 & 420 \\ 2184 & 1944 & 1212 \\ 1652 & 1472 & 916 \end{pmatrix}, \quad A = \begin{pmatrix} 81 & 108 & 270 \\ 234 & 312 & 780 \\ 177 & 236 & 590 \end{pmatrix}, \quad B = \begin{pmatrix} 11 & 8 & 7 \\ 8 & 8 & 4 \\ 16 & 16 & 8 \end{pmatrix}$$

and

$$D = \begin{pmatrix} 1 & 2 & 4 \\ 1 & 1 & 3 \\ 2 & 2 & 6 \end{pmatrix}.$$

Then there exists matrices $X = \begin{pmatrix} 1 & 0 & 1 \\ 1 & -4 & 3 \\ 2 & 4 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 3 \\ 1 & 2 & 2 \end{pmatrix}$ and $M_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ satisfing $M_1 = AX + YB + M_0$, and it is easy to verify the existence for group inverse of block matrices $M$.

According to Theorem 2.1, we calculate

$$D^\# = \begin{pmatrix} -2.3333 & 4.2222 & -0.4444 \\ 0.3333 & -0.5556 & -0.1111 \\ 0.6667 & -1.1111 & 0.2222 \end{pmatrix},$$

$$G_1 = \begin{pmatrix} 0.6198 & -1.1041 & 1.1719 \\ -0.0339 & 0.0729 & -0.0703 \\ -1.0781 & 1.9374 & -2.0468 \end{pmatrix}.$$

Then we calculate the subblock $G_2$, $G_3$, $G_4$ of matrix $M^\#$ in the same method.

So we have $M^\# = \begin{pmatrix} 0.6198 & -1.1041 & 1.1719 & 0.2500 & -0.1667 & 0.3333 \\ -0.0338 & 0.0729 & -0.0703 & -0.6250 & 0.4167 & -0.8333 \\ -1.0781 & 1.9374 & -2.0468 & -1.2500 & 0.8333 & -1.6667 \\ -3.5832 & 6.3332 & -6.7498 & -1.3333 & 3.5555 & 0.8889 \\ 0.4583 & -0.8333 & 0.8749 & 1.3333 & -1.2222 & 1.4445 \\ 0.9166 & -1.6665 & 1.7499 & 2.6666 & -2.4444 & 2.8889 \end{pmatrix}.$

EXAMPLE FOR THEOREM 2.5. Let

$$A=\begin{pmatrix}6&3&1&2\\4&1&1&2\\2&0&0&1\\2&1&1&1\end{pmatrix},\quad B=\begin{pmatrix}29&27\\19&19\\5&9\\14&10\end{pmatrix},\quad C=\begin{pmatrix}16&5&3&7\\22&10&6&9\end{pmatrix}$$

and

$$D=\begin{pmatrix}0&0\\47&29\end{pmatrix}.$$

Then there exists matrices $X=\begin{pmatrix}1&0&3&2\\2&1&0&3\end{pmatrix}$ and $Y=\begin{pmatrix}2&3\\3&1\\6&0\\1&3\end{pmatrix}$ satisfing $XA=C$, $AY=B$.

We have $G_1=\begin{pmatrix}-3.6168&17.6555&-12.3376&-16.8050\\5.2484&-25.7145&17.9756&24.4693\\1.2779&-6.3999&4.4547&6.0662\\3.9705&-19.3146&13.5209&18.4030\end{pmatrix}$, and calculate the subblock $G_2$, $G_3$, $G_4$ of matrix $M^{\#}$ in the same method.

Then

$$M^{\#}=\begin{pmatrix}-3.6168&17.6555&-12.3376&-16.8050&-01.2879&1.4168\\5.2484&-25.7145&17.9756&24.4693&1.9589&-2.0484\\1.2779&-6.3999&4.4547&6.0662&0.4753&-0.4779\\3.9705&-19.3146&13.5209&18.4030&1.4837&-1.5705\\-0.8895&4.1904&-2.9641&-4.0220&-0.3763&0.3895\\0.8789&-3.9834&2.8308&3.8466&0.3526&-0.3789\end{pmatrix}.$$

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