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A CANONICAL FORM FOR H–UNITARY MATRICES

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Dedicated to the memory of our dear friend and mentor Leiba Rodman, whose work also inspired the present paper. We miss his open mind, his inspiration, his keen interest in mathematical matters that were of joint interest to us, but most of all we miss his warm friendship

(*Communicated by I. M. Spitkovsky*)

Abstract. In this paper matrices A are considered that have the property that $A^*HA = H$, where $H = H^*$ is invertible. A canonical form is given for the pair of matrices (A,H) under transformations $(A,H) \to (S^{-1}AS, S^*HS)$, where S is invertible, in which the canonical form for the A-matrix is the usual Jordan canonical form. The real case is studied as well.

1. Introduction and preliminaries

Let $H = H^*$ be an invertible Hermitian matrix. A matrix A is called H-unitary if $A^*HA = H$. Note that for an invertible matrix S the matrix $S^{-1}AS$ is S^*HS-unitary. The class of H-unitary matrices has been studied extensively, and canonical forms for pairs (A,H) under the transformation $(A,H) \to (S^{-1}AS, S^*HS)$ exist both for the complex case (see [6]) and for the real case (see [13] and [16]).

Such canonical forms were developed from the canonical form of pairs (A,H) where $H = H^*$ is invertible and A is H-selfadjoint using the idea of the Cayley transform, see [6], [14], and by direct methods in [17, 18], see also the early treatments in [15] and [8]. These canonical forms do not have the matrix A in Jordan canonical form, and even then the canonical form for the matrix H may be fairly complicated. There are different approaches as well, for special cases. In [1], besides a canonical form for the general case, the diagonalizable case was treated. Starting from a classification of indecomposable blocks for H-normal matrices for the case when H has only one negative eigenvalue, a canonical form for H-unitary matrices for that special case has been developed in [7]. In [12, 13] H-unitary matrices are treated as a subclass of polynomially H-normal matrices. The class of H-unitary matrices is a special case of H-expansive matrices: a matrix A is called H-expansive if $A^*HA - H \geqslant 0$. It is therefore possible to simplify the simple form for H-expansive matrices as described in [4] even further to arrive at a canonical form for H-unitary matrices. Closest to the spirit of our approach comes [11], although that paper does not result in a complete canonical

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form. However, several of the intermediate results we shall use in the treatment of the real case are already available from [11].

We will consider both the real and complex cases. More precisely, we firstly discuss the real case, i.e., $H = H^T$ real symmetric and invertible and A real. Recall that a real matrix A is called H-*orthogonal* if $A^T HA - H = 0$. Clearly this is the real case of H-unitary matrices, and we shall use the term H-unitary throughout the paper also for the real case. We consider the complex case: $H = H^*$ complex Hermitian and invertible and A complex in the final section of the paper, but start with an analysis of the real case.

Obviously the H-unitary matrices form a subclass of the H-expansive matrices for which $A^T HA - H \geqslant 0$. It is the aim of this paper to show that the methods we developed to arrive at a simple form for H-expansive matrices (see [4] and [10]) may be carried further for H-unitary matrices. We shall do this all the way through to a canonical form for H, with A already in Jordan canonical form. This novel approach leads to a more transparent canonical form for the matrix pair (A,H), with A in standard real Jordan canonical form. The methods we use in the complex case are similar to the ones in the real case.

We shall denote by $J_n(\lambda)$ the $n \times n$ Jordan block with eigenvalue λ. Since we are interested in the real Jordan canonical form, we shall denote the real Jordan block with eigenvalues $\alpha \pm i\beta$ by $J_n(\gamma)$, where $\gamma = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$, so

$$J_n(\gamma) = \begin{bmatrix} \gamma & I_2 & & \\ & \ddots & \ddots & \\ & & \ddots & I_2 \\ & & & \gamma \end{bmatrix}. \tag{1}$$

Throughout, we shall also denote by $toep(s_1, \cdots, s_n)$ the upper triangular Toeplitz matrix with the numbers $s_1, \cdots, s_n$ on the first row.

Let us first recall some general properties of H-orthogonal and H-unitary matrices. If A is H-unitary and $\lambda \in \sigma(A)$ then $(\overline{\lambda})^{-1}$ is an eigenvalue of A as well. If we are considering a real matrix A, eigenvalues come in four different kinds:

1. Quadruples λ, $\overline{\lambda}$, $\frac{1}{\lambda}$, $(\overline{\lambda})^{-1}$ with $\lambda \notin \mathbb{R}$ and $|\lambda| \neq 1$.

2. Pairs λ, $\frac{1}{\lambda}$ with $\lambda \in \mathbb{R}$, $\lambda \neq \pm 1$.

3. Pairs $\lambda = \alpha \pm i\beta$, $\alpha^2 + \beta^2 = 1$, $\beta \neq 0$, so $|\lambda| = 1$.

4. $\lambda = 1$ or $\lambda = -1$.

Let us agree to denote the (sum of the) root spaces corresponding to such a quadruple, resp. pairs, resp. eigenvalue by $\mathscr{R}_\lambda(A)$. Then these spaces are mutually H-orthogonal (see e.g. [2, 3, 6, 9] or [16]).

Thus we may restrict attention to matrices for which the spectrum is one of these four kinds.

It is proved in [13], see also [6, 16] and [8], that a pair (A,H), where $H=H^T$ is invertible and A is H-unitary may be decomposed as follows.

PROPOSITION 1.1. *Let $H=H^T$ be invertible and let A be H-unitary. Then there is an invertible matrix S such that*

$$S^{-1}AS=\oplus_{j=1}^{k}A_j,\qquad S^THS=\oplus_{j=1}^{k}H_j, \tag{2}$$

where in each pair (A_j,H_j) the matrix $H_j=H_j^T$ is invertible, and A_j is H_j-unitary, and each pair is of one of the following five indecomposable forms

(i) *(real non-unimodular eigenvalues)*

$$A_j=J_{n_j}(\lambda)\oplus J_{n_j}\left(\frac{1}{\lambda}\right)\ \textit{with}\ \lambda\neq\pm1,\ \lambda\in\mathbb{R},\qquad H_j=\begin{bmatrix}0 & H_{12}\\ H_{12}^T & 0\end{bmatrix} \tag{3}$$

(ii) *(complex non-unimodular eigenvalues)*

$$A_j=J_{n_j}(\gamma)\oplus J_{n_j}(\gamma^{-1}) \tag{4}$$

$$\textit{where}\ \gamma=\begin{bmatrix}\alpha & \beta\\ -\beta & \alpha\end{bmatrix}\ \textit{with}\ \alpha^2+\beta^2\neq1,\ \beta\neq0,$$

$$H_j=\begin{bmatrix}0 & H_{12}\\ H_{12}^T & 0\end{bmatrix}$$

(iii) *(unimodular non-real eigenvalues)*

$$A_j=J_{n_j}(\gamma)\ \textit{where}\ \gamma=\begin{bmatrix}\alpha & \beta\\ -\beta & \alpha\end{bmatrix}\ \textit{with}\ \alpha^2+\beta^2=1,\ \beta\neq0, \tag{5}$$

(iv) *(±1, odd partial multiplicity)*

$$A_j=J_{n_j}(\pm1),\ \textit{with}\ \ n_j\ \textit{odd}, \tag{6}$$

(v) *(±1, even partial multiplicities)*

$$A_j=J_{n_j}(\pm1)\oplus J_{n_j}(\pm1)\ \textit{with}\ n_j\ \textit{even}\qquad H_j=\begin{bmatrix}0 & H_{12}\\ H_{12}^T & 0\end{bmatrix}. \tag{7}$$

The matrices H_j, and in particular the form of the matrices H_{12} in (3), (4), (7) may be further reduced to a canonical form as is described in the main results of this paper.

In particular, note that even sized blocks with eigenvalue one or minus one come in pairs. This was proved in e.g. [12], see in particular Proposition 3.4 there and its proof, and also in [1], Proposition 3.1.

As a consequence of this, all one needs to do to arrive at a canonical form for the pair (A,H) is to derive canonical forms for each of these five indecomposable blocks.

In this approach, we begin with case (5), continuing with cases (6) and (7), which will be treated together in one section, after which we discuss case (3) and finally case (4).

Before stating the result on the general canonical form, let us introduce some special matrices. Whenever $n > 1$ is an odd integer, we define the following matrices $P_n, \hat{P}_n, P_n(\lambda), \hat{P}_n(\gamma)$ and $P_n(\gamma)$ in Definitions 1,2 and 3, respectively.

DEFINITION 1. We define for odd $n > 1$ the $\frac{n+1}{2} \times \frac{n-1}{2}$ matrix $P_n = \left[p_{ij}\right]_{i=1,j=1}^{\frac{n+1}{2},\frac{n-1}{2}}$ as follows:

$$
\begin{aligned}
&p_{i\,j} = 0 && \text{when } i+j \leqslant \frac{n-1}{2},\\
&p_{i\,\frac{n-1}{2}-i+1} = (-1)^{\frac{n-1}{2}-i+1} && \text{for } i = 1,\dots,\frac{n+1}{2}-1,\\
&p_{\frac{n+1}{2}\,j} = (-1)^j \cdot \frac{1}{2} && \text{for } j = 1,\dots,\frac{n-1}{2},
\end{aligned}
$$

and all other entries are defined by $p_{i\,j+1} = -(p_{i\,j} + p_{i+1\,j})$.

Next, define $\hat{P}_n = \left[\hat{p}_{ij}\right]_{i=1,j=1}^{\frac{n+1}{2},\frac{n-1}{2}}$ by $\hat{p}_{ij} = |p_{ij}|$. Observe that for the entries of $\hat{P}_n$ we have $\hat{p}_{i\,j+1} = \hat{p}_{i\,j} + \hat{p}_{i+1\,j}$.

To get a feeling for how such a matrix looks, we give P_{11} below:

$$
P_{11} = \begin{bmatrix}
0 & 0 & 0 & 0 & -1\\
0 & 0 & 0 & 1 & -\frac{9}{2}\\
0 & 0 & -1 & \frac{7}{2} & -\frac{16}{2}\\
0 & 1 & -\frac{5}{2} & \frac{9}{2} & -\frac{14}{2}\\
-1 & \frac{3}{2} & -\frac{4}{2} & \frac{5}{2} & -\frac{6}{2}\\
-\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2}
\end{bmatrix}.
$$

Also, P_9 is the submatrix of P_{11} formed by deleting the last column and first row.

Observe that the recursion for the entries of P_n actually holds for all its entries, provided the first column and last row are given, or the last column and first row. Also note that the recursion is the same as the one for Pascal's triangle, modulo a minus sign. The recursion $\hat{p}_{i\,j+1} = \hat{p}_{i\,j} + \hat{p}_{i+1\,j}$ for the entries in $\hat{P}_n$ is exactly the same as the one for Pascal's triangle, but the entries in $\hat{P}_n$ are not the numbers in the Pascal triangle because the starting values are different: if we consider the entries in the first column and last row of $\hat{P}_n$ as the starting values, then the nonzero starting numbers are $1, \frac{1}{2}$ rather than $1, 1$ as would be the case for the Pascal triangle.

For the numbers p_{ij} and $\hat{p}_{ij}$ we can give somewhat more complicated but still explicit formulas. With the understanding that $\binom{j}{k} = 0$ whenever $k < 0$ or $j < k$.

$$
p_{ij} = \frac{(-1)^j}{2}\left(\binom{j+1}{\frac{n+1}{2}-i} - \binom{j-1}{\frac{n+1}{2}-i-2}\right),
$$

and also

$$\hat{p}_{ij} = \frac{1}{2}\left(\binom{j+1}{\frac{n+1}{2}-i} - \binom{j-1}{\frac{n+1}{2}-i-2}\right).$$

Indeed, it can be easily checked that these numbers satisfy the recursion and the initial values given in Definition 1.

DEFINITION 2. For $\lambda \in \mathbb{R}\setminus\{-1,1\}$ and $n>1$ an odd integer, we define the $\frac{n+1}{2}\times\frac{n-1}{2}$ matrix $P_n(\lambda)$ as follows:

$$P_n(\lambda) = \left[p_{ij}\lambda^{\frac{n+1}{2}+j-i}\right]_{i=1,\ j=1}^{\frac{n+1}{2},\ \frac{n-1}{2}} \tag{8}$$

where p_{ij} are the entries of the matrix P_n introduced above.

For example, $P_5(\lambda)$ is the 3×2 matrix given by

$$P_5(\lambda) = \begin{bmatrix} 0 & \lambda^4 \\ -\lambda^2 & \frac{3}{2}\lambda^3 \\ -\frac{1}{2}\lambda & \frac{1}{2}\lambda^2 \end{bmatrix}.$$

For $\gamma = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$ we define the following matrices:

DEFINITION 3. Let $n>1$ be an odd integer, then the $\frac{n+1}{2}\times\frac{n-1}{2}$ block matrices $\hat{P}_n(\gamma)$ and $P_n(\gamma)$ with two by two matrix blocks are defined as:

$$\hat{P}_n(\gamma) = \left[p_{ij}(\gamma^T)^{\frac{n+1}{2}+j-i}\right]_{i=1,\ j=1}^{\frac{n+1}{2},\ \frac{n-1}{2}} \quad \text{with } \alpha^2+\beta^2=1,\ \beta\neq 0; \tag{9}$$

$$P_n(\gamma) = \left[p_{ij}K_1\gamma^{\frac{n+1}{2}+j-i}\right]_{i=1,\ j=1}^{\frac{n+1}{2},\ \frac{n-1}{2}}, \quad \text{with } \alpha^2+\beta^2\neq 1; \tag{10}$$

where p_{ij} are the entries of the matrix P_n introduced earlier, and $K_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

For example, $P_5(\gamma)$ is the 3×2 block matrix given by

$$P_5(\gamma) = \begin{bmatrix} 0 & K_1\gamma^4 \\ -K_1\gamma^2 & \frac{3}{2}K_1\gamma^3 \\ -\frac{1}{2}K_1\gamma & \frac{1}{2}K_1\gamma^2 \end{bmatrix}.$$

We also introduce for odd n the $\frac{n+1}{2}\times\frac{n+1}{2}$ matrix Z_n which has zeros everywhere, except in the $(\frac{n+1}{2},\frac{n+1}{2})$-entry, where it has a one. For instance, Z_5 is given by

$$Z_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We shall also make use of the matrices $Z_n \otimes I_2$, which is the $(n+1)\times(n+1)$ matrix which has zeros everywhere except in the two by two lower right block where it has I_2, and $Z_n \otimes K_1$, where $K_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, i.e., the $(n+1)\times(n+1)$ matrix which has zeros everywhere except in the two by two lower right block where it has K_1.

Whenever $n > 1$ is an even integer, we define analogously the following matrices Q_n, $\hat{Q}_n$, $Q_n(\lambda)$, $\hat{Q}_n(\gamma)$ and $Q_n(\gamma)$ in Definitions 4,5 and 6, respectively

DEFINITION 4. For even n the $\frac{n}{2}\times\frac{n}{2}$ matrix $Q_n = \left[q_{ij}\right]_{i=1,j=1}^{\frac{n}{2},\frac{n}{2}}$ is defined as follows:

$$
\begin{aligned}
q_{ij} &= 0 && \text{when } i+j \leqslant \frac{n}{2}, \\
q_{i\,\frac{n}{2}-i+1} &= (-1)^{\frac{n}{2}-i} && \text{for } i = 1,\dots,\frac{n}{2}, \\
q_{\frac{n}{2}\,j} &= (-1)^{j-1} && \text{for } j = 1,\dots,\frac{n}{2},
\end{aligned}
$$

and all other entries are defined by $q_{i\;j+1} = -(q_{i\,j} + q_{i+1\,j})$.

Also define $\hat{Q}_n = \left[\hat{q}_{ij}\right]_{i=1,j=1}^{\frac{n}{2},\frac{n}{2}}$ by $\hat{q}_{ij} = |q_{ij}|$. Observe that $\hat{q}_{i\;j+1} = \hat{q}_{i\,j} + \hat{q}_{i+1\,j}$.

Again, we give an example: Q_{10} is given by

$$
Q_{10} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 4 \\ 0 & 0 & 1 & -3 & 6 \\ 0 & -1 & 2 & -3 & 4 \\ 1 & -1 & 1 & -1 & 1 \end{bmatrix}.
$$

Also, Q_8 is formed from this by deleting the first row and last column.

Note that the numbers involved, apart from a minus sign, are exactly the numbers from Pascal's triangle, so in this case, with n even, we can give a precise formula: when $i+j \geqslant \frac{n}{2}+1$ we have

$$
q_{ij} = (-1)^{j-1}\binom{j-1}{\frac{n}{2}-i},
$$

and also

$$
\hat{q}_{ij} = \binom{j-1}{\frac{n}{2}-i}.
$$

DEFINITION 5. For $\lambda \in \mathbb{R}\setminus\{-1,1\}$ and $n > 1$ an even integer, then the $\frac{n}{2}\times\frac{n}{2}$ matrix $Q_n(\lambda)$ is defined as follows:

$$
Q_n(\lambda) = \left[q_{ij}\lambda^{\frac{n}{2}+j-i-1}\right]_{i=1,\;j=1}^{\frac{n}{2},\;\frac{n}{2}} \tag{11}
$$

where q_{ij} are the entries of the matrix Q_n introduced earlier.

For $\gamma = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$ we define the following matrices:

DEFINITION 6. Let $n > 1$ be an even integer, then the $\frac{n}{2} \times \frac{n}{2}$ block matrix $\hat{Q}_n(\gamma)$ is defined as follows:

$$\hat{Q}_n(\gamma) = \left[q_{ij} H_0 (\gamma^T)^{\frac{n}{2}+j-i}\right]_{i=1,\ j=1}^{\frac{n}{2},\ \frac{n}{2}} \quad \text{with} \ \ \alpha^2+\beta^2 = 1,\ \beta \neq 0; \tag{12}$$

and the $\frac{n}{2} \times \frac{n}{2}$ block matrix $Q_n(\gamma)$ is defined as:

$$Q_n(\gamma) = \left[q_{ij} K_1 \gamma^{\frac{n}{2}+j-i-1}\right]_{i=1,\ j=1}^{\frac{n}{2},\ \frac{n}{2}} \quad \text{with} \ \ \alpha^2+\beta^2 \neq 1; \tag{13}$$

where q_{ij} are the entries of the matrix Q_n introduced earlier, $H_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $K_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

With all these definitions in hand we are now ready to state the main theorem of the paper, giving a canonical form for a pair of real matrices (A,H), where A is H-unitary, with A in real Jordan canonical form.

THEOREM 1.1. *Let A be H-unitary, with both A and H real. Then the pair (A,H) can be decomposed as follows. There is an invertible real matrix S such that*

$$S^{-1}AS = \oplus_{l=1}^{p} A_l, \qquad S^T HS = \oplus_{l=1}^{p} H_l,$$

where each pair (A_l,H_l) is of one of the following forms for some n depending on l

(i) $\sigma(A_l) = \{1\}$ *and the pair (A_l,H_l) has one of the following two forms:*

Case 1 $\left(J_n(1), \varepsilon \begin{bmatrix} Z_n & P_n \\ P_n^T & 0 \end{bmatrix}\right)$ *with n odd, and $\varepsilon = \pm 1$.*

Case 2 $\left(J_n(1) \oplus J_n(1), \begin{bmatrix} 0 & 0 & 0 & Q_n \\ 0 & 0 & -Q_n^T & 0 \\ 0 & -Q_n & 0 & 0 \\ Q_n^T & 0 & 0 & 0 \end{bmatrix}\right)$ *with n even.*

(ii) $\sigma(A_l) = \{-1\}$ *and the pair (A_l,H_l) has one of the following two forms:*

Case 1 $\left(J_n(-1), \varepsilon \begin{bmatrix} Z_n & P_n(-1) \\ P_n(-1)^T & 0 \end{bmatrix}\right)$ *with n odd, and $\varepsilon = \pm 1$.*

Case 2 $\left(J_n(-1) \oplus J_n(-1), \begin{bmatrix} 0 & 0 & 0 & Q_n(-1) \\ 0 & 0 & -Q_n(-1)^T & 0 \\ 0 & -Q_n(-1) & 0 & 0 \\ Q_n(-1)^T & 0 & 0 & 0 \end{bmatrix}\right)$
with n even.

(iii) $\sigma(A_l) = \{\alpha \pm i\beta\}$ *with $\alpha^2+\beta^2 = 1$ and $\beta \neq 0$, and the pair (A_l,H_l) has one of the following two forms with* $\gamma = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$*:*

Case 1 $\left(J_n(\gamma), \varepsilon \begin{bmatrix} Z_n \otimes I_2 & \hat{P}_n(\gamma) \\ \hat{P}_n(\gamma)^T & 0 \end{bmatrix}\right)$ *with n odd, and $\varepsilon = \pm 1$.*

Case 2 $\left(J_n(\gamma), \varepsilon \begin{bmatrix} 0 & \hat{Q}_n(\gamma) \\ \hat{Q}_n(\gamma)^T & 0 \end{bmatrix}\right)$ *with n even, and $\varepsilon = \pm 1$.*

(iv) $\sigma(A_l) = \{\lambda, \frac{1}{\lambda}\}$ *with $\lambda \in \mathbb{R} \setminus \{-1, 1\}$ and the pair (A_l, H_l) is of the form* $\left(J_n(\lambda) \oplus J_n(\frac{1}{\lambda}), \begin{bmatrix} 0 & H_{12} \\ H_{12}^T & 0 \end{bmatrix}\right)$*, where H_{12} is of one of the following two forms, depending on whether n is odd or even:*

Case 1 *n is odd:* $H_{12} = \begin{bmatrix} Z_n & P_n(\lambda) \\ P_n(\frac{1}{\lambda})^T & 0 \end{bmatrix}.$

Case 2 *n is even:* $H_{12} = \begin{bmatrix} 0 & Q_n(\lambda) \\ -\frac{1}{\lambda^2} Q_n(\frac{1}{\lambda})^T & 0 \end{bmatrix}.$

(v) $\sigma(A_l) = \{\alpha \pm i\beta, (\alpha \pm i\beta)^{-1}\}$ *with $\alpha^2 + \beta^2 \neq 1$ and $\beta \neq 0$, and the pair (A_l, H_l) is of the form* $\left(J_n(\gamma) \oplus J_n(\gamma^{-1}), \begin{bmatrix} 0 & H_{12} \\ H_{12}^T & 0 \end{bmatrix}\right)$*, where H_{12} is of one of the following two forms, depending on whether n is odd or even:*

Case 1 *n is odd:* $H_{12} = \begin{bmatrix} Z_n \otimes K_1 & P_n(\gamma) \\ P_n(\gamma^{-1})^T & 0 \end{bmatrix}.$

Case 2 *n is even:* $H_{12} = \begin{bmatrix} 0 & Q_n(\gamma) \\ -\gamma^{-2} Q_n(\gamma^{-1})^T & 0 \end{bmatrix}.$

Note that the columns of the matrix S in the theorem form a special real Jordan basis for A.

The theorem should be compared with the canonical form obtained in [13], in particular with Theorem 5.5 there.

The proof of the theorem will be given in the next sections. In Section 2 the case of part (iii) is treated, in Section 3 parts (i) and (ii) are proved, in Section 4 part (iv) is dealt with, and finally in Section 5 part (v) is proved.

The final section of the paper, Section 6, deals with complex H-unitary matrices, i.e., the case where $H = H^*$ is an invertible Hermitian matrix and A satisfies $A^*HA = H$. A canonical form for this case, similar to the one in Theorem 1.1 is presented in Theorem 6.1.

2. The case of non-real eigenvalues on the unit circle

In the construction of the simple form for the H-expansive case a number of lemmas were stated and proved and the lemmas in [10] and [4] are also applicable in the H-unitary case. For the convenience of the reader, we state the lemmas without proofs. The proofs can be found in Section 2 of [4] (see also Section 3.2 in [10]).

2.1. Preliminary lemmas

For a matrix X, we usually use the notation X_{ij} to denote the (i,j)-th scalar entry. We denote a block matrix X by $X=(X_{i,j})$, where $X_{i,j}$ is the (i,j)-th block entry of X.

Let A be H-expansive and assume that A is in real Jordan canonical form (e.g., see [6], Section A2 and [19], Chapter 6). Furthermore, let $\lambda=\alpha\pm i\beta$ be a pair of complex conjugate eigenvalues of A with $|\lambda|=1$ (i.e. $\alpha^2+\beta^2=1$) and assume $\beta\neq 0$. We therefore look at the case where the eigenvalues lie on the unit circle, but are not ± 1.

With this notation we have the following lemmas which will be used in the derivation of the canonical form.

LEMMA 2.1. *Let* $\gamma=\begin{bmatrix}\alpha & \beta\\ -\beta & \alpha\end{bmatrix}$ *be such that* $\beta\neq 0$ *and* $\alpha^2+\beta^2=1$. *If*

$$H=\begin{bmatrix}h_{11} & h_{12}\\ h_{12} & h_{22}\end{bmatrix}$$

and $\gamma^T H\gamma-H\geqslant 0$, *then* $\gamma^T H\gamma-H=0$ *and* $H=h_{11}I$.

The following notation is used in the sequel:

$$\mathscr{E}=\{aI+bH_0\,|\,a,b\in\mathbb{R}\},\quad H_0=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}.$$

LEMMA 2.2. *If* $K,L\in\mathscr{E}$ *then:*

$$KL=LK \tag{14}$$
$$KK^T=K^TK=(k_{11}^2+k_{12}^2)I. \tag{15}$$

LEMMA 2.3. *Let* $\gamma=\begin{bmatrix}\alpha & \beta\\ -\beta & \alpha\end{bmatrix}$ *be such that* $\beta\neq 0$ *and* $\alpha^2+\beta^2=1$. *If*

$$Y=\begin{bmatrix}y_{11} & y_{12}\\ y_{21} & y_{22}\end{bmatrix}$$

is any non-symmetric real matrix with $\gamma^TY\gamma-Y=0$, *then* $Y\in\mathscr{E}$. *Conversely, if* $Y\in\mathscr{E}$ *then* $\gamma^TY\gamma-Y=0$.

For convenience we introduce $K_0=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$ and $K_1=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$.

LEMMA 2.4. *Let* γ *be defined as before, let* X, Y *be any* 2×2 *real matrices not necessarily symmetric and suppose that* $\gamma^TX+\gamma^TY\gamma-Y=0$. *Then* $X=x_{11}K_0+x_{12}K_1$. *Furthermore if* $X=x_{11}I+x_{12}H_0$, *then it follows that* $X=0$ *and* $\gamma^TY\gamma-Y=0$ *and thus* $Y\in\mathscr{E}$.

LEMMA 2.5. *Let* $\gamma = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$ *be such that* $\alpha^2+\beta^2=1$, *and let* X, Y *and* W *be of the form* $aI+bH_0$ *and let* Z *be any* 2×2 *real matrix such that*

$$W+\gamma^T Y+X\gamma+\gamma^T Z\gamma-Z=0. \tag{16}$$

Put $\Psi=-\gamma^2$. *Then,*

$$X=\Psi^T Y-\gamma^T W, \qquad X\gamma+\gamma^T Y=-W, \qquad \gamma^T Z\gamma-Z=0 \tag{17}$$

and $Z\in\mathscr{E}$.

Setting $W=0$ in Lemma 2.5 yields the following corollary.

COROLLARY 2.1. *Let* $\gamma = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$ *be such that* $\alpha^2+\beta^2=1$ *and* $X=x_{11}I+x_{12}H_0$, $Y=y_{11}I+y_{12}H_0$ *and let* Z *be any* 2×2 *real matrix, satisfying*

$$X\gamma+\gamma^T Y+\gamma^T Z\gamma-Z=0. \tag{18}$$

Put $\Psi=-\gamma^2$. *Then* $X=\Psi^T Y$ *and* $Y=\Psi X$. *Furthermore,* $X\gamma+\gamma^T Y=0$ *and* $\gamma^T Z\gamma-Z=0$, *hence* $Z\in\mathscr{E}$. *In particular, if* Z *is symmetric, then* $Z=cI$ *for some constant* c.

Observe that an H-unitary matrix A is in particular also H-expansive. Recall that in order to derive a canonical form for H-unitary matrices we only need to consider the indecomposable blocks. In this section we are concerned with the case where A consist of one Jordan block with non-real eigenvalues $\alpha\pm i\beta$ on the unit circle, i.e., with $\gamma = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$, where $\alpha^2+\beta^2=1$ and $\beta\neq 0$, and

$$A=\begin{bmatrix} \gamma & I_2 & & \\ & \ddots & \ddots & \\ & & \ddots & I_2 \\ & & & \gamma \end{bmatrix}. \tag{19}$$

Furthermore, let H be denoted by

$$H=\left[H_{i,j}\right]_{i,j=1}^n \tag{20}$$

where each $H_{i,j}$ is a 2×2 matrix. Then the simple form obtained for the matrix H in the H-expansive case can now be simplified even further. From the H-expansive case we have, for example when $n=5$, that our matrix H has the following form (see Section 3.1.1 in [10]; also [4]):

$$H=\begin{bmatrix} 0 & 0 & 0 & 0 & c_3(\Psi^T)^2 \\ 0 & 0 & 0 & c_3\Psi^T & * \\ 0 & 0 & c_3 I & * & * \\ 0 & c_3\Psi & * & * & * \\ c_3\Psi^2 & * & * & * & * \end{bmatrix} \tag{21}$$

However, using Lemma 2.1, Lemma 2.3, Lemma 2.4, and Corollary 2.1 the entries denoted by $*$ can now be determined in the H-unitary case to obtain a canonical form for H.

2.2. On the structure of H-unitary matrices: results that hold for any Jordan basis

The following relations for the block entries in the matrix H in (20) can be derived from Lemma 2.1, Lemma 2.3, Lemma 2.4, and Corollary 2.1, namely

$$H_{i,j} = a_{ij}I + b_{ij}H_0, \qquad a_{ij}, b_{ij} \in \mathbb{R}, \tag{22}$$

$$H_{i,j+1} = \Psi^T H_{i+1,j} - \gamma^T H_{i,j} \tag{23}$$

and

$$H_{i,i} = c_i I, \tag{24}$$

where $c_i = (H_{\left[\frac{n+1}{2}\right],\left[\frac{n+1}{2}\right]})_{ii}$ and $i \leqslant \left[\frac{n+1}{2}\right]$ or $j \leqslant \left[\frac{n+1}{2}\right]$. Moreover, it was shown in [4] and [10] that Lemmas 2.1, 2.3, 2.4, and Corollary 2.1 imply that $H_{i,j} = 0$ for $i+j \leqslant n$.

The following lemma gives a useful expression for the (i,j)-th entry of $A^THA - H$, which is needed in the sequel.

LEMMA 2.6. *For an H-unitary matrix A, with A as in (19), and H as in (20) we have for $i > 1$, $j > 1$*

$$0 = (A^THA - H)_{i,j} = \begin{bmatrix} 0 \cdots 0 \; I \; \gamma^T \; 0 \cdots 0 \end{bmatrix} H \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I \\ \gamma \\ 0 \\ \vdots \\ 0 \end{bmatrix} - H_{i,j}$$
$$= H_{i-1,j-1} + H_{i-1,j}\gamma + \gamma^T H_{i,j-1} + \gamma^T H_{i,j}\gamma - H_{i,j}. \tag{25}$$

We will show by induction that the relations (22),(23) and (24) hold for all $i,j = 1,\ldots,n$ in the H-unitary case. Relation (24) follows from (22) and the fact that H must be symmetric. Assume (22) holds for $i+j \leqslant k$ (and $k < n$). We now show that (22) also holds for $i+j \leqslant k+1$. By Lemma 2.6, replacing j by $j+1$, we have

$$0 = (A^THA - H)_{i,j+1} = H_{i-1,j} + H_{i-1,j+1}\gamma + \gamma^T H_{i,j} + \gamma^T H_{i,j+1}\gamma - H_{i,j+1}.$$

From our induction hypothesis we know that $H_{i-1,j}, H_{i-1,j+1}$ and $H_{i,j}$ are in $\mathscr{E}$, i.e., they are of the form $aI + bH_0$, for some a and b. Hence we have from Lemma 2.5 that

$$\gamma^T H_{i,j+1}\gamma - H_{i,j+1} = 0.$$

Thus, from Lemma 2.3 we have $H_{i,j+1} \in \mathscr{E}$, i.e.,

$$H_{i,j+1} = a_{i,j+1}I + b_{i,j+1}H_0.$$

Now consider (23). This relation holds for all blocks, but it is only of interest for the nonzero block entries. Assume (23) holds for $i+j \leqslant k$. We now show that (23) also holds for $i+j \leqslant k+1$. Again by Lemma 2.6, replacing i with $i+1$ and j with $j+1$, we have

$$0 = (A^T HA - H)_{i+1,j+1} = H_{i,j} + H_{i,j+1}\gamma + \gamma^T H_{i+1,j} + \gamma^T H_{i+1,j+1}\gamma - H_{i+1,j+1}.$$

Since (22) holds, (we have that $H_{i,j}$, $H_{i,j+1}$ and $H_{i+1,j}$ are in $\mathscr{E}$) we immediately have by Lemma 2.5 that

$$H_{i,j+1} = \Psi^T H_{i+1,j} - \gamma^T H_{i,j}.$$

Hence (23) holds for $i+j \leqslant k+1$. This establishes (22), (23) and (24) for all i and j.

The blocks $H_{i,i+1}$ have a particular structure because of (23). Indeed, for those blocks we have $H_{i,i+1} = \Psi^T H_{i+1,i} - \gamma^T H_{i,i}$. Using (24) and the fact that $H_{i+1,i} = H_{i,i+1}^T$ we see that $H_{i,i+1}$ satisfies

$$H_{i,i+1} = \Psi^T H_{i,i+1}^T - \gamma^T c_i.$$

This leads us to consider matrices that satisfy this equation.

LEMMA 2.7. *If $H = aI + bH_0$ satisfies $H = \Psi^T H^T - \gamma^T c$ for some $c \in \mathbb{R}$ then $b = \frac{c}{2\beta} + \frac{\alpha}{\beta}a$. In particular, in that case*

$$H = aI + \left(\frac{c}{2\beta} + \frac{\alpha}{\beta}a\right) H_0 = \left(\frac{c}{2\beta} + \frac{a}{\beta}\gamma^T\right) H_0.$$

Proof. Insert $H = aI + bH_0$ in $H = \Psi^T H^T - \gamma^T c$, using $\Psi^T = -(\gamma^T)^2 = (\beta^2 - \alpha^2)I + 2\alpha\beta H_0$, $\gamma^T = \alpha I - \beta H_0$, to see that

$$\begin{aligned} aI + bH_0 &= ((\beta^2 - \alpha^2)I + 2\alpha\beta H_0)(aI - bH_0) - (\alpha I - \beta H_0)c \\ &= ((\beta^2 - \alpha^2)a + 2\alpha\beta b - \alpha c)I + (-(\beta^2 - \alpha^2)b + 2\alpha\beta a + \beta c)H_0, \end{aligned}$$

where we also used $H_0^2 = -I$. It follows that

$$\begin{aligned} a &= (\beta^2 - \alpha^2)a + 2\alpha\beta b - \alpha c \\ b &= -(\beta^2 - \alpha^2)b + 2\alpha\beta a + \beta c. \end{aligned}$$

Now recall $\alpha^2 + \beta^2 = 1$, so

$$\begin{aligned} (\alpha^2 + \beta^2)a &= (\beta^2 - \alpha^2)a + 2\alpha\beta b - \alpha c \\ (\alpha^2 + \beta^2)b &= -(\beta^2 - \alpha^2)b + 2\alpha\beta a + \beta c. \end{aligned}$$

This is equivalent to

$$2\alpha^2 a - 2\alpha\beta b = -\alpha c$$
$$2\beta^2 b - 2\alpha\beta a = \beta c.$$

Dividing the first equation by α and the second by $-\beta$, we see $2\alpha a - 2\beta b = -c$ from both equations. This gives $b = \frac{c}{2\beta} + \frac{\alpha}{\beta} a$ as claimed.

Next, the first form of H now follows directly. To see the second, consider $\beta H H_0 = a\beta H_0 - (\frac{1}{2}c + a\alpha)I = -a(\alpha I - \beta H_0) - \frac{1}{2}cI = -a\gamma^T - \frac{1}{2}cI$. Thus,

$$H = \frac{a}{\beta}\gamma^T H_0 + \frac{c}{2\beta} H_0$$

as desired. □

When n is even, we have $H_{\frac{n}{2},\frac{n}{2}} = 0$, and so from (24) $c_{\frac{n}{2}} = 0$. Hence by (23) $H_{\frac{n}{2},\frac{n}{2}+1} = \Psi^T H^T_{\frac{n}{2},\frac{n}{2}+1} - c_{\frac{n}{2}}\gamma^T = \Psi^T H^T_{\frac{n}{2},\frac{n}{2}+1}$. Thus we have, applying Lemma 2.7 with $c_{\frac{n}{2}} = 0$ that

$$H_{\frac{n}{2},\frac{n}{2}+1} = \begin{bmatrix} \beta & \alpha \\ -\alpha & \beta \end{bmatrix} d, \tag{26}$$

for some real number $d = \frac{a}{\beta}$.

Since every $H_{i,j}$ is determined by blocks in the antidiagonal on which it is, and on the antidiagonal above it by (23), we see that from (24) and (26) the matrix H is completely determined by at most n real parameters, one for each non-zero antidiagonal.

We now rephrase Lemma 2.7 in the following manner.

LEMMA 2.8. *Let* $\gamma = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$*, and let* X *be of the form* $aI + bH_0$ *such that*

$$\gamma X + \gamma^T X^T + cI = 0, \tag{27}$$

for any real c*, then* $c = 2(b\beta - a\alpha)$.

Proof. Let γ and X be as in the lemma. A straightforward calculation yields

$$\gamma X + \gamma^T X^T = \begin{bmatrix} 2(a\alpha - b\beta) & 0 \\ 0 & 2(a\alpha - b\beta) \end{bmatrix}.$$

Hence, $c = 2(b\beta - a\alpha)$. □

2.3. Towards a canonical form

Up to this point we have considered the form that H has for any real Jordan basis for A. However, by choosing an appropriate Jordan basis it is possible to achieve a canonical form for H.

First we give a more general form for the new Jordan basis to be constructed. Again we consider the case of one Jordan block $A = J_n(\gamma)$, where $\gamma = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$ with $\alpha^2+\beta^2 = 1$, $\beta \neq 0$. Let $x_1, x_2, \ldots, x_{2n-1}, x_{2n}$ be the basis. We denote by $X_1, X_2, \ldots, X_n$ the $2n\times 2$ matrices with columns $x_1, x_2; x_3, x_4; \ldots; x_{2n-1}, x_{2n}$, respectively. Note that

$$AX_1 = X_1\gamma,\ AX_j = X_j\gamma + X_{j-1} \text{ for } j = 2, \ldots, n.$$

Construct a new Jordan basis as follows: Let

$$Z_1 = X_1h_1,\ Z_2 = X_2h_1 + X_1h_2,\ Z_3 = X_3h_1 + X_2h_2 + X_1h_3, \text{ etc},$$

where the h_j's are 2×2 matrices of the form $aI + bH_0$. So, in general

$$Z_j = \sum_{i=1}^{j} X_i h_{j-i+1}.$$

From the specific form of h_j we have that $\gamma h_j = h_j\gamma$ (see Lemma 2.2). Thus indeed for $j = 2, \ldots, n$ we have

$$\begin{aligned} AZ_j &= AX_jh_1 + \sum_{i=1}^{j-1} AX_ih_{j-i+1} \\ &= X_j\gamma h_1 + X_{j-1}h_1 + \sum_{i=2}^{j-1}(X_i\gamma + X_{i-1})h_{j-i+1} + X_1\gamma h_{j-1+1} \\ &= X_jh_1\gamma + X_{j-1}h_1 + \sum_{i=1}^{j-1} X_ih_{j-i+1}\gamma + \sum_{i=2}^{j-1} X_{i-1}h_{j-i+1} \\ &= (X_jh_1 + \sum_{i=1}^{j-1} X_ih_{j-i+1})\gamma + (X_{j-1}h_1 + \sum_{i=1}^{j-2} X_ih_{j-i}) \\ &= Z_j\gamma + Z_{j-1}. \end{aligned}$$

With respect to this new Jordan basis we have a new form for H. Indeed, let $S = [Z_1\, Z_2 \ldots Z_n]$ be the $2n \times 2n$ matrix formed with the Z_i's as block columns. Then $S^{-1}AS = A$ and we compute $S^THS = [(S^THS)_{i,j}]_{i,j=1}^n$. The (i,j)-th entry then is

$$\begin{aligned} (S^THS)_{i,j} &= Z_i^THZ_j \\ &= \left(\sum_{k=1}^{i} X_kh_{i-k+1}\right)^T H \left(\sum_{l=1}^{j} X_lh_{j-l+1}\right) \\ &= \sum_{k=1}^{i}\sum_{l=1}^{j} h_{i-k+1}^T H_{k,l} h_{j-l+1}. \end{aligned} \tag{28}$$

Using these relations it may be possible to choose the matrices h_j, for $j = 1, \ldots, n-1$, such that S^THS is brought into a canonical form.

Observe that the relation of the new basis to the old one is well-understood: $Z = X \cdot toep(h_1, h_2, \cdots, h_n)$, i.e., the basis transformation is achieved by multiplying with a 2×2 block Toeplitz matrix.

We now state two more new lemmas that are of importance in the derivation of the canonical form of H.

LEMMA 2.9. *If $H \in \mathscr{E}$ and the number c is such that $diag(H\gamma) = -\frac{1}{2}cI$, and $H = \Psi^T H^T - \gamma^T c$, then there exists a matrix $h \in \mathscr{E}$ so that*

$$H + (h + h^T \Psi^T) = -\frac{1}{2} c\gamma^T$$

and any such h is given by

$$h = -\frac{1}{2}H + d\gamma^T$$

for any $d \in \mathbb{R}$.

Proof. Consider, for $h \in \mathscr{E}$ the expression

$$H\gamma + (h + h^T\Psi^T)\gamma = H\gamma + (h\gamma - h^T\gamma^T) = H\gamma + (h\gamma - \gamma^T h^T).$$

Now $h\gamma - \gamma^T h^T$ is skew symmetric, so the diagonal of the matrix above is the same as the diagonal of $H\gamma$, which is $-\frac{1}{2}cI$.

It remains to show that the h may be chosen so that the off-diagonal of $H\gamma + (h\gamma - \gamma^T h^T)$ is zero. Setting $H = aI + bH_0$ and $h = a_1 I + b_1 H_0$ then the $(1,2)$-entry of $H\gamma + (h\gamma - \gamma^T h^T)$ becomes

$$a\beta + b\alpha + 2(b_1\alpha + a_1\beta) = \alpha(b + 2b_1) + \beta(a + 2a_1). \tag{29}$$

Obviously, one choice of a_1, b_1 that makes this zero is $a_1 = -\frac{1}{2}a$ and $b_1 = -\frac{1}{2}b$ so that $h = -\frac{1}{2}H$. Since (29) is a linear equation in two unknowns, there is one degree of freedom. One checks that setting $h = -\frac{1}{2}H + d\gamma^T$ gives all solutions. $\square$

LEMMA 2.10. *Given a diagonal matrix $c_1 I$ and a number $0 \neq c_2 \in \mathbb{R}$, then there exists an $h \in \mathscr{E}$ such that*

$$c_1 I + c_2(\Psi h + h^T \Psi^T) = 0. \tag{30}$$

Proof. Let $h = aI + bH_0$, then

$$\Psi h + h^T \Psi^T = 2((\beta^2 - \alpha^2)a + 2\alpha\beta b)I.$$

In order to obtain equation (30), we have to choose a, b such that

$$c_1 + 2c_2((\beta^2 - \alpha^2)a + 2\alpha\beta b) = 0.$$

We may take either $a=0$, then $b=-\frac{c_1}{4c_2\alpha\beta}$ providing that $\alpha\beta\neq 0$. Or, if $b=0$ then $a=-\frac{c_1}{2c_2(\beta^2-\alpha^2)}$ if $\beta\neq\pm\alpha$. Obviously, still one degree of freedom remains in the matrix h. $\square$

From here on we distinguish between the cases where n is odd or n is even. We start with the case that n is odd.

Proof of Case 1, part (iii) of Theorem 1.1. An important observation to be made is that from equation (23) it follows that in order to prove part (iii) in Theorem 1.1 it suffices to show that, by an appropriate change of basis,we can make the block $H_{\frac{n+1}{2},\frac{n+1}{2}}$ equal to εI_2, make the blocks $H_{j,j}$ with $j>\frac{n+1}{2}$ equal to zero, make the block $H_{\frac{n+1}{2},\frac{n+3}{2}}$ equal to $-\varepsilon\frac{1}{2}\gamma^T$, and finally, make the blocks $H_{j+1,j}$ with $j>\frac{n+1}{2}$ also equal to zero. Indeed, once this is accomplished, all other entries can be deduced using (23).

Recall that we already know that the central 2×2 block entry $H_{\frac{n+1}{2},\frac{n+1}{2}}$ is a multiple of the identity. Now take $S_1=toep(h_1,0,\cdots,0)$, where $h_1\in\mathscr{E}$ is invertible, and put $H^{(1)}=S_1^THS_1$. Then by (28) we have that

$$H^{(1)}_{\frac{n+1}{2},\frac{n+1}{2}}=h_1^TH_{\frac{n+1}{2},\frac{n+1}{2}}h_1=h_1h_1^TH_{\frac{n+1}{2},\frac{n+1}{2}}.$$

So we see that we can scale the entry in the $\left(\frac{n+1}{2},\frac{n+1}{2}\right)$-position to εI_2, where $\varepsilon=\pm1$. Once that is done, we can pull out ε in front of the whole matrix H, and this way we may assume without loss of generality that $H^{(1)}_{\frac{n+1}{2},\frac{n+1}{2}}=I_2$. From now on we shall assume that this is the case.

Next, we shall construct in a sequential way matrices

$$S_j=toep(I_2,0,\cdots,0,h_j,0,\cdots 0)$$

with a matrix $h_j\in\mathscr{E}$ on the jth diagonal such that $H^{(j)}=S_j^TH^{(j-1)}S_j$ has the form as described in part (iii) of Theorem 1.1 at least for the block entries $H^{(j)}_{k,l}$ with $k+l\leqslant n+j$.

First we consider the block entry in the position $\left(\frac{n+1}{2},\frac{n+3}{2}\right)$. By (23) we know

$$H^{(1)}_{\frac{n+1}{2},\frac{n+3}{2}}=\Psi^TH^{(1)}_{\frac{n+3}{2},\frac{n+1}{2}}-\gamma^TH^{(1)}_{\frac{n+1}{2},\frac{n+1}{2}}.$$

Using $H^{(1)}_{\frac{n+3}{2},\frac{n+1}{2}}=(H^{(1)}_{\frac{n+1}{2},\frac{n+3}{2}})^T$ and $H^{(1)}_{\frac{n+1}{2},\frac{n+1}{2}}=I_2$, this becomes

$$H^{(1)}_{\frac{n+1}{2},\frac{n+3}{2}}=\Psi^T(H^{(1)}_{\frac{n+1}{2},\frac{n+3}{2}})^T-\gamma^T.$$

By Lemma 2.7 it follows that $H^{(1)}_{\frac{n+1}{2},\frac{n+3}{2}}=(\frac{1}{2\beta}+\frac{a}{\beta}\gamma^T)H_0$ for some real number a. In particular

$$H^{(1)}_{\frac{n+1}{2},\frac{n+3}{2}}\gamma=\frac{1}{2\beta}H_0\gamma+\frac{a}{\beta}H_0,$$

so $diag\,(H^{(1)}_{\frac{n+1}{2},\frac{n+3}{2}}\gamma) = -\frac{1}{2}I_2$. Hence $H^{(1)}_{\frac{n+1}{2},\frac{n+3}{2}}$ satisfies all conditions of Lemma 2.9.

By (28), with $S_2 = toep(I_2, h_2, 0, \cdots, 0)$ we have for $H^{(2)} = S_2^T H^{(1)} S_2$

$$H^{(2)}_{\frac{n+1}{2},\frac{n+3}{2}} = h_2 + H^{(1)}_{\frac{n+1}{2},\frac{n+3}{2}} + h_2^T H^{(1)}_{\frac{n-1}{2},\frac{n+3}{2}}.$$

Using (23) again to determine $H^{(1)}_{\frac{n-1}{2},\frac{n+3}{2}} = \Psi^T$, we have that

$$H^{(2)}_{\frac{n+1}{2},\frac{n+3}{2}} = h_2 + H^{(1)}_{\frac{n+1}{2},\frac{n+3}{2}} + h_2^T \Psi^T.$$

By Lemma 2.9 (with $c=1$), there is a choice of h_2 such that

$$H^{(2)}_{\frac{n+1}{2},\frac{n+3}{2}} = -\frac{1}{2}\gamma^T.$$

One also easily checks that for $i+j \leqslant n+1$ the entries $H^{(2)}_{i,j} = H^{(1)}_{i,j}$.

Next, we turn our attention to the block entry in the $\left(\frac{n+3}{2}, \frac{n+3}{2}\right)$ position. Consider $S_3 = toep(I_2, 0, h_3, 0 \cdots, 0)$ and put $H^{(3)} = S_3^T H^{(2)} S_3$. By (28) we have

$$\begin{aligned} H^{(3)}_{\frac{n+3}{2},\frac{n+3}{2}} &= H^{(2)}_{\frac{n+3}{2},\frac{n+3}{2}} + H^{(2)}_{\frac{n+3}{2},\frac{n-1}{2}} h_3 + h_3^T H^{(2)}_{\frac{n-1}{2},\frac{n+3}{2}} \\ &= H^{(2)}_{\frac{n+3}{2},\frac{n+3}{2}} + H^{(2)}_{\frac{n+3}{2},\frac{n-1}{2}} h_3 + h_3^T (H^{(2)}_{\frac{n+3}{2},\frac{n-1}{2}})^T. \end{aligned}$$

Now $H^{(2)}_{\frac{n+3}{2},\frac{n-1}{2}} = H^{(1)}_{\frac{n+3}{2},\frac{n-1}{2}} = \Psi$, so

$$H^{(3)}_{\frac{n+3}{2},\frac{n+3}{2}} = H^{(2)}_{\frac{n+3}{2},\frac{n+3}{2}} + \Psi h_3 + h_3^T \Psi^T.$$

Since $H^{(2)}_{\frac{n+3}{2},\frac{n+3}{2}}$ is a symmetric matrix in $\mathscr{E}$, it is a multiple of I_2, say $c_1 I_2$ for some real number c_1. Hence, applying Lemma 2.10, we see that h_3 can be chosen so that $H^{(3)}_{\frac{n+3}{2},\frac{n+3}{2}} = 0$. In addition, by (28) $H^{(3)}_{i,j} = H^{(2)}_{i,j}$ for $i+j \leqslant n+2$.

The next step is to consider the block entry in the position $\left(\frac{n+5}{2}, \frac{n+3}{2}\right)$. Put

$$S_4 = toep(I_2, 0, 0, h_4, 0, \cdots, 0),$$

and $H^{(4)} = S_4^T H^{(3)} S_4$. Then by (28) we have

$$H^{(4)}_{\frac{n+5}{2},\frac{n+3}{2}} = H^{(3)}_{\frac{n+5}{2},\frac{n+3}{2}} + H^{(3)}_{\frac{n+5}{2},\frac{n-3}{2}} h_4 + h_4^T H^{(3)}_{\frac{n-1}{2},\frac{n+3}{2}}.$$

We have already established that $H^{(3)}_{\frac{n-1}{2},\frac{n+3}{2}} = \Psi^T$, and by (23) we have

$$\Psi^T H^{(3)}_{\frac{n+5}{2},\frac{n-3}{2}} - \gamma^T H^{(3)}_{\frac{n+3}{2},\frac{n-3}{2}} = H^{(3)}_{\frac{n+3}{2},\frac{n-1}{2}} = \Psi.$$

Since $H^{(3)}_{\frac{n+3}{2},\frac{n-3}{2}}=0$, this gives $H^{(3)}_{\frac{n+5}{2},\frac{n-3}{2}}=\Psi^2$. Hence

$$H^{(4)}_{\frac{n+5}{2},\frac{n+3}{2}}=H^{(3)}_{\frac{n+5}{2},\frac{n+3}{2}}+\Psi^2 h_4+h_4^T\Psi^T.$$

Equivalently,

$$H^{(4)}_{\frac{n+3}{2},\frac{n+5}{2}}=H^{(3)}_{\frac{n+3}{2},\frac{n+5}{2}}+\Psi h_4+h_4^T(\Psi^T)^2. \tag{31}$$

In addition, we know by (23) that

$$(H^{(3)}_{\frac{n+5}{2},\frac{n+3}{2}})^T=H^{(3)}_{\frac{n+3}{2},\frac{n+5}{2}}=\Psi^T H^{(3)}_{\frac{n+5}{2},\frac{n+3}{2}}-\gamma^T H^{(3)}_{\frac{n+3}{2},\frac{n+3}{2}},$$

and since $H^{(3)}_{\frac{n+3}{2},\frac{n+3}{2}}=0$, this gives $(H^{(3)}_{\frac{n+5}{2},\frac{n+3}{2}})^T=\Psi^T H^{(3)}_{\frac{n+5}{2},\frac{n+3}{2}}$. So,

$$H^{(3)}_{\frac{n+3}{2},\frac{n+5}{2}}\gamma=-\gamma^T H^{(3)}_{\frac{n+5}{2},\frac{n+3}{2}}=-(H^{(3)}_{\frac{n+3}{2},\frac{n+5}{2}}\gamma)^T,$$

which shows that the diagonal of $H^{(3)}_{\frac{n+3}{2},\frac{n+5}{2}}\gamma$ is zero. So, $H^{(3)}_{\frac{n+3}{2},\frac{n+5}{2}}$ satisfies the conditions of Lemma 2.9 with $c=0$. Hence, there is an $h\in\mathscr{E}$ such that

$$H^{(3)}_{\frac{n+3}{2},\frac{n+5}{2}}+h+h\Psi^T=0.$$

Comparing with (31) we see that taking $\Psi h_4=h$, i.e., $h_4=\Psi^T h$, yields $H^{(4)}_{\frac{n+3}{2},\frac{n+5}{2}}=0$ as desired. Also note that for $i+j\leqslant n+3$ we have $H^{(4)}_{i,j}=H^{(3)}_{i,j}$.

Finally, we provide an induction argument to finish the proof for the case where n is odd. Suppose $S_1,S_2,\dots,S_k$ have been chosen so that $H^{(k)}$ has all the entries $H^{(k)}_{i,j}$ with $i+j\leqslant n+k-1$ as in part (iii), Case 1, of Theorem 1.1. Put

$$S_{k+1}=toep(I_2,0,\cdots,0,h_{k+1},0,\cdots,0),$$

and $H^{(k+1)}=S_{k+1}^T H^{(k)} S_{k+1}$. Then by (28) we have

$$\begin{aligned} H^{(k+1)}_{i,j} &= (S_{k+1}^T H^{(k)} S_{+1})_{i,j} \\ &= H^{(k)}_{i,j}+H^{(k)}_{i,j-k}h_{k+1}+h_{k+1}^T H^{(k)}_{i-k,j}+h_{k+1}^T H^{(k)}_{i-k,j-k}h_{k+1}. \end{aligned} \tag{32}$$

We claim that $H^{(k+1)}_{i,j}=H^{(k)}_{i,j}$ for $i+j\leqslant n+k$. Indeed, if $i+j\leqslant n+k$, then $i+j-k\leqslant n$ and so $H^{(k)}_{i,j-k}=0$, $H^{(k)}_{i-k,j}=0$ and $H^{(k)}_{i-k,j-k}=0$. Now consider $i+j=n+k+1$. Then still $H^{(k)}_{i-k,j-k}=0$. Also $i+j-k=n+1$, so the entries $H^{(k)}_{i,j-k}=H^{(k)}_{i,n-i+1}$, and $H^{(k)}_{i-k,j}=H^{(k)}_{i-k,n+k-i+1}$ are on the main anti-diagonal. By the argument above they are equal to the corresponding entries of $H^{(1)}$, and so they are equal to the corresponding entries in the form of part (iii), Case 1 of Theorem 1.1. From this, one has that

$$H^{(k)}_{i,n-i+1}=(\Psi^T)^{\frac{n+1}{2}-i}$$

$$H^{(k)}_{i-k,n+k-i+1}=(\Psi^T)^{\frac{n+1}{2}-i+k}.$$

Then (32) becomes

$$H^{(k+1)}_{i,n+k+1-i} = H^{(k)}_{i,n+k+1-i} + (\Psi^T)^{\frac{n+1}{2}-i} h_{k+1} + h^T_{k+1}(\Psi^T)^{\frac{n+1}{2}-i+k}. \tag{33}$$

Now we distinguish between k even and k odd. When k is odd we are interested in the entry with $n+k+1-i = i+1$, i.e., the entry just above the main diagonal. For this entry we have $i = \frac{n+k}{2}$, so the equation above becomes

$$\begin{aligned} H^{(k+1)}_{\frac{n+k}{2},\frac{n+k+2}{2}} &= H^{(k)}_{\frac{n+k}{2},\frac{n+k+2}{2}} + (\Psi^T)^{\frac{1-k}{2}} h_{k+1} + h^T_{k+1}(\Psi^T)^{\frac{k+1}{2}} \\ &= H^{(k)}_{\frac{n+k}{2},\frac{n+k+2}{2}} + \Psi^{\frac{k-1}{2}} h_{k+1} + h^T_{k+1}(\Psi^T)^{\frac{k-1}{2}}\Psi^T. \end{aligned}$$

In addition, because $H^{(k)}_{\frac{n+k}{2},\frac{n+k}{2}} = 0$ by the induction hypothesis, we have that the diagonal of $H^{(k)}_{\frac{n+k}{2},\frac{n+k+2}{2}}\gamma$ is zero. Then we can apply Lemma 2.9 with $c = 0$ to ensure the existence of an h such that

$$H^{(k)}_{\frac{n+k}{2},\frac{n+k+2}{2}} + h + h^T\Psi^T = 0.$$

Taking $h = \Psi^{\frac{k-1}{2}} h_{k+1}$, we arrive at the desired zero entry.

When k is even we are interested in the entry with $n+k+1-i = i$, i.e., the entry on the main diagonal. For this entry we have $i = \frac{n+k+1}{2}$, so the equation (33) becomes

$$\begin{aligned} H^{(k+1)}_{\frac{n+k+1}{2},\frac{n+k+1}{2}} &= H^{(k)}_{\frac{n+k+1}{2},\frac{n+k+1}{2}} + (\Psi^T)^{\frac{-k}{2}} h_{k+1} + h^T_{k+1}(\Psi^T)^{\frac{k}{2}} \\ &= H^{(k)}_{\frac{n+k+1}{2},\frac{n+k+1}{2}} + \Psi^{\frac{k}{2}} h_{k+1} + h^T_{k+1}(\Psi^T)^{\frac{k}{2}}. \end{aligned}$$

Since $H^{(k)}_{\frac{n+k+1}{2},\frac{n+k+1}{2}}$ is on the main diagonal, it is a symmetric matrix in $\mathscr{E}$, and therefore it is of the form $c_1 I_2$ for some real number c_1. By Lemma 2.10 there is a matrix h such that

$$H^{(k)}_{\frac{n+k+1}{2},\frac{n+k+1}{2}} + \Psi h + h^T\Psi^T = 0.$$

Now take h_{k+1} so that $\Psi^{\frac{k-2}{2}} h_{k+1} = h$, i.e., $h_{k+1} = (\Psi^T)^{\frac{k-2}{2}} h$, to obtain that for this choice of h_{k+1} we have the desired $H^{(k+1)}_{\frac{n+k+1}{2},\frac{n+k+1}{2}} = 0$.

This proves part (iii), Case 1 in Theorem 1.1. □

Before turning to the proof in the case when n is even, we prove a lemma that will be useful.

LEMMA 2.11. *Given a diagonal matrix cI_2 there is a matrix $h \in \mathscr{E}$ such that*

$$cI + (\gamma^T h - h^T\gamma)H_0 = 0.$$

Proof. Let $h = aI + bH_0$, then one computes

$$(\gamma^T h - h^T \gamma)H_0 = 2(\beta a - \alpha b)I_2.$$

Taking $a = -\beta \frac{c}{2}$ and $b = \alpha \frac{c}{2}$ this becomes zero. $\square$

Proof of Case 2, part (iii) of Theorem 1.1. For the case when n is even, it suffices to prove that we can make the block-entry $H_{\frac{n}{2},\frac{n}{2}+1}$ equal to $\varepsilon H_0 \gamma^T$, make the blocks $H_{j,j}$ with $j > \frac{n}{2}$ equal to zero, and finally, make the blocks $H_{j+1,j}$ with $j > \frac{n}{2}$ also equal to zero. Indeed, again all other entries can then be deduced from (23).

The equation (23) relates the block entries $H_{\frac{n}{2},\frac{n}{2}+1}$ and $H_{\frac{n}{2}+1,\frac{n}{2}} = H^T_{\frac{n}{2},\frac{n}{2}+1}$:

$$H_{\frac{n}{2},\frac{n}{2}+1} = \Psi^T H_{\frac{n}{2}+1,\frac{n}{2}} - \gamma^T H_{\frac{n}{2},\frac{n}{2}} = \Psi^T H_{\frac{n}{2}+1,\frac{n}{2}}$$

because $H_{\frac{n}{2},\frac{n}{2}} = 0$. In particular, multiplying by γ and using the commutativity of $\mathscr{E}$, we obtain

$$\gamma H_{\frac{n}{2},\frac{n}{2}+1} = -\gamma^T H_{\frac{n}{2}+1,\frac{n}{2}} = -(\gamma H_{\frac{n}{2},\frac{n}{2}+1})^T.$$

So, $\gamma H_{\frac{n}{2},\frac{n}{2}+1}$ is skew-symmetric. It follows that $\gamma H_{\frac{n}{2},\frac{n}{2}+1} = cH_0$ for some real number c, i.e.,

$$H_{\frac{n}{2},\frac{n}{2}+1} = cH_0\gamma^T.$$

To achieve the form in Case 2 of part (iii) in the theorem for this entry we need to show that it is possible to make $c = \pm 1$. This can be achieved as follows.

Put $S_1 = toep(h_1, 0\cdots, 0)$, for some h_1 in $\mathscr{E}$ and $H^{(1)} = S_1^T H S_1$. Then $H^{(1)}_{i,j} = h_1 h_1^T H_{i,j}$. Taking $h_1 = \frac{1}{\sqrt{|c|}} I$ we obtain $H^{(1)}_{\frac{n}{2},\frac{n}{2}+1} = \varepsilon H_0 \gamma^T$, with $\varepsilon = \pm 1$. Taking ε in front of the matrix we may assume in the remainder of the proof that $H^{(1)}_{\frac{n}{2},\frac{n}{2}+1} = H_0\gamma^T$.

Next, we consider the entry of $H^{(1)}$ in the position $(\frac{n}{2}+1, \frac{n}{2}+1)$. Since this is on the main diagonal it is a multiple of the identity. Put $S_2 = toep(I_2, h_2, 0, \cdots, 0)$ and $H^{(2)} = S_2^T H^{(1)} S_2$. Then by (28) we have

$$H^{(2)}_{i,j} = H^{(1)}_{i,j} + H^{(1)}_{i,j-1} h_2 + h_2^T H^{(1)}_{i-1,j} + h_2^T H^{(1)}_{i-1,j-1} h_2.$$

As in the case when n is odd we see from this that $H^{(2)}_{i,j} = H^{(1)}_{i,j}$ for $i + j \leqslant n+1$, as in this case $H^{(1)}_{i-1,j} = 0, H^{(1)}_{i,j-1} = 0$ and $H^{(1)}_{i-1,j-1} = 0$. For $i = j = \frac{n}{2}+1$ we obtain that

$$\begin{aligned}
H^{(2)}_{\frac{n}{2}+1,\frac{n}{2}+1} &= H^{(1)}_{\frac{n}{2}+1,\frac{n}{2}+1} + H^{(1)}_{\frac{n}{2}+1,\frac{n}{2}} h_2 + h_2^T H^{(1)}_{\frac{n}{2},\frac{n}{2}+1} \\
&= H^{(1)}_{\frac{n}{2}+1,\frac{n}{2}+1} + \gamma H_0^T h_2 + h_2^T H_0 \gamma^T \\
&= H^{(1)}_{\frac{n}{2}+1,\frac{n}{2}+1} + h_2^T \gamma^T H_0 - \gamma h_2 H_0 \\
&= H^{(1)}_{\frac{n}{2}+1,\frac{n}{2}+1} + (\gamma^T h_2^T - h_2 \gamma) H_0.
\end{aligned}$$

It is an easy check that $(\gamma^T h_2^T - h_2\gamma)H_0$, being the product of two skew-symmetric matrices in $\mathscr{E}$ is actually a diagonal matrix. Then by Lemma 2.11 it is possible to choose h_2 so that $H^{(2)}_{\frac{n}{2}+1,\frac{n}{2}+1} = 0$.

Next, let us consider the block entry in the position $(\frac{n}{2}+1,\frac{n}{2}+2)$. First, by (23)

$$H^{(2)}_{\frac{n}{2}+1,\frac{n}{2}+2}=\Psi^T H^{(2)}_{\frac{n}{2}+2,\frac{n}{2}+1}-\gamma^T H^{(2)}_{\frac{n}{2}+1,\frac{n}{2}+1}=\Psi^T H^{(2)}_{\frac{n}{2}+2,\frac{n}{2}+1}.$$

Multiplying by γ to obtain

$$\gamma H^{(2)}_{\frac{n}{2}+1,\frac{n}{2}+2}=-\gamma^T H^{(2)}_{\frac{n}{2}+2,\frac{n}{2}+1}=-(\gamma H^{(2)}_{\frac{n}{2}+1,\frac{n}{2}+2})^T.$$

So, $H^{(2)}_{\frac{n}{2}+1,\frac{n}{2}+2}=cH_0\gamma^T$ for some real number c. Put $S_3=toep(I_2,0,h_3,0,\cdots,0)$ and $H^{(3)}=S_3^T H^{(2)}S_3$. Again by (28)

$$H^{(3)}_{i,j}=H^{(2)}_{i,j}+H^{(2)}_{i,j-2}h_3+h_3^T H^{(2)}_{i-2,j}+h_3^T H^{(2)}_{i-2,j-2}h_3.$$

It follows that for $i+j\leqslant n+2$ we have $H^{(3)}_{i,j}=H^{(2)}_{i,j}$. Also,

$$H^{(3)}_{\frac{n}{2}+1,\frac{n}{2}+2}=H^{(2)}_{\frac{n}{2}+1,\frac{n}{2}+2}+H^{(2)}_{\frac{n}{2}+1,\frac{n}{2}}h_3+h_3^T H^{(2)}_{\frac{n}{2}-1,\frac{n}{2}+2}.$$

Now $H^{(2)}_{\frac{n}{2}+1,\frac{n}{2}}=-H_0\gamma$, and by (23) we have

$$H^{(2)}_{\frac{n}{2}-1,\frac{n}{2}+2}=\Psi^T H^{(2)}_{\frac{n}{2},\frac{n}{2}+1}=H_0\gamma^T\Psi^T=-H_0(\gamma^3)^T.$$

So

$$H^{(3)}_{\frac{n}{2}+1,\frac{n}{2}+2}=H^{(2)}_{\frac{n}{2}+1,\frac{n}{2}+2}-H_0\gamma h_3+h_3^T H_0\gamma^T\Psi^T.$$

Now use the fact that $H^{(2)}_{\frac{n}{2}+1,\frac{n}{2}+2}=cH_0\gamma^T$ to rewrite this as

$$H^{(3)}_{\frac{n}{2}+1,\frac{n}{2}+2}=cH_0\gamma^T-H_0\gamma h_3+h_3^T H_0\gamma^T\Psi^T=H_0\gamma^T(cI+\Psi h_3+h_3^T\Psi^T).$$

Applying Lemma 2.10 we see that there exists an $h_3\in\mathscr{E}$ such that $H^{(3)}_{\frac{n}{2}+1,\frac{n}{2}+2}=0$.

As in the case when n is odd, we now proceed with induction. Suppose $S_1,S_2,\ldots,S_k$ have been chosen so that $H^{(k)}$ has all the entries $H^{(k)}_{i,j}$ with $i+j\leqslant n+k-1$ as in part (iii), Case 2, of Theorem 1.1. Put

$$S_{k+1}=toep(I_2,0,\cdots,0,h_{k+1},0,\cdots,0),$$

and $H^{(k+1)}=S_{k+1}^T H^{(k)}S_{k+1}$. Then (32) holds again, and so $H^{(k+1)}_{i,j}=H^{(k)}_{i,j}$ for $i+j\leqslant n+k$. Now consider $i+j=n+k+1$. Then still $H^{(k)}_{i-k,j-k}=0$. Also $i+j-k=n+1$, so the entries $H^{(k)}_{i,j-k}=H^{(k)}_{i,n-i+1}$, and $H^{(k)}_{i-k,j}=H^{(k)}_{i-k,n+k-i+1}$ are on the main anti-diagonal. By the argument above they are equal to the corresponding entries of $H^{(1)}$, and so they are equal to the corresponding entries in the form of part (iii), Case 2 of Theorem 1.1. As before, one has that

$$H^{(k)}_{i,n-i+1}=(-1)^{\frac{n}{2}-i}H_0(\gamma^T)^{n+1-2i}$$
$$H^{(k)}_{i-k,n+k-i+1}=(-1)^{\frac{n}{2}-i+k}H_0(\gamma^T)^{n+1-2i+2k}$$

Then (32) becomes

$$H^{(k+1)}_{i,n+k+1-i} = H^{(k)}_{i,n+k+1-i} + (-1)^{\frac{n}{2}-i} H_0 (\gamma^T)^{n+1-2i} h_{k+1} + h^T_{k+1} (-1)^{\frac{n}{2}-i+k} H_0 (\gamma^T)^{n+1-2i+2k}.$$

Now we distinguish between k even and k odd. For k even we are interested in the off-diagonal block with $i+1 = n+k+1-i$, i.e., $i = \frac{n+k}{2}$ and the block entry is $H^{(k+1)}_{\frac{n+k}{2},\frac{n+k}{2}+1}$. For this entry (32) becomes

$$\begin{aligned}
H^{(k+1)}_{\frac{n+k}{2},\frac{n+k}{2}+1} &= H^{(k)}_{\frac{n+k}{2},\frac{n+k}{2}+1} + (-1)^{-\frac{k}{2}} H_0 (\gamma^T)^{1-k} h_{k+1} + h^T_{k+1} (-1)^{\frac{k}{2}} H_0 (\gamma^T)^{k+1} \\
&= H^{(k)}_{\frac{n+k}{2},\frac{n+k}{2}+1} + (-1)^{\frac{k}{2}} (H_0 \gamma^{k-1} h_{k+1} - h^T_{k+1} (\gamma^T)^{k-1} H_0 \Psi^T) \\
&= H^{(k)}_{\frac{n+k}{2},\frac{n+k}{2}+1} + (-1)^{\frac{k}{2}} (H_0 \gamma^{k-1} h_{k+1} + (H_0 \gamma^{k-1} h_{k+1})^T \Psi^T).
\end{aligned}$$

Also, because of (23) applied to $H^{(k)}_{\frac{n+k}{2},\frac{n+k}{2}+1}$ one easily sees that $H^{(k)}_{\frac{n+k}{2},\frac{n+k}{2}+1}$ satisfies the conditions of Lemma 2.9. Applying that lemma there is an $h \in \mathscr{E}$ such that $H^{(k)}_{\frac{n+k}{2},\frac{n+k}{2}+1} + (h + h^T \Psi^T) = 0$. Now take h_{k+1} so that $h = (-1)^{\frac{k}{2}} H_0 \gamma^{k-1} h_{k+1}$, which is possible, to achieve $H^{(k+1)}_{\frac{n+k}{2},\frac{n+k}{2}+1} = 0$ as desired.

When k is odd, we are interested in the diagonal block with $i = n+k+1-i$, i.e., $i = \frac{n+k+1}{2}$. For this entry (32) becomes

$$\begin{aligned}
H^{(k+1)}_{\frac{n+k+1}{2},\frac{n+k+1}{2}} &= H^{(k)}_{\frac{n+k+1}{2},\frac{n+k+1}{2}} + (-1)^{\frac{-k-1}{2}} H_0 (\gamma^T)^{-k} h_{k+1} + h^T_{k+1} (-1)^{\frac{k-1}{2}} H_0 (\gamma^T)^{k} \\
&= H^{(k)}_{\frac{n+k+1}{2},\frac{n+k+1}{2}} + (-1)^{\frac{-k-1}{2}} ((\gamma^k h_{k+1} - h^T_{k+1} (\gamma^T)^k) H_0.
\end{aligned}$$

Applying Lemma 2.11 we see that it is possible to choose h_{k+1} so that $H^{(k+1)}_{\frac{n+k+1}{2},\frac{n+k+1}{2}} = 0$ as desired. This concludes the proof of Case 2, part (iii) of Theorem 1.1. $\square$

3. The case of eigenvalues ± 1

We consider now the case where A is H-unitary and $\sigma(A) \subset \{\pm 1\}$. As stated in the introduction, the indecomposable blocks are the ones where either A is similar to one Jordan block with odd size, or to a pair of Jordan blocks with equal even size.

Recall from the introduction the definition of the matrices P_n, $P_n(-1)$, Q_n and $Q_n(-1)$.

THEOREM 3.1. *(i) Let A be H-unitary and $\sigma(A) = \{1\}$. Then the pair (A,H) can be decomposed as*

$$S^{-1}AS = \oplus_{l=1}^{p} A_l, \qquad S^T HS = \oplus_{l=1}^{p} H_l,$$

where the pair (A_l, H_l) *is of one of the following two forms for some* n *depending on* l

Case 1 $\left(J_n(1), \varepsilon \begin{bmatrix} Z_n & P_n \\ P_n^T & 0 \end{bmatrix}\right)$ *with* n *odd, and* $\varepsilon = \pm 1$.

Case 2 $\left(J_n(1) \oplus J_n(1), \begin{bmatrix} 0 & 0 & 0 & Q_n \\ 0 & 0 & -Q_n^T & 0 \\ 0 & -Q_n & 0 & 0 \\ Q_n^T & 0 & 0 & 0 \end{bmatrix}\right)$ *with* n *even.*

(ii) Let A *be* H*-unitary and* $\sigma(A) = \{-1\}$. *Then the pair* (A, H) *can be decomposed as*

$$S^{-1}AS = \oplus_{l=1}^{p} A_l, \qquad S^T HS = \oplus_{l=1}^{p} H_l,$$

where the pair (A_l, H_l) *is of one of the following two forms for some* n *depending on* l

Case 1 $\left(J_n(-1), \varepsilon \begin{bmatrix} Z_n & P_n(-1) \\ P_n(-1)^T & 0 \end{bmatrix}\right)$ *with* n *odd, and* $\varepsilon = \pm 1$.

Case 2 $\left(J_n(-1) \oplus J_n(-1), \begin{bmatrix} 0 & 0 & 0 & Q_n(-1) \\ 0 & 0 & -Q_n(-1)^T & 0 \\ 0 & -Q_n(-1) & 0 & 0 \\ Q_n(-1)^T & 0 & 0 & 0 \end{bmatrix}\right)$

with n *even.*

Proof. (i), *Case* 1. There is an invertible matrix S that decomposes the pair to a block diagonal where the blocks with odd size n in A are similar to $J_n(1)$, and the remaining blocks of even size are similar to $J_n(1) \oplus J_n(1)$.

We start with the case n odd. We may assume that $A = J_n(1)$ with respect to a basis $\{x_1, \ldots, x_n\}$. Denote $H = \left[H_{ij}\right]_{i,j=1}^{n}$ (so $H_{ij} = \langle Hx_j, x_i \rangle$). From [10], see also [4], we know the following:

$$H_{ij} = 0 \qquad \text{when } i + j \leqslant n,$$

and

$$H_{i\,j} + H_{i\,j+1} + H_{i+1\,j} = 0. \tag{34}$$

For sake of convenience, denote $c := H_{\frac{n+1}{2}\,\frac{n+1}{2}}$. Then c is real as H is real symmetric. Also, since H is invertible, $c \neq 0$. By repeated application of (34) we have that along the main anti-diagonal of H the entries alternate between c and $-c$, that is, $H_{i\,n+1-i} = (-1)^{\frac{n+1}{2}-i} c$. This determines all entries H_{ij} for $i + j < n + 2$. In particular, the $\frac{n+1}{2} \times \frac{n+1}{2}$ upper left corner of H is given by $c \cdot Z_n$.

By (34) we have

$$H_{\frac{n+1}{2}\,\frac{n+1}{2}} + H_{\frac{n+1}{2}+1\,\frac{n+1}{2}} + H_{\frac{n+1}{2}\,\frac{n+1}{2}+1} = 0.$$

By symmetry of H we have that $H_{\frac{n+1}{2}+1\,\frac{n+1}{2}} = H_{\frac{n+1}{2}\,\frac{n+1}{2}+1}$. Combining these two brings up that

$$H_{\frac{n+1}{2}\,\frac{n+1}{2}+1} = -\frac{1}{2} c.$$

Repeated application of (34), combined with knowledge of the entries on the antidiagonal $i+j=n+1$ now determines the entries on the antidiagonal $i+j=n+2$. We obtain for $i=2,\dots,\frac{n+1}{2}$ that $H_{in+2-i}=p_{i\,\frac{n+3}{2}-i}c$. This determines all entries of H_{ij} for $i+j<n+3$.

If we show that there is an invertible S such that $S^{-1}AS=A$ and the right lower corner of S^THS is zero, then by repeated application of (34) the bottom row of the right upper corner of S^THS has entries alternating between $-\frac{1}{2}c$ and $\frac{1}{2}c$. From there on going up, again by repeated application of (34) proves that the upper right corner of S^THS is given by $c\cdot P_n$. This proves Case 1, upon realising that by scaling of the Jordan chain we can make $c=\pm 1$.

So it remains to find such an S. We do this by changing the Jordan basis step by step. First we define a new Jordan basis as follows: let

$$z_j^{(2)}=x_j,\text{ for } j=1,2,\text{ and } z_j^{(2)}=x_j+h_2x_{j-2}\text{ for } j>2.$$

Here h_2 is a real number to be determined. Observe that this is indeed a Jordan basis. Set

$$S_2=\left[z_1^{(2)}\ z_2^{(2)}\ \cdots\ z_n^{(2)}\right].$$

Then $S_2^{-1}AS_2=A$. Put $H^{(2)}=S_2^THS_2$. Then for $i+j<n+3$ one checks that $H_{ij}^{(2)}=H_{ij}$. Moreover, by the scalar analogue of (28) we have

$$\begin{aligned}H^{(2)}_{\frac{n+3}{2}\,\frac{n+3}{2}}&=H_{\frac{n+3}{2}\,\frac{n+3}{2}}+h_2\left(H_{\frac{n+3}{2}\,\frac{n+3}{2}-2}+H_{\frac{n+3}{2}-2\,\frac{n+3}{2}}\right)\\&=H_{\frac{n+3}{2}\,\frac{n+3}{2}}-2ch_2.\end{aligned}$$

Obviously, taking $h_2=\frac{1}{2}cH_{\frac{n+3}{2}\,\frac{n+3}{2}}$ we obtain that $H^{(2)}_{\frac{n+3}{2}\,\frac{n+3}{2}}=0$.

Using (34) we also have

$$\begin{aligned}0=&H^{(2)}_{\frac{n+3}{2}\,\frac{n+3}{2}}+H^{(2)}_{\frac{n+3}{2}\,\frac{n+3}{2}+1}+H^{(2)}_{\frac{n+3}{2}+1\,\frac{n+3}{2}}\\=&2H^{(2)}_{\frac{n+3}{2}\,\frac{n+3}{2}+1}.\end{aligned}$$

It follows that $H^{(2)}_{\frac{n+3}{2}\,\frac{n+3}{2}+1}=0$. Using (34) this determines all entries $H_{ij}^{(2)}$ with $i+j<n+5$.

Next, define a new Jordan basis by setting

$$z_j^{(4)}=z_j^{(2)},\text{ for }\ j=1,2,3,4,\text{ and } z_j^{(4)}=z_j^{(2)}+h_4z_{j-4}^{(2)}\text{ for } j>4.$$

Put $S_4=\left[z_1^{(4)}\ \cdots\ z_n^{(4)}\right]$. Then $S_4^{-1}AS_4=A$. We define $H^{(4)}=S_4^TH^{(2)}S_4$. Then one easily checks that $H_{ij}^{(4)}=H_{ij}^{(2)}$ for all entries with $i+j<n+5$, and that by the scalar analogue of (28)

$$\begin{aligned}H^{(4)}_{\frac{n+5}{2}\,\frac{n+5}{2}}&=H^{(2)}_{\frac{n+5}{2}\,\frac{n+5}{2}}+h_4\left(H^{(2)}_{\frac{n+5}{2}\,\frac{n+5}{2}-4}+H^{(2)}_{\frac{n+5}{2}-4\,\frac{n+5}{2}}\right)\\&=H^{(2)}_{\frac{n+5}{2}\,\frac{n+5}{2}}+2ch_4.\end{aligned}$$

Obviously, taking $h_4 = -\frac{1}{2} c H^{(2)}_{\frac{n+5}{2}\,\frac{n+5}{2}}$ we obtain that $H^{(4)}_{\frac{n+5}{2}\,\frac{n+5}{2}} = 0$. Now we can continue as above to show that $H^{(4)}_{\frac{n+5}{2}\,\frac{n+5}{2}+1} = 0$. Using (34) this determines all entries $H^{(4)}_{ij}$ with $i+j < n+7$.

Continuing this way, by induction suppose that for some j we have already made the diagonal entries $h_{\frac{n+2j+1}{2}\,\frac{n+2j+1}{2}}$ zero by consecutively using $S_2, S_4, \dots, S_{2j}$. It then follows as above that also $h_{\frac{n+2j+1}{2}\,\frac{n+2j+1}{2}+1} = 0$. Then construct S_{2j+2} such that the next diagonal entry is zero, and hence also the entry adjacent to it on the first super-diagonal. Then $S = S_{n-1} S_{n-3} \cdots S_4 S_2$ such that the lower right corner of $S^T H S$ is zero. Indeed, by construction the diagonal and first super-diagonal are zero. Repeated application of (34) then proves that the second super-diagonal is zero. But then again applying (34) this can be lifted to the third super-diagonal. By induction we get that the full lower right corner of $S^T H S$ is zero.

(ii), *Case* 1. This case is proved analogously, replacing everywhere the use of (34) by the use of

$$H_{ij} - H_{i\,j+1} - H_{i+1\,j} = 0, \tag{35}$$

which follows immediately from $A^T H A = H$.

(i), *Case* 2. We may assume, based on [12] that in this case

$$A = J_n(1) \oplus J_n(1), \qquad H = \begin{bmatrix} 0 & H^{(0)} \\ (H^{(0)})^T & 0 \end{bmatrix}$$

for an invertible $n \times n$ matrix $H^{(0)}$. By [4], see also [10], we have that $H^{(0)}_{ij} = 0$ for $i + j \leqslant n$. Moreover, from $A^T H A = H$ we have that

$$J_n(1)^T H^{(0)} J_n(1) = H^{(0)},$$

and so

$$H^{(0)}_{ij} + H^{(0)}_{i\,j+1} + H^{(0)}_{i+1\,j} = 0, \tag{36}$$

(compare (34)). However, in contrast with the case when n is odd, here $H^{(0)}$ is not a symmetric matrix.

Consider $S = \hat{S} \oplus \tilde{S}$, where $\hat{S}$ and $\tilde{S}$ are $n \times n$ upper triangular Toeplitz matrices. Then $S^{-1} A S = A$, and

$$S^T H S = \begin{bmatrix} 0 & \hat{S}^T H^{(0)} \tilde{S} \\ \tilde{S}^T (H^{(0)})^T \hat{S} & 0 \end{bmatrix}.$$

We shall show that it is possible to take $\tilde{S} = I$ and to choose $\hat{S}$ so that $\hat{S}^T H^{(0)}$ has the form given in part (i), Case 2 of the theorem, that is,

$$\tilde{H} = \hat{S}^T H^{(0)} = \begin{bmatrix} 0 & Q_n \\ -Q_n^T & 0 \end{bmatrix}.$$

In order to achieve this, it suffices, in view of (36) to show that $\hat{S}$ can be chosen such that

$$\begin{aligned}&\tilde{H}_{\frac{n}{2}\,\frac{n}{2}+1}=1,\\&\tilde{H}_{\frac{n}{2}+j\,\frac{n}{2}+j}=0,\ \text{for } j=1,\dots,\frac{n}{2},\\&\tilde{H}_{\frac{n}{2}+j\,\frac{n}{2}+j+1}=0,\ \text{for } j=1,\dots,\frac{n}{2}-1.\end{aligned}$$

Let $\hat{S}=toep(h_1,\cdots,h_n)$, and compute $\tilde{H}_{ij}=(\hat{S}^T H^{(0)})_{ij}$:

$$\tilde{H}_{ij}=\begin{cases}0 & \text{if } i+j\leqslant n,\\ h_1H^{(0)}_{ij}+h_2H^{(0)}_{i-1\,j}+\cdots+h_{i+j-n}H^{(0)}_{n+1-j\,j} & \text{if } i+j>n.\end{cases} \tag{37}$$

Put $S_1=h_1I_n$, and $H^{(1)}=S_1^TH^{(0)}$. Take $i+j=n+1$ and $i=\frac{n}{2}$, $j=\frac{n}{2}+1$. Then

$$H^{(1)}_{\frac{n}{2}\,\frac{n}{2}+1}=h_1H^{(0)}_{\frac{n}{2}\,\frac{n}{2}+1}.$$

Because of the invertibility of $H^{(0)}$ we can take h_1 so that $\frac{1}{h_1}=H^{(0)}_{\frac{n}{2}\,\frac{n}{2}+1}$, and hence $H^{(1)}_{\frac{n}{2}\,\frac{n}{2}+1}=1$. By (36) this ensures that the entries of $H^{(1)}$ on the anti-diagonal $i+j=n+1$ are alternatingly $+1$ and -1.

Now put $S_2=toep(1,h_2,0,\cdots,0)$, and let $H^{(2)}=S_2^TH^{(1)}$. Then for $i+j>n$

$$H^{(2)}_{ij}=H^{(1)}_{ij}+h_2H^{(1)}_{ij}.$$

In particular, if $i+j=n+1$ we have

$$H^{(2)}_{in+1-i}=H^{(1)}_{in+1-i}+h_2H^{(1)}_{i-1\,n+1-i}=H^{(1)}_{in+1-i},$$

and so $H^{(2)}_{ij}=H^{(1)}_{ij}$ for $i+j\leqslant n+1$. Further, for $i+j=n+2$ we have

$$H^{(2)}_{in+2-i}=H^{(1)}_{in+2-i}+h_2H^{(1)}_{i-1\,n+2-i}.$$

Now $H^{(2)}_{i-1\,n+2-i}=\pm1$. In particular, for $i=\frac{n}{2}+1$, we have

$$H^{(2)}_{\frac{n}{2}+1\,\frac{n}{2}+1}=H^{(1)}_{\frac{n}{2}+1\,\frac{n}{2}+1}+h_2H^{(1)}_{\frac{n}{2}\,\frac{n}{2}+1}=H^{(1)}_{\frac{n}{2}+1\,\frac{n}{2}+1}+h_2.$$

Choosing $h_2=-H^{(1)}_{\frac{n}{2}+1\,\frac{n}{2}+1}$ we see that $H^{(2)}_{\frac{n}{2}+1\,\frac{n}{2}+1}=0$. Together with (36) this also determines the values $H^{(2)}_{ij}$ with $i+j=n+2$, and they are exactly as in the statement of the theorem.

Next, put $S_3=toep(1,0,h_3,0,\cdots,0)$ and $H^{(3)}=S_3^TH^{(2)}$. By (37) we have for $i+j>n$

$$H^{(3)}_{ij}=H^{(2)}_{ij}+h_3H^{(2)}_{i-2\,j}.$$

If $i+j \leqslant n+2$ this gives $H^{(3)}_{ij} = H^{(2)}_{ij}$, (since $i+j-2 \leqslant n$, so that $H^{(2)}_{i-2\,j} = 0$). For $i+j = n+3$ we obtain

$$H^{(3)}_{i\,n+3-i} = H^{(2)}_{i\,n+3-i} + h_3 H^{(2)}_{i-2\,n+3-i}.$$

Take $i = \frac{n}{2}+1$, then

$$H^{(3)}_{\frac{n}{2}+1\,\frac{n}{2}+2} = H^{(2)}_{\frac{n}{2}+1\,\frac{n}{2}+2} + h_3 H^{(2)}_{\frac{n}{2}-1\,\frac{n}{2}+2}.$$

Since $H^{(2)}_{\frac{n}{2}-1\,\frac{n}{2}+2} = -1$ we can take $h_3 = H^{(2)}_{\frac{n}{2}+1\,\frac{n}{2}+2}$ to obtain $H^{(3)}_{\frac{n}{2}+1\,\frac{n}{2}+2} = 0$. Note that this also determines all entries $H^{(3)}_{ij}$ with $i+j = n+3$ by (36).

Now we can continue as before by induction to finish the proof.

(ii) *Case* 2 can be proved in the same way. □

4. Real non-unimodular eigenvalues

We consider now the case where A is H-unitary and $\sigma(A) = \{\lambda, \frac{1}{\lambda}\}$, where λ is real and $\lambda \neq 0, \pm 1$. Since A is invertible, $\lambda \neq 0$. The indecomposable blocks are the ones where A is similar to a direct sum of two Jordan blocks of the same size, one with eigenvalue λ, the other with eigenvalue $\frac{1}{\lambda}$. So we may assume that A is of the form

$$A = J_n(\lambda) \oplus J_n\left(\frac{1}{\lambda}\right).$$

As is known from [9, 2, 5] the spectral subspaces corresponding to the eigenvalues λ and $\frac{1}{\lambda}$ are H-neutral, but their sum is H-nondegenerate. Thus H has the following form

$$H = \begin{bmatrix} 0 & H_{12} \\ H_{12}^T & 0 \end{bmatrix}$$

for some $n \times n$ matrix H_{12}. Writing out $A^T H A = H$ in the blocks, we obtain that H_{12} satisfies the following

$$J_n(\lambda)^T H_{12} J_n\left(\frac{1}{\lambda}\right) = H_{12}. \tag{38}$$

EXAMPLE. Consider the case $n = 3$. Then, denoting $H_{12} = \left[h_{ij}\right]_{i,j=1}^{3}$, (38) becomes

$$\begin{bmatrix} \lambda & 0 & 0 \\ 1 & \lambda & 0 \\ 0 & 1 & \lambda \end{bmatrix} \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} \frac{1}{\lambda} & 1 & 0 \\ 0 & \frac{1}{\lambda} & 1 \\ 0 & 0 & \frac{1}{\lambda} \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}.$$

This gives nine relations for the entries, which we consider row-by-row. The three identities for the entries in the first row are

$$\begin{aligned} h_{11} &= h_{11}, \\ \lambda h_{11} + h_{12} &= h_{12}, \\ \lambda h_{12} + h_{13} &= h_{13}. \end{aligned}$$

While the first of these identities is uninteresting, the second and third give us

$$h_{11} = 0, \quad h_{12} = 0. \tag{39}$$

Next, we consider the entries in the second row, again giving three identities:

$$\begin{aligned} \frac{1}{\lambda}h_{11} + h_{21} &= h_{21}, \\ h_{11} + \lambda h_{21} + \frac{1}{\lambda}h_{12} + h_{22} &= h_{22}, \\ h_{12} + \lambda h_{22} + \frac{1}{\lambda}h_{13} + h_{23} &= h_{23}. \end{aligned}$$

Using (39) the second of these equations implies

$$h_{21} = 0, \tag{40}$$

while the third gives

$$h_{13} = -\lambda^2 h_{22}. \tag{41}$$

Finally, the entries in the third row again give rise to three identities:

$$\begin{aligned} \frac{1}{\lambda}h_{21} + h_{31} &= h_{31}, \\ h_{21} + \lambda h_{31} + \frac{1}{\lambda}h_{22} + h_{32} &= h_{32}, \\ h_{22} + \lambda h_{32} + \frac{1}{\lambda}h_{23} + h_{33} &= h_{33}. \end{aligned}$$

Using (40) the second of these identities gives

$$h_{31} = -\frac{1}{\lambda^2}h_{22}. \tag{42}$$

The third identity finally can be rewritten as

$$h_{22} + \lambda h_{32} + \frac{1}{\lambda}h_{23} = 0. \tag{43}$$

So, for any Jordan basis of A we have that $H = \begin{bmatrix} 0 & H_{12} \\ H_{12}^T & 0 \end{bmatrix}$, where

$$H_{12} = \begin{bmatrix} 0 & 0 & -\lambda^2 h_{22} \\ 0 & h_{22} & h_{23} \\ -\frac{1}{\lambda^2}h_{22} & -\frac{1}{\lambda}h_{22} - \frac{1}{\lambda^2}h_{23} & h_{33} \end{bmatrix}$$

for some real numbers h_{22}, h_{23}, h_{33}. Conversely, for any choice of these numbers the matrix A is H-unitary. $\square$

Returning to the general case, we may now consider what canonical form we can obtain for H by choosing a particular Jordan basis. Suppose S is an invertible $2n\times 2n$ matrix such that $S^{-1}AS=A$. and write

$$S=\begin{bmatrix} S_{11} & S_{12}\\ S_{21} & S_{22}\end{bmatrix}.$$

Then from $SA=AS$ we obtain four relations: $S_{11}J_n(\lambda)=J_n(\lambda)S_{11}$, $S_{22}J_n(\frac{1}{\lambda})=J_n(\frac{1}{\lambda})S_{22}$, $S_{21}J_n(\lambda)=J_n(\frac{1}{\lambda})S_{21}$, and $S_{12}J_n(\frac{1}{\lambda})=J_n(\lambda)S_{12}$. The last two equations imply that $S_{12}=0$ and $S_{21}=0$, while the first two equations imply that S_{11} and S_{22} are upper triangular Toeplitz matrices.

Now consider $H_1=S^THS$. Then it is easily seen that A is H_1-unitary. Also,

$$H_1=S^THS=\begin{bmatrix} 0 & S_{11}^TH_{12}S_{22}\\ S_{22}^TH_{12}S_{11} & 0\end{bmatrix}.$$

Thus we see that

$$H_{12}^{(1)}=S_{11}^TH_{12}S_{22} \tag{44}$$

and we can use the freedom in selecting S_{11} and S_{22} to obtain a desired form for H_{12}. Note that this gives us $2n$ variables, as each of S_{11} and S_{22} is determined by n real numbers.

Example continued. We continue the analysis of the case $n=3$. Take $S_{11}=I$, and let $S_{22}=Toep(s_1,s_2,s_3)$. Then $H_{12}^{(1)}=H_{12}S_{22}$. Let us denote

$$H_{12}^{(1)}=\begin{bmatrix} 0 & 0 & -\lambda^2h_{22}^{(1)}\\ 0 & h_{22}^{(1)} & h_{23}^{(1)}\\ -\frac{1}{\lambda^2}h_{22}^{(1)} & h_{32}^{(1)} & h_{33}^{(1)}\end{bmatrix}$$

where according to (43)

$$h_{22}^{(1)}+\lambda h_{32}^{(1)}+\frac{1}{\lambda}h_{23}^{(1)}=0.$$

We shall show that it is possible to take s_1,s_2,s_3 such that the following three conditions are satisfied:

$$h_{33}^{(1)}=0, h_{22}^{(1)}=1, h_{23}^{(1)}=-\frac{1}{2}\lambda, h_{32}^{(1)}=-\frac{1}{2\lambda}.$$

First observe that the latter three of these conditions are cosistent with (43).

Indeed, in order to show this, we consider the third column of $H_{12}^{(1)}=H_{12}S_{22}$, which gives

$$\begin{bmatrix} -\lambda^2h_{22}^{(1)}\\ h_{23}^{(1)}\\ h_{33}^{(1)}\end{bmatrix}=\begin{bmatrix} -\lambda^2\\ -\frac{1}{2}\lambda\\ 0\end{bmatrix}=H_{12}\begin{bmatrix} s_3\\ s_2\\ s_1\end{bmatrix}. \tag{45}$$

Since H_{12} is invertible, it is obvious that this equation is solvable for s_1,s_2,s_3.

Thus we arrive at the following canoncial form for the pair (A,H):

$$A = J_3(\lambda) \oplus J_3\left(\frac{1}{\lambda}\right), \qquad H = \begin{bmatrix} 0 & H_{12} \\ H_{12}^T & 0 \end{bmatrix},$$

where

$$H_{12} = \begin{bmatrix} 0 & 0 & -\lambda^2 \\ 0 & 1 & -\frac{1}{2}\lambda \\ -\frac{1}{\lambda^2} & -\frac{1}{2\lambda} & 0 \end{bmatrix}.$$

Compare this with the case $A = J_3(1)$, where we can achieve a canonical form for the corresponding H as

$$H = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & -\frac{1}{2} \\ -1 & -\frac{1}{2} & 0 \end{bmatrix},$$

and observe the resemblance.

Also compare with the case of two unimodular non-real eigenvalues, and observe the resemblance. □

Returning to the general case, consider the equation (38), and denote $H_{12} = \left[h_{ij}\right]_{i,j=1}^{n}$. The equality (38) becomes for the entries in the first row:

$$\left[\lambda\ 0 \cdots 0\right] H_{12} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \frac{1}{\lambda} \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

where there are $j-2$ zeros on the top. This is equal to

$$\lambda \left[h_{11} \cdots h_{1n}\right] \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \frac{1}{\lambda} \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{cases} h_{11} & \text{if } j=1 \\ \lambda h_{1\,j-1} + h_{1j} & \text{if } j>1 \end{cases} = h_{1j}. \tag{46}$$

This implies

$$h_{1j} = 0 \text{ for } j = 1,2,\ldots,n-1. \tag{47}$$

Next, consider row i with $i>1$, then $\left(J_n(\lambda)^T H_{12} J_n(\frac{1}{\lambda})\right)_{ij} = h_{ij}$ becomes

$$\begin{bmatrix}0 \cdots 0 \; 1 \; \lambda \; 0 \cdots 0\end{bmatrix} H_{12} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \frac{1}{\lambda} \\ 0 \\ \vdots \\ 0 \end{bmatrix} =$$

$$= \quad \begin{bmatrix} h_{i-1\,1} \cdots h_{i-1\,n} \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \frac{1}{\lambda} \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} h_{i1} \cdots h_{in} \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \frac{1}{\lambda} \\ 0 \\ \vdots \\ 0 \end{bmatrix} =$$

$$= \begin{cases} \frac{1}{\lambda} h_{i-1\,1} + h_{i1} = h_{i1} & \text{if } j=1 \\ h_{i-1\,j-1} + \frac{1}{\lambda} h_{i-1\,j} + \lambda h_{i\,j-1} + h_{ij} = h_{ij} & \text{if } j>1. \end{cases} \tag{48}$$

This implies

$$h_{i1} = 0, \text{ for } i = 1, \ldots, n-1, \tag{49}$$

$$h_{i-1\,j-1} + \frac{1}{\lambda} h_{i-1\,j} + \lambda h_{i\,j-1} = 0. \tag{50}$$

Observe that (47) and (50) together determine H_{12} once the last column of H_{12} is given. Observe also that these two relations hold for any Jordan basis of A. We can now change the Jordan basis in such a way that a canonical form for H is obtained, using (44). We can do so by using S_{22} only, taking $S_{11} = I_n$. Then $H_{12}^{(1)} = H_{12} S_{22}$. Since $S_{22} = Toep(s_1, \cdots, s_n)$ is completely determined by its last column, and $H_{12}^{(1)} = \left[h_{ij}^{(1)}\right]_{i,j=1}^n$ is also determined by its last column, in principle we can select a canonical form by specifying the last column, and solving the equation

$$H_{12} \begin{bmatrix} s_n \\ \vdots \\ s_1 \end{bmatrix} = \begin{bmatrix} h_{1n}^{(1)} \\ \vdots \\ h_{nn}^{(1)} \end{bmatrix}.$$

Note that by invertibility of H_{12} there is a unique solution.

However, in order to stay close to the canonical forms developed for the cases where the eigenvalues are on the unit circle, we shall make a different choice. We shall take n conditions on the entries in $H_{12}^{(1)}$: for n odd these conditions are

$$\begin{aligned} h_{in} &= 0 \qquad \text{for } i = \frac{n+3}{2}, \dots, n, \\ h_{\frac{n+3}{2}\, j} &= 0 \qquad \text{for } j = \frac{n+3}{2}, \dots, n, \\ h_{\frac{n+1}{2}\, \frac{n+1}{2}} &= 1, \\ h_{\frac{n+1}{2}\, \frac{n+3}{2}} &= -\tfrac{1}{2}\lambda \end{aligned}$$

while for n even the conditions are

$$\begin{aligned} h_{in} &= 0 \quad \text{for } i = \frac{n}{2}+1, \dots, n, \\ h_{\frac{n}{2}+1\, j} &= 0 \quad \text{for } j = \frac{n}{2}+1, \dots, n, \\ h_{\frac{n}{2}\, \frac{n}{2}+1} &= 1. \end{aligned}$$

Note that in both cases there are indeed precisely n conditions which are linear in the entries of $H_{12}^{(1)}$, leading to n linear relations on $s_1, \dots, s_n$. (Which we do not state explicitly here!)

EXAMPLE. For $n = 4$ this would mean that

$$H_{12}^{(1)} = \begin{bmatrix} 0 & 0 & 0 & -\lambda^2 \\ 0 & 0 & 1 & -\lambda \\ 0 & -\frac{1}{\lambda^2} & 0 & 0 \\ \frac{1}{\lambda^4} & -\frac{1}{\lambda^3} & 0 & 0 \end{bmatrix}.$$

Again, compare with the case where the eigenvalues are on the unit circle to see the resemblance. □

It remains to show that it is possible to choose $s_1, \dots, s_n$ so that the conditions formulated just before the previous example are satisfied. However, this is in principle easy: we just retrace what these conditions imply for the entries in the last column of $H_{12}^{(1)}$. We have already seen that the last column of $H_{12}^{(1)}$ can be stipulated at will, and that given any choice for that there is a unique choice for $s_1, \dots, s_n$ that will give the desired $H_{12}^{(1)}$.

To make this argument precise, we first recall from the introduction the matrices $P_n(\lambda)$ and $Q_n(\lambda)$ given by (2) and (5).

Recall also the definition of the matrix Z_n as the $\frac{n+1}{2} \times \frac{n+1}{2}$ matrix with zero entries everywhere except for a one in the right lower corner.

THEOREM 4.1. *Let A be H-unitary and $\sigma(A) = \{\lambda, \frac{1}{\lambda}\}$, with $\lambda \in \mathbb{R} \setminus \{0, 1, -1\}$. Then the pair (A, H) can be decomposed as*

$$S^{-1}AS = \oplus_{l=1}^{p} A_l, \qquad S^T HS = \oplus_{l=1}^{p} H_l,$$

where the pair (A_l,H_l) *is of the form* $\left(J_n(\lambda)\oplus J_n(\frac{1}{\lambda}), \begin{bmatrix} 0 & H_{12} \\ H_{12}^T & 0 \end{bmatrix}\right)$, *with* n *depending on* l, *and where* H_{12} *is of one of the following two forms, depending on whether* n *is odd or even:*

Case 1 n *is odd:* $H_{12} = \begin{bmatrix} Z_n & P_n(\lambda) \\ P_n(\frac{1}{\lambda})^T & 0 \end{bmatrix}$.

Case 2 n *is even:* $H_{12} = \begin{bmatrix} 0 & Q_n(\lambda) \\ -\frac{1}{\lambda^2}Q_n(\frac{1}{\lambda})^T & 0 \end{bmatrix}$.

Proof. First observe that the entries in the matrices given in Case 1 and Case 2, respectively do satisfy (50). This is easily checked by induction.

Case 1. We continue the argument given before the definition of $P_n(\lambda)$. Let h_n be the last column of $P_n(\lambda)$, and let y be the vector in $\mathbb{R}^n$ given by $y = \begin{bmatrix} h_n \\ 0 \end{bmatrix}$. Solve $H_{12}s = y$ and set $S_{22} = Toep(s_1,\cdots,s_n)$, where s_j is the j-th coordinate of the vector s. Then by our arguments from before the proof we get that

$$H_{12}^{(1)} = H_{12}S_{22} = \begin{bmatrix} Z_n & P_n(\lambda) \\ P_n(\frac{1}{\lambda})^T & 0 \end{bmatrix},$$

as desired.

Case 2 is proved in the same way. □

5. Non-real non-unimodular eigenvalues

We consider now the case where A is H-unitary and $\sigma(A) = \{\lambda, \frac{1}{\lambda}, \bar{\lambda}, \frac{1}{\bar{\lambda}}\}$, where λ is non-real and non-unimodular. In this case, the indecomposable blocks are the ones where A is similar to a direct sum of two real Jordan blocks of the same size, one corresponding to the two eigenvalue $\lambda, \bar{\lambda}$, and the other corresponding to the two eigenvalues λ^{-1}, $\bar{\lambda}^{-1}$. Let $\lambda = \alpha + i\beta$, and set as usual $\gamma = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$. Note that $\frac{1}{\lambda} = \frac{1}{\alpha^2+\beta^2}(\alpha - i\beta)$ and that $\gamma^{-1} = \frac{1}{\alpha^2+\beta^2}\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$ is exactly the two by two matrix with eigenvalues λ^{-1}, $\bar{\lambda}^{-1}$.

So we may assume that A is a $4n \times 4n$ matrix of the form

$$A = J_n(\gamma) \oplus J_n(\gamma^{-1}).$$

As is known from [9, 2, 5] the spectral subspace corresponding to the pair of eigenvalues λ, $\bar{\lambda}$, as well as the one corresponding to the pair of eigenvalues $\frac{1}{\lambda}$ and $\frac{1}{\bar{\lambda}}$ is H-neutral, while the sum of these two spectral subspaces is H-nondegenerate. Thus H has the following form

$$H = \begin{bmatrix} 0 & H_{12} \\ H_{12}^T & 0 \end{bmatrix}$$

for some $2n\times 2n$ matrix H_{12}. Writing out $A^THA=H$ in the blocks, we obtain that H_{12} satisfies the following

$$J_n(\gamma)^T H_{12} J_n(\gamma^{-1}) = H_{12}. \tag{51}$$

Denoting $H_{12}=\left[h_{i,j}\right]_{i,j=1}^n$, where each $h_{i,j}$ is a two by two matrix, (38) gives n^2 relations for the entries. The identities for the entries in the first row are

$$\gamma^T h_{1,1}\gamma^{-1} = h_{1,1},$$
$$\gamma^T \begin{bmatrix} h_{1,j-1} & h_{1,j}\end{bmatrix}\begin{bmatrix} I \\ \gamma^{-1}\end{bmatrix} = h_{1,j} \text{ for } j>1.$$

The last of these conditions can be rewritten as follows by working out the matrix product and then multiplying on the right by γ:

$$\gamma^T h_{1,j-1}\gamma + \gamma^T h_{1,j} - h_{1,j}\gamma = 0. \tag{52}$$

Next, we consider the identities for the first column, except the first entry:

$$\begin{bmatrix} I & \gamma^T\end{bmatrix}\begin{bmatrix} h_{i-1,1} \\ h_{i,1}\end{bmatrix}\gamma^{-1} = h_{i,1},$$

which, after working out the matrix product and multiplying by γ on the right leads to

$$h_{i-1,1}\gamma + \gamma^T h_{i,1} - h_{i,1}\gamma = 0. \tag{53}$$

Finally, the identity for an entry not in the first row or column becomes

$$\begin{bmatrix} I & \gamma^T\end{bmatrix}\begin{bmatrix} h_{i-1,j-1} & h_{i-1,j} \\ h_{i,j-1} & h_{i,j}\end{bmatrix}\begin{bmatrix} I \\ \gamma^{-1}\end{bmatrix} = h_{i,j}$$

for $i>1$ and $j>1$. Again, working out the matrix product and multiplying by γ on the right gives

$$h_{i-1,j-1}\gamma + \gamma^T h_{i,j-1}\gamma + h_{i-1,j} + \gamma^T h_{i,j} - h_{i,j}\gamma = 0. \tag{54}$$

5.1. Intermezzo

Before proceeding, we need several lemmas, most of which are easily proved by brute force computation on two by two matrices. Recall that $K_0=\begin{bmatrix} 1 & 0 \\ 0 & -1\end{bmatrix}$ and $K_1=\begin{bmatrix} 0 & 1 \\ 1 & 0\end{bmatrix}$, while $H_0=\begin{bmatrix} 0 & 1 \\ -1 & 0\end{bmatrix}$. The set of matrices of the form $x_1K_0+x_2K_1$ where x_1 and x_2 are real, will be denoted by $\mathscr{K}$, the set of matrices of the form $a_1I+a_2H_0$ is denoted by $\mathscr{E}$ as before.

LEMMA 5.1. *Let $\gamma=aI+bH_0$ be in $\mathscr{E}$, and let $X=x_0K_0+x_1K_1$ be in $\mathscr{K}$. Then*

a. *$X\gamma$ is in $\mathscr{K}$,*

b. $\gamma^T X = X\gamma$.

c. *For any 2×2 matrix X (not necessarily in $\mathscr{K}$), if $\gamma^T X = X\gamma$, then $X \in \mathscr{K}$.*

d. *For any 2×2 matrix X (not necessarily in $\mathscr{K}$), $\gamma^T X - X\gamma$ is in $\mathscr{E}$.*

LEMMA 5.2. *Suppose X is in $\mathscr{K}$ and suppose Y is a 2×2 matrix satisfying $\gamma^T X\gamma + \gamma^T Y - Y\gamma = 0$, with $\gamma \neq 0$ in $\mathscr{E}$. Then $X = 0$ and Y is in $\mathscr{K}$.*

Proof. By part d in the previous lemma, $\gamma^T Y - Y\gamma \in \mathscr{E}$, while $\gamma^T X\gamma$ is in $\mathscr{K}$ by part a of the previous lemma. Since the intersection of $\mathscr{E}$ and $\mathscr{K}$ consists of only the zero matrix, and since γ is not zero and in $\mathscr{E}$ implies that γ is invertible, we obtain that X is zero, and that $\gamma^T Y - Y\gamma = 0$. Then part c of the previous lemma implies that Y is in $\mathscr{K}$. □

LEMMA 5.3. *Let X, Y, Z be in $\mathscr{K}$, let γ be invertible and in $\mathscr{E}$, and let W be any 2×2 matrix such that*

$$\begin{bmatrix} I & \gamma^T \end{bmatrix} \begin{bmatrix} X & Y \\ Z & W \end{bmatrix} \begin{bmatrix} I \\ \gamma^{-1} \end{bmatrix} = W.$$

Then W is in $\mathscr{K}$ and

$$X\gamma + Z\gamma^2 + Y = 0. \tag{55}$$

Proof. The condition is equivalent to

$$X\gamma + \gamma^T Z\gamma + Y = -(\gamma^T W - W\gamma).$$

By the assumptions on X, Y, Z and by Lemma 5.1 the left hand side is in $\mathscr{K}$ and the right hand side is in $\mathscr{E}$. Hence both parts are zero. Again by Lemma 5.1 we then have $W \in \mathscr{K}$. Also $\gamma^T Z\gamma = Z\gamma^2$, giving (55). □

5.2. Results for general Jordan basis

We resume the argument we started before the intermezzo. Consider the entries of H_{12} in the first row, and remember that for those entries we have (52):

$$\gamma^T h_{1,j-1}\gamma + \gamma^T h_{1,j} - h_{1,j}\gamma = 0.$$

Applying Lemma 5.2 inductively we see that

$$h_{1,j} = 0 \text{ for } j = 1, \ldots, n-1,$$

and that $h_{1,n}$ is in $\mathscr{K}$.

Likewise, considering the entries in the first column we have (53):

$$h_{i-1,1}\gamma + \gamma^T h_{i,1} - h_{i,1}\gamma = 0.$$

Again, by induction and using Lemma 5.1, part b, and Lemma 5.2, we have that $h_{i,1}$ is in $\mathscr{K}$. As in the proof of Lemma 5.1 we see that

$$h_{i,1} = 0 \text{ for } i = 1,\dots,n-1,$$

and $h_{n,1}$ is in $\mathscr{K}$.

Now consider entries that are not in the first row or column. For those we have (54)

$$h_{i-1,j-1}\gamma + \gamma^T h_{i,j-1}\gamma + h_{i-1,j} + \gamma^T h_{i,j} - h_{i,j}\gamma = 0.$$

Using the fact that entries in the first row and column are in $\mathscr{K}$ (and most of them are zero anyway), we can use induction on the row index and column index to obtain from Lemma 5.3 that $h_{i,j}$ is in $\mathscr{K}$ for all i and j, and that

$$h_{i-1,j-1}\gamma + h_{i,j-1}\gamma^2 + h_{i-1,j} = 0, \tag{56}$$

or equivalently

$$h_{i-1,j-1} + h_{i,j-1}\gamma + h_{i-1,j}\gamma^{-1} = 0. \tag{57}$$

Using this, and the fact that all but the last entries in the first row and column are zero we have

$$h_{i,j} = 0 \text{ for } i+j < n+1.$$

Also, on the main skew-diagonal, we have $h_{i-1,n-i+2} = -h_{i,n-i+1}\gamma^2$, so that $h_{i,n-i+1} = (-1)^{n-i}h_{n,1}\gamma^{2(n-i)}$.

5.3. Results for specific Jordan basis

Returning to the general case, we may now consider what canonical form we can obtain for H by choosing a particular Jordan basis. Suppose S is an invertible $2n \times 2n$ matrix such that $S^{-1}AS = A$. and write

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}.$$

Then from $SA = AS$ we obtain the following four relations: $S_{11}J_n(\gamma) = J_n(\gamma)S_{11}$, $S_{22}J_n(\gamma^{-1}) = J_n(\gamma^{-1})S_{22}$, $S_{21}J_n(\gamma = J_n(\gamma^{-1})S_{21}$, and $S_{12}J_n(\gamma^{-1}) = J_n(\gamma)S_{12}$. The last two equations imply that $S_{12} = 0$ and $S_{21} = 0$, while the first two equations imply that S_{11} and S_{22} are upper triangular block Toeplitz matrices, with entries that are 2×2 matrices in $\mathscr{E}$. The latter statement holds because for a complex Jordan block the matrices that commute with it are upper triangular Toeplitz matrices with complex entries.

Now consider $H_1 = S^T HS$. Then it is easily seen that A is H_1-unitary. Also,

$$H_1 = S^T HS = \begin{bmatrix} 0 & S_{11}^T H_{12} S_{22} \\ S_{22}^T H_{12} S_{11} & 0 \end{bmatrix}.$$

Thus we see that

$$H_{12}^{(1)} = S_{11}^T H_{12} S_{22} \tag{58}$$

and we can use the freedom in selecting S_{11} and S_{22} to obtain a desired form for H_{12}. Note that this gives us $4n$ variables, as each of S_{11} and S_{22} is determined by n real two by two matrices in $\mathscr{E}$.

Observe that H_{12} is determined by (57) as soon as the last block column of H_{12} is specified, i.e., as soon as $h_{i,n}$ are given, we can retrieve the entries $h_{i,j}$ from the last column and the first row. Note also that these two relations hold for any Jordan basis of A. We can now change the Jordan basis in such a way that a canonical form for H is obtained, using (58). We can do so by using S_{22} only, taking $S_{11}=I_{2n}$. Then $H_{12}^{(1)}=H_{12}S_{22}$. Take $S_{22}=Toep(s_1,\cdots,s_n)$, where each of the s_j's is a matrix in $\mathscr{E}$. Then S_{22} is completely determined by its last column, and $H_{12}^{(1)}=\left[h_{i,j}^{(1)}\right]_{i,j=1}^{n}$ is also determined by its last column. So in principle we can select a canonical form by specifying the last column of S_{22}, and solving the equation

$$H_{12}\begin{bmatrix} s_n \\ \vdots \\ s_1 \end{bmatrix}=\begin{bmatrix} h_{1,n}^{(1)} \\ \vdots \\ h_{n,n}^{(1)} \end{bmatrix}.$$

Note that by invertibility of H_{12} there is a unique solution.

This is a similar argument as in the case of two real non-unimodular eigenvalues.

Again, we wish to stay close to the canonical forms developed for the cases where the eigenvalues are on the unit circle. We shall take n conditions on the block entries in $H_{12}^{(1)}$: for n odd these conditions are

$$\begin{aligned} h_{i,n}&=0 \qquad\qquad \text{for } i=\frac{n+3}{2},\dots,n,\\ h_{\frac{n+3}{2},j}&=0 \qquad\qquad \text{for } j=\frac{n+3}{2},\dots,n,\\ h_{\frac{n+1}{2},\frac{n+1}{2}}&=K_1=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\\ h_{\frac{n+1}{2},\frac{n+3}{2}}&=-\tfrac{1}{2}K_1\gamma. \end{aligned}$$

Recall that we have already shown that each $h_{i,j}$ is in $\mathscr{K}$, motivating our choice for $h_{\frac{n+1}{2},\frac{n+1}{2}}$.

As an aside, let us compute what the latter two conditions imply for $h_{\frac{n+3}{2},\frac{n+1}{2}}$. By (57) we have

$$h_{\frac{n+3}{2},\frac{n+1}{2}}\gamma+h_{\frac{n+1}{2},\frac{n+3}{2}}\gamma^{-1}+h_{\frac{n+1}{2},\frac{n+1}{2}}=0,$$

which implies that $h_{\frac{n+3}{2},\frac{n+1}{2}}\gamma=-\frac{1}{2}K_1$. Hence $h_{\frac{n+3}{2},\frac{n+1}{2}}=-\frac{1}{2}K_1\gamma^{-1}$.

For n even the conditions are

$$\begin{aligned} h_{i,n}&=0 \quad \text{for } i=\frac{n}{2}+1,\dots,n,\\ h_{\frac{n}{2}+1,j}&=0 \quad \text{for } j=\frac{n}{2}+1,\dots,n,\\ h_{\frac{n}{2},\frac{n}{2}+1}&=K_1. \end{aligned}$$

Again as an aside, let us calculate what the last two of these conditions imply for $h_{\frac{n}{2}+1,\frac{n}{2}}$. By (57) we have

$$h_{\frac{n}{2}+1,\frac{n}{2}}\gamma+h_{\frac{n}{2},\frac{n}{2}+1}\gamma^{-1}+h_{\frac{n}{2},\frac{n}{2}}=0$$

which gives $h_{\frac{n}{2}+1,\frac{n}{2}}\gamma+K_1\gamma^{-1}=0$, so that $h_{\frac{n}{2}+1,\frac{n}{2}}=-K_1\gamma^{-2}$.

Note that in both cases n is odd and n is even there are indeed precisely n conditions which are linear in the entries of $H_{12}^{(1)}$, leading to n linear relations on $s_1,\ldots,s_n$, which, again, we do not state explicitly.

It remains to show that it is possible to choose $s_1,\ldots,s_n$ so that these conditions are satisfied. However, this is in principle easy: we just retrace what these conditions imply for the entries in the last column of $H_{12}^{(1)}$.

To make this argument precise, we first recall the following notations.

Let $n>1$ be an odd integer, then the $\frac{n+1}{2}\times\frac{n-1}{2}$ blok matrix $P_n(\lambda)$ with two by two matrix blocks is defined as follows:

$$P_n(\gamma)=\left[p_{ij}K_1\gamma^{\frac{n+1}{2}+j-i}\right]_{i=1,\ j=1}^{\frac{n+1}{2},\ \frac{n-1}{2}} \tag{59}$$

where p_{ij} are the entries of the matrix P_n introduced earlier.

Let $n>1$ be an even integer, then the $\frac{n}{2}\times\frac{n}{2}$ matrix $Q_n(\gamma)$ is defined as follows:

$$Q_n(\gamma)=\left[q_{ij}K_1\gamma^{\frac{n}{2}+j-i-1}\right]_{i=1,\ j=1}^{\frac{n}{2},\ \frac{n}{2}} \tag{60}$$

where q_{ij} are the entries of the matrix Q_n introduced earlier.

Recall also the definition of the matrix Z_n as the $\frac{n+1}{2}\times\frac{n+1}{2}$ matrix with zero entries everywhere except for a one in the right lower corner.

THEOREM 5.1. *Let A be H-unitary and $\sigma(A)=\{\lambda,\bar{\lambda},\lambda^{-1},\bar{\lambda}^{-1}\}$, with λ non-real and non-unimodular. Then the pair (A,H) can be decomposed as*

$$S^{-1}AS=\oplus_{l=1}^{p}A_l,\qquad S^THS=\oplus_{l=1}^{p}H_l,$$

where the pair (A_l,H_l) is of the form $\left(J_n(\gamma)\oplus J_n(\gamma^{-1}),\begin{bmatrix}0 & H_{12}\\ H_{12}^T & 0\end{bmatrix}\right)$, with n depending on l, and where H_{12} is of one of the following two forms, depending on whether n is odd or even:

Case 1 n is odd: $H_{12}=\begin{bmatrix}Z_n\otimes K_1 & P_n(\gamma)\\ P_n(\gamma^{-1})^T & 0\end{bmatrix}$.

Case 2 n is even: $H_{12}=\begin{bmatrix}0 & Q_n(\gamma)\\ -\gamma^{-2}Q_n(\gamma^{-1})^T & 0\end{bmatrix}$.

Proof. First observe that the entries in the matrices given in Case 1 and Case 2, respectively do satisfy (57). This is easily checked by induction for the upper right and lower left corners. The fact that these two match has been proved already by showing that $h_{\frac{n+3}{2},\frac{n+1}{2}}=-\frac{1}{2}K_1\gamma^{-1}$ for odd n, and that $h_{\frac{n}{2}+1,\frac{n}{2}}=-K_1\gamma^{-2}$ for even n.

Case 1. We continue the argument given before the definition of $P_n(\gamma)$. Let h_n be the last column of $P_n(\lambda)$, and let y be the vector in $\mathbb{R}^n$ given by $y = \begin{bmatrix} h_n \\ 0 \end{bmatrix}$. Solve $H_{12}s = y$ and set $S_{22} = Toep(s_1, \cdots, s_n)$, where s_j is the j-th block two by two matrix-coordinate of the block-vector s. Then by our arguments from before the proof we get that

$$H_{12}^{(1)} = H_{12}S_{22} = \begin{bmatrix} Z_n \otimes K_1 & P_n(\gamma) \\ P_n(\gamma^{-1})^T & 0 \end{bmatrix},$$

as desired.

Case 2 is proved in the same way. □

6. The complex case

The case where $H = H^*$ is an invertible complex Hermitian matrix, and A is a complex H-unitary matrix ($A^*HA = H$) has been studied using Cayley transform techniques in [5, 6, 13, 16]. In all these papers the goal is to bring H in as simple a form as possible, at the expense of losing the Jordan canonical form for A. As in the real case we shall treat here an approach that keeps A in Jordan canonical form, leading to a transparent canonical form for the pair (A, H).

The first observation we make is that now there are only two types of indecomposable blocks, instead of the multitude of cases we had to study in the real case: one with a pair of eigenvalues $\lambda, \bar{\lambda}^{-1}$ with $|\lambda| \neq 1$, where A is similar to $J_n(\lambda) \oplus J_n(\bar{\lambda}^{-1})$, while H is simultaneously congruent to $\begin{bmatrix} 0 & H_{12} \\ H_{12}^* & 0 \end{bmatrix}$, and the other one with a single eigenvalue λ on the unit circle, where A is similar to a single Jordan block $J_n(\lambda)$.

As in the real case, we shall treat these two cases separately. The final result is the following theorem.

THEOREM 6.1. *Let H be a complex Hermitian invertible matrix, and let A be H-unitary. Then the pair (A, H) can be decomposed as*

$$S^{-1}AS = \oplus_{l=1}^{p} A_l, \qquad S^*HS = \oplus_{l=1}^{p} H_l,$$

where the pairs (A_l, H_l) have one of the following forms with n depending on l

(i) $\sigma(A_l) = \{\lambda, \bar{\lambda}^{-1}\}$ *with* $|\lambda| \neq 1$, *and*

$$A_l = J_n(\lambda) \oplus J_n(\bar{\lambda}^{-1}) \qquad H_l = \begin{bmatrix} 0 & H_{12} \\ H_{12}^* & 0 \end{bmatrix},$$

where H_{12} has one of the following two forms depending on whether n is odd or even:

Case 1 $\quad H_{12} = \begin{bmatrix} Z_n & P_n(\bar{\lambda}) \\ P_n(\bar{\lambda}^{-1})^T & 0 \end{bmatrix}$ *when n is odd,* *Case* 2 $H_{12} = \begin{bmatrix} 0 & Q_n(\bar{\lambda}) \\ -\frac{1}{\bar{\lambda}^2} Q_n(\bar{\lambda}^{-1})^T & 0 \end{bmatrix}$ *when n is even.*

(ii) $\sigma(A_l) = \{\lambda\}$ *with* $|\lambda| = 1$*, and the pair* (A_l, H_l) *has one of the following two forms*

Case 1 $\left(J_n(\lambda), \varepsilon \begin{bmatrix} Z_n & P_n(\bar{\lambda}) \\ P_n(\lambda)^T & 0 \end{bmatrix}\right)$ *with* $\varepsilon = \pm 1$ *and n is odd,*

Case 2 $\left(J_n(\lambda), \varepsilon \begin{bmatrix} 0 & i\bar{\lambda} Q_n(\bar{\lambda}) \\ -i\lambda Q_n(\lambda)^T & 0 \end{bmatrix}\right)$ *with* $\varepsilon = \pm 1$ *and n is even.*

The columns of the matrix S in the theorem form a special Jordan basis for A.

To give an idea of how the matrix H looks when n is even in the canonical form in part (ii), consider the case $n = 8$, then

$$H = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i\bar{\lambda}^{-7} \\ 0 & 0 & 0 & 0 & 0 & 0 & i\bar{\lambda}^{-5} & -3i\bar{\lambda}^{-6} \\ 0 & 0 & 0 & 0 & 0 & -i\bar{\lambda}^{-3} & 2i\bar{\lambda}^{-4} & -3i\bar{\lambda}^{-5} \\ 0 & 0 & 0 & 0 & i\bar{\lambda} & -i\bar{\lambda}^{-2} & i\bar{\lambda}^{-3} & -i\bar{\lambda}^{-4} \\ 0 & 0 & 0 & -i\lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & i\lambda^3 & i\lambda^2 & 0 & 0 & 0 & 0 \\ 0 & -i\lambda^5 & -2i\lambda^4 & -i\lambda^3 & 0 & 0 & 0 & 0 \\ i\lambda^7 & 3i\lambda^6 & 3i\lambda^5 & i\lambda^4 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Proof. (i) From $A^*HA = H$, with $A = J_n(\lambda) \oplus J_n(\bar{\lambda}^{-1})$ and $H = \begin{bmatrix} 0 & H_{12} \\ H_{12}^* & 0 \end{bmatrix}$ we compute that

$$J_n(\bar{\lambda})^T H_{12} J_n(\bar{\lambda}^{-1}) = H_{12}. \tag{61}$$

Write $H_{12} = \left[h_{ij}\right]_{i,j=1}^n$. Then (61) means that for $i > 1, j > 1$

$$h_{i-1\,j-1} + h_{i-1\,j}\bar{\lambda}^{-1} + \bar{\lambda} h_{i\,j-1} = 0, \tag{62}$$

while $h_{1\,j} = 0$ for $j = 1, \cdots, n-1$ and $h_{i\,1} = 0$ for $i = 1, \cdots, n-1$.

Now compare (61) with (38), and (62) with (50), also compare (47) and (49) for the entries in the first row and column of H_{12}. Observe that the equations are exactly the same, replacing λ by $\bar{\lambda}$. The proof for the first part now runs exactly as in Section 4.

(ii) For the case where $\sigma(A) = \{\lambda\}$ with $|\lambda| = 1$, we first make an observation that holds independently of whether n is odd or n is even. Let $A = J_n(\lambda)$, then one easily computes that

$$(A^*HA)_{ij} = H_{i-1\,j-1} + \lambda H_{i-1\,j} + \bar{\lambda} H_{i\,j-1} + |\lambda|^2 H_{ij}.$$

Using $A^*HA = H$, and $|\lambda|^2 = 1$, we obtain

$$H_{i-1\,j-1} + \lambda H_{i-1\,j} + \bar{\lambda} H_{i\,j-1} = 0. \tag{63}$$

Also recall that we already know that $H_{ij} = 0$ whenever $i + j < n + 1$.

For n odd the entry $H_{\frac{n+1}{2}\,\frac{n+1}{2}}$ is a real number, because H is a Hermitian matrix. By scaling (which is in fact a similarity transformation with $S = s_1 I$) we can take that number to be either plus one or minus one. We denote that entry by ε, and put that in front of the matrix H, so that we may assume without loss of generality from now on $H_{\frac{n+1}{2}\,\frac{n+1}{2}} = 1$. Remaining with n odd, we then see from (63) that

$$1 + \lambda H_{\frac{n+1}{2}\,\frac{n+1}{2}+1} + \bar{\lambda} H_{\frac{n+1}{2}+1\,\frac{n+1}{2}} = 0.$$

In other words, $\mathrm{Re}\,(\lambda H_{\frac{n+1}{2}\,\frac{n+1}{2}+1}) = -\frac{1}{2}$, so that $H_{\frac{n+1}{2}\,\frac{n+1}{2}+1} = (-\frac{1}{2} + ir)\bar{\lambda}$ for some real number r.

For n even, (63) with $i - 1 = j - 1 = \frac{n}{2}$, using $H_{\frac{n}{2}\,\frac{n}{2}+1} = \bar{H}_{\frac{n}{2}+1\,\frac{n}{2}}$ (because of the fact that H is Hermitian), and $H_{\frac{n}{2}\,\frac{n}{2}} = 0$, we obtain $\mathrm{Re}\,(\lambda H_{\frac{n}{2}\,\frac{n}{2}+1}) = 0$. This means that $H_{\frac{n}{2}\,\frac{n}{2}+1} = \varepsilon i\bar{\lambda}(1 + r)$, where r is some real number, and $\varepsilon = \pm 1$. Again, we put ε in front of the matrix, so that we can take from now on $H_{\frac{n}{2}\,\frac{n}{2}+1} = i\bar{\lambda}(1 + r)$.

Next, the proof proceeds by constructing inductively a special Jordan basis for which the pair (A, H) has the form specified in part (ii) of the theorem with respect to this basis. Recall that changing one Jordan basis to another is equivalent to the transformation $(A, H) \to (S^{-1}AS, S^*HS)$ with an upper triangular Toeplitz S. In such an upper triangular Toeplitz matrix $S = toep(s_1, s_2, \cdots, s_n)$ we have n numbers at our disposal for the construction.

We shall use these in the case n is odd to make $H_{\frac{n+1}{2}\,\frac{n+1}{2}} = \varepsilon$, $H_{\frac{n+1}{2}\,\frac{n+1}{2}+1} = -\frac{1}{2}\bar{\lambda}$, and then for $j = \frac{n+1}{2} + 1, \cdots, n$ to make $H_{jj} = 0$ and for $j = \frac{n+1}{2} + 1, \cdots, n - 1$ to make $H_{j\,j+1} = 0$. Observe that these are precisely n entries in the matrix H, and that together with (63) these choices determine H completely. Note that in fact the first step has already been taken when we scaled $H_{\frac{n+1}{2}\,\frac{n+1}{2}}$ to ε, using s_1.

For n even, we use these n degrees of freedom in the choice of S to make $H_{\frac{n}{2}\,\frac{n}{2}+1} = i\bar{\lambda}$, for $j = \frac{n}{2} + 1, \cdots n$ to make $H_{jj} = 0$, and for $j = \frac{n}{2} + 1, \cdots n - 1$ to make $H_{j\,j+1} = 0$. Again, observe that these are precisely n entries in the matrix H, and that together with (63) they determine H completely.

From now on we shall stay with n is even until further notice. Let $S_1 = s_1 I$, and consider $H^{(1)} = S_1^* H S_1$. Then for all i, j we have

$$H^{(1)}_{ij} = |s_1|^2 H_{ij}.$$

In particular, $H^{(1)}_{\frac{n}{2}\,\frac{n}{2}+1} = |s_1|^2 i\bar{\lambda}(1 + r)$. Taking $s_1 = \frac{1}{\sqrt{1+r}}$ we obtain $H^{(1)}_{\frac{n}{2}\,\frac{n}{2}+1} = i\bar{\lambda}$ as desired. This fixes all entries $H^{(1)}_{ij}$ with $i + j \leqslant n + 1$ via (63).

Next, take $S_2 = toep(1, s_2, 0, \cdots, 0)$ and consider $H^{(2)} = S_2^* H^{(1)} S_2$. Then one computes that

$$H^{(2)}_{i\,j} = |s_2|^2 H^{(1)}_{i-1\,j-1} + \bar{s}_2 H^{(1)}_{i-1\,j} + s_2 H^{(1)}_{i\,j-1} + H^{(1)}_{i\,j}.$$

Observe that for $i + j \leqslant n+1$ we have $H^{(2)}_{i\,j} = H^{(1)}_{i\,j}$, in particular this holds for $i = \frac{n}{2}, j = \frac{n}{2}+1$. We shall choose s_2 so that $H^{(2)}_{\frac{n}{2}+1\,\frac{n}{2}+1} = 0$. Indeed, using the fact that $H^{(1)}$ is Hermitian

$$H^{(2)}_{\frac{n}{2}+1\,\frac{n}{2}+1} = 2\mathrm{Re}\,(s_2 H^{(1)}_{\frac{n}{2}+1\,\frac{n}{2}}) + H^{(1)}_{\frac{n}{2}+1\,\frac{n}{2}+1} = 2\mathrm{Re}\,(-s_2 i\lambda) + H^{(1)}_{\frac{n}{2}+1\,\frac{n}{2}+1}.$$

Obviously, this can be made zero by an appropriate choice of s_2 (to be precise, we may take $s_2 = -\frac{1}{2} i\bar{\lambda} H^{(1)}_{\frac{n}{2}+1\,\frac{n}{2}+1}$).

Having $H^{(2)}_{\frac{n}{2}+1\,\frac{n}{2}+1} = 0$ also fixes $H^{(2)}_{\frac{n}{2}\,\frac{n}{2}+2}$ by (63) and $H^{(2)}_{\frac{n}{2}\,\frac{n}{2}+1} = i\bar{\lambda}$, in fact we will have

$$H^{(2)}_{\frac{n}{2}\,\frac{n}{2}+2} = -i\bar{\lambda}^{2}.$$

This now fixes all entries $H^{(2)}_{i\,j}$ with $i + j \leqslant n+2$ via (63).

In the next step we take $S_3 = toep(1, 0, s_3, 0, \cdots, 0)$ and consider $H^{(3)} = S_3^* H^{(2)} S_3$. We have

$$H^{(3)}_{i\,j} = |s_3|^2 H^{(2)}_{i-2\,j-2} + \bar{s}_3 H^{(2)}_{i-2\,j} + s_3 H^{(2)}_{i\,j-2} + H^{(2)}_{i\,j}.$$

Observe that for $i + j \leqslant n+2$ we have $H^{(3)}_{i\,j} = H^{(2)}_{i\,j}$, so that entries fixed in the previous steps retain their values. We shall use s_3 to consider $H^{(3)}_{\frac{n}{2}+1\,\frac{n}{2}+2}$, and show that it is possible to choose s_3 so that this is zero.

$$H^{(3)}_{\frac{n}{2}+1\,\frac{n}{2}+2} = |s_3|^2 H^{(2)}_{\frac{n}{2}-1\,\frac{n}{2}} + \bar{s}_3 H^{(2)}_{\frac{n}{2}-1\,\frac{n}{2}+2} + s_3 H^{(2)}_{\frac{n}{2}+1\,\frac{n}{2}} + H^{(2)}_{\frac{n}{2}+1\,\frac{n}{2}+2}.$$

The first of the four terms on the right hand side is zero, $H^{(2)}_{\frac{n}{2}-1\,\frac{n}{2}+2} = -i\bar{\lambda}^{3}$ (using (63) this is easily computed), and $H^{(2)}_{\frac{n}{2}+1\,\frac{n}{2}} = -i\lambda$. So the equation above becomes

$$H^{(3)}_{\frac{n}{2}+1\,\frac{n}{2}+2} = -i\bar{s}_3\bar{\lambda}^{3} - s_3 i\lambda + H^{(2)}_{\frac{n}{2}+1\,\frac{n}{2}+2}.$$

It is readily seen that s_3 may be chosen so that this becomes zero (e.g., setting this to zero and mutiplying the resulting equation with λ gives $-2i\mathrm{Re}\,(\lambda^2 s_3) + \lambda H^{(2)}_{\frac{n}{2}+1\,\frac{n}{2}+2} = 0$, which is solvable). Having $H^{(3)}_{\frac{n}{2}+1\,\frac{n}{2}+2} = 0$, and all the entries with $i + j \leqslant n+2$ fixed as well in (63) determines all entries $H^{(3)}_{i\,j}$ with $i + j \leqslant n+3$. In particular, $H^{(3)}_{\frac{n}{2}\,\frac{n}{2}+3}$ can now be computed to be equal to $i\bar{\lambda}^{3}$.

Next, consider $S_4 = toep(1, 0, 0, s_4, 0, \cdots, 0)$, and put $H^{(4)} = S_4^* H^{(3)} S_4$. As in the previous steps one checks that

$$H^{(4)}_{i\,j} = |s_4|^2 H^{(3)}_{i-3\,j-3} + \bar{s}_4 H^{(3)}_{i-3\,j} + s_4 H^{(3)}_{i\,j-3} + H^{(3)}_{i\,j},$$

and in particular it follows from this that $H^{(4)}_{ij} = H^{(3)}_{ij}$ for $i+j \leqslant n+3$. We use s_4 to make $H^{(4)}_{\frac{n}{2}+2\,\frac{n}{2}+2} = 0$. Indeed,

$$H^{(4)}_{\frac{n}{2}+2\,\frac{n}{2}+2} = |s_4|^2 H^{(3)}_{\frac{n}{2}-1\,\frac{n}{2}-1} + \bar{s}_4 H^{(3)}_{\frac{n}{2}-1\,\frac{n}{2}+2} + s_4 H^{(3)}_{\frac{n}{2}+2\,\frac{n}{2}-1} + H^{(3)}_{\frac{n}{2}+2\,\frac{n}{2}+2}.$$

The first term being zero again, this becomes, using $H^{(3)}_{\frac{n}{2}-1\,\frac{n}{2}+2} = -i\bar{\lambda}^{3}$,

$$H^{(4)}_{\frac{n}{2}+2\,\frac{n}{2}+2} = s_4 i\lambda^3 - \bar{s}_4 i\bar{\lambda}^{3} + H^{(3)}_{\frac{n}{2}+2\,\frac{n}{2}+2} = 2\mathrm{Re}\,(s_4 i\lambda^3) + H^{(3)}_{\frac{n}{2}+2\,\frac{n}{2}+2},$$

and it is easily seen that s_4 may be chosen so that this is zero. Using (63) this determines all entries $H^{(4)}_{ij}$ with $i+j \leqslant n+4$. In particular $H^{(4)}_{\frac{n}{2}+1\,\frac{n}{2}+3} = 0$, and $H^{(4)}_{\frac{n}{2}\,\frac{n}{2}+4} = -i\bar{\lambda}^{4}$.

Continuing in this way results in the form described in the theorem, part (ii), Case 2.

Finally, we consider the case where n is odd. In this case, we consider $S_2 = toep(1, s_2, 0, \cdots, 0)$ as above, and put $H^{(2)} = S_2^* H^{(1)} S_2$, assuming that $H^{(1)} = |s_1|^2 H$ already has been modified to have 1 in the central entry (after pulling out $\varepsilon = \pm 1$ in front of the matrix). We shall select s_2 so that $H^{(2)}_{\frac{n+1}{2}\,\frac{n+1}{2}+1} = -\frac{1}{2}\bar{\lambda}$, in agreement with the statement of the theorem. Indeed, as in the case where n is even we have

$$H^{(2)}_{ij} = |s_2|^2 H^{(1)}_{i-1\,j-1} + \bar{s}_2 H^{(1)}_{i-1\,j} + s_2 H^{(1)}_{i\,j-1} + H^{(1)}_{ij}.$$

For $i = \frac{n+1}{2}$, $j = \frac{n+1}{2}+1$ this amounts to

$$H^{(2)}_{\frac{n+1}{2}\,\frac{n+1}{2}+1} = |s_2|^2 H^{(1)}_{\frac{n+1}{2}-1\,\frac{n+1}{2}} + \bar{s}_2 H^{(1)}_{\frac{n+1}{2}-1\,\frac{n+1}{2}+1} + s_2 H^{(1)}_{\frac{n+1}{2}\,\frac{n+1}{2}} + H^{(1)}_{\frac{n+1}{2}\,\frac{n+1}{2}+1}.$$

The first of these four terms is zero, for the second we obtain from (63) that $H^{(1)}_{\frac{n+1}{2}-1\,\frac{n+1}{2}+1}$ $= -\bar{\lambda}^{2} H^{(1)}_{\frac{n+1}{2}\,\frac{n+1}{2}} = -\bar{\lambda}^{2}$. So the equation becomes

$$H^{(2)}_{\frac{n+1}{2}\,\frac{n+1}{2}+1} = -\bar{s}_2\bar{\lambda}^{2} + s_2 + H^{(1)}_{\frac{n+1}{2}\,\frac{n+1}{2}+1}.$$

Recall that for some real number r we have that $H^{(1)}_{\frac{n+1}{2}\,\frac{n+1}{2}+1} = (-\frac{1}{2}+ir)\bar{\lambda}$, so that we arrive at

$$H^{(2)}_{\frac{n+1}{2}\,\frac{n+1}{2}+1} = -\bar{s}_2\bar{\lambda}^{2} + s_2 + \left(-\frac{1}{2}+ir\right)\bar{\lambda}.$$

This is equal to $-\frac{1}{2}\bar{\lambda}$ if and only if

$$-\bar{s}_2\bar{\lambda}^{2} + s_2 + ir\bar{\lambda} = 0.$$

Multiplying this equation by λ we see that this in turn becomes equivalent to $2\mathrm{Im}\,(s_2\lambda) + r = 0$, which is solved by taking for instance $s_2 = -\frac{1}{2}i\bar{\lambda}r$.

Next, take again $S_3 = toep(1,0,s_3,0,\cdots,0)$ and consider $H^{(3)} = S_3^* H^{(2)} S_3$. Choose s_3 so that $H^{(3)}_{\frac{n+1}{2}+1\, \frac{n+1}{2}+1} = 0$. Without going into details: this can be done. Continue by taking $S_4 = toep(1,0,0,s_4,0,\cdots,0)$, and considering $H^{(4)} = S_4^* H^{(3)} S_4$. Choose s_4 so that $H^{(4)}_{\frac{n+1}{2}+1\, \frac{n+1}{2}+2} = 0$. Now take S_5 in the obvious way, define $H^{(5)}$ as usual, and prove s_5 can be chosen such that $H^{(5)}_{\frac{n+1}{2}+2\, \frac{n+1}{2}+2} = 0$. Continue in this way proving that the entries along the main diagonal and the one above it are zero. In that case by (63) the entries of the final H will be given as in the theorem part (ii), Case 1. $\square$

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