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THE BIRKHOFF ORTHOGONALITY IN PRE–HILBERT C^*–MODULES

PAWEŁ WÓJCIK

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Abstract. In this work we characterize the Birkhoff orthogonality for elements and finite dimensional subspaces of a pre-Hilbert C^*-module in terms of a convex hull of continuous linear functionals. The aim of the paper is to present results concerning the B-orthogonality and its applications. We also present the results concerning smoothness. Moreover, we give a new proof of the Bhatia–Šemrl theorem.

1. Introduction

Let $(X,\|\cdot\|)$ be a normed space over $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$. If the norm comes from an inner product $\langle\cdot|\cdot\rangle$, there is one natural orthogonality relation: $x\perp y :\Leftrightarrow \langle x|y\rangle = 0$. In general case, there are several notions of orthogonality and one of the most outstanding is the definition introduced by Birkhoff [4] (cf. also James [6]). For $x,y\in X$ we define:

$$x\perp_{\mathrm{B}} y \quad :\Leftrightarrow \quad \forall_{\lambda\in\mathbb{K}}: \ \|x\|\leqslant\|x+\lambda y\|.$$

This relation is clearly homogeneous, but neither symmetric nor additive, unless the norm comes from an inner product. Of course, in an inner product space we have $\perp_{\mathrm{B}}=\perp$.

An element x is *B-orthogonal to a subspace* $M\subset X$ (i.e.,$x\perp_{\mathrm{B}}M$) if and only if $x\perp_{\mathrm{B}}m$ for all $m\in M$.

Let us recall some basic facts about C^*-algebras and Hilbert C^*-modules and introduce our notation. A C^*-algebra $\mathscr{A}$ is a Banach *-algebra with the norm satisfying the C^*-condition $\|a^*a\|_{\mathscr{A}}=\|a\|^2_{\mathscr{A}}$ for all $a\in\mathscr{A}$. Let V be a normed space over $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$. Let $\mathscr{A}$ be a C^*-algebra over the same field. A *positive* element a of a C^*-algebra $\mathscr{A}$ is a self-adjoint element such that $\sigma(a)\subset[0,1)$. If $a\in\mathscr{A}$ is positive, we write $a\geqslant 0$.

A *right pre-Hilbert C^*-module X over a C^*-algebra $\mathscr{A}$* is a linear space which is a right $\mathscr{A}$-module equipped with an $\mathscr{A}$-valued inner-product $\langle\cdot|\cdot\rangle V\times V\to\mathscr{A}$ that is sesquilinear, positive definite and respects the module action, i.e.,

(C1) $\forall_{\alpha,\beta\in\mathbb{K}}\ \forall_{x,y,z\in V}\ \ \langle x|\alpha y+\beta z\rangle=\alpha\langle x|y\rangle+\beta\langle x|z\rangle$,
(C2) $\forall_{x,y\in V}\ \forall_{a\in\mathscr{A}}\ \ \langle x|ya\rangle=\langle x|y\rangle a$,
(C3) $\forall_{x,y\in V}\ \ \langle x|y\rangle=\langle y|x\rangle^*$,
(C4) $\forall_{x\in V}\ \ \langle x|x\rangle\geqslant 0$; if $\langle x|x\rangle=0$ then $x=0$,

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For a pre-Hilbert C^*-module V the Cauchy–Schwarz inequality holds: $\| \langle x|y\rangle \|_{\mathscr{A}}^2 \leqslant \| \langle x|x\rangle \|_{\mathscr{A}} \cdot \| \langle y|y\rangle \|_{\mathscr{A}}$. In particular, $\|x\| := \sqrt{\| \langle x|x\rangle \|_{\mathscr{A}}}$ defines a norm on V. A pre-Hilbert $\mathscr{A}$-module which is complete with respect to this norm is called a *Hilbert C^*-module over $\mathscr{A}$*, or a *Hilbert A-module*.

Obviously, every Hilbert space is a Hilbert C^*-module. Also, every C^*-algebra $\mathscr{A}$ can be regarded as a Hilbert C^*-module over itself with the inner product $\langle x|y\rangle := x^*y$, and the corresponding norm is just the norm on $\mathscr{A}$ because of the C^*-condition.

2. Preliminaries

Let X be a normed space over $\mathbb{R}$ or $\mathbb{C}$. We write B_X for the closed unit ball. Let S_X denote the unit sphere in X. The dual space is denoted by X^*. It is easy to see that for two elements x,y of a normed linear space X, it holds $x \perp_B y$ if and only if there is a norm one linear functional $f \in X^*$ such that $f(x) = \|x\|$ and $f(y) = 0$. If we have additional structures on a normed linear space X, then we get other characterizations of the Birkhoff orthogonality. One of the first results of this form is the result obtained by Bhatia and Šemrl [2] for the C^*-algebra $\mathscr{B}(\mathscr{H})$ of all bounded linear operators on a Hilbert space $\mathscr{H}$.

THEOREM 1. [2] *Let $A,B \in \mathscr{B}(\mathscr{H})$.*

(a) If $\dim \mathscr{H} < \infty$, then $A \perp_B B$ if and only if there is a unit vector $x \in \mathscr{H}$ such that $\|Ax\| = \|A\|$ and $\langle Ax|Bx\rangle = 0$.

(b) If $\dim \mathscr{H} = \infty$, then $A \perp_B B$ if and only if there is a sequence of unit vectors $(x_n) \subset \mathscr{H}$ such that $\lim_{n\to\infty} \|Ax_n\| = \|A\|$ and $\lim_{n\to\infty} \langle Ax_n|Bx_n\rangle = 0$.

The characterization of the Birkhoff orthogonality for elements of a Hilbert C^*-module by means of the states of the underlying C^*-algebra was obtained by Arambašić and Rajić [1]. Here, a *state* is a positive linear functional with norm 1.

THEOREM 2. [1] *Let V be a Hilbert $\mathscr{A}$-module, and $x,y \in V$. Then $x \perp_B y$ if and only if there is a positive linear functional $\varphi \in \mathscr{A}^*$ such that $\|\varphi\| = 1$ and $\varphi(\langle x|x\rangle) = \|x\|^2$ and $\varphi(\langle x|y\rangle) = 0$.*

In particular, Theorem 2 implies the following.

THEOREM 3. *Let V be a Hilbert $\mathscr{A}$-module, and $x \in V$. Assume that $Y \subset V$ is a finite dimensional linear subspace, and let $x \in V \setminus Y$. Then $x \perp_B Y$ if and only if for every $y \in Y$ there is a positive linear functional $\varphi \in \mathscr{A}^*$ such that $\|\varphi\| = 1$ and $\varphi(\langle x|x\rangle) = \|x\|^2$ and $y \in \ker \varphi(\langle x|\cdot\rangle)$.*

In the next section, we will show a result similar to Theorem 3.

Let Ext$K$ denote the set of all extremal points of a given set K. The dual space is denoted by X^*. The next result is known.

THEOREM 4. [7, p. 170] *Let X be a normed linear space, $Y=\operatorname{span}\{x_1,\dots,x_n\}$ an n-dimensional subspace of X, $x\in X\subset Y$ and $y_o\in Y$. The following statements are equivalent:*

(a) $x\perp_{\mathrm{B}}Y$;

(b) *There exist h extremal points $f_1,\dots,f_h$ of S_{X^*}, where $1\leqslant h\leqslant n+1$ if the scalars are real and $1\leqslant h\leqslant 2n+1$ if the scalars are complex and h numbers $\lambda_1,\dots,\lambda_h>0$ with $\sum_{j=1}^{h}\lambda_j=1$, such that*

$$\sum_{j=1}^{h}\lambda_j f_j(y)=0 \quad \textit{for all} \quad y\in Y, \qquad \textit{and} \qquad \sum_{j=1}^{h}\lambda_j f_j(x)=\|x\|,$$

(c) *There exist h extremal points $f_1,\dots,f_h$ of S_{X^*}, where $1\leqslant h\leqslant n+1$ if the scalars are real and $1\leqslant h\leqslant 2n+1$ if the scalars are complex and h numbers $\lambda_1,\dots,\lambda_h>0$ with $\sum_{j=1}^{h}\lambda_j=1$, such that*

$$\sum_{j=1}^{h}\lambda_j f_j(y)=0 \quad \textit{for all} \quad y\in Y, \qquad \textit{and} \qquad f_j(x)=\|x\| \;\; \textit{for} \;\; j=1,\dots,h.$$

3. Main result

The four theorems above motivate the following section. Let X be a normed space over $\mathbb{R}$ or $\mathbb{C}$. In this paper, the set $D\subset X$ is called *symmetric*, if

$$\text{for all } x\in A, \text{ and for all } \gamma\in\mathbb{K} \text{ with } |\gamma|=1, \text{ we have } \gamma x\in A.$$

A set $\mathscr{N}\subset S_{X^*}$ is called a **-norming set in X* if

$$\|x\|=\sup\{|x^*(x)| : x^*\in\mathscr{N}\}$$

for all $x\in X$. Let V be a pre-Hilbert $\mathscr{A}$-module. A set $\mathscr{D}\subset S_V$ is called a *norming set in V* if

$$\|x\|=\sup\{\|\langle b|x\rangle\|_{\mathscr{A}} : b\in\mathscr{D}\}$$

for all $x\in V$. For example, S_{X^*} is a *-norming set in X. Similarly, $\operatorname{Ext}S_{X^*}$ is also *-norming set in X. In general, *-norming set $\mathscr{N}$ need not consist of extreme points and it is even possible to have $\mathscr{N}\cap\operatorname{Ext}S_{X^*}=\emptyset$.

If K is dense in S_V, then K is a norming set in V. We say that $E\subset X^*$ is *total over X*, if for all $x\in X\setminus\{0\}$, there exists $\varphi\in E$ such that $\varphi(x)\neq 0$.

LEMMA 1. *Let X be a normed space. Suppose $M\subset X^*$. If M is a *-norming set in X, then M is total over X.*

Proof. Assume, contrary to our claim, that M is not total over X. Then, there exists $x_o\in X\setminus\{0\}$ such that $\varphi(x_o)=0$ for all $\varphi\in M$. It follows that $0<\|x_o\|=\sup\{\varphi(x_o):\varphi\in M\}=\sup\{0:\varphi\in M\}=0$, which is a contradiction. $\square$

LEMMA 2. *Let U be a vector space. Let $M \subset U^*$ be a total set over U. Assume that $\dim U^* = p < \infty$. Then there are functionals $x_1^*, \ldots, x_p^*$ in M such that $K := \{x_1^*, \ldots, x_p^*\}$ forms a Hamel basis of U^*.*

Proof. It is enough to show that $\dim(\mathrm{span} M) = \dim U^*$. Assume, contrary to our claim, that $\dim(\mathrm{span} M) < \dim U^*$. Then there are functionals $y_1^*, \ldots, y_k^*$ in M such that $\mathrm{span}\{y_1^*, \ldots, y_k^*\} = \mathrm{span} M$ and $k < p$. It follows that $\bigcap_{j=1}^{k} \ker y_j^* \neq \{0\}$. Let us fix x_o in $\left(\bigcap_{j=1}^{k} \ker y_j^*\right) \setminus \{0\}$. Since $M \subset \mathrm{span}\{y_1^*, \ldots, y_k^*\}$, we have $z^*(x_o) = 0$ for all $z^* \in M$. Moreover, $x_o \neq 0$. It means that M is not total over U, and we have a contradiction. □

The following considerations have been inspired by Theorems 1, 2 (in particular Theorem 3) and 4. We will prove a new type of characterization of B-orthogonality in pre-Hilbert C^*-modules. Namely, we will consider a condition $x \perp_B Y$ instead of $x \perp_B y$ and moreover we will apply the norming sets. Furthermore, we will consider the case over $\mathbb{R}$ and the case over $\mathbb{C}$ simultaneously. What is more, we will consider the Birkhoff orthogonality in pre-Hilbert C^*-modules instead of in Hilbert C^*-modules. We will obtain a characterization of the B-orthogonality in which only the norming sets are involved.

Let $\mathscr{N} \subset S_{\mathscr{A}^*}$ be a fixed *-norming set (in $\mathscr{A}$) and let $\mathscr{D} \subset S_V$ be a fixed norming set (in V). We define the the following set:

$$\mathscr{F}_h := \left\{ \sum_{k=1}^{h} \lambda_k a_k^*(\langle u_k | \cdot \rangle) \in V^* : \ a_k^* \in \mathscr{N}, u_k \in \mathscr{D}, \lambda_k \geqslant 0, \ \sum_{k=1}^{h} \lambda_k = 1 \right\}. \tag{1}$$

It is obvious that

$$\mathscr{F}_h \subset \mathrm{conv}\{ a^*(\langle u | \cdot \rangle) \in V^* : \ a^* \in \mathscr{N}, u \in \mathscr{D}\} \subset B_{V^*}$$

The symbol B_{V^*} denotes the closed unit ball.

Suppose that Y is a n-dimensional subspace of X. We define a new constant:

$$\vartheta(x) := \inf\{ \ | \ \|x\| - v^*(x) \ | : \ v^* \in \mathscr{F}_{n+1}, \ Y \subset \ker v^* \}$$

in the real case. In a similar way we define

$$\vartheta(x) := \inf\{ \ | \ \|x\| - v^*(x) \ | : \ v^* \in \mathscr{F}_{2n+1}, \ Y \subset \ker v^* \}$$

in the complex case. Clearly $\vartheta(x) \geqslant 0$ for all $x \in V \setminus Y$. Now we prove the main result of this paper.

THEOREM 5. *Let V be a pre-Hilbert $\mathscr{A}$-module, where $V, \mathscr{A}$ are over the same field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Assume that $\mathscr{N} \subset S_{\mathscr{A}^*}$ is a *-norming set (in $\mathscr{A}$). Suppose that $\mathscr{D} \subset S_V$ is a norming set (in V). Moreover, suppose that $\mathscr{N}$ is symmetric. Let $Y \subset V$*

be an n-dimensional subspace. Assume $x \in V \setminus Y$. Then the following conditions are equivalent:

(A) $x \perp_{\mathrm{B}} Y$,

(B) $\vartheta(x) = 0$.

Proof. We start with proving (B)$\Rightarrow$(A). Assume, contrary to our claim, that $x \not\perp_{\mathrm{B}} Y$. Let $W := \operatorname{span}(Y \cup \{x\})$. Since W is finite dimensional, there is $w \in W \setminus Y$ such that $w \perp_{\mathrm{B}} Y$ (by Riesz's Lemma and the compactness of S_W). It follows that $w = \alpha x + y_1$ for some $\alpha \in \mathbb{K}$, $y_1 \in Y$. It is clear that $\frac{1}{\alpha} w \perp_{\mathrm{B}} Y$. We show that $\|x\| > \|\frac{1}{\alpha} w\|$. If we had $\|x\| \leqslant \|\frac{1}{\alpha} w\|$, then we would obtain

$$\|x\| \leqslant \|\tfrac{1}{\alpha} w\| \leqslant \|\tfrac{1}{\alpha} w + y\| \text{ for all } y \in Y.$$

In particular, putting $-\frac{1}{\alpha} y_1 + y$ in place of y, we would obtain

$$\|x\| \leqslant \|\tfrac{1}{\alpha} w - \tfrac{1}{\alpha} y_1 + y\| = \|x + y\| \text{ for all } y \in Y.$$

But then $x \perp_{\mathrm{B}} Y$ and we would obtain a contradiction.

Define now $\varepsilon := \frac{1}{2}\|x\| - \frac{1}{2}\|\frac{1}{\alpha} w\| > 0$. Directly from the definition of $\vartheta(x)$, we have

$$\|x\| \leqslant \varepsilon + \left| \sum_{k=1}^{h} \lambda_k a_k^*(\langle u_k | x \rangle) \right| \tag{2}$$

for some $\lambda_1, \ldots, \lambda_h \geqslant 0$, $a_1^* \ldots, a_h^* \in \mathscr{N}$, $u_1 \ldots, u_h \in \mathscr{D}$ (where $h = n+1$ in real case or $h = 2n+1$ in complex case), such that $\sum_{k=1}^{h} \lambda_k = 1$ and

$$\sum_{k=1}^{h} \lambda_k a_k^*(\langle u_k | y \rangle) = 0 \text{ for all } y \in Y. \tag{3}$$

Thus we have

$$\begin{aligned}
\varepsilon + \left\| \frac{1}{\alpha} w \right\| &= \frac{1}{2}\|x\| - \frac{1}{2}\left\| \frac{1}{\alpha} w \right\| + \left\| \frac{1}{\alpha} w \right\| = \frac{1}{2}\|x\| + \frac{1}{2}\left\| \frac{1}{\alpha} w \right\| < \frac{1}{2}\|x\| + \frac{1}{2}\|x\| \\
&= \|x\| \overset{(2)}{\leqslant} \varepsilon + \left| \sum_{k=1}^{h} \lambda_k a_k^*(\langle u_k | x \rangle) \right| \overset{\left(x = \frac{1}{\alpha} w - \frac{1}{\alpha} y_1\right)}{=} \\
&= \varepsilon + \left| \sum_{k=1}^{h} \lambda_k a_k^* \left(\left\langle u_k \middle| \frac{1}{\alpha} w - \frac{1}{\alpha} y_1 \right\rangle \right) \right| \overset{(3)}{=} \\
&= \varepsilon + \left| \sum_{k=1}^{h} \lambda_k a_k^* \left(\left\langle u_k \middle| \frac{1}{\alpha} w \right\rangle \right) \right| \leqslant \varepsilon + \sum_{k=1}^{h} \lambda_k \left| a_k^* \left(\left\langle u_k \middle| \frac{1}{\alpha} w \right\rangle \right) \right| \\
&\leqslant \varepsilon + \sum_{k=1}^{h} \lambda_k \|a_k^*\| \cdot \left\| \left\langle u_k \middle| \frac{1}{\alpha} w \right\rangle \right\| \leqslant \varepsilon + \sum_{k=1}^{h} \lambda_k \|u_k\| \cdot \left\| \frac{1}{\alpha} w \right\| \\
&\leqslant \varepsilon + \sum_{k=1}^{h} \lambda_k \left\| \frac{1}{\alpha} w \right\| = \varepsilon + \left\| \frac{1}{\alpha} w \right\| \sum_{k=1}^{h} \lambda_k \leqslant \varepsilon + \left\| \frac{1}{\alpha} w \right\|.
\end{aligned}$$

We get $\varepsilon + \|\frac{1}{\alpha}w\| < \varepsilon + \|\frac{1}{\alpha}w\|$, which is a contradiction.

Now we prove (A)$\Rightarrow$(B). Let us now define $U := \operatorname{span}(Y \cup \{x\})$ and let us consider any $\varepsilon \in (0,1)$. It is clear that $\dim U = n+1 < \infty$. Then the compactness of S_U implies that there are $w_1, \ldots, w_m \in S_U$ such that

$$S_U \subset \bigcup_{k=1}^{m} B\Big(w_k; \frac{\varepsilon}{4}\Big), \tag{4}$$

where $B\big(w_k; \frac{\varepsilon}{4}\big) = \{z \in U : \|z - w_k\| < \frac{\varepsilon}{4}\}$. It is easy to check that the set

$$M := \{a^*(\langle u|\cdot\rangle)|_U \in U^* : a^* \in \mathscr{N}, u \in \mathscr{D}\}.$$

is *-norming in U. Indeed, for every $p \in U$, from the assumptions we have

$$\begin{aligned} \|p\| &= \sup\{\|\langle u|p\rangle\| : u \in \mathscr{D}\} \\ &= \sup\{\sup\{|a^*(\langle u|p\rangle)| : a^* \in \mathscr{N}\} : u \in \mathscr{D}\} \\ &= \sup\{|a^*(\langle u|p\rangle)| : a^* \in \mathscr{N},\ u \in \mathscr{D}\} \\ &= \sup\{|v^*(p)| : v^* \in M\}, \end{aligned}$$

This means that the set $\{a^*(\langle u|\cdot\rangle)|_U \in U^* : a^* \in \mathscr{N}, u \in \mathscr{D}\}$ is *-norming in U. Therefore there exist $a_1^*, \ldots, a_m^* \in \mathscr{N}$, $u_1, \ldots, u_m \in \mathscr{D}$ such that

$$|\ \|w_k\| - |a_k^*(\langle u_k|w_k\rangle)|\ | < \frac{\varepsilon}{4} \text{ for } k = 1, \ldots, m.$$

The set $\mathscr{N}$ is symmetric. Thus, without loss of generality, we may assume that

$$|\ \|w_k\| - a_k^*(\langle u_k|w_k\rangle)\ | < \frac{\varepsilon}{4} \text{ for } k = 1, \ldots, m. \tag{5}$$

Then we define $L := \{a_1^*(\langle u_1|\cdot\rangle)|_U, \ldots, a_m^*(\langle u_m|\cdot\rangle)|_U\}$. We have already defined the set

$$M = \{a^*(\langle u|\cdot\rangle)|_U \in U^* : a^* \in \mathscr{N}, u \in \mathscr{D}\}.$$

We have shown that M is *-norming in U. By Lemma 1, M is total over U. It follows from Lemma 2 that there is $K \subset M$ such that K forms a Hamel basis of U^*.

Without loss of generality, we may assume that

$$K = \{b_1^*(\langle z_1|\cdot\rangle)|_U, \ldots, b_n^*(\langle z_n|\cdot\rangle)|_U, b_{n+1}^*(\langle z_{n+1}|\cdot\rangle)|_U\},$$

for some $b_1^*(\langle z_1|\cdot\rangle)|_U, \ldots, b_n^*(\langle z_n|\cdot\rangle)|_U, b_{n+1}^*(\langle z_{n+1}|\cdot\rangle)|_U \in M$.

Let us now define the sets

$$E := \{\gamma a_k^*(\langle u_k|\cdot\rangle)|_U \in U^* \ :\ k = 1, \ldots, m,\ \gamma \in \mathbb{K} \text{ and } |\gamma| = 1\}$$

and

$$F := \{\gamma b_k^*(\langle z_k|\cdot\rangle)|_U \in U^* \ :\ k = 1, \ldots, n, n+1,\ \gamma \in \mathbb{K} \text{ and } |\gamma| = 1\}$$

and let us introduce the set P defined by

$$P := \operatorname{conv}(E \cup F). \tag{6}$$

The set P is convex, absorbing and balanced. Moreover, P is compact. Hence this set introduces a new norm $\|\cdot\|_P$ in U^* by the Minkowski functional.

Now we can define the function $\|\cdot\|_T : U \to \mathbb{R}$ by $\|x\|_T := \max\{|\varphi(x)| : \varphi \in E \cup F\}$. It is a norm. Indeed, it is easy to show that $\|p+r\|_T \leqslant \|p\|_T + \|r\|_T$ and $|\alpha| \cdot \|p\|_T = \|\alpha p\|_T$. We prove only an implication $\|p\|_T = 0 \Rightarrow p = 0$. If $p \in U$ and $\|p\|_T = 0$, then $\varphi(p) = 0$ for all $\varphi \in E \cup F$. Moreover $K \subset F$, whence $\varphi(p) = 0$ for all $\varphi \in K$. It is helpful to recall that K forms the Hamel basis in U^*. Therefore we obtain $p = 0$.

It is easy to see that $\|\psi\|_P = \sup\{|\psi(x)| : \|x\|_T \leqslant 1\}$. Thus we may say that $\|\cdot\|_P$ is the dual norm for $\|\cdot\|_T$, i.e., $(U, \|\cdot\|_T)^* = (U^*, \|\cdot\|_P)$. Directly from the definition of $\|\cdot\|_T$, we have the following inequality

$$\|v\|_T \leqslant \|v\| \text{ for all } v \in U \setminus \{0\}. \tag{7}$$

Next we will prove that

$$(1-\varepsilon)\|v\| \leqslant \|v\|_T \leqslant (1+\varepsilon)\|v\| \text{ for all } v \in U \setminus \{0\}. \tag{8}$$

It follows from (6) and $a_k^*(\langle u_k|\cdot\rangle)|_U \in E \subset E \cup F$ that

$$|a_k^*(\langle u_k|v\rangle)| \leqslant \max\{|\varphi(v)| : \varphi \in E \cup F\} = \|v\|_T \text{ for all } v \in U. \tag{9}$$

It follows that

$$\|w_k\| \overset{(5)}{<} \frac{\varepsilon}{4} + |a_k^*(\langle u_k|w_k\rangle)| \overset{(9)}{\leqslant} \frac{\varepsilon}{4} + \|w_k\|_T \tag{10}$$

and by (7) we get $0 \leqslant \|w_k\| - \|w_k\|_T$. Then, by (10) we have

$$|\, \|w_k\| - \|w_k\|_T| \leqslant \frac{\varepsilon}{4} \text{ for all } k = 1, \ldots, m. \tag{11}$$

Fix $v \in U$ such that $v \neq 0$. It is clear that $\frac{v}{\|v\|} \in S_U$. Applying (4) we have

$$\left\| w_{k_o} - \frac{v}{\|v\|} \right\| < \frac{\varepsilon}{4} \tag{12}$$

for some $w_{k_o} \in \{w_1, \ldots, w_m\}$.

Then, we have

$$\begin{aligned}
\left| \left\| \frac{v}{\|v\|} \right\| - \left\| \frac{v}{\|v\|} \right\|_T \right| &= \left| 1 - \left\| \frac{v}{\|v\|} \right\|_T \right| = \left| \|w_{k_o}\| - \left\| \frac{v}{\|v\|} \right\|_T \right| \\
&\leqslant |\, \|w_{k_o}\| - \|w_{k_o}\|_T| + \left| \|w_{k_o}\|_T - \left\| \frac{v}{\|v\|} \right\|_T \right| \\
&\leqslant |\, \|w_{k_o}\| - \|w_{k_o}\|_T| + \left\| w_{k_o} - \frac{v}{\|v\|} \right\|_T \overset{(11),(7)}{\leqslant} \\
&\leqslant \frac{\varepsilon}{4} + \left\| w_{k_o} - \frac{v}{\|v\|} \right\| \overset{(12)}{\leqslant} \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2} \leqslant \varepsilon.
\end{aligned}$$

Thus we get $\left| \left\| \frac{v}{\|v\|} \right\| - \left\| \frac{v}{\|v\|} \right\|_T \right| \leqslant \varepsilon$ for all $v \in U \setminus \{0\}$. From this it is very easy to prove that

$$(1-\varepsilon)\|v\| \leqslant \|v\|_T \leqslant (1+\varepsilon)\|v\| \text{ for all } v \in U \setminus \{0\}. \tag{13}$$

Let $\perp_B^T$ denote the Birkhoff orthogonality with respect to the norm $\|\cdot\|_T$. The space $(U, \|\cdot\|_T)$ is finite dimensional, whence (applying again Riesz's Lemma and the compactness of the unit sphere $S_{(U,\|\cdot\|_T)}$) there is $\widehat{x} \in U \setminus Y$ such that $\widehat{x} \perp_B^T Y$. It follows that $\widehat{x} = \beta x + y_1$ for some $\beta \in \mathbb{K}$, $y_1 \in Y$.

Since $\widehat{x} \perp_B^T Y$, it follows that $\frac{1}{\beta}\widehat{x} \perp_B^T Y$. Now we can apply Theorem 4. There exist h extremal points $f_1, \ldots, f_h$ of $S_{(U,\|\cdot\|_T)^*}$, where $1 \leqslant h \leqslant n+1$ if the scalars are real and $1 \leqslant h \leqslant 2n+1$ if the scalars are complex and h numbers $\lambda_1, \ldots, \lambda_h > 0$ with $\sum_{j=1}^{h} \lambda_j = 1$, such that

$$\sum_{j=1}^{h} \lambda_j f_j(y) = 0 \text{ for all } y \in Y, \quad \text{and} \quad f_j\left(\frac{1}{\beta}\widehat{x}\right) = \left\| \frac{1}{\beta}\widehat{x} \right\|_T \text{ for } j = 1, \ldots, h. \tag{14}$$

It follows directly from the definition of $\|\cdot\|_T$ that $\operatorname{Ext} S_{(U,\|\cdot\|_T)^*} \subset E \cup F$. In fact, this means that

$$f_1, \ldots, f_h \in \bigcup_{|\gamma|=1} \gamma \cdot \{a_t^*(\langle u_t|\cdot\rangle)|_U,\ b_j^*(\langle z_j|\cdot\rangle)|_U : t=1,\ldots,m,\ j=1,\ldots,n,n+1\},$$

and hence

$$f_1 = \gamma_1 c_1^*(\langle s_1|\cdot\rangle)|_U,\ \ldots,\ f_h = \gamma_h c_h^*(\langle s_h|\cdot\rangle)|_U \tag{15}$$

for some $|\gamma_j| = 1$, and $c_j^* \in \{a_j^*, b_j^*\}$, and $s_j \in \{u_j, z_j\}$. It follows from (14), (15) that

$$0 = \sum_{j=1}^{h} \lambda_j c_j^*(\langle s_j|y\rangle) \text{ for } y \in Y \quad \text{and} \quad \gamma_j c_j^*\left(\left\langle s_j|\frac{1}{\beta}\widehat{x}\right\rangle\right)\Big|_U = \left\| \frac{1}{\beta}\widehat{x} \right\|_T \text{ for all } j. \tag{16}$$

It follows from $x \perp_B Y$ that

$$\|x\| \leqslant \left\| x + \frac{1}{\beta} y \right\| \text{ for all } y \in Y. \tag{17}$$

Since $\widehat{x} \perp_B^T Y$, we deduce that $\frac{1}{\beta}\widehat{x} \perp_B^T Y$ and

$$\left\| \frac{1}{\beta}\widehat{x} \right\|_T \leqslant \left\| \frac{1}{\beta}\widehat{x} - \frac{1}{\beta} y \right\|_T \text{ for all } y \in Y. \tag{18}$$

Thus we have

$$\left\| \frac{1}{\beta}\widehat{x} \right\|_T \overset{(18)}{\leqslant} \left\| \frac{1}{\beta}\widehat{x} - \frac{1}{\beta} y_1 \right\|_T \overset{(\widehat{x}=\beta x+y_1)}{=} \|x\|_T \overset{(13)}{\leqslant} (1+\varepsilon)\|x\| = \|x\| + \varepsilon\|x\| \tag{19}$$

and

$$\|x\|-\varepsilon\|x\|=(1-\varepsilon)\|x\|\overset{(17)}{\leqslant}(1-\varepsilon)\left\|x+\frac{1}{\beta}y_1\right\|=(1-\varepsilon)\left\|\frac{1}{\beta}\widehat{x}\right\|\overset{(13)}{\leqslant}\left\|\frac{1}{\beta}\widehat{x}\right\|_T. \tag{20}$$

It follows from (19), (20) that

$$\left|\,\|x\|-\left\|\frac{1}{\beta}\widehat{x}\right\|_T\,\right|\leqslant\varepsilon\|x\|. \tag{21}$$

Finally, we will show $\vartheta(x)=0$. Summarizing, we have $\lambda_j\geqslant 0$, $\sum_{j=1}^{h}\lambda_j=1$.

Moreover $\gamma_j c_j^*\in\mathscr{N}$ and $s_j\in\mathscr{D}$, which yields $\sum_{j=1}^{h}\lambda_j\gamma_j c_j^*(\langle s_j|\cdot\rangle)\in\mathscr{F}_h$. By (16), we can conclude that $Y\subset\ker\left(\sum_{j=1}^{h}\lambda_j\gamma_j c_j^*(\langle s_j|\cdot\rangle)\right)$. Finally, we deduce

$$\begin{aligned}\left|\|x\|-\sum_{j=1}^{h}\lambda_j\gamma_j c_j^*(\langle s_j|x\rangle)\right| &\overset{(\widehat{x}=\beta x+y_1)}{=}\left|\|x\|-\sum_{j=1}^{h}\lambda_j\gamma_j c_j^*\left(\left\langle s_j\Big|\frac{1}{\beta}\widehat{x}-\frac{1}{\beta}y_1\right\rangle\right)\right|\\ &\overset{(16)}{=}\left|\|x\|-\sum_{j=1}^{h}\lambda_j\gamma_j c_j^*\left(\left\langle s_j\Big|\frac{1}{\beta}\widehat{x}\right\rangle\right)\right|\\ &\overset{(16)}{=}\left|\|x\|-\sum_{j=1}^{h}\lambda_j\left\|\frac{1}{\beta}\widehat{x}\right\|_T\right|\\ &=\left|\|x\|-\left\|\frac{1}{\beta}\widehat{x}\right\|_T\right|\overset{(21)}{\leqslant}\varepsilon\|x\|.\end{aligned}$$

Thus we get $\left|\|x\|-\sum_{j=1}^{h}\lambda_j\gamma_j c_j^*(\langle s_j|x\rangle)\right|\leqslant\varepsilon\|x\|$. Since ε was arbitrary, this implies that $0=\inf\{|\,\|x\|-v^*(x)\,|:\ v^*\in\mathscr{F}_h,\ Y\subset\ker v^*\}=\vartheta(x)$. □

4. Approximation

We are interested in the applications of Theorem 5. In approximation theory the condition that x is Birkhoff orthogonal to Y can be interpreted as follows. Suppose $x\in X\setminus Y$. Then the zero vector is the best approximation to x among all vectors in Y.

LEMMA 3. *Let V be a pre-Hilbert $\mathscr{A}$-module, where $V,\mathscr{A}$ are over the field $\mathbb{R}$ (or $\mathbb{C}$). Assume that $\mathscr{N}\subset S_{\mathscr{A}^*}$ is a *-norming set (in $\mathscr{A}$). Suppose that $\mathscr{D}\subset S_V$ is a norming set (in V). Moreover, suppose that $\mathscr{N}$ is symmetric. Let $Y\subset V$ be an n-dimensional subspace. Assume $x\in V\setminus Y$. Then the following condition holds:*

(i) if $x\perp_B Y$, then $\|x\|=\sup\{|v^*(x)|:\ Y\subset\ker v^*,\ v^*\in\mathscr{F}_{n+1}\}$ (or $\mathscr{F}_{2n+1}$).

If $V,\mathscr{A}$ are over the field $\mathbb{R}$, then the following condition also holds:

(ii) if $x\perp_B Y$, then $\|x\|=\sup\{v^*(x):\ Y\subset\ker v^*,\ v^*\in\mathscr{F}_{n+1}\}$.

Proof. It is obvious that $\|x\| \geqslant \sup\{|v^*(x)| : v^* \in \mathscr{F}_{n+1},\ Y \subset \ker v^*\}$. By Theorem 5 let us choose a sequence $(v_n^*)_{n=1,2,\ldots} \subset \mathscr{F}_{n+1}$ (or $\mathscr{F}_{2n+1}$) such that $Y \subset \ker v_n^*$ and

$$|\ \|x_n\| - v_n^*(x)\ | < \tfrac{1}{n}.$$

It follows from this inequality that

$$\|x\| - |v_n^*(x)| \leqslant |\ \|x\| - |v_n^*(x)|\ | \leqslant |\ \|x\| - v_n^*(x)\ | < \tfrac{1}{n}.$$

Now, suppose that $V, \mathscr{A}$ are over the field $\mathbb{R}$. Since $\mathscr{N}$ is symmetric, we may assume that $v_n^*(x) \in [0, +\infty)$. Thus $\|x\| < \frac{1}{n} + v_n^*(x)$, which means that

$$\|x\| = \sup\{v^*(x) : v^* \in \mathscr{F}_{n+1},\ Y \subset \ker v^*\}. \quad \square$$

Now we present a different expression for the formula for distance from a given $x \in V$ to a finite dimensional subspace $Y \subset V$.

For $x \in V \setminus Y$ put

$$P_Y(x) := \{y \in Y : \|x - y\| = \mathrm{dist}(x, Y)\}.$$

THEOREM 6. *Let V, $\mathscr{N}$, D, x, Y be such as in Lemma 3. Then*

$$\mathrm{dist}(x, Y) = \sup\{|v^*(x)| : v^* \in \mathscr{F}_{n+1},\ Y \subset \ker v^*\}.$$

In the complex case, we have $\mathrm{dist}(x, Y) = \sup\{|v^*(x)| : v^* \in \mathscr{F}_{2n+1},\ Y \subset \ker v^*\}$.

Proof. Since the proofs are similar we present only the real case. Since Y is a finite-dimensional subspace, there exists $y_o \in P_Y(x)$. It is easy to check that

$$y_o \in P_Y(x) \ \Leftrightarrow\ (x - y_o) \perp_{\mathrm{B}} Y \tag{22}$$

Now by applying (22) and Lemma 3 we arrive at the desired assertion. Indeed, we obtain

$$\begin{aligned} \mathrm{dist}(x, Y) = \|x - y_o\| &= \sup\{|v^*(x - y_o)| : v^* \in \mathscr{F}_{n+1},\ Y \subset \ker v^*\} \\ &= \sup\{|v^*(x)| : v^* \in \mathscr{F}_{n+1},\ Y \subset \ker v^*\}. \end{aligned}$$

In the complex case we obtain $\mathrm{dist}(x, Y) = \sup\{v^*(x) : v^* \in \mathscr{F}_{2n+1},\ Y \subset \ker v^*\}$. $\square$

5. Smoothness in spaces $\mathscr{C}(\Omega; \mathscr{A})$

Our aim in this section is to consider smoothness and B-orthogonal relation in situations where we use function spaces. Let $\mathscr{A}$ be a real (or complex) C^*-algebra with identity $\mathbf{1}$. For a compact topological space Ω we denote by $\mathscr{C}(\Omega; \mathscr{A})$ the normed space of all $\mathscr{A}$-valued continuous functions with the usual sup-norm $\|f\|_\infty := \sup\{\|f(t)\|_{\mathscr{A}} : t \in \Omega\}$. It is obvious that $\mathscr{C}(\Omega; \mathscr{A})$ is also a real (or complex) C^*-algebra. In particular, $\mathscr{C}(\Omega; \mathscr{A})$ is also a Hilbert C^*-module over itself with the inner product $\langle f|g\rangle := f^* g$, where $f^*(t) := f(t)^*$, $t \in \Omega$. Now, we obtain another characterization of B-orthogonality. For $f \in \mathscr{C}(\Omega; \mathscr{A})$ put $M(f) := \{t \in \Omega : \|f(t)\|_{\mathscr{A}} = \|f\|_\infty\}$. We consider only the case when $M(f) \neq \emptyset$.

PROPOSITION 1. *Let $\mathscr{A}$ be a real (or complex) C^*-algebra. Let $f,g\in\mathscr{C}(\Omega;\mathscr{A})$. Assume that $M(f)=\{t_o\}$. Then*

$$f\perp_{\mathrm{B}} g \quad \Leftrightarrow \quad f(t_o)\perp_{\mathrm{B}} g(t_o).$$

Proof. We start with proving "$\Leftarrow$". Suppose that $f(t_o)\perp_{\mathrm{B}} g(t_o)$. Directly from the definition of $\perp_{\mathrm{B}}$, we have $\|f(t_o)\|_{\mathscr{A}}\leqslant\|f(t_o)+\lambda g(t_o)\|_{\mathscr{A}}$ for $\lambda\in\mathbb{K}$. Thus we have

$$\|f\|_\infty=\|f(t_o)\|_{\mathscr{A}}\leqslant\|f(t_o)+\lambda g(t_o)\|_{\mathscr{A}}\leqslant\|f+\lambda g\|_\infty$$

Therefore $\|f\|_\infty\leqslant\|f+\lambda g\|_\infty$ for $\lambda\in\mathbb{K}$.

Now we prove the converse. Let us consider $V:=\mathscr{C}(\Omega;\mathscr{A})$ and $\widehat{\mathscr{A}}:=\mathscr{C}(\Omega;\mathscr{A})$. We define a inner product

$$\langle\cdot|\cdot\rangle:\mathscr{C}(\Omega;\mathscr{A})\times\mathscr{C}(\Omega;\mathscr{A})\to\mathscr{C}(\Omega;\mathscr{A}),\qquad \langle h|p\rangle:=h^*p.$$

Then, the space $V=\mathscr{C}(\Omega;\mathscr{A})$ can be regarded as a Hilbert C^*-module over itself with $\widehat{\mathscr{A}}=\mathscr{C}(\Omega;\mathscr{A})$. Fix an arbitrary element $t\in\Omega$. Fix an arbitrary functional $x^*\in S_{\mathscr{A}^*}$. Define a linear and continuous functional

$$a^*_{t,x^*}\in\mathscr{C}(\Omega;\mathscr{A})\to\mathbb{K}\quad\text{by}\quad a^*_{t,x^*}:=x^*(f(t)),\quad f\in\mathscr{C}(\Omega;\mathscr{A}).$$

Now we define a set $\mathscr{N}:=S_{\mathscr{A}^*}$. It is easy to check that the set $\mathscr{N}$ is *-norming and symmetric. Let us consider a set $\mathscr{D}:=\{e\}$, where $e\in\mathscr{C}(\Omega;\mathscr{A})$, $e(t)=\mathbf{1}$ for $t\in\Omega$. The set D is norming. Next, we define the the following set:

$$\mathscr{F}_2:=\left\{\lambda a^*_{t,x^*}(\langle e|\cdot\rangle)+(1-\lambda)a^*_{u,y^*}(\langle e|\cdot\rangle)\in\mathscr{C}(\Omega;\mathscr{A})^*:t,u\in\Omega,x^*,y^*\in\mathscr{N},\lambda\in[0,1]\right\}$$

Note that $f\perp_{\mathrm{B}}g$ yields $\alpha f\perp_{\mathrm{B}}\beta g$ for all $\alpha,\beta\in\mathbb{K}$ (i.e., $\perp_{\mathrm{B}}$ is full homogeneous). Without loss of generality, we may assume that $\|f\|=1$.

Now, suppose that $f\perp_{\mathrm{B}}g$. Applying Lemma 3 we obtain

$$\|f\|=\sup\{|v^*(f)|:\ g\in\ker v^*,\ v^*\in\mathscr{F}_2\}.$$

There are $v_n^*\in\mathscr{F}_2$ such that $|v_n^*(f)|\to\|f\|_\infty$ and $g\in\ker v_n^*$. By symmetry of $\mathscr{N}$, we may assume that $v_n^*(f)\in[0,+\infty)$, and then $v_n^*(f)\to\|f\|_\infty$. This means that

$$\begin{aligned}v_n^*(f)&=\lambda_n a^*_{t_n,x_n^*}(\langle e|f\rangle)+(1-\lambda_n)a^*_{u_n,y_n^*}(\langle e|f\rangle)\\&=\lambda_n a^*_{t_n,x_n^*}(e^*f)+(1-\lambda_n)a^*_{u_n,y_n^*}(e^*f)\\&=\lambda_n a^*_{t_n,x_n^*}(f)+(1-\lambda_n)a^*_{u_n,y_n^*}(f)\\&=\lambda_n x_n^*(f(t_n))+(1-\lambda_n)y_n^*(f(u_n))\end{aligned}$$

for some $x_1^*,x_2^*,x_3^*\ldots\in\mathscr{N}$, $t_1,t_2,t_3,\ldots\in\Omega$. Therefore, we obtain

$$\lambda_n x_n^*(f(t_n))+(1-\lambda_n)y_n^*(f(u_n))\to\|f\|_\infty. \tag{23}$$

Moreover, $v_n^*(g)=0$. In a similar way one can prove

$$\lambda_n x_n^*(g(t_n))+(1-\lambda_n)y_n^*(g(u_n))=0. \tag{24}$$

The sets $[0,1]$ (or $\mathbb{T}:=\{z\in\mathbb{C} : |z|\leqslant 1\}$) and Ω are compact. The closed unit ball $\overline{B}_{\mathscr{A}^*}$ is weak*-compact. Therefore, without loss of generality, we may assume that there are elements t,u in Ω, functionals $x^*,y^*\in\overline{B}_{\mathscr{A}^*}$, a number $\lambda\in[0,1]$ (or $\lambda\in\mathbb{T}$) and subsequences $\{t_{n_k}\}$, $\{u_{n_k}\}$, $\{x^*_{n_k}\}$, $\{y^*_{n_k}\}$, $\{\lambda_{n_k}\}$ such that $t_{n_k}\to t$, $u_{n_k}\to u$, $x^*_{n_k}\xrightarrow{w^*}x^*$, $y^*_{n_k}\xrightarrow{w^*}y^*$, $\lambda_{n_k}\to\lambda$. Clearly $f(t_{n_k})\to f(t)$. Now the condition (23) becomes

$$\lambda x^*(f(t))+(1-\lambda)y^*(f(u))=\|f\|_\infty. \tag{25}$$

It is clear that $g(t_{n_k})\to g(t)$. We get from (24)

$$\lambda x^*(g(t))+(1-\lambda)y^*(g(u))=0. \tag{26}$$

We will show that $x^*(f(t))=1=y^*(f(u))$. Now, we obtain the equality

$$1=\|f\|_\infty\overset{(25)}{=}\lambda x^*(f(t))+(1-\lambda)y^*(f(u))$$

and $x^*(f(t)),y^*(f(u))\in[-1,1]$ (or $x^*(f(t)),y^*(f(u))\in\mathbb{T}$). It is easy to check that $1\in\operatorname{Ext}[-1,1]$ (or in complex case $1\in\operatorname{Ext}\mathbb{T}$). It yields $x^*(f(t))=1$ and $y^*(f(u))=1$, whence $\|x^*\|=\|f(t)\|=1$ and $\|y^*\|=\|f(u)\|=1$.

Bearing in mind that $M(f)=\{t_o\}$, we have $t_o=t=u$, so $x^*(f(t_o))=1$ and $y^*(f(t_o))=1$. We can rewrite (25) and (26) in the form

$$\lambda x^*(f(t_o))+(1-\lambda)y^*(f(t_o))=\|f\|_\infty \quad\text{and}\quad \lambda x^*(g(t_o))+(1-\lambda)y^*(g(t_o))=0. \tag{27}$$

Let us define $w^*:=\lambda x^*+(1-\lambda)y^*$. It follows from (27) that

$$w^*(f(t_o))=\|f\|_\infty \quad\text{and}\quad \|w^*\|=1 \quad\text{and}\quad w^*(g(t_o))=0. \tag{28}$$

Then for $\lambda\in\mathbb{K}$ we have

$$\begin{aligned}\|f(t_o)\|_{\mathscr{A}} &= \|f\|_\infty \overset{(28)}{=} w^*(f(t_o))=|w^*(f(t_o))+0|\\ &\overset{(28)}{=} |w^*(f(t_o))+\lambda w^*(g(t_o))|=|w^*(f(t_o)+\lambda g(t_o))|\\ &\overset{(28)}{\leqslant} \|f(t_o)+\lambda g(t_o)\|_{\mathscr{A}},\end{aligned}$$

thus finally we get $f(t_o)\perp_B g(t_o)$. □

A normed space $(X,\|\cdot\|)$ is said to be *smooth at the point* $x_o\in X\setminus\{0\}$, if there is a unique $x^*\in X^*$ such that $x^*(x_o)=\|x_o\|$ and $\|x^*\|=1$. Now, we consider a set $D_{sm}(X):=\{x\in X : X \text{ is smooth at } x\}$. It is well known that the set $D_{sm}(\mathscr{C}(\Omega))$ is dense in $\mathscr{C}(\Omega)$. Moreover, if X is a separable real Banach space, then $D_{sm}(X)$ is dense. Now we will give a characterization of smoothness at a point in terms of the Birkhoff orthogonality (see [5]).

THEOREM 7. [5] *Let X be a normed space let $x_o\in X\setminus\{0\}$. Then the following statements are equivalent:*

(i) *X is smooth at x_o, i.e., $x_o \in D_{sm}(X)$;*

(ii) *the Birkhoff orthogonality is x_o-additive at right, i.e., for every $y,z \in X$ with $x_o \perp y$ $x_o \perp_{\mathrm{B}} z$, we have also $x_o \perp_{\mathrm{B}} y+z$.*

The next result may be also known, but for the convenience of the readers we present it here.

THEOREM 8. *If $f \in \mathscr{C}(\Omega)$, then $f \in D_{sm}(\mathscr{C}(\Omega))$ if and only if there is a unique $t_1 \in \Omega$ such that $|f(t_1)| = \|f\|_\infty$.*

There is a natural question. What happens in a general C^*-algebra $\mathscr{C}(\Omega;\mathscr{A})$ where $\mathscr{A} \neq \mathbb{R}$? Namely, we want to explore the set $D_{sm}(\mathscr{C}(\Omega;\mathscr{A}))$ instead of $D_{sm}(\mathscr{C}(\Omega))$. For $f \in \mathscr{C}(\Omega;\mathscr{A})$ put $M(f) := \{t \in \Omega : \|f(t)\|_{\mathscr{A}} = \|f\|_\infty\}$.

A *semi-ideal* of $\mathscr{C}(\Omega;\mathscr{A})$ is a linear subspace $\mathscr{X}$ of $\mathscr{C}(\Omega;\mathscr{A})$ such that $\varphi\cdot h \in \mathscr{X}$ whenever $\varphi \in \mathscr{C}(\Omega;\mathbb{R})$, $h \in \mathscr{X}$.

PROPOSITION 2. *Let $\mathscr{X} \subset \mathscr{C}(\Omega;\mathscr{A})$ be a semi-ideal (not necessarily closed). If $f \in \mathscr{X}$, $f \neq 0$ and $\overline{\overline{M(f)}} > 1$, then $f \notin D_{sm}(\mathscr{X})$.*

Proof. Fix arbitrarily $t_1, t_2 \in M(f)$ such that $t_1 \neq t_2$. By Urysohn's Lemma there is a continuous function $\rho : \Omega \to [0,1]$ such that $\rho(t_1) = 0$ and $\rho(t_2) = 1$. It is obvious that $\rho\cdot f$, $(1-\rho)\cdot f \in \mathscr{C}(\Omega;\mathscr{A})$. Then for $\lambda \in \mathbb{K}$ we have

$$\|f\|_\infty = \|f(t_1)\|_{\mathscr{A}} = \|f(t_1) + \lambda\rho(t_1)\cdot f(t_1)\|_{\mathscr{A}} \leqslant \|f + \lambda\rho\cdot f\|_\infty,$$
$$\|f\|_\infty = \|f(t_2)\|_{\mathscr{A}} = \|f(t_2) + \lambda(1-\rho(t_2))\cdot f(t_2)\|_{\mathscr{A}} \leqslant \|f + \lambda(1-\rho)\cdot f\|_\infty,$$

which means that $f \perp_{\mathrm{B}} \rho\cdot f$ and $f \perp_{\mathrm{B}} (1-\rho)\cdot f$. Since $\mathscr{X}$ is a semi-ideal,

$$\rho f,\ (1-\rho) f \in \mathscr{X}.$$

On the other hand it is easy to verify that f is not B-orthogonal to f. Thus, f is not B-orthogonal to $\rho\cdot f + (1-\rho)\cdot f$, and Theorem 7 yields $f \notin D_{sm}(\mathscr{X})$. □

COROLLARY 1. *If $f \in \mathscr{C}(\Omega;\mathscr{A})$ and $\overline{\overline{M(f)}} > 1$, then $f \notin D_{sm}(\mathscr{C}(\Omega;\mathscr{A}))$.*

So, the case of $\overline{\overline{M(f)}} > 1$ is clear. Now we will investigate the case where $\overline{\overline{M(f)}} = 1$. Fix $t_o \in \Omega$. We say that subspace $U \subset \mathscr{C}(\Omega;\mathscr{A})$ is *t_o-surjective*, if for all $a \in \mathscr{A}$, there exists $g \in U$ such that

$$g(t_o) = a, \quad \text{or, equivalently,} \quad \mathscr{A} = \bigcup_{g \in U} \{g(t_o)\}.$$

It is clear that $\mathscr{C}(\Omega;\mathscr{A})$ is t_o-surjective. On the other hand, t_o-sujective subspace may be small.

EXAMPLE 1. Let us consider $\mathscr{C}([0,1];\mathbb{R}^2)$ (with some normed space $\mathbb{R}^2$). Fix $t_o \in (0,1]$. We define $f, g \in \mathscr{C}([0,1];\mathbb{R}^2)$ by $f(t) := (t,0)$, $g(t) := (0,t)$. It is easy to check that the space $U := \operatorname{span}\{f,g\}$ is t_o-surjective and $\dim U = 2 < \dim \mathscr{C}([0,1];\mathbb{R}^2)$.

The next result establishes the connection between $D_{sm}(\mathscr{C}(\Omega;\mathscr{A}))$ and $D_{sm}(\mathscr{A})$ and $D_{sm}(U)$.

THEOREM 9. *Let $\mathscr{A}$ be a real (or complex) C^*-algebra. Suppose that Ω is a compact topological space. Assume that $f \in \mathscr{C}(\Omega;\mathscr{A})$ and $M(f) = \{t_1\}$. The following conditions are equivalent:*

(a) *$f \in D_{sm}(\mathscr{C}(\Omega;\mathscr{A}))$;*
(b) *$f(t_1) \in D_{sm}(\mathscr{A})$;*
(c) *there is a t_1-surjective subspace $U \subset \mathscr{C}(\Omega;\mathscr{A})$ such that $f \in D_{sm}(U)$.*

Proof. We start with proving (b)$\Rightarrow$(a). Fix arbitrarily $g,h \in \mathscr{C}(\Omega;\mathscr{A})$ such that $f\perp_B g$ and $f\perp_B h$. By Proposition 1 we have $f(t_1)\perp_B g(t_1)$ and $f(t_1)\perp_B h(t_1)$. It follows from (b) and Theorem 7 that $f(t_1)\perp_B g(t_1)+h(t_1)$. Using again Proposition 1 we get $f\perp_B g+h$ and Theorem 7 yields $f \in D_{sm}(\mathscr{C}(\Omega;\mathscr{A}))$.

The implications (a)$\Rightarrow$(c) is obvious. Finally, we prove (c)$\Rightarrow$(b). Fix arbitrarily $x,y \in \mathscr{A}$ such that $f(t_1)\perp_B x$ and $f(t_1)\perp_B y$. Since U is t_1-surjective, there are $g,h \in U$ such that $g(t_1) = x$, $h(t_1) = y$. It follows from (c) and Theorem 7 that $f\perp_B g+h$. Using again Proposition 1 we get $f(t_1)\perp_B g(t_1)+h(t_1)$, which means $f(t_1)\perp_B x+y$. Theorem 7 yields $f(t_1) \in D_{sm}(\mathscr{A})$. $\square$

6. Bhatia–Šemrl theorem

Now, we will show a new proof of the Bhatia–Šemrl theorem using Lemma 3. We will use again the new method to obtain the following characterization of B-orthogonality.

THEOREM 10. [2] *Let $\mathscr{H}$ be a real Hilbert space. Suppose that* $\dim\mathscr{H} < \infty$. *Let $A,B \in \mathscr{B}(\mathscr{H})$. Then*

$$A\perp_B B \quad \Leftrightarrow \quad \exists_{x\in S_{\mathscr{H}}}\ \|Ax\| = \|A\|,\ Ax\perp Bx.$$

We are now ready to prove a real version of the Bhatia–Šemrl theorem using these concepts. Our approach will revolve around Lemma 3 (in particular *(ii)*).

Proof. We start with proving "$\Leftarrow$". Suppose that there is a vector x such that $\|Ax\| = \|A\|$, $Ax\perp_B Bx$. Thus we have $\langle Ax|Bx\rangle = 0$ hence

$$\begin{aligned}\|A\|^2 &= \|Ax\|^2 = \langle Ax|Ax\rangle = \langle Ax|Ax\rangle + 0\\ &= \langle Ax|Ax\rangle + \lambda\,\langle Ax|Bx\rangle = \langle Ax|Ax+\lambda Bx\rangle\\ &\leqslant \|Ax\|\cdot\|Ax+\lambda Bx\| \leqslant \|A\|\cdot\|A+\lambda B\|.\end{aligned}$$

Therefore $\|A\| \leqslant \|A+\lambda B\|$ for $\lambda \in \mathbb{R}$.

Now we prove the converse. Let us consider $V := \mathscr{B}(\mathscr{H})$ and $\mathscr{A} := \mathscr{B}(\mathscr{H})$. We define a inner product

$$\langle\cdot|\cdot\rangle_{\mathscr{B}(\mathscr{H})} : \mathscr{B}(\mathscr{H})\times\mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{H}), \qquad \langle M|N\rangle_{\mathscr{B}(\mathscr{H})} := M^*N.$$

Then, the space $V = \mathscr{B}(\mathscr{H})$ can be regarded as a Hilbert C^*-module over itself with $\mathscr{A} = \mathscr{B}(\mathscr{H})$. Fix arbitrarily two vectors $x, y \in \mathscr{H}$. Define a linear and continuous functional

$$a^*_{x,y} \in \mathscr{B}(\mathscr{H}) \to \mathbb{R} \quad \text{by} \quad a^*_{x,y}(T) := \langle Tx|y\rangle_{\mathscr{H}}, \quad T \in \mathscr{B}(\mathscr{H}).$$

Now we define a set $\mathscr{N} := \{a^*_{x,y} \in \mathscr{B}(\mathscr{H})^* : \|x\| = \|y\| = 1\}$. It is easy to check that the set $\mathscr{N}$ is *-norming. Let us consider a set $\mathscr{D} := \{I\}$, where $I \in \mathscr{B}(\mathscr{H})$, $I(x) = x$. The set D is norming.

We will consider only inner products $\langle\cdot|\cdot\rangle_{\mathscr{B}(\mathscr{H})} : \mathscr{B}(\mathscr{H}) \times \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{H})$, $\langle\cdot|\cdot\rangle_{\mathscr{H}} : \mathscr{H} \times \mathscr{H} \to \mathbb{R}$, and to shorten the notation we will write $\langle\cdot|\cdot\rangle_{\mathscr{B}} := \langle\cdot|\cdot\rangle_{\mathscr{B}(\mathscr{H})}$ and $\langle\cdot|\cdot\rangle := \langle\cdot|\cdot\rangle_{\mathscr{H}}$.

We define the the following set:

$$\mathscr{F}_2 := \left\{\lambda a^*_{x,y}(\langle I|\cdot\rangle_{\mathscr{B}}) + (1-\lambda) a^*_{u,w}(\langle I|\cdot\rangle_{\mathscr{B}}) \in \mathscr{B}(\mathscr{H})^* : x, y, u, w \in S_{\mathscr{H}},\ \lambda \in [0,1]\right\};$$

compare (1).

Note that $A \perp_{\rm B} B$ yields $\alpha A \perp_{\rm B} \beta B$ for all $\alpha, \beta \in \mathbb{R}$ (i.e., $\perp_{\rm B}$ is full homogeneous). Without loss of generality, we may assume that $\|A\| = 1$.

Now, suppose that $A \perp_{\rm B} B$. Applying Lemma 3 (in particular *(ii)*) we obtain

$$\|A\| = \sup\{v^*(A) : B \in \ker v^*,\ v^* \in \mathscr{F}_2\}.$$

There are $v^*_n \in \mathscr{F}_2$ such that $v^*_n(A) \to \|A\|$ and $B \in \ker v^*_n$. This means that

$$\begin{aligned} v^*_n(A) &= \lambda_n a^*_{x_n,y_n}(\langle I|A\rangle_{\mathscr{B}}) + (1-\lambda_n) a^*_{u_n,w_n}(\langle I|A\rangle_{\mathscr{B}}) \\ &= \lambda_n a^*_{x_n,y_n}(I^*A) + (1-\lambda_n) a^*_{u_n,w_n}(I^*A) \\ &= \lambda_n a^*_{x_n,y_n}(A) + (1-\lambda_n) a^*_{u_n,w_n}(A) \\ &= \lambda_n \langle Ax_n|y_n\rangle + (1-\lambda_n)\langle Au_n|w_n\rangle. \end{aligned}$$

Therefore, we obtain

$$\lambda_n \langle Ax_n|y_n\rangle + (1-\lambda_n)\langle Au_n|w_n\rangle \to \|A\|. \tag{29}$$

Moreover, $v^*_n(B) = 0$. In a similar way one can prove

$$\lambda_n \langle Bx_n|y_n\rangle + (1-\lambda_n)\langle Bu_n|w_n\rangle = 0. \tag{30}$$

The set $[0,1]$ is compact. Since $\mathscr{H}$ is finite dimensional, $S_{\mathscr{H}}$ is a compact set. Therefore, without loss of generality, we may assume that there are vectors x, y, u, w in $S_{\mathscr{H}}$, a number $\lambda \in [0,1]$ and subsequences $\{x_{n_k}\}$, $\{y_{n_k}\}$, $\{u_{n_k}\}$, $\{w_{n_k}\}$, $\{\lambda_{n_k}\}$ such that $x_{n_k} \to x$, $y_{n_k} \to y$, $u_{n_k} \to u$, $w_{n_k} \to w$, $\lambda_{n_k} \to \lambda$. Now the condition (29) becomes

$$\lambda \langle Ax|y\rangle + (1-\lambda)\langle Au|w\rangle = \|A\|. \tag{31}$$

We get from (30)

$$\lambda \langle Bx|y\rangle + (1-\lambda)\langle Bu|w\rangle = 0. \tag{32}$$

We will show that $\langle Ax|y\rangle = 1 = \langle Au|w\rangle$. We have

$$1 = \|A\| = \lambda \langle Ax|y\rangle + (1-\lambda)\langle Au|w\rangle$$

and $\langle Ax|y\rangle, \langle Au|w\rangle \in [-1,1]$. It is easy to check that $1 \in \mathrm{Ext}[-1,1]$. So, $\langle Ax|y\rangle = 1$ and $\langle Au|w\rangle = 1$.

We have $\langle Ax|y\rangle = 1$ and $\langle Ax|y\rangle \leqslant \|Ax\|\cdot\|y\| \leqslant 1\cdot 1 = 1$. This implies $\langle Ax|y\rangle = \|Ax\|\cdot\|y\|$ and $\|Ax\| = 1 = \|y\|$. So, $Ax = y$, and hence $\|Ax\| = \|A\|$.

In a similar way, one checks that $Au = w$ and $\|Au\| = \|A\|$. Putting $Ax = y$, $Au = w$ into (32), we get

$$\lambda \langle Bx|Ax\rangle + (1-\lambda)\langle Bu|Au\rangle = 0. \tag{33}$$

It follows from (33) that $\langle Bx|Ax\rangle \leqslant 0 \leqslant \langle Bu|Au\rangle$ or $\langle Bu|Au\rangle \leqslant 0 \leqslant \langle Bx|Ax\rangle$. Without loss of generality, we may assume that

$$\langle Bx|Ax\rangle \leqslant 0 \leqslant \langle Bu|Au\rangle. \tag{34}$$

We define a set $\mathscr{M}(A) := \{x \in S_{\mathscr{H}} : \|Ax\| = \|A\|\}$. It is obvious that $\overline{\overline{\mathscr{M}(A)}} \geqslant 2$. We have two possibilities:

Possibility 1: If $\mathscr{M}(A) = 2$, then $x = u$ or $x = -u$. By (34) we obtain

$$\langle Bx|Ax\rangle \leqslant 0 \leqslant \langle Bu|Au\rangle = \langle B(\pm x)|A(\pm x)\rangle = \langle Bx|Ax\rangle,$$

hence $\langle Bx|Ax\rangle = 0$.

Possibility 2: If $\overline{\overline{\mathscr{M}(A)}} > 2$, then $\mathscr{M}(A)$ is connected. Indeed, fix arbitrarily two linearly independent vectors $a, b \in \mathscr{M}(A)$, i.e., $a \neq b \neq -a$. Define $\eta\colon [0,1] \to S_{\mathscr{H}}$ by $\eta(t) := \frac{(1-t)a+tb}{\|(1-t)a+tb\|}$. It is easy to check that η is a path and $\|A(\eta(t))\| = \|A\|$ for all $t \in [0,1]$. This means that $\eta([0,1]) \subset \mathscr{M}(A)$, and therefore $\mathscr{M}(A)$ is connected.

Now, we define a mapping $\varphi\colon \mathscr{M}(A) \to \mathbb{R}$ by $\varphi(v) := \langle Bv|Av\rangle$, $v \in \mathscr{M}(A)$. Inequalities (34) yield $\varphi(x) \leqslant 0 \leqslant \varphi(u)$. The mapping φ is continuous. Moreover, the set $\mathscr{M}(A)$ is connected. Using the Darboux property we get $\varphi(x_o) = 0$ for some $x_o \in \mathscr{M}(A)$. Thus for the vector $x_o \in \mathscr{M}(A)$ we have $\langle Bx_o|Ax_o\rangle = 0$ and $\|Ax_o\| = \|A\|$, i.e., $\|Ax_o\| = \|A\|$ and $Ax_o \perp Bx_o$. Whence, in any case $\exists_{x\in S_{\mathscr{H}}}\ \|Ax\| = \|A\|$, $Ax \perp Bx$. □

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Paweł Wójcik
Institute of Mathematics
Pedagogical University of Cracow
Podchorążych 2, 30-084 Kraków, Poland
e-mail: pwojcik@up.krakow.pl