Operators
and
Matrices
Volume 10, Number 4 (2016), 785–828
doi:10.7153/oam-10-47

MEROMORPHIC MATRIX TRIVIALIZATIONS OF FACTORS OF AUTOMORPHY OVER A RIEMANN SURFACE

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In memory of Leiba Rodman,
a dear friend and dedicated colleague

(*Communicated by A. Böttcher*)

Abstract. It is a consequence of the Jacobi Inversion Theorem that a line bundle over a Riemann surface M of genus g has a meromorphic section having at most g poles, or equivalently, the divisor class of a divisor over M contains a divisor having at most g poles (counting multiplicities). We explore various analogues of these ideas for vector bundles and associated matrix divisors over M. The most explicit results are for the genus 1 case. We also review and improve earlier results concerning the construction of automorphic or relatively automorphic meromorphic matrix functions having a prescribed null/pole structure.

1. Introduction

Let M be a closed Riemann surface of genus $g \geqslant 1$ and $\rho : \widehat{M} \to M$ the universal cover with $\mathscr{G}$ the group of covering transformations. The collection of equivalence classes of rank r holomorphic vector bundles over M is equivalent to the collection of equivalence classes of rank r holomorphic factors of automorphy on $\widehat{M}$. Recall that a rank r holomorphic factor of automorphy on $\widehat{M}$ is a map $\zeta : \mathscr{G} \times \widehat{M} \to GL(r,\mathbb{C})$ which is holomorphic on $\widehat{M}$ for T fixed in $\mathscr{G}$ and satisfies

$$\zeta(ST,u) = \zeta(S,Tu)\zeta(T,u) \text{ for } S,T \in \mathscr{G} \text{ and } u \in \widehat{M}.$$

Two rank r factors of automorphy ζ and η are holomorphically equivalent in case there is a holomorphic function $h : \widehat{M} \to GL(r,\mathbb{C})$ satisfying

$$h(Tu)\zeta(T,u) = \eta(T,u)h(u) \text{ for } T \in \mathscr{G} \text{ and } u \in \widehat{M}.$$

A rank r factor of automorphy that is given by a representation $\zeta : \mathscr{G} \to GL(r,\mathbb{C})$ is called a rank r flat factor of automorphy.

Mathematics subject classification (2010): 14D20, 30F10, 30F35, 47A48.

Keywords and phrases: holomorphic vector bundle, factor of automorphy, zero/pole interpolation problems, Abel-Jacobi map, theta functions, transfer-function realization.

Every rank r factor of automorphy ζ on $\widehat{M}$ is meromorphically trivial in the sense that there is an $r \times r$-meromorphic matrix function $\widehat{F}$ on $\widehat{M}$, nondegenerate in the sense that $\det \widehat{F} \not\equiv 0$, such that

$$\widehat{F}(Tu) = \zeta(T,u)\widehat{F}(u), T \in \mathscr{G}, u \in \widehat{M}. \tag{1.1}$$

A nondegenerate $r \times r$-meromorphic matrix function $\widehat{F}$ satisfying (1.1) will be called a meromorphic matrix trivialization, or simply a trivialization, of ζ. When (1.1) holds, we also say that $\widehat{F}$ is a *relatively automorphic meromorphic matrix function* on $\widehat{M}$ (with respect to the factor of automorphy ζ). In case the factor of automorphic is trivial (i.e., $\zeta(T,u) = I_r$ for all T and u), we say simply that F is an *automorphic meromorphic matrix function*. If F is a meromorphic matrix trivialization of ζ, then any other meromorphic matrix trivialization of ζ will have the form FK, where K is a nondegenerate $r \times r$-automorphic meromorphic matrix function on $\widehat{M}$, or what amounts to the same, a (single-valued) global meromorphic function on M. Equivalently, the columns of a meromorphic matrix trivialization form a basis for the space of meromorphic sections of ζ over the field of meromorphic functions on M.

In the scalar case ($r = 1$), there is a well-known correspondence between three types of objects: (1) factors of automorphy (flat or general) on $\widehat{M}$, (2) holomorphic line bundles on M, and (3) divisors (i.e., pole/zero multiplicity specifications) on M. While to some extent the analogue of this structure is understood for the higher rank case (see in particular [19, 18]), one goal of this paper is to delineate more concretely the higher rank analogue of this correspondence between (1) (left) matrix factors of automorphy on $\widehat{M}$, (2) holomorphic (row) vector bundles on M, and (3) (right) matrix null-pole divisors (or right nondegenerate matrix germs) on M. In particular, we continue the work of [4] by using the parametrization of a null-pole subspace from [6] to get a more explicit analogue of a divisor (i.e., an encoding of zero and pole data) for the higher rank case.

If one starts with a divisor $\mathscr{D}$ (i.e., an encoding of pole/zero structure which amounts to simply pole/zero multiplicity in the scalar case – precise definitions can be found in Section 4 below), one can pose three basic Interpolation Problems:

(I) **First Interpolation Problem:** Find an automorphic meromorphic matrix function F on M (i.e., an automorphic meromorphic matrix function $\widehat{F}$ on the covering surface $\widehat{M}$ with trivial factor of automorphy) with right null/pole structure prescribed by $\mathscr{D}$;

(II) **Second Interpolation Problem:** Find a relatively automorphic meromorphic matrix function $\widehat{F}$ on $\widehat{M}$ with flat (left) factor of automorphy and with right null/pole structure on M prescribed by $\mathscr{D}$;

(III) **Third Interpolation Problem:** Find a relatively automorphic meromorphic matrix function $\widehat{F}$ on $\widehat{M}$ (with not necessarily flat factor of automorphy) with right null/pole structure on M prescribed by $\mathscr{D}$.

Problem (I) was addressed in [4] for the case of "simple" null-pole structure (for precise definitions see Section 3 below) while Problem (II) was addressed in [5]. Here we revisit Problems (I) and (II) and complete the results obtained there to a more satisfactory form (see Theorem 4.10 below). We also make explicit the connections with the solution of a variant of Problem (I) studied in [10]. It turns out that Problem (III) always has a solution; we give a simple direct proof of this result based on the theorem of Grauert [17] guaranteeing the triviality of any holomorphic vector bundle over a simply-connected domain (see the end of Section 4.1 below).

In the line bundle case ($r = 1$), a well known consequence of Abel-Jacobi theory is that one can find a meromorphic trivialization F of a flat line bundle having a divisor with total zero multiplicity equal to the pole multiplicity with multiplicity value equal to at most g (the genus of M). One purpose of this paper is to find an analogue of this result for the vector bundle case, i.e., to analyze to what extent it is possible to find a meromorphic trivialization F of a given flat matrix factor of automorphy so that the matrix null-pole structure is as simple as possible (see Theorem 4.8 and Corollary 4.9 below).

In Section 2 we present two explicit trivializations of a flat factor of automorphy over an elliptic curve M (i.e., Riemann surface of genus 1) when the curve is presented in the form $M = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$. In Section 3, we demonstrate how to construct trivializations of a flat factor of automorphy on an elliptic curve that has simple null-pole structure while maintaining a count on the number of poles and zeros in such trivializations. These results use the fact that the uniformization of M in the elliptic case is the complex plane $\mathbb{C}$ where one can make use of theta functions (including theta functions with matrix arguments) to give explicit constructions of relatively automorphic functions on $\widehat{M} = \mathbb{C}$.

For generalizations to higher genus (the setting of Section 4), there are two distinct approaches which coalesce to a single approach for the case of genus g equal to 1.

- The first approach uses the Abel-Jacobi map to embed the Riemann surface M into the Jacobian variety $\mathbb{C}^g/\Lambda$ (where Λ is the period lattice). The universal cover is thereby embedded into $\mathbb{C}^g$ where one can use the function theory of $\mathbb{C}^g$ (in particular theta functions in g variables). We use this approach in Subsection 4.4 to obtain explicit formulas for the canonical functions used earlier in Section 4 to adapt the theory of null/pole subspaces from [6] to the Riemann-surface setting. From these explicit formulas one can read off continuous-dependence results which are needed to guarantee that a certain bundle-trivialization procedure leads to simple pole structure (Theorem 4.9 below).

- The second approach works directly with the uniformization $\widehat{M}$ which can be taken to be either the unit disk $\mathbb{D}$ or the upper or right half plane $\mathbb{H}$ contained in $\mathbb{C}$, with the group of deck transformations concretely identified with a Fuchsian group of linear-fractional transformations on $\widehat{M}$. The analogue of a theta function (constructed by adding up over all terms obtained via the action of a group of Möbius transformations) is what is called a Poincaré series (see e.g. [3, 11, 12, 15]). It would be of interest to find higher-genus analogues of the genus-1 theta-function constructions in Sections 2 and 3 by using this second approach.

In the first approach an explicit description of the universal cover (an embedded submanifold in $\mathbb{C}^g$) is somewhat cumbersome while the group of deck transformations is simply the group of translations by elements of the period lattice; in the second approach the universal cover is simple but the group of deck transformations (linear-fractional maps rather than translations) is less explicit.

The present study can be viewed as the latest installment in developing the higher-genus analogue of the theory of interpolation for rational matrix functions (meromorphic matrix functions on the Riemann sphere $\mathbb{C} \cup \infty$, the essentially unique compact Riemann surface of genus $g = 0$), a topic developed in much of the work of Leiba Rodman, especially in the monograph [6]. It is with sadness and respect that we dedicate this paper to our dear friend and collaborator Leiba Rodman.

2. Explicit trivializations of flat bundles when $g = 1$

In this section, we describe two natural meromorphic matrix trivializations of a rank r flat factors of automorphy over an elliptic curve. In spite of the natural forms of these trivializations, the associated meromorphic matrix functions do not have simple null-pole divisor structure. This complication is addressed in the next section.

I. Theta function trivializations: Let M be an elliptic curve presented in the form $M = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$, where as is customary $\operatorname{Im} \tau > 0$. The universal cover for M can be taken to be $\widehat{M} = \mathbb{C}$ and the group of deck transformations can be identified with $G = \mathbb{Z} + \tau\mathbb{Z}$ in a natural way. Then a rank r flat vector bundle ζ over M as discussed in the Introduction corresponds to a representation $\zeta : \mathbb{Z} + \tau\mathbb{Z} \to GL(r, \mathbb{C})$. A flat factor of automorphy ζ is equivalent to a normalized flat factor of automorphy ξ_V where the representation has the form

$$\xi_V(1) = I_r \ : \ \xi_V(\tau) = V, \tag{2.1}$$

with V an invertible $r \times r$-matrix. The form of V will be further simplified below. Meromorphic matrix trivializations of ξ_V correspond to $r \times r$-meromorphic matrix functions F on $\mathbb{C}$ that satisfy

$$F(u + m + n\tau) = \xi_V(m + n\tau)F(u) = V^n F(u).$$

The scalar case: As is well known, in the case $r = 1$, meromorphic trivializations of flat factors of automorphy on $\mathbb{C}$ (or, equivalently, line bundles) can be given explicitly using theta functions. To see this, for $\alpha \neq 0$, let ξ_α be the degree-zero line bundle corresponding to the character $\xi_\alpha(m + \tau n) = \alpha^n$ and let θ be the classical theta function defined for $u \in \mathbb{C}$ by

$$\theta(u) = \sum_{n \in \mathbb{Z}} \exp 2\pi i \left(\frac{n^2 \tau}{2} + nu \right). \tag{2.2}$$

The function θ has the automorphic behavior

$$\theta(u + 1) = \theta(u) \ : \ \theta(u + \tau) = \exp(-\pi i \tau - 2\pi i u)\theta(u).$$

As a consequence, when $\alpha \neq 1$, the function

$$f_\alpha(u) = \frac{\theta(u - l_\alpha)}{\theta(u)}, \tag{2.3}$$

where $l_\alpha = \frac{\log \alpha}{2\pi i}$, provides a meromorphic trivialization of the factor of automorphy ξ_α. The function f_α has a simple zero at points in $\Delta - l_\alpha + \mathbb{Z} + \tau\mathbb{Z}$ and simple poles at points in $\Delta + \mathbb{Z} + \tau\mathbb{Z}$, where $\Delta = \frac{1}{2} + \frac{1}{2}\tau$. In the case $\alpha = 1$, nonzero meromorphic functions on M provide trivializations. In particular, if

$$z_1, \cdots, z_N \; : \; w_1 \cdots, w_N$$

are $2N$ points on M with

$$z_1 + \cdots + z_N - w_1 - \cdots - w_N \equiv 0 \mod (\mathbb{Z} + \tau\mathbb{Z}),$$

then when considered on $\mathbb{C}$ the function

$$f(u) = \prod_{i=1}^{N} \frac{\theta(u + \Delta - z_i)}{\theta(u + \Delta - w_i)}$$

provides a trivialization of ξ_1 with zeros and poles at points over the divisor (f) given on M by

$$(f) = z_1 + \cdots + z_N - w_1 - \cdots - w_N.$$

The matrix case: Less well known is that one can use theta functions with matrix arguments to obtain trivializations of flat matrix factors of automorphy over $M = \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$ that is analogous to the line bundle case as follows.

For A an $r \times r$-matrix, define the matrix-valued theta function on $\mathbb{C}$ by

$$\Theta_A(u) = \theta(uI_r + A) = \sum_{n \in \mathbb{Z}} \exp 2\pi i \left(\frac{n^2\tau}{2} I_r + n(uI_r + A) \right). \tag{2.4}$$

The function Θ_A has the following automorphic behavior:

$$\Theta_A(u+1) = \Theta_A(u) \; : \; \Theta_A(u+\tau) = \exp(-2\pi i A)\exp(-\pi i \tau - 2\pi i u)\Theta_A(u).$$

As a result the meromorphic matrix function

$$\overline{\Theta}_A = (\theta)^{-1}\Theta_A \tag{2.5}$$

is a trivialization of the rank r flat factor of automorphy ξ_A corresponding to the representation of $\mathbb{Z} + \tau\mathbb{Z}$ given by

$$\xi_A(1) = I_r \; : \; \xi_A(\tau) = \exp(-2\pi i A). \tag{2.6}$$

If one sets $V = \exp(-2\pi i A)$, then $G_V = \overline{\Theta}_A$ is a trivialization of the factor of automorphy ξ_V corresponding to the representation (2.1). Note with these identifications,

$$G_V(u) = \theta^{-1}(u) \sum_{n \in \mathbb{Z}} V^{-n} \exp 2\pi i \left(\frac{n^2\tau}{2} + nu \right). \tag{2.7}$$

The representation ξ_V can be assumed to be in the form

$$\xi_J(1) = I_r \ : \ \xi_J(\tau) = J, \tag{2.8}$$

where the matrix J is in Jordan canonical form. In particular, J is a direct sum of matrices of the form

$$J_\alpha = \begin{bmatrix} \alpha & 1 & & \\ & \ddots & \ddots & \\ & & \alpha & 1 \\ & & & \alpha \end{bmatrix} \tag{2.9}$$

(with unspecified entries equal to 0) where $\alpha \neq 0$.

Providing trivializations of flat factors of automorphy ξ on $\mathbb{C}$ can be accomplished by providing trivializations of the representations of $\mathbb{Z} + \tau\mathbb{Z}$ corresponding to irreducible factors, i.e., to trivializations of ξ_{J_α} where J_α is the $r \times r$-matrix given by (2.9). It should be noted that the factor of automorphy ξ_{J_1} corresponds to the unique equivalence class of indecomposable rank r vector bundles of degree 0 on M that have a holomorphic section. These bundles form the basic building blocks used by Atiyah in the classic paper [2] describing vector bundles over an elliptic curve.

The trivialization G_{J_α} given by (2.7) can be expressed explicitly as follows. Write $J_\alpha = \alpha I + Q$, where Q is the $r \times r$ nilpotent matrix

$$Q = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix}$$

Using the binomial expansion for $(\alpha + x)^{-n}$, one obtains

$$\begin{aligned} J_\alpha^{-n} &= (\alpha I + Q)^{-n} = \alpha^{-n}(I - (-\alpha^{-1}Q))^{-n} \\ &= \sum_{j=0}^{r-1} \frac{(-1)^j n(n+1)\cdots(n+j-1)}{\alpha^{n+j} j!} Q^j. \end{aligned}$$

Substitution of this expression for J_α^{-n} as V^{-n} into (2.7) combined with an interchange of the order of summation then gives $G_{J_\alpha}(u) = (\theta(u))^{-1} \cdot \widetilde{G}_{J_\alpha}(u)$ where

$$\widetilde{G}_{J_\alpha}(u) = \sum_{j=0}^{r-1} \frac{(-1)^j}{\alpha^j j!} \left(\sum_{n \in \mathbb{Z}} n(n+1)\cdots(n+j-1) \exp 2\pi i \left(\frac{n^2\tau}{2} + n(u - l_\alpha) \right) \right) Q^j, \tag{2.10}$$

where $l_\alpha = \frac{\log \alpha}{2\pi i}$.

For $j = 0, \ldots, r-1$, introduce the differential operators L_j, with $L_0 = I$ and

$$L_j = \frac{(-1)^j}{\alpha^j j!} D'(D'+1)\cdots(D'+j-1)$$

where $D' = \frac{1}{2\pi i}\frac{d}{du}$. Since

$$L_j \exp(2\pi i n u) = \frac{(-1)^j n(n+1)\cdots(n+j-1)}{\alpha^j j!}\exp(2\pi i n u)$$

one concludes that

$$G_{J_\alpha}(u) = (\theta)^{-1}(u)\sum_{j=0}^{r-1}(L_j(D')\theta)(u-l_\alpha)Q^j$$

or, equivalently,

$$G_{J_\alpha}(u) = (\theta)^{-1}(u)\begin{bmatrix} \theta & L_1[\theta] & L_2[\theta] & \cdots & L_{r-1}[\theta] \\ & \theta & L_1[\theta] & \ddots & \vdots \\ & & \ddots & \ddots & L_2[\theta] \\ & & & \ddots & L_1[\theta] \\ & & & & \theta \end{bmatrix}(u-l_\alpha). \tag{2.11}$$

The determinant of the trivialization G_{J_α} of ξ_{J_α} has a zero of order r at points in $\Delta + l_\alpha + \mathbb{Z} + \tau\mathbb{Z}$ and a pole of order r at points in $\Delta + \mathbb{Z} + \tau\mathbb{Z}$. These zeros and poles correspond to the simple zeros and poles of the diagonal elements at these points. Any other trivialization of ξ_α will have the form $G_\alpha K$, where K is a nondegenerate meromorphic $r \times r$-matrix function on M.

II. Single pole trivializations: Another natural trivialization of the rank r vector bundle ξ_{J_α} can be constructed as follows. For $a \in \mathbb{C}$, let

$$\lambda_a(u) = -\frac{1}{2\pi i}\frac{\theta'(u-a-\frac{1}{2}-\frac{\tau}{2})}{\theta(u-a-\frac{1}{2}-\frac{\tau}{2})},\ u \in \mathbb{C}.$$

The salient properties of λ_a are that the poles of this function are simple poles at points in $a + \mathbb{Z} + \tau\mathbb{Z}$, and that

$$\lambda_a(u+1) - \lambda_a(u) = 0, \quad \lambda_a(u+\tau) - \lambda_a(u) = 1. \tag{2.12}$$

Note that λ_a is a (multiple of) the translate $\lambda_a(u) = \zeta(u-a)$ of the Weierstrass function

$$\zeta(u) = \frac{1}{u} + \sum_{(m,n)\neq(0,0)}\left(\frac{1}{u-\Omega_{m,n}} + \frac{1}{\Omega_{m,n}} + \frac{u}{\Omega_{m,n}^2}\right),$$

where $\Omega_{m,n} = m + \tau n$ (see e.g. Problem #5 page 279 and Problem #1 page 309 in [25]).

For fixed $\alpha \neq 0$ we will use a sequence of polynomials $\{p_n\}_{n\geqslant 0}$ with p_n of degree n, $p_0(u) = 1$, which satisfy

$$p_{n+1}(u+1) - p_{n+1}(u) = \alpha^{-1}p_n(u),\ u \in \mathbb{C}. \tag{2.13}$$

Such a sequence of polynomials is uniquely determined if one requires that $p_n(0) = 0$ for $n \geqslant 1$, and $p_0 = 1$. These polynomials are then given by

$$p_n(u) = \frac{\alpha^{-n}}{n!}(u-(n-1))\cdots(u-2)(u-1)u,\ n \geqslant 1. \tag{2.14}$$

Note that the relations (2.12) satisfied by λ_a implies that the composite functions $p_n \circ \lambda_a$ satisfy the relations

$$\begin{aligned} p_{n+1}(\lambda_a(u+1)) &= p_n(\lambda_a(u)),\\ p_{n+1}(\lambda_a(u+\tau)) &= p_{n+1}(\lambda_a(u)) + \alpha^{-1}p_n(\lambda_a(u)). \end{aligned} \tag{2.15}$$

If we define the $r\times r$-matrix function $P_r(\lambda)$ by

$$P_r(\lambda) = \begin{bmatrix} p_0(\lambda) & p_1(\lambda) & p_2(\lambda) & \cdots & p_{r-1}(\lambda) \\ & p_0(\lambda) & p_1(\lambda) & \ddots & \vdots \\ & & \ddots & \ddots & p_2(\lambda) \\ & & & p_0(\lambda) & p_1(\lambda) \\ & & & & p_0(\lambda) \end{bmatrix},$$

then, for $a \in \mathbb{C}$ and $\alpha \neq 0$ fixed, it is easily seen from the relations (2.15) that the meromorphic matrix function

$$G_r(u) = P_r(\lambda_a(u)) \tag{2.16}$$

satisfies

$$G_r(u+1) = G_r(u), \quad G_r(u+\tau) = \alpha^{-1}J_\alpha G_r(u),$$

and hence, more generally,

$$G_r(u+m+n\tau) = \alpha^{-n}\xi_{J_\alpha}(m+n\tau)G_r(u).$$

Thus any trivialization F_r of ξ_{J_α} will have the form

$$F_r = G_r S_r \tag{2.17}$$

where S_r is a nondegenerate $r\times r$-meromorphic matrix function satisfying

$$S_r(u+m+n\tau) = \alpha^n S_r(u). \tag{2.18}$$

When $\alpha = 1$, the matrix function G_r given by (2.16) already provides a trivialization of ξ_{J_1}. When $\alpha \neq 1$ a trivialization of ξ_{J_α} can be given by F_r of the form (2.17), where S_r is a diagonal meromorphic matrix function on $\mathbb{C}$ with diagonal entries $s_{11},\dots,s_{rr}$ satisfying $s_{ii}(u+m+n\tau) = \alpha^n s_{ii}(u)$, $i = 1,\dots,r$. These trivializations of ξ_{J_α} (when $r > 2$) have high order poles at points in $a+\mathbb{Z}+\tau\mathbb{Z}$.

3. Meromorphic trivializations with simple null-pole structure when $g = 1$

As mentioned in the Introduction, one of the goals of the present paper is to find trivializations of flat factors of automorphy on $\widehat{M}$ which have simple null-pole structure. Throughout this section we again assume that M is an elliptic curve (i.e., M has genus 1).

Suppose that the meromorphic matrix function F is a trivialization of the flat factor of automorphy ξ. The condition that F have *simple null-pole structure* must be defined precisely. Let $z_1,\dots,z_N : w_1,\dots,w_N$ be $2N$ distinct points in “the” fundamental domain for $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$. Let $\mathbf{x}_1,\dots,\mathbf{x}_N : \mathbf{y}_1,\dots,\mathbf{y}_N$ be nonzero (column) vectors in $\mathbb{C}^r$. An $r\times r$-meromorphic matrix function F on $\mathbb{C}$ is said to interpolate the simple null-pole data

$$\mathscr{D} : (z_1,\mathbf{x}_1),\dots,(z_N,\mathbf{x}_N) : (w_1,\mathbf{y}_1),\dots,(w_N,\mathbf{y}_N) \tag{3.1}$$

if the following conditions are satisfied:

1. The null-pole divisor of $\det F$ on the fundamental domain is
$$(\det F) = z_1+\cdots+z_N - w_1-\cdots-w_N.$$
2. The only poles of any entry of F are at most simple poles at points of $w_i+\mathbb{Z}+\tau\mathbb{Z}$, $i=1,\dots,N$.
3. The matrix function F is analytic at points in $z_i+\mathbb{Z}+\tau\mathbb{Z}$ (already a consequence of condition (2) above) and the vector $\mathbf{x}_j$ spans the right kernel of F at these points, $j=1,\dots,N$.
4. The matrix function F^{-1} is analytic at points in $w_i+\mathbb{Z}+\tau\mathbb{Z}$ and $\mathbf{y}_i^\top$ (here $\top$ denotes transpose) spans the left kernel of F^{-1}, at these points $i=1,\dots,N$.

When all these conditions are satisfied, it can be shown that the poles of F are all simple and occur precisely at the points $w_1,\dots,w_N$ with rank 1 residue at w_i having left image spanned by the vector $\mathbf{y}_i^\top$ for each i, while the poles of F^{-1} are all simple and occur precisely at the points $z_1,\dots,z_N$ with rank 1 residue at z_i having right image spanned by the vector $\mathbf{x}_i$ for each i. For further information, see [6, 20].

If F satisfies (1)–(4), then we say that F has *simple null-pole data structure* and that $\mathscr{D}$ given by (4.10) is the *null-pole divisor of* F, or that F interpolates the data set $\mathscr{D}$.

We will establish the following:

THEOREM 3.1. *Given a rank r flat factor of automorphy ξ over the elliptic curve $M=\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$, there exist a simple null-pole data set $\mathscr{D}$ of the form (3.1) and a trivialization F of ξ such that F interpolates $\mathscr{D}$. In general, the size $N=N_r$ of $\mathscr{D}$ can be taken to satisfy $N_r<2r$. Moreover, if ξ does not have a direct summand equivalent to a representation ξ_{J_1}, then it is possible to take $N_r=r$.*

This theorem follows directly from the next proposition which is established via a constructive inductive proof based on the rank r.

PROPOSITION 3.2. *For $\alpha \neq 0$ there exist a simple null-pole data set $\mathcal{D}$ of the form (4.10) and a trivialization F_r of the form (2.17) of $\xi_{J_\alpha^{(r)}}$ which interpolates $\mathcal{D}$. If $\alpha = 1$, $N = N_r$ can be taken in the form $N_r = 2(r-1)$. If $\alpha \neq 1$, then it is possible to take $N_r = r$.*

Proof. Consider first the case $r = 1$. If $\alpha = 1$, the constant function $f(p) = 1$ is a trivialization of ξ_1. If $\alpha \neq 1$, as shown in (2.3), one can use theta functions to obtain a scalar function having one simple zero and one simple pole with factor of automorphy equal to α.

Inductively suppose now that the proposition has been established for $\xi_{J_\alpha^{(r)}}$ where $J_\alpha^{(r)}$ is the $r \times r$ matrix of the form (2.9). In more detail, we suppose that it has been shown that there is a trivialization $F_r = G_r S_r$ as in (2.17) with G_r of the form (2.16) and S_r satisfying (2.18) such that F_r interpolates a simple null-pole data set $\mathcal{D}_r$ as in (4.10) (so $F = F_r$ satisfies conditions (1)–(4)) with $N = N_r = r$ if $\alpha \neq 1$, and $N = N_r = 2(r-1)$, if $\alpha = 1$. It will be further assumed that the interpolation points $z_1, \dots, z_{N_r}$: $w_1, \dots, w_{N_r}$ are different from a. For convenience, we also assume that $a = 0$.

Write $S = S_r$ in the form

$$S_r = \begin{bmatrix} \mathbf{s}_r \\ \mathbf{s}_{r-1} \\ \vdots \\ \mathbf{s}_1 \end{bmatrix},$$

where $\mathbf{s}_1, \dots, \mathbf{s}_{r-1}, \mathbf{s}_r$ are r-dimensional row vector functions.

We wish to construct a trivialization F_{r+1} of $\xi_{J_\alpha^{(r+1)}}$ of the form

$$F_{r+1} = G_{r+1} S_{r+1} \tag{3.2}$$

also as in (2.17). Thus G_{r+1} should have the form

$$G_{r+1}(u) = \begin{bmatrix} 1 & \mathbf{p}(u) \\ 0 & G_r(u) \end{bmatrix} \tag{3.3}$$

with $\mathbf{p}$ equal to the r-dimensional row vector function

$$\mathbf{p}(u) = \begin{bmatrix} p_1(\lambda_0(u)) & p_2(\lambda_0(u)) & \dots & p_r(\lambda_0(u)) \end{bmatrix}$$

where p_j ($j = 1, 2, \dots$) are the polynomials as in (2.13) and (2.14), and $S_{r+1}(u)$ should be an $(r+1) \times (r+1)$ matrix function satisfying (2.18) which we assume to have the form

$$S_{r+1} = \begin{bmatrix} s_0 & \mathbf{s}_{r+1} \\ 0 & S_r \end{bmatrix}. \tag{3.4}$$

From (3.2) combined with (3.3) and (3.4) we get

$$F_{r+1} = G_{r+1} \begin{bmatrix} s_0 & \mathbf{s}_{r+1} \\ 0 & S_r \end{bmatrix} = \begin{bmatrix} s_0 & \mathbf{s}_{r+1} + \mathbf{p} S_r \\ 0 & G_r S_r \end{bmatrix} \tag{3.5}$$

where here we use the fact that $p_0(u) = 1$.

By assumption $F_r = G_r S_r$ trivializes $\xi_\alpha^{(r)}$ and has a simple matrix null-pole divisor supported on $z_1 + \cdots + z_{N_r} - w_1 - \cdots - w_{N_r}$. The goal is to choose s_0 and $\mathbf{s}_{r+1}$ so that

(i) F_{r+1} trivializes $\xi_{J_\alpha^{(r+1)}}$, and

(ii) F_{r+1} has a simple matrix null-pole divisor with one additional pole and zero, if $\alpha \neq 1$ and two additional poles and zeros, if $\alpha = 1$.

To achieve condition (i), by the analysis in Section 2 we need only guarantee that S_{r+1} satisfies (2.18). As S_r satisfies (2.18) by the induction hypothesis, this condition reduces to the two conditions

$$s_0(u+1) = s_0(u), \quad s_0(u+\tau) = \alpha s_0(u) \tag{3.6}$$

and

$$\mathbf{s}_{r+1}(u+1) = \mathbf{s}_{r+1}(u), \quad \mathbf{s}_{r+1}(u+\tau) = \alpha \mathbf{s}_{r+1}(u). \tag{3.7}$$

We therefore take the scalar function s_0 to be a meromorphic function on $\mathbb{C}$ satisfying (3.6). To help meet requirement (ii), we arrange that the null-pole divisor (s_0) of s_0 is as small as possible; in case $\alpha = 1$, we can arrange that

$$(s_0) = \zeta_1 + \zeta_2 - \pi_1 - \pi_2$$

where the points $\zeta_1, \zeta_2, \pi_1, \pi_2$ are chosen disjoint from points in the data $\mathscr{D}_r$ and $a = 0$. In the case where $\alpha \neq 1$, we shall take the divisor (s_0) to be of the simpler form

$$(s_0) = \zeta_1 - \pi_1.$$

The vector function $\mathbf{s}_{r+1}$ is an r dimensional meromorphic row vector function on $\mathbb{C}$ that satisfies

$$\mathbf{s}_{r+1}(u+1) = \mathbf{s}_{r+1}(u), \;\; \mathbf{s}_{r+1}(u+\tau) = \alpha \mathbf{s}_{r+1}(u) \tag{3.8}$$

that remains to be chosen.

From the automorphic properties of F_{r+1} (as also can be seen directly from (3.6) and (3.7)) we have

$$\mathbf{s}_{r+1}(u+1) + \mathbf{p}(u+1)S_r(u+1) = \mathbf{s}_{r+1}(u) + \mathbf{p}(u)S_r(u) \tag{3.9}$$

and

$$\mathbf{s}_{r+1}(u+\tau) + \mathbf{p}(u+\tau)S_r(u+\tau) = \alpha \mathbf{s}_{r+1}(u) + \alpha \mathbf{p}(\mathbf{u})S_r(u) + \overline{\mathbf{p}}(u)S_r(u). \tag{3.10}$$

where

$$\overline{\mathbf{p}}(u) = \left[p_0(\lambda_0(u)) \; p_1(\lambda_0(u)) \; \ldots \; p_{r-1}(\lambda_0(u))\right].$$

For later reference, note that the term $\overline{\mathbf{p}}S_r$ appearing in equation (3.10) is the first row of $F_r = G_r S_r$. In particular, the induction assumption guarantees that this term does not have a pole at $u = 0$.

To complete the achievement of requirement (ii), the entries $s_{r+1,j}$, $(j=1\ldots,r)$ of $\mathbf{s}_{r+1}$ must be chosen so that the null-pole divisor of F_{r+1} is simple.

For $j \geqslant 1$ the $(1,j+1)$-entry in F_{r+1} is

$$s_{r+1,j}+(p_1\circ\lambda_0)[S_r]_{1,j}+\cdots+(p_r\circ\lambda_0)[S_r]_{r,j}. \tag{3.11}$$

We first choose $s_{r+1,j}$ to remove any pole of $(p_1\circ\lambda_0)[S_r]_{1,j}+\cdots+(p_r\circ\lambda_0)[S_r]_{r,j}$ at $a=0$. In the case $\alpha\neq 1$, the removal of poles at $a=0$ can be accomplished by using theta functions. In more detail, given a polynomial p with $p(0)=0$, there exists a meromorphic function $s=s(u)$ with pole only at 0 satisfying

$$s(u+m+\tau n)=\alpha^n s(u),\ p(1/u)-s(u) \text{ is analytic at } u=0. \tag{3.12}$$

This statement can be established as follows: Let $\Delta=\frac{1}{2}+\frac{1}{2}\tau$. First assume $\alpha\neq 1$ and let $\mu_k=\frac{1}{2\pi ik}\log\alpha+\Delta$. For an appropriate choice of C_k the function

$$q_k(u)=C_k\left(\frac{\theta(u-\mu_k)}{\theta(u-\Delta)}\right)^k$$

satisfies $q_k(u+m+n\tau)=\alpha^n q_k(u)$ and has a Laurent expansion about $u=0$ which begins with the term $\frac{1}{u^k}$. By choosing appropriate linear combinations of the functions $q_k,\ldots,q_1$, for $k\geqslant 1$, one can construct functions p_k satisfying $p_k(u+m+n\tau)=\alpha^n p_k(u)$ with principal part $\frac{1}{u^k}$ at $u=0$. Therefore, in the case $\alpha\neq 1$, the existence of a function s with property (3.12) follows.

When $\alpha=1$, given a polynomial $p=p(u)$ with $p(0)=0$, there exists an elliptic function s such that the principal part of s at $u=0$ is $p(\frac{1}{u})$. This is easily established using the Weierstrass functions $\varsigma,\wp,\wp',\ldots$ associated with the lattice $\mathbb{Z}+\tau\mathbb{Z}$. Thus the existence of a function s satisfying (3.12) follows.

The existence of s satisfying (3.12) allows one to choose $s_{r+1,j}$ so that the entries (3.11) do not have poles at $a=0$. It follows from equations (3.9) and (3.10) that F_{r+1} does not have poles at points in $0+\mathbb{Z}+\tau\mathbb{Z}$.

At this point the (1,2)-entry

$$-s_0^{-1}(\mathbf{s}_{r+1}+\mathbf{p}S_r)F_r^{-1}$$

of F_{r+1}^{-1} may have poles at points in $\{w_1,\ldots,w_N\}+\mathbb{Z}+\tau\mathbb{Z}$. As above, by suitably modifying the entries in $\mathbf{s}_{r+1}$, one can remove any of these poles at fixed representatives of the points, while maintaining (3.7) . The identities (3.9) and (3.10) along with the fact that $\overline{\mathbf{p}}S_rF_r^{-1}$ is the row of the identity matrix, imply that the first row of F_r^{-1} does not have poles at any point in $\{w_1,\ldots,w_N\}+\mathbb{Z}+\tau\mathbb{Z}$. It follows that F_{r+1} has the correct pole structure at

$$\{w_1,\ldots,w_N,\pi_1,\pi_2\}+\mathbb{Z}+\tau\mathbb{Z}$$

when $\alpha=1$ and at

$$\{w_1,\ldots,w_N,\pi_1\}+\mathbb{Z}+\tau\mathbb{Z},$$

when $\alpha\neq 1$.

There remains to check that F_{r+1} and F_{r+1}^{-1} have the correct zero structure. The zero divisor of the determinant of F_{r+1} is $z_1+\cdots+z_N+\zeta_1$ in the case $\alpha\neq 1$ and $z_1+\cdots+z_N+\zeta_1+\zeta_2$ in the case $\alpha=1$. The form

$$F_{r+1}^{-1}=\begin{bmatrix} s_0^{-1} & -s_0^{-1}(\mathbf{s}_{r+1}+\mathbf{p}S_r)F_r^{-1} \\ 0 & F_r^{-1}\end{bmatrix}$$

allows one to conclude that the entries of F_{r+1}^{-1} have at most simple poles and only at the zeros of the determinant of F_{r+1}. At z_j, $j=1,\ldots,N$ the right kernel of $F_{r+1}(z_j)$ is spanned by

$$\begin{bmatrix} -s_0^{-1}(z_j)(\mathbf{s}_{r+1}+\mathbf{p}S_r)(z_j)(\mathbf{x}_j) \\ \mathbf{x}_j\end{bmatrix}.$$

The zero divisor of the determinant of F_{r+1}^{-1} is $w_1+\cdots+w_N+\pi_1$ in the case $\alpha\neq 1$ and $w_1+\cdots+w_N+\pi_1+\pi_2$ in the case $\alpha=1$. At w_j, $j=1,\ldots,N$ the right kernel of $F_{r+1}^{-1}(w_j)$ is spanned by $[0,\mathbf{y}_j^{\top}]$. At π_i the right kernel is spanned by the $r+1$-dimensional row vector

$$\begin{bmatrix} 1 & -(s_0^{-1}(\mathbf{s}_{r+1}+\mathbf{p}S_r)F_r^{-1})(\pi_i)F_r(\pi_i)\end{bmatrix}$$

for $i=1,2$. The proof is complete. $\square$

4. Trivialization of factors of automorphy via matrix-divisor constructions

4.1. Right matrix null/pole divisors, matrix-divisor spaces, vector bundles, and factors of automorphy

A general theory of null-pole divisors of meromorphic matrix functions, extending the case of simple null-pole structure described in the preceding section, is presented in [6]. This general theory can be used to elucidate the concrete results on trivializations described above, and furthermore applies equally well to the higher genus case. Suppose F is a meromorphic $r\times r$-matrix function defined in a neighborhood of a point q_0 on a Riemann surface with $\det F\not\equiv 0$. In local coordinates (z,U) near q_0 with $z(q_0)=0$ one introduces a local (right) null-pole triple Υ, that captures the null-pole behavior of F. This null-pole triple has the form

$$\Upsilon=((B_\zeta,A_\zeta),(A_\pi,C_\pi),S). \tag{4.1}$$

In this triple, the pair of matrices, (A_π,C_π), where A_π is $n_\pi\times n_\pi$ and C_π is $n_\pi\times r$ captures the pole behavior of F at q_0 in the sense that for some matrix $\widetilde{B}$ the matrix function

$$F(q)-\widetilde{B}(z(q)I-A_\pi)^{-1}C_\pi \tag{4.2}$$

is analytic at q_0 and the matrix size n_π is as small as possible so that (4.2) holds. The pair (A_π,C_π) is called a *left pole pair* for F since it is a *left null pair* for F^{-1} in the sense that $(z(q)I-A_\pi)^{-1}C_\pi F(q)^{-1}$ has analytic continuation to q_0 (the zero of F^{-1}, i.e., pole of F, is canceled by the pole of $(z(q)I-A_{pi})^{-1}C_\pi$ at q_0). In a similar manner, the pair (B_ζ, A_ζ), where A_ζ is $n_\zeta \times n_\zeta$ and B_ζ is $r \times n_\zeta$, captures the zero behavior of F at q_0 in the sense that for some matrix $\widetilde{C}$,

$$F^{-1}(q) - B_\zeta (z(q)I - A_\zeta)^{-1} \widetilde{C} \tag{4.3}$$

is analytic at q_0 with the matrix size n_ζ again as small as possible. We then refer to (B_ζ, A_ζ) as a *right null pair* for F since $F(q)B_\zeta(z(q)I - A_\zeta)^{-1}$ has analytic continuation to q_0 (the zero of F is cancelled on the right by the pole of $B_\zeta(z(q)I - A_\zeta)^{-1}$ at q_0).

The $n_\pi \times n_\zeta$ matrix S, called the coupling matrix, is that solution of the Sylvester equation

$$A_\pi S - S A_\zeta = C_\pi B_\zeta \tag{4.4}$$

which encodes the additional information needed to completely specify the $\mathscr{O}_{q_0}$-row module $\mathscr{O}_{q_0}^{1\times r} \cdot F$ (where $\mathscr{O}_{q_0}$ is the space of germs of functions holomorphic on a neighborhood of q_0). Information equivalent to knowledge of the row module $\mathscr{O}_{q_0}^{1\times r} \cdot F$ is knowledge of *right germ* of the nondegenerate meromorphic matrix function F at q_0. Here the *right germ* of F at q_0 is the equivalence class of nondegenerate meromorphic matrix functions on a neighborhood of q_0, where two such functions F and F' are considered equivalent if there is matrix function H holomorphic and invertible on a neighborhood of q_0 such that $F' = HF$.

The precise connection between the right null-pole triple Υ given as in (4.1) and the module $\mathscr{O}_{q_0}^{1\times r} \cdot F$ is as follows (see [6] as well as [7, 8] for the genus 0 case, [20] for an expository treatment of the simple-multiplicity genus 0 case, and [5] for the Riemann surface case). Suppose the local null-pole triple Υ of F at $q_0 \in M_0$ has the form (4.1) given in terms of the local coordinate z at q_0. Then

$$\mathscr{O}_{q_0}^{1\times r} \cdot F = \mathscr{S}(\Upsilon, q_0, z) \tag{4.5}$$

where we set

$$\begin{aligned} \mathscr{S}(\Upsilon, q_0, z) = \{ &\mathbf{x}(z(q)I - A_\pi))^{-1} C_\pi + \mathbf{h}(q) \colon \mathbf{x} \in \mathbb{C}^{1\times n_\pi},\ \mathbf{h} \in \mathscr{O}_{q_0}^{1\times r} \\ &\text{such that } \mathbf{x}S = \operatorname{res}_{q_0}[\mathbf{h}(q) B_\zeta (z(q)I - A_\zeta)^{-1}] \}. \end{aligned} \tag{4.6}$$

The set $\mathscr{S}(\Upsilon, q_0, z)$ defined by (4.6) will be referred to as the singular subspace of the 0-admissible Sylvester data set Υ at the point q_0 with respect to local coordinate z. For simplicity we shall assume that it is understood that a choice of local coordinate has bee made and write simply $\mathscr{S}(\Upsilon, q_0)$. It should be noted that the residue appearing in this last equation is a matrix residue. We also point out that the role of the Sylvester equation (4.4) is to guarantee that the set $\mathscr{S}(\Upsilon, q_0, z)$ is indeed a left module over $\mathscr{O}_{q_0}$.

Moreover those quintuples of matrices Υ (4.1) which can arise as the local null-pole triple for a nondegenerate meromorphic matrix function F at a point q_0 are characterized as the 0-*admissible Sylvester data sets*, i.e., the quintuples of matrices Υ (4.1) such that:

(a) A_π is nilpotent (i.e., $\sigma(A_\pi) = \{0\}$) and the input pair (A_π, C_π) is *controllable* (i.e., $\operatorname{span}\{\operatorname{Ran} A_\pi^j C_\pi \colon 0 \leqslant j \leqslant n_\pi - 1\} = \mathbb{C}^{n_\pi}$),

(b) A_ζ is nilpotent (i.e., $\sigma(A_\zeta) = \{0\}$) and the output pair (B_ζ, A_ζ) is *observable* (i.e., $\bigcap_{j=0}^{n_\zeta - 1} \operatorname{Ker} B_\zeta A_\zeta^j = \{0\}$),

(c) S satisfies the Sylvester equation (4.4).

Different local null-pole triples for F are related by a pair of similarities of A_π and A_ζ. More specifically, if U and V are invertible matrices of appropriate respective sizes, then

$$\widetilde{\Upsilon} = ((B_\zeta U, U^{-1} A_\zeta U), (V^{-1} A_\pi V, V^{-1} C_\pi), V^{-1} S U) \tag{4.7}$$

is also a local null-pole triple for F and any other local null-pole triple (with respect to the same local coordinate z at q_0) has this form. The 0-admissible Sylvester data sets Υ and $\widetilde{\Upsilon}$ given by (4.1) and (4.7) are then said to be similar Sylvester data sets.

REMARK 4.1. In the discussion that follows we will work mainly with relatively automorphic meromorphic matrix functions on the universal cover $\widehat{M}$ of the Riemann surfaces M. If we fix a choice of coordinate for a sufficiently small neighborhood at a point q_0 in a fundamental domain $M_0 \subset \widehat{M}$ and then use a deck transformation T to lift this coordinate to a coordinate for a neighborhood of the point $T(q_0)$ lying above q_0, the relative-automorphy property of F guarantees that the local null-pole triple at $T(q_0)$ for F is exactly the same as the local null-pole triple at q_0 for F (since the invertible left factor of automorphy $\xi(T,u)$ can be absorbed into the free invertible matrix left factor H in the definition of right matrix germ). In summary, this compatible choice of local coordinates leads to identical null-pole triples for points over the same base point in the fundamental domain M_0.

The reader should be aware however that our definitions here correspond to *right null-pole triples* (describing the *row* module $\mathscr{O}_{q_0}^{1\times r} \cdot F$ or equivalently the *right* matrix germ $\{H \cdot F \colon H^{\pm 1} \in \mathscr{O}_{q_0}^{r\times r}\}$) whereas the main focus in [6] is on *left null-pole triples* which describe the *column* module $F \cdot \mathscr{O}_{q_0}^{r\times 1}$ or equivalently the *left* matrix germ $\{F \cdot H \colon H^{\pm 1} \in \mathscr{O}_{q_0}^{r\times r}\}$.

We now illustrate the characterization (4.5) with a simple example.

EXAMPLE 4.2. Let f be the scalar function defined in local coordinates near zero by $f(u) = u^2$. Note that

$$\mathscr{O}_0 \cdot f = \{h(u) = \sum_{j=0}^{\infty} h_j u^j \text{ holomorphic at } 0 \colon h_0 = h_1 = 0\}. \tag{4.8}$$

On the other hand, any local null-pole triple for f at $u = 0$ consists only of a zero pair, i.e., $n_\pi = 0$, and the condition in (4.5) collapses to

$$\operatorname{res}_0 [h(u) B_\zeta (uI - A_\zeta)^{-1}] = 0. \tag{4.9}$$

One choice of null pair for f at $u=0$ is the pair $(B_\zeta, A_\zeta) = \left(\left[\begin{smallmatrix}1 & 0\end{smallmatrix}\right], \left[\begin{smallmatrix}0 & 1\\ 0 & 0\end{smallmatrix}\right]\right)$, where equation (4.3) holds with $\widetilde{C} = \left[\begin{smallmatrix}0\\ 1\end{smallmatrix}\right]$. If $h(u) = h_0 + h_1 u + \cdots$ is analytic at zero, then

$$\begin{aligned}\operatorname{res}_0\left\{h(u)\left[\begin{smallmatrix}1 & 0\end{smallmatrix}\right]\left(\left[\begin{smallmatrix}u & 0\\ 0 & u\end{smallmatrix}\right] - \left[\begin{smallmatrix}0 & 1\\ 0 & 0\end{smallmatrix}\right]\right)^{-1}\right\} &= \operatorname{res}_0\left\{h(u)\left[\begin{smallmatrix}1 & 0\end{smallmatrix}\right]\begin{bmatrix}u^{-1} & u^{-2}\\ 0 & u^{-1}\end{bmatrix}\right\}\\ &= \begin{bmatrix}h_0 & h_1\end{bmatrix}.\end{aligned}$$

Combining (4.8) with (4.9), we arrive at a verification of the characterization (4.5) for this case.

REMARK 4.3. We define the *adjoint* Υ^* of a 0-admissible Sylvester data set

$$\Upsilon = ((B_\zeta, A_\zeta), (A_\pi, C_\pi), S)$$

as the Sylvester data set

$$\Upsilon^* = ((C_\pi^\top, A_\pi^\top), (A_\zeta^\top, B_\zeta^\top), -S^\top)$$

where $X^\top$ denotes the *transpose* of the matrix X. If the meromorphic matrix F locally interpolates Υ at q_0, then $(F^\top)^{-1}$ locally interpolates the divisor Υ^* at q_0 (with respect to the same local coordinate).

Let us introduce the notation

$$\mathscr{D} = \{(\Upsilon_{q_0}, q_0) \colon q_0 \in M_0\} \tag{4.10}$$

for a collection of 0-admissible Sylvester data sets

$$\Upsilon_{q_0} = ((B_{\zeta_{q_0}}, A_{\zeta_{q_0}}), (A_{\pi_{q_0}}, C_{\pi_{q_0}}), S_{q_0}) \tag{4.11}$$

tagged to each point $q_0 \in M_0$. Here it is understood that for each point q_0 there is also specified a choice z of local coordinate at q_0 in order for the formula (4.6) for the singular subspace $\mathscr{S}(\Upsilon, q_0, z_0)$ to be well defined. For definiteness, the sizes of the matrices in Υ_{u_0} are specified as follows: $A_{\pi_{u_0}}$ is $n_{\pi_{u_0}} \times n_{\pi_{u_0}}$ and $C_{\pi_{u_0}}$ is $n_{\pi_{u_0}} \times r$, $A_{\zeta_{u_0}}$ is $n_{\zeta_{u_0}} \times n_{\zeta_{u_0}}$ and $B_{\zeta_{u_0}}$ is $r \times n_{\zeta_{u_0}}$, and S_{u_0} is a $n_{\pi_{u_0}} \times n_{\zeta_{u_0}}$ matrix for $i = 1, \ldots, k$. We shall also impose the restriction that the local data (Υ_{u_0}, u_0) are trivial (i.e., both $n_{\pi_{u_0}} = 0$ and $n_{\zeta_{u_0}} = 0$) for all but finitely many points $u_0 = u_1, \ldots, u_k$ in M_0. In the sequel a data set $\mathscr{D}$ as in (4.10) subject to this finite-support restriction will be referred to as a *(right matrix) null-pole divisor*. The integer

$$\deg(\mathscr{D}) = \sum_{u_0 \in M_0} (n_{\zeta_{u_0}} - n_{\pi_{u_0}})$$

is called the *degree* of the divisor $\mathscr{D}$. In what follows, it will often be assumed that the degree of $\mathscr{D}$ is zero. Thus in this case $N := \sum_{u_0 \in M_0} n_{\zeta_{u_0}} = \sum_{u_0 \in M_0} n_{\pi_{u_0}}$.

For later purposes, it is useful to have a partitioning of the index set $\{1,\ldots,k\}$ into three types:

$$\begin{aligned}\mathrm{I} &= \{i\colon n_{\pi_{u_i}} > 0 \text{ while } n_{\zeta_{u_i}} = 0\},\\ \mathrm{II} &= \{i\colon \text{ both } n_{\pi_{u_i}} > 0 \text{ and } n_{\zeta_{u_i}} > 0\},\\ \mathrm{III} &= \{i\colon n_{\pi_{u_i}} = 0 \text{ while } n_{\zeta_{u_i}} > 0\}.\end{aligned} \tag{4.12}$$

Without loss of generality we may assume that

$$\begin{aligned}\mathrm{I} &= \{1,\ldots,n_\infty\},\\ \mathrm{II} &= \{n_\infty+1,\ldots,n_\infty+n_c\},\\ \mathrm{III} &= \{n_\infty+n_c+1,\ldots,n_\infty+n_c+n_0\}.\end{aligned}$$

Then $k = n_\infty + n_c + n_0$ where n_∞ counts the number of points where there is a pole but no zero, n_0 counts the number of points where there is a zero but no pole, $n_Z = n_0 + n_c$ counts the number of points where there is a zero, and $n_P = n_\infty + n_c$ counts the number of points where there is a pole and n_c counts the number of points in M_0 where there there is both a zero and a pole.

One can lift the data to $\widehat{\mathscr{D}}$ on the universal cover $\widehat{M}$ (as in Remark 4.1) and ask whether there are automorphic meromorphic matrix functions or relatively automorphic meromorphic matrix functions with respect to a flat factor of automorphy interpolating the data $\widehat{\mathscr{D}}$ on $\widehat{M}$. Such questions were addressed in [5, 10] and will be discussed below.

Given a null-pole divisor (4.10), we associate a linear matrix-divisor space $\mathscr{L}_*(\mathscr{D})$ of meromorphic $(1\times r)$-row vector functions on M by

$$\mathscr{L}_*(\mathscr{D}) = \{f \in \mathscr{M}(M)^{1\times r}\colon f(q) \in \mathscr{S}(\Upsilon_{q_0}, q_0) \text{ for all } q_0 \in M\} \tag{4.13}$$

or the more general sheaf version: for U equal to any open subset of M, define $\mathscr{L}_*(\mathscr{D})|U$ by

$$\mathscr{L}_*(\mathscr{D})|_U = \{f \in \mathscr{M}(U)^{1\times r}\colon f(q) \in \mathscr{S}(\Upsilon_{q_0}, q_0) \text{ for all } q_0 \in U\}.$$

The matrix-divisor space $\mathscr{L}_*(\mathscr{D})$ is a matrix analogue of the space $L(-D)$ for $D = \sum_{p\in M_0} n_p p$ a classical scalar divisor (formal sum of points in M_0 with multiplicities n_{q_0} such that $n_{q_0} = 0$ for all but finitely many q_0) given by

$$L(-D) = \{f \in \mathscr{M}(M) : (f) \geqslant D\}$$

where (f) denotes the pole-zero divisor of f. Analogous to what is done for the scalar-valued case (see [16, Section 29.11]) where one associates a holomorphic line bundle with a scalar divisor, one can associate a holomorphic vector bundle $E_{\mathscr{D}}$ with the right matrix null-pole divisor $\mathscr{D}$ in such a way that the space of holomorphic sections of the adjoint bundle $E_{\mathscr{D}}^*$ is isomorphic to the space of meromorphic functions $\mathscr{L}_*(\mathscr{D})$ (4.13) as follows.

Given a data set $\mathscr{D}$ as in (4.10), by the results from [6] we can find a open covering $\mathfrak{U} = \{U_\alpha\}_{\alpha \in A}$ of M and invertible $r \times r$-matrix-valued meromorphic functions L_α on U_α so that L_α solves the zero-pole interpolation problem for $\mathscr{D}$ restricted to U_α:

$$\mathscr{O}_{q_0}^{1\times r} \cdot L_\alpha = \mathscr{S}(\Upsilon_{u_0}, u_0) \text{ for all } u_0 \in U_\alpha.$$

The collection of transition functions

$$\Phi_{\alpha,\beta} = L_\alpha L_\beta^{-1} \in \mathscr{O}_{U_\alpha \cap U_\beta}^{r\times r}$$

defines the equivalence class of an r-dimensional holomorphic vector bundle $E_{\mathscr{D}}^*$ on M (with sections locally identified with row vector functions). In view of the connection (4.5) between interpolants and null-pole subspaces, we see that: *if U is any open subset of M, then the row-vector function f is in $\mathscr{L}_*(\mathscr{D})|_U$ if and only if*

$$f|_{U\cap U_\alpha} \in \mathscr{O}_{U\cap U_\alpha}^{1\times r} \cdot L_\alpha|_{U\cap U_\alpha} \text{ for all } \alpha \in A,$$

i.e., if and only if *the function $h_\alpha(u) := f(u)L_\alpha(u)^{-1}$ is holomorphic on $U \cap U_\alpha$ for all α.* It then follows that $h_\alpha L_\alpha = h_\beta L_\beta$ $(= f)$ on $U \cap U_\alpha \cap U_\beta$, and hence

$$h_\beta = h_\alpha L_\alpha L_\beta^{-1} \text{ on } U \cap U_\alpha \cap U_\beta. \tag{4.14}$$

This observation has the implication that the functions $\{h_\alpha^\top\}_{\alpha\in A}$ piece together to form a holomorphic section for the bundle $E_{\mathscr{D}}$ (with sections locally identified with column vector functions) dual to $E_{\mathscr{D}}^*$ and defined via the collection of transition functions

$$\Phi_{\alpha,\beta}^* = (\Phi_{\alpha,\beta}^\top)^{-1} = (L_\alpha^\top)^{-1}((L_\beta^\top)^{-1})^{-1} \in \mathscr{O}_{U_\alpha\cap U_\beta}^{r\times r}.$$

From the correspondence $f|_{U_\alpha} \mapsto h_\alpha^\top = (L_\alpha^\top)^{-1}\,(f|_{U_\alpha})^\top$ derived above, we see that then the matrix-divisor space $\mathscr{L}_*(\mathscr{D})$ is in one-to-one correspondence with holomorphic sections of the bundle $E_{\mathscr{D}}$ (with local sections given in the more conventional form of column-vector functions).

So far, for a given right matrix divisor $\mathscr{D}$, we have obtained an equivalence between the matrix-divisor space $\mathscr{L}_*(\mathscr{D})$ and the vector bundle $E_{\mathscr{D}}^*$. We now explain how one can associate a factor of automorphy $\zeta_{\mathscr{D}}$ on the universal cover $\widehat{M}$ of M with any divisor $\mathscr{D}$. We let $\widehat{M}$ be the universal cover of M with projection map $\rho \colon \widehat{M} \to M$. Let $\{U_\alpha\}$ be the cover described above determining the transition functions $\Phi_{\alpha,\beta} = L_\alpha L_\beta^{-1}$ defining the bundle $E_{\mathscr{D}}$. It can be assumed that this cover is chosen so that for all α, the set $\widehat{U}_\alpha = \rho^{-1}(U_\alpha)$ is the disjoint union $\cup\{T\widehat{V}_\alpha : T \in \mathscr{G}\}$, where $\widehat{V}_\alpha$ is a fixed component of $\widehat{U}_\alpha$. For $s \in \widehat{U}_\alpha$ define $\widehat{L}_\alpha(s) = L_\alpha(\rho(s))$. Then the collection of transition functions $\widehat{\Phi}_{\alpha,\beta} = \widehat{L}_\alpha \widehat{L}_\beta^{-1}$ defines a vector bundle $\widehat{E}_{\mathscr{D}}$ on $\widehat{M}$. By a theorem of Grauert [17], the bundle $\widehat{E}_{\mathscr{D}}$ is holomorphically equivalent to the trivial bundle. Thus for each α there exists an invertible holomorphic matrix functions H_α on U_α such that

$$H_\alpha \widehat{L}_\alpha \widehat{L}_\beta^{-1} H_\beta^{-1} = I.$$

The meromorphic matrix function defined on $\widehat{M}$ by $F_{\mathscr{D}}(u) = H_\alpha(u)\widehat{L}_\alpha(u)$ for $u \in \widehat{U}_\alpha$ is a trivialization of the left factor of automorphy

$$\zeta_{\mathscr{D}}(T,u) = F_{\mathscr{D}}(Tu)F_{\mathscr{D}}^{-1}(u) \tag{4.15}$$

associated with the divisor $\mathscr{D}$, i.e., one can verify: *$F_{\mathscr{D}}$ is a well-defined invertible $r \times r$ matrix-valued meromorphic function on $\widehat{M}$ satisfying (1) $F_{\mathscr{D}}(Tu) = \zeta_{\mathscr{D}}(T,u)F_{\mathscr{D}}(u)$ for all $u \in \widehat{M}$, and (2) $F_{\mathscr{D}}$ has right null-pole divisor $\mathscr{D}$ at points in $\rho^{-1}(q_0)$ for all* $q_0 \in M_0$ (with compatible choice of local coordinates as in Remark 4.1), i.e., we have obtained an (albeit not particularly constructive) solution of the **Third Interpolation Problem** mentioned in the Introduction.

Thus the matrix-divisor space $\mathscr{L}_*(\mathscr{D})$ lifts to the space $\widehat{\mathscr{L}_*}(\mathscr{D})$ consisting of all meromorphic row-vector functions $\widehat{f}$ on $\widehat{M}$ such that (1) $\widehat{f}(q) \in \mathscr{S}(\Upsilon_{q_0}, q_0)$ for all $q_0 \in \widehat{M}$, and (2) $\widehat{f}$ is relatively automorphic with factor of automorphy $\zeta_{\mathscr{D}}$:

$$\widehat{f}(Tu) = \zeta_{\mathscr{D}}(T,u)\widehat{f}(u) \text{ for all } T \in \mathscr{G}.$$

Now that we have identified a global solution $F_{\mathscr{D}}$ of the interpolation problem for divisor $\mathscr{D}$ (lifted to $\widehat{M}$), we may adjust the correspondence (4.14) between $\mathscr{L}_*(\mathscr{D})$ and the space of holomorphic section of $E_{\mathscr{D}}^*$ to the global form

$$\widehat{f} \mapsto \widehat{h} := (F_{\mathscr{D}}^{\top})^{-1}\widehat{f}^{\top} \tag{4.16}$$

which puts the elements of $\widehat{L}_*(\mathscr{D})$ in one-to-one correspondence with global holomorphic column-vector functions $\widehat{h}$ which are relatively automorphic with (left) factor of automorphy $\zeta_{\mathscr{D}}^* := (\zeta_{\mathscr{D}}^{\top})^{-1}$: $\widehat{h}(Tu) = \zeta_{\mathscr{D}}^*(T,u)\widehat{h}(u)$.

In short, the bundle $E_{\mathscr{D}}^*$ corresponds to the factor of automorphy $\zeta_{\mathscr{D}}$ on $\widehat{M}$, given as in (4.15) in terms of the trivialization $F_{\mathscr{D}}$, with the space of holomorphic sections $H^0(\zeta_{\mathscr{D}}^*)$ of the dual factor of automorphy $\zeta_{\mathscr{D}}^*$ being holomorphic vector functions as in (4.16).

4.2. Trivializations of flat factors of automorphy via divisor constructions

In the case where $\deg \mathscr{D} = 0$, a method developed in [5] leads to a condition sufficient for the existence of a trivializations of $\zeta_{\mathscr{D}}$ where the entries have only limited poles over points on M in a nonspecial divisor of degree g. In order to formulate this result, we first recall some basic results about holomorphic vector bundles.

A nonnegative degree g divisor $D = p_1 + \cdots + p_g$ on the closed Riemann surface of genus g is called a *nonspecial divisor* in case $i(D) = 0$, where $i(D)$ is the dimension of the space of meromorphic 1-forms ω whose divisor (ω) satisfies $(\omega) \geqslant D$. As is customary, the collection of nonnegative divisors of degree g can be identified with the (topological) g-fold symmetric product $M^{(g)}$. The collection of nonspecial divisors forms an open subset of $M^{(g)}$ and given the nonspecial divisor D, there is a nonspecial divisor $D' = p_1 + \cdots + p_g$ close to D where the points $p'_1, \ldots, p'_g$ are distinct [13, page 91]. It follows from the Riemann-Roch Theorem [13, page 73] that the nonnegative degree g divisor is nonspecial if and only if the degree $g-1$ divisor $D_0 = D - p_0$

satisfies $h^0(\lambda_{D_0}) = 0$, where as is customary, $h^0(\lambda_{D_0})$ is used to denote the dimension of the space of holomorphic sections of the line bundle λ_{D_0} associated with the divisor D_0 or, equivalently, the dimension $l(D_0)$ of the linear space $L(D_0)$ of meromorphic functions f on M whose divisors satisfy $(f) + D_0 \geqslant 0$; here and in the sequel, p_0 is any point disjoint from $p_1, \ldots, p_g$. In case D is nonspecial, then by the Riemann-Roch Theorem the degree $g-1$ divisor $D_0 = D - p_0$ satisfies

$$h^0(\lambda_{D_0}) = h^0(\lambda_{D_0}^{-1}\kappa) = 0,$$

where κ is the canonical line bundle. Applying the Jacobi inversion theorem (see e.g. [13, page 97]) to the divisor $(\lambda_{D_0}^{-1}\kappa) + p_0$, one obtains a linearly equivalent degree g divisor $\widetilde{D} = \widetilde{p}_1 + \cdots + \widetilde{p}_g$. The degree $g-1$ divisor

$$\widetilde{D}_0 = \widetilde{p}_1 + \cdots + \widetilde{p}_g - p_0$$

is equivalent to the divisor $(\lambda_{D_0}^{-1}\kappa)$ and therefore also satisfies $l(\widetilde{D}_0) = 0$. As noted above, it follows that $\widetilde{D} = \widetilde{p}_1 + \cdots + \widetilde{p}_g$ is nonspecial.

It is a consequence of Weil's characterization of flat bundles [26] (see also [18, page 110]) that a rank r bundle E of degree zero over M will be flat in case, for some line bundle λ of degree $g-1$, one has $h^0(\lambda \otimes E) = 0$. In case of genus one, this condition is also sufficient. In particular, if the line bundle λ has the form λ_{D_0} where $D_0 = p_1 + \cdots + p_g - p_0$ with $p_1 + \cdots + p_g$ nonspecial and E is a rank r vector bundle of degree zero over M such that

$$h^0(\lambda_{D_0} \otimes E) = 0, \tag{4.17}$$

it follows that E is flat. In such a case we will say that the degree zero vector bundle E has the property NSF (nonspecial flat). To summarize the preceding discussion in terms of the notion of NSF bundles, we see that any NSF bundle is flat and in the genus 1 case the classes of flat and NSF bundles coincide. The goal of this section (see Theorem 4.8 below) is to show that, given a flat factor of automorphy ζ associated with an NSF bundle E_ζ, there is a meromorphic matrix function F automorphic with respect to ζ having matrix divisor supported on only $g+1$ points. Before arriving at this result, we need to go through a number of preliminaries.

The property NSF is symmetric with respect to bundle adjoints on the collection of degree zero bundles. Indeed, if E has property NSF and (4.17) holds, then it follows from the Riemann-Roch Theorem for vector bundles (see [18, page 64]) that

$$h^0((\lambda_{D_0})^{-1}\kappa \otimes E^*) = 0.$$

As mentioned earlier, the line bundle $\lambda_{D_0}^{-1}\kappa$ is equivalent to a degree $g-1$ line bundle $\tilde{\lambda}$ associated with a divisor of the form

$$\widetilde{D}_0 = \widetilde{p}_1 + \cdots + \widetilde{p}_g - p_0,$$

where the nonnegative degree g divisor $\widetilde{p}_1 + \cdots + \widetilde{p}_g$ is nonspecial. Thus E^* has property NSF.

REMARK 4.4. As noted in [5], there are examples in higher genus of degree 0 (even semi-stable) rank 2 bundles where $h^0(\lambda \otimes E) \neq 0$ for all degree $g-1$ line bundles λ.

Following [5], we introduce automorphic meromorphic matrix functions on $\widehat{M}$ that have a prescribed (left) pole pair (A_π, C_π) at the point $\mathcal{G}w$ where w is a fixed point in the fundamental domain M_0. To this end let D_0 be a divisor of degree $g-1$ on M of the form $D_0 = p_1 + \cdots + p_g - p_0$, where $p_1 + \cdots + p_g$ is nonspecial and such that the points $p_0, p_1, \ldots, p_g$ are distinct from w. We also fix a local coordinate z for M_0 centered at w (so $z(w) = 0$). Since $h^0(\lambda_{D_0}) = 0$, it follows that for any integer $k \geqslant 1$ there is a unique meromorphic function $f_{kw}^{D_0}$ on M, equivalently, an automorphic meromorphic function on $\widehat{M}$, whose divisor satisfies $(f_{kw}^{D_0}) + D_0 + kw \geqslant 0$ and which is normalized so that the principal part of the Laurent series at w with respect to the local coordinate z at w has the form $z(u)^{-k}$ (see [5, page 147]). Suppose that A is the $n \times n$ Jordan cell

$$A = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix}.$$

Introduce the matrix function

$$f_{w,A}^{D_0} = \begin{bmatrix} f_w^{D_0} & f_{2w}^{D_0} & f_{3w}^{D_0} & \cdots & f_{nw}^{D_0} \\ & f_w^{D_0} & f_{2w}^{D_0} & \ddots & \vdots \\ & & \ddots & \ddots & f_{3w}^{D_0} \\ & & & \ddots & f_{2w}^{D_0} \\ & & & & f_w^{D_0} \end{bmatrix} \tag{4.18}$$

where as usual unspecified entries are equal to 0. This definition is extended to an arbitrary nilpotent matrix A by setting $f_{w,SAS^{-1}}^{D_0} = S f_{w,A}^{D_0} S^{-1}$ and $f_{w,A_1 \oplus A_2}^{D_0} = f_{w,A_1}^{D_0} \oplus f_{w,A_2}^{D_0}$, where S is an invertible matrix. It follows that for any nilpotent matrix N the difference

$$f_{w,N}^{D_0}(u) - ((z(u)I - N)^{-1}$$

is analytic at $u = w$. This last statement follows from the fact that, when N is $r \times r$ Jordan cell with eigenvalue 0, the local (left) pole pair of $f_{w,N}^{D_0}$ at w has the form $(A_\pi, C_\pi) = (N, I_r)$. The only other poles of entries of the meromorphic matrix function $f_{w,N}^{D_0}$ are at the points $p_1, \ldots, p_s$ of the divisor D_0. In fact, if the divisor D_0 is written in the form

$$D_0 = n_1 p_1 + \cdots + n_s p_s - p_0,$$

where the distinct points $p_1, \ldots, p_s$ appear with the positive multiplicities $n_1, \ldots, n_s$, respectively, then at $p_1, \ldots, p_s$ the entries of $f_{w,N}^{D_0}$ have poles of order at most $n_1, \ldots, n_s$, respectively.

Let the degree $g-1$ divisor $D_0 = p_1 + \cdots + p_g - p_0$ be as above with points $p_0, p_1, \ldots, p_g$ distinct from points $u_1, \ldots, u_k$ on which $\mathscr{D}$ is supported. For $\mathscr{D}$ a right matrix null-pole divisor supported at the points $u_1, \ldots, u_k$ as in (4.10) with the set of indices $\{1, \ldots, k\}$ partitioned into subsets I, II, III as in (4.12), we introduce the space $\mathscr{M}_{\mathscr{D}}^{D_0}$ of r-dimensional row-vector meromorphic functions on M given by

$$\mathbf{k} = \mathbf{u}_0 + \sum_{i \in \mathrm{I} \cup \mathrm{II}} \mathbf{u}_i f^{D_0}_{u_i, A_{\pi_{u_i}}} C_{\pi_{u_i}}, \tag{4.19}$$

where $\mathbf{u}_0 \in \mathbb{C}^{1\times r}$, $\mathbf{u}_i \in \mathbb{C}^{1\times n_{\pi_{u_i}}}$, $i = 1, \ldots, n_P$. The linear map $\mathbf{T}$ from $\mathbb{C}^{1\times(r+n_{u_1}+\cdots+n_{u_{n_P}})}$ to $\mathscr{M}_{\mathscr{D}}^{\mathscr{D}_0}$ that associates the vector $\begin{bmatrix}\mathbf{u}_0 \ \mathbf{u}_1 \ \cdots \ \mathbf{u}_{n_P}\end{bmatrix}$ with the vector $\mathbf{k}$ given by (4.19) is one-to-one. Indeed, if

$$\mathbf{k} = \mathbf{T}(\begin{bmatrix}\mathbf{u}_0 \ \mathbf{u}_1 \ \cdots \ \mathbf{u}_{n_P}\end{bmatrix}) = \mathbf{0},$$

then the singular part of $\mathbf{k}$ at u_i or, equivalently, the singular part of

$$\mathbf{u}_i (z_i(u) I - A_{\pi i})^{-1} C_{\pi_{u_i}}$$

(where z_i is the local coordinate at u_i) at u_i, is zero. This leads one to conclude that $\mathbf{u}_i A^j_{\pi_{u_i}} C_{\pi_{u_i}} = \mathbf{0}$ for $j = 0, 1, \ldots, n_{\pi_{u_i}} - 1$. The controllability of the pair $(A_{\pi_{u_i}}, C_{\pi u_i})$ for each $i = 1, \ldots, n_P$ implies that $\mathbf{u}_i = \mathbf{0}$. As $\mathbf{u}_0$ is the value of $\mathbf{k}$ at p_0 (since each $f^{D_0}_{u_i, A}$ vanishes at p_0), we get $\mathbf{u}_0 = 0$ as well. We conclude that $\mathbf{T}$ is one-to-one.

When one assumes that the meromorphic matrix function $\mathbf{k}$ has the form (4.19), there is a convenient test for identifying when $\mathbf{k} \in \mathscr{O}^{1\times r}_{u_i} \cdot F_{\mathscr{D}}$ for each of the points u_i ($i = 1, \ldots, k = n_\infty + n_c + n_0$) in the support of the matrix divisor $\mathscr{D}$. Before presenting this result we need some additional notation.

For $i \in \mathrm{I} \cup \mathrm{II}$ and $j \in \mathrm{II} \cup \mathrm{III}$ and $i \neq j$ in case both i and j are in II, let

$$\Gamma^{D_0}_{\mathscr{D}, ij} = -\mathrm{res}_{u=u_j} [f^{D_0}_{u_i, A_{\pi_{u_i}}}(u) C_{\pi_{u_i}} B_{\zeta_{u_j}} ((z_j(u) I - A_{\zeta_{u_j}})^{-1}] \tag{4.20}$$

and for i and j both in II with $i = j$, let

$$\begin{aligned} \Gamma^{D_0}_{\mathscr{D}, ij} &= \Gamma^{D_0}_{\mathscr{D}, jj} \\ &=: S_{u_j} - \mathrm{res}_{u_j} \left[\left(f^{D_0}_{u_j, A_{\pi_{u_j}}}(u) - (z_j(u) I - A_{\pi_{u_j}})^{-1} \right) C_{\pi_{u_j}} B_{\zeta_{u_j}} (z_j(u) I - A_{\zeta_{u_j}})^{-1} \right]. \end{aligned} \tag{4.21}$$

These matrices form the block entries for an $n_P \times n_Z$ block matrix $\Gamma^{D_0}_{\mathscr{D}}$ having rows indexed by $i \in \mathrm{I} \cup \mathrm{II}$ and columns indexed by $j \in \mathrm{II} \cup \mathrm{III}$:

$$\Gamma^{D_0}_{\mathscr{D}} = [\Gamma^{D_0}_{\mathscr{D}, ij}]_{i \in \mathrm{I} \cup \mathrm{II}, j \in \mathrm{II} \cup \mathrm{III}}. \tag{4.22}$$

We are now able to formulate and prove the following result.

PROPOSITION 4.5. *Assume that the meromorphic row-vector function* $\mathbf{k}$ *has the form* (4.19)*. Then* $\mathbf{k} \in \mathscr{O}^{1\times r}_{u_i} F_{\mathscr{D}}$ *for each point* u_i *in the support of* $\mathscr{D}$ *if and only if*

$$\mathbf{u}_0 B_\zeta = \mathrm{row}_{i \in \mathrm{I} \cup \mathrm{II}} [\mathbf{u}_i] \cdot \Gamma^{D_0}_{\mathscr{D}} \tag{4.23}$$

where $B_\zeta = \mathrm{row}_{i \in \mathrm{II} \cup \mathrm{III}} [B_{\zeta_i}]$.

Proof. Suppose $\mathbf{k}$ is given in the form (4.19) and one wishes to investigate if this meromorphic row-vector function belongs to $\mathscr{O}_u^{1\times r}\cdot F_{\mathscr{D}}$ at $u=u_j$ for $j\in \mathrm{I}\cup\mathrm{II}\cup\mathrm{III}$.

If $j\in\mathrm{I}$, no extra condition is required: $\mathbf{k}$ is already of the correct form to be in the $\mathscr{O}_{u_j}^{1\times r}\cdot F_{\mathscr{D}}$.

For $j\in\mathrm{III}$, then using the description (4.5) one sees that the germ of $\mathbf{k}$ belongs to $\mathscr{O}^{1\times r}(\{u_j\})\cdot F_{\mathscr{D}}$ if and only if

$$\operatorname{res}_{u=u_j}\left\{\left[\mathbf{u}_0+\sum_{i=1}^{n_P}\mathbf{u}_i f^{D_0}_{w_i,A_{\pi_i}}(u)C_{\pi_i}\right]B_{\zeta_j}(z_j(u)I-A_{\zeta_j})^{-1}\right\}=0,$$

or, equivalently,

$$\mathbf{u}_0 B_{\zeta_j}=\sum_{i=1}^{n_P}\mathbf{u}_i\Gamma^{D_0}_{\mathscr{D},ij}. \tag{4.24}$$

For $j\in\mathrm{II}$, we rewrite $\mathbf{k}$ near u_j as

$$\begin{aligned}\mathbf{k}=&[\mathbf{u}_j(z_j(u)I-A_{\pi_{u_j}})^{-1}C_{\pi_j}]\\&+\left[\left(\mathbf{u}_0+\sum_{i\in\mathrm{I}\cup\mathrm{II}}\mathbf{u}_i f^{D_0}_{u_i,A_{\pi_{u_i}}}C_{\pi_{u_i}}\right)-\mathbf{u}_j(z_j(u)I-A_{\pi_{u_j}})^{-1}C_{\pi_j}\right].\end{aligned}$$

Note that the second bracketed term in this decomposition is holomorphic at u_j since the singular parts at u_j cancel out. We then apply the characterization (4.5) to this decomposition to see that $\mathbf{k}\in\mathscr{O}_{u_j}^{1\times k}F_{\mathscr{D}}$ if and only if

$$\begin{aligned}\operatorname{res}_{u=u_j}\Bigg\{&\left[\left(\mathbf{u}_0+\sum_{i\in\mathrm{I}\cup\mathrm{II}}\mathbf{u}_i f^{D_0}_{u_i,A_{\pi_{u_i}}}C_{\pi_{u_i}}\right)-\mathbf{u}_j(z_j(u)I-A_{\pi_{u_j}})^{-1}C_{\pi_j}\right]\cdot\\&\cdot B_{\zeta_{u_j}}(z_j(u)I-A_{\zeta_{u_j}})^{-1}\Bigg\}=\mathbf{u}_jS_{u_j}.\end{aligned}$$

This condition collapses to (4.24) for the case where $j\in\mathrm{II}$.

When we arrange the conditions (4.24) as an equality of two block row matrices (with block-row entries indexed by $j\in\mathrm{II}\cup\mathrm{III}$), we arrive at the single matrix equation (4.23). As the analysis is necessary and sufficient, the result of Proposition 4.5 follows. $\square$

The subspace $(\mathscr{M}^{D_0}_{\mathscr{D}})_0$ of $\mathscr{M}^{D_0}_{\mathscr{D}}$ consisting of the meromorphic functions $\mathbf{k}$ on M of the form (4.19) with $\mathbf{u}_0=\mathbf{0}$ was used in [5] to describe the holomorphic sections of $(\lambda_{-D_0}\otimes E_{\mathscr{D}})^*\cong\lambda_{D_0}\otimes E^*_{\mathscr{D}}$. In fact, what is described directly is the matrix-divisor subspace $\mathscr{L}_*((-D_0)\otimes\mathscr{D})$ for the right matrix divisor $(-D_0)\otimes\mathscr{D}$. By the discussion in Section 4.1, we see that the space $\mathscr{L}_*((-D_0)\otimes\mathscr{D})$ is isomorphic to the space of holomorphic sections for the bundle $(\lambda_{-D_0}\otimes E_{\mathscr{D}})^*$. That is, after taking transposes, the space of sections of the bundle $\lambda_{D_0}\otimes E^*_{\mathscr{D}}$ is seen to be equivalent to the space $\mathscr{L}_*((-D_0)\otimes\mathscr{D})$ of multivalued functions $\mathbf{k}$ on M satisfying $\mathbf{k}\in\mathscr{O}_u^{1\times r}\cdot F_{\mathscr{D}}f_{D_0}^{-1}$ where f_{D_0} is a trivialization of the line bundle λ_{D_0} (with classical line bundle conventions). As noted in [5, page 149], $\mathscr{L}_*((-D_0)\otimes\mathscr{D})$ can be identified concretely as a subspace of $(\mathscr{M}^{D_0}_{\mathscr{D}})_0$; the result is as follows.

PROPOSITION 4.6. *Let D_0 be a degree $g-1$ divisor of the form $n_1p_1+\cdots+n_sp_s-p_0$, where $D=n_1p_1+\cdots+n_sp_s$ is a nonspecial divisor and let the degree zero null-pole divisor $\mathscr{D}$ be of the form (4.10) with partitioning of indices as in (4.12). Define the block matrix $\Gamma_{\mathscr{D}}^{D_0}$ as in (4.22). Then the matrix-divisor space $\mathscr{L}_*((-D_0)\otimes\mathscr{D})$ of the divisor $(-D_0)\otimes\mathscr{D}$, an isomorphic copy of the space of holomorphic sections of the bundle $\lambda_{D_0}\otimes E_{\mathscr{D}}^*$, is equal to the subspace of row vector meromorphic functions $\mathbf{k}$ in $(\mathscr{M}_{\mathscr{D}}^{D_0})_0$, i.e., $\mathbf{k}$ has the form (4.19) with $\mathbf{u}_0=0$*

$$\mathbf{k}=\sum_{i\in\mathrm{I}\cup\mathrm{II}}\mathbf{u}_i f_{u_i,A_{\pi u_i}}^{D_0}C_{\pi_{u_i}}, \tag{4.25}$$

where the row vector $\mathbf{u}=\mathrm{row}_{i\in\mathrm{I}\cup\mathrm{II}}[\mathbf{u}_i]$ satisfies

$$\mathbf{u}\Gamma_{\mathscr{D}}^{D_0}=\mathbf{0}. \tag{4.26}$$

In particular, $h^0(\lambda_{D_0}\otimes E_{\mathscr{D}}^*)$ equals the dimension of the left-kernel of $\Gamma_{\mathscr{D}}^{D_0}$ acting on $\mathbb{C}^{1\times(n_{\pi u_1}+\cdots+n_{\pi u_{np}})}$.

Proof. By assumption the points $p_0,p_1,\dots,p_g$ are assumed to be disjoint from the points u_i $(i\in\mathrm{I}\cup\mathrm{II}\cup\mathrm{III})$ of the support of $\mathscr{D}$. Thus the criterion (4.23) from Proposition 4.5 (applied with $\mathbf{u}_0=0$) informs us that any $\mathbf{k}$ of the form (4.25) is in $\mathscr{O}_{u_i}^{1\times r}\cdot F_{\mathscr{D}}f_{D_0}^{-1}$ for each point u_i in the support of $\mathscr{D}$. Moreover, since each function $f_{u_i,A_{\pi u_i}}^{D_0}$ $(i\in\mathrm{I}\cup\mathrm{II})$ has a simple pole at each p_i and a simple zero at p_0, we see that any $\mathbf{k}$ of the form (4.25) is in $\mathscr{O}_{p_j}^{1\times r}\cdot F_{\mathscr{D}}f_{D_0^{-1}}$ for $j=0,1,\dots,g$ as well.

Conversely we argue that any function $\mathbf{k}'$ in $\mathscr{L}_*((-D_0)\otimes\mathscr{D})$ necessarily has the form (4.25) as follows. We choose vectors $\mathbf{u}_1,\dots,\mathbf{u}_{np}$ so that the function $\mathbf{k}$ given by (4.25) has the property that its singularities at the points $u_1,\dots,u_{np}$ exactly cancel the singularities of $\mathbf{k}'$ at these points, i.e., so that

$$\mathbf{k}-\mathbf{k}' \text{ is analytic on } M\setminus\{p_0,p_1,\dots,p_g\}.$$

From the assumptions on $\mathbf{k}$ and $\mathbf{k}'$, it follows that each matrix entry of $\mathbf{k}-\mathbf{k}'$ is in the classical divisor space $\mathscr{L}(D_0)$. As D is nonspecial, it follows that $\mathbf{k}-\mathbf{k}'=0$, i.e., $\mathbf{k}=\mathbf{k}'$ is in the space $(\mathscr{M}_{\mathscr{D}}^{D)})_0$. $\square$

The content of Proposition 4.6 is essentially the same as Theorem 6 from [5]; we note that our analysis here also corrects some misprints in the formula for Γ^λ given in [5, page 149].

If the divisor $\mathscr{D}$ given as in (4.10) has degree zero, then it follows from Proposition 4.6 that a necessary and sufficient condition for the bundle $E_{\mathscr{D}}$ (or, equivalently, $E_{\mathscr{D}}^*$) to have property NSF is that there be a choice of nonspecial divisor $D=p_1+\cdots+p_g$ so that, with $D_0=p_1+\cdots+p_g-p_0$, the associated matrix $\Gamma_{\mathscr{D}}^{D_0}$ is invertible.

The matrix $\Gamma_{\mathscr{D}}^{D_0}$ can be used to give criteria for the existence of a *nondegenerate* (i.e., with determinant not vanishing identically) $r\times r$-meromorphic matrix function with controlled pole behavior. To see this let $\mathscr{D}$ be given as in (4.10) and $D_0 = p_1 + \cdots + p_g - p_0$ be a divisor of degree $g-1$ with $D = p_1 + \cdots + p_g$ a nonspecial divisor. As usual we assume that the points $p_0, p_1 \ldots, p_g$ of D_0 are distinct from the points $u_1, \ldots, u_k$ $(k = n_\infty + n_c + n_0)$ of the divisor $\mathscr{D}$. Introduce the $r \times r$ meromorphic matrix function

$$K = U_0 + \sum_{i \in \mathrm{I} \cup \mathrm{II}} U_{u_i} f^{D_0}_{u_i, A_{\pi_{u_i}}} C_{\pi_{u_i}}, \tag{4.27}$$

where the matrix $U_i \in \mathbb{C}^{r \times n_{\pi_{u_i}}}$ for $i = 1, \ldots, n_P$ and U_0 is $r \times r$.

PROPOSITION 4.7. *Let $\mathscr{D}$ be a rank r degree zero right matrix null-pole divisor on M of the form (4.10) and let D_0 be a divisor of degree $g-1$ of the form $D_0 = n_1 p_1 + \cdots + n_s p_s - p_0$ ($p_0, p_1, \ldots, p_s$ taken to be distinct with multiplicities $n_j \geqslant 0$, $j = 1, \ldots, s$ satisfying $n_1 + \cdots + n_s = g$) such that $n_1 p_1 + \cdots + n_s p_s$ is a nonspecial divisor and the points $p_0, p_1, \ldots, p_s$ are all distinct from the points $u_1, \ldots, u_k (k = n_\infty + n_c + n_0)$. Let K be the meromorphic matrix function given in (4.27). The condition*

$$U_0 \cdot \mathrm{row}_{j \in \mathrm{II} \cup \mathrm{III}} [B_{\zeta_{n_{\zeta_j}}}] = \mathrm{row}_{i \in \mathrm{I} \cup \mathrm{II}} [U_i] \cdot \Gamma^{D_0}_{\mathscr{D}} \tag{4.28}$$

is necessary and sufficient for the germ of the meromorphic matrix function K given by (4.27) at u_0 to satisfy

$$\mathscr{O}^{r \times r}_{u_0} K \subset \mathscr{O}^{r \times r}_{u_0} F_{\mathscr{D}} \text{ for all } u_0 \in M \setminus \{p_1, \ldots, p_g\}. \tag{4.29}$$

In particular, if $\Gamma^{D_0}_{\mathscr{D}}$ is invertible, then for a fixed $r \times r$-matrix U_0 there exists a unique meromorphic matrix function K satisfying

1. *$K(p_0) = U_0$,*
2. *K satisfies (4.29), and*
3. *each entry of K has a possible pole at p_j of order at most n_j for $j = 1, \ldots, s$.*

Proof. The existence part of the result follows from the condition (4.24) which gives necessary and sufficient conditions for the rows of K to belong to $\mathscr{O}^{1 \times r}_{u_0} \cdot F_{\mathscr{D}}$. We omit further detail. As for the uniqueness, the conditions imply that each entry of K has its only poles in $\{u_i \colon i \in \mathrm{I} \cup \mathrm{II}\} \cup \{p_1, \ldots, p_s\}$ with the various multiplicities controlled by $\mathscr{D}$ and D_0. This forces K to have the form (4.27). Specifying the value of K at p_0 determines the matrix U_0. The fact that $K \in \mathscr{O}^{r \times r}_{u_j} F_{\mathscr{D}}$ for $j \in \mathrm{II} \cup \mathrm{III}$ then leads to the system of equations (4.28) which is just a matrix version of (4.23). The assumption that $\Gamma^{D_0}_{\mathscr{D}}$ is invertible then leads to the remaining coefficients $U_1, \ldots, U_{n_P}$ being uniquely determined. $\square$

Proposition 4.7 as presented here is a corrected version of Proposition 8 from [5, p. 152]. (The assertion that F^{-1} is analytic off $\{z_1, \ldots, z_k, p_0, \ldots, p_g\}$ should not have been included in the statement of Proposition 8 in [5, p. 152].)

We can now establish the following result:

THEOREM 4.8. *Let E be a rank r vector bundle of degree zero over the closed Riemann surface M of genus g and let ζ be the corresponding factor of automorphy on $\widehat{M}$. If the vector bundle E has property NSF, then there exists a nonspecial divisor D_{ns}, which we write out in the more detailed form $D_{ns} = n_1 p_1 + \cdots + n_s p_s$ where $p_1, \ldots, p_s$ are distinct with respective multiplicities $n_1, \ldots, n_s$ adding up to g, and a trivialization F of ζ such that the only poles of entries of F are at points in $\widehat{M}$ over the points p_j in the support of D_{ns}, with pole order at p_j at most equal to the multiplicity n_j of p_j in D_{ns} for $j = 1, \ldots s$.*

Proof. Assume that E has property NSF. Then we may choose a nonspecial divisor D so that $h^0(\lambda_D \otimes E) = 0$ Let G be a trivialization of the factor of automorphy ζ^* associated with E^*, so that $G_* = (G^\top)^{-1}$ is a trivialization of the factor of automorphy ζ associated to E. Let $\mathscr{D}$ be the right matrix divisor of G restricted to M_0. Note that $E^* \cong E^*_{\mathscr{D}}$. Since $h^0(\lambda_{D_0} \otimes E_{\mathscr{D}}) = h^0(\lambda_{D_0} \otimes E) = 0$, it follows from Proposition 4.6 that the matrix $\Gamma^{D_0}_{\mathscr{D}}$ is invertible. We may then apply Proposition 4.7 to get a uniquely determined nondegenerate meromorphic matrix function K satisfying properties (1), (2), (3) with $U_0 = I_r$ (or any invertible matrix). By taking transposes, we may conclude from property (2) that the germ of $K^\top$ belongs to $G^\top \cdot \mathscr{O}^{r\times r}_{u_0} = (G_*)^{-1} \cdot \mathscr{O}^{r\times r}_{u_0}$ for $u_0 \neq p_1, \ldots, p_g$. Thus $G_* K^\top$ is analytic off $p_1, \ldots, p_g$. Since K is a nondegenerate meromorphic function, $F = G_* K^\top$ of ζ is also a trivialization of ζ. The only poles of entries of F are at the points p_j, with multiplicity at most n_j, $j = 1, \ldots, s$. Thus F is a trivialization of ζ with the desired properties. $\square$

The next result assures us that degree zero vector bundles having property NSF always have trivializations with pole behavior analogous to that of the trivializations (2.11) when $g = 1$. The proof of this result depends on the fact that the entries of the matrix $\Gamma^{D_0}_{\mathscr{D}}$ depend continuously on the divisor D; we postpone the proof of this latter result to Section 4.4 where we discuss explicit formulas for the entries of the matrix $\Gamma^{D_0}_{\mathscr{D}}$.

THEOREM 4.9. *Let E be a rank r vector bundle of degree zero over the closed Riemann surface M of genus g and let ζ be the corresponding factor of automorphy on $\widehat{M}$. If the vector bundle E has property NSF, then there exists a non-special divisor $p_1 + \cdots + p_g$, where the points $p_1, \ldots, p_g$ are distinct and a trivialization F of ζ such that the only poles of entries of F are simple poles at points in $\widehat{M}$ over the points $p_1, \ldots, p_g$.*

Proof. Suppose that the bundle E has property NSF with associated nonspecial divisor $D = p_1 + \cdots + p_g$ and $D_0 = p_1 + \cdots + p_g - p_0$ such that $h^0(\lambda_{D_0} \otimes E) = 0$. Again by Proposition 4.6 we get that the matrix $\Gamma^{D_0}_{\mathscr{D}}$ is invertible. A consequence of [13, page 91] already noted is that the nonspecial divisor $D = g_1 + \cdots + p_g$ can be approximated arbitrarily well (in the topology of $M^{(g)}$) by a nonspecial divisor $D' = p'_1 + \cdots + p'_g$ with $p'_1, \ldots, p'_g$ distinct. The block entries of the matrix $\Gamma^{D_0}_{\mathscr{D}}$ involve the building blocks $f^{D_0}_{kw}$ as in formula (4.18). We keep the local admissible Sylvester data sets Υ_{q_0} (4.11) fixed and perturb only the divisor D_0. Then the matrix entries of $\Gamma_{\mathscr{D}}^{D_0}$ move continuously as long as the canonical scalar functions $f_{kw}^{D_0}$ are continuous with respect to the support $p_1+\cdots+p_g \in M^{(g)}$ of D_0, which is precisely the content of Corollary 4.16 discussed below. Since invertibility is an open condition, it follows that $\Gamma_{\mathscr{D}}^{D_0'}$ is again invertible as long as the point $(p_1',\dots,p_g')$ is arranged to be sufficiently close to $(p_1,\dots,p_g)$ in $M^{(g)}$. We now use the construction in Theorem 4.8 to see that E has a trivialization F having only possible poles occurring at the points $p_1',\dots,p_g'$ with pole order at p_j' at most 1. $\square$

4.3. Automorphic interpolants with given divisor

Let $\mathscr{D}$ be a degree zero null-pole divisor. We say that the meromorphic $r\times r$ matrix-valued function F *interpolates the right matrix null-pole divisor* $\mathscr{D}$ if

$$\mathscr{O}_{u_0}^{r\times r}F = \mathscr{O}_{u_0}^{r\times r}F_{\mathscr{D}} \text{ for all } u_0 \in M. \tag{4.30}$$

An equivalent condition is that

$$\mathscr{O}_{u_0}^{1\times r}F = \mathscr{S}(\Upsilon_{u_0},u_0) \text{ for all } u_0 \in M$$

if the divisor $\mathscr{D}$ is given in terms of tagged 0-admissible Sylvester data sets as in (4.10). We now present our solutions of the First and Second Interpolation Problems from the Introduction.

To formulate the solution, it is useful to introduce another definition and some additional notation. Given the degree zero null-pole divisor $\mathscr{D}$ as in (4.10), let us say that a divisor D_0 is *$\mathscr{D}$-admissible* if D_0 is a degree $(g-1)$ divisor of the form

$$D_0 = p_1+\ldots+p_g-p_0 \tag{4.31}$$

where $D=p_1+\cdots+p_g$ is nonspecial with distinct points $p_0,\dots,p_g$ distinct from $u_1,\dots,u_k$ $(k=n_\infty+n_c+n_0)$. The additional notation is:

$$\begin{aligned}
&R_{ij} = \operatorname{res}_{p_j}[f_{u_i,A_{\pi_i}}^{D_0}(u)C_{\pi_i}] \text{ for } i=1,\dots,n_P,\ j=1,\dots,g,\\
&R = \begin{bmatrix} R_{11} & \cdots & R_{1g}\\ \vdots & & \vdots\\ R_{n_p1} & \cdots & R_{n_pg}\end{bmatrix}, \quad F_{A_\pi}^{D_0}(u) = \operatorname{diag.}_{i\in \mathrm{I}\cup\mathrm{II}}[f_{u_i,A_{\pi_i}}^{D_0}(u)],\\
&B_\zeta = \operatorname{row}_{j\in \mathrm{II}\cup\mathrm{III}}[B_{\zeta_i}], \quad C_\pi = \operatorname{col.}_{j\in \mathrm{I}\cup\mathrm{II}}[C_{\pi_j}].
\end{aligned} \tag{4.32}$$

THEOREM 4.10. *Let $\mathscr{D}$ be a degree zero null/pole divisor as in* (4.10). *Then:*

1. **Solution of the First Interpolation Problem:** *The First Interpolation Problem has a solution, i.e., there exists a (single-valued) meromorphic function F on M which interpolates the divisor $\mathscr{D}$, if and only if, for any choice of $\mathscr{D}$-admissible*

divisor D_0 as in (4.31), the matrix $\Gamma_{\mathscr{D}}^{D_0}$ given by (4.22) is invertible with inverse $(\Gamma_{\mathscr{D}}^{D_0})^{-1}$ satisfying the side constraint (with notation as in (4.32))

$$B_\zeta(\Gamma_{\mathscr{D}}^{D_0})^{-1}R = 0. \tag{4.33}$$

When this is the case, then the unique interpolant with invertible value U_0 at p_0 is given by

$$K(u) = U_0(I_r + B_\zeta(\Gamma_{\mathscr{D}}^{D_0})^{-1}F_{A_\pi}^{D_0}(u)C_\pi) \tag{4.34}$$

2. **Solution of the Second Interpolation Problem:** *A sufficient condition for the Second Interpolation Problem to have a solution, i.e., for the existence a relatively automorphic meromorphic matrix function $\widehat{F}$ on $\widehat{M}$ with a flat factor of automorphy $\zeta_{\widehat{F}}$ which interpolates the null/pole divisor $\mathscr{D}$, is that there exist a $\mathscr{D}$-admissible divisor D_0 as in (4.31) so that the matrix $\Gamma_{\mathscr{D}}^{D_0}$ given by (4.22) is invertible. If M has genus $g=1$, then this sufficient condition is also necessary.*

Proof. Statement (2) is the content of Corollary 7 from [5]. It remains to verify statement (1).

Assume first that there is an $r\times r$-meromorphic matrix function G on M interpolating the null-pole data $\mathscr{D}$. We first note that the existence of such a G is equivalent to the holomorphic triviality of the the bundle $E_{\mathscr{D}}$ and the factor of automorphy $\zeta_{\mathscr{D}}$. As a consequence, $h^0(\lambda_{D_0}\otimes E_{\mathscr{D}}^*) = 0$. It follows from Proposition 4.6 that $\Gamma_{\mathscr{D}}^{D_0}$ is invertible.

For each $i\in \mathrm{I}\cup\mathrm{II}$, near u_i $(i\in \mathrm{I}\cup\mathrm{II})$ there is a $r\times n_{\pi_i}$- matrix $\widetilde{B}_i$ such that

$$G(u) - \widetilde{B}_i((u-u_i)-A_{\pi_i})^{-1} \tag{4.35}$$

is analytic. Let K be the meromorphic matrix function

$$K = G(p_0) + \sum_{i=1}^{n_P}\widetilde{B}_i f_{w_i,A_{\pi_i}}^{D_0} C_{\pi_i},$$

which is of the form (4.27). It follows that the matrix function $H = G-K$ can only have poles at $p_1,\dots,p_g$. Moreover, the divisors (H_{ij}) of the entries H_{ij} of $H = G-K$ satisfy $(H_{ij})+D_0 \geqslant 0$. Thus $G=K$. Since G does not have poles at the points $p_1,\dots,p_g$, then

$$\left[\widetilde{B}_1 \ \dots \ \widetilde{B}_{n_P}\right]R = 0. \tag{4.36}$$

From the fact that $K\in \mathscr{O}_{u_i}^{r\times r}F_{\mathscr{D}}$ for $i\in \mathrm{II}\cup\mathrm{III}$, application of the criterion (4.23) to each row of K shows that

$$\left[\widetilde{B}_1 \cdots \widetilde{B}_{n_P}\right]\Gamma_{\mathscr{D}}^{D_0} = G(p_0)B_\zeta.$$

As both $\Gamma_{\mathscr{D}}^{D_0}$ and $G(p_0) := U_0$ are invertible, we may solve uniquely for $\left[\widetilde{B}_1 \cdots \widetilde{B}_{n_P}\right]$ to get

$$\left[\widetilde{B}_1 \cdots \widetilde{B}_{n_P}\right] = U_0B_\zeta(\Gamma_{\mathscr{D}}^{D_0})^{-1}.$$

Substituting this expression for $\widetilde{B}_i$ into (4.36) and (4.35) leads us to the validity of the side condition (4.33) and to the formula (4.34) for G.

We next turn to the sufficiency of the stated conditions. We define $K(u)$ by (4.34). Note that K has the form (4.27) with

$$\begin{bmatrix} U_{u_1} & \cdots & U_{u_{nP}} \end{bmatrix} = U_0 \begin{bmatrix} B_{\zeta_1} & \cdots & B_{\zeta_{nP}} \end{bmatrix} (\Gamma_{\mathscr{D}}^{D_0})^{-1}.$$

By Proposition 4.7, K is the unique meromorphic matrix function satisfying conditions (1), (2), (3) in Proposition 4.7. The fact that condition (4.33) is satisfied tells us that K has no poles in $\{p_1, \dots, p_g\}$. We conclude that condition (4.29) actually holds at all $u_0 \in M$. It remains to show that (4.29) actually holds with equality on all of M.

To this end we let $F_{\mathscr{D}}$ be a trivialization of $\zeta_{\mathscr{D}}$. We now view K as an automorphic meromorphic matrix function on all of $\widehat{M}$. From condition (4.29) we read off that the $r \times r$ matrix function $H := KF_{\mathscr{D}}^{-1}$ is holomorphic on all of $\widehat{M}$. Note that $K(p_0) = U_0$ is invertible, so $\det K$ does not vanish identically. As $\det K$ is a single-valued meromorphic matrix function on M, the winding number of $\det K$ around the boundary of the fundamental domain M_0 is zero. The function $F_{\mathscr{D}}$ a priori is multivalued when considered as a function on M, but since it is interpolating the divisor $\mathscr{D}$ which has degree equal to 0, it follows that the winding number of $\det F_{\mathscr{D}}$ around the boundary of M_0 is also zero. As a consequence, $\det H$ has no zeros on M_0. This implies H is invertible on $\widehat{M}$ and hence

$$\mathscr{O}_{u_0}^{r\times r} K = \mathscr{O}_{u_0}^{r\times r} H F_{\mathscr{D}} = \mathscr{O}_{u_0}^{r\times r} F_{\mathscr{D}}$$

for all $u_0 \in \widehat{M}$, i.e., equality holds in (4.29) for all u_0 as required. This completes the proof. $\square$

REMARK 4.11. The First Interpolation Problem was addressed in [4] for the case $g = 1$ and in [5] (see Theorem 9 there) for the case of arbitrary genus. The result in [4] stated the result only for the simple multiplicity case; the actual statement was somewhat more cumbersome since it was missed there that the matrix $\Gamma_{\mathscr{D}}^{D_0}$ necessarily is invertible. Theorem 9 in [5] handles the general multiplicity case but the proof there has a gap since it appears to rely on the misstatements in Proposition 8 there mentioned above. The present proof uses the corrected version (Proposition 4.7 above) of Proposition 8 from [5].

REMARK 4.12. Interpolation problems for relatively automorphic meromorphic matrix functions on Riemann surfaces closely related to those considered here were also studied in [9, 10]. The problem considered in [10] was as follows:

(IV) **Fourth Interpolation Problem:** Given two flat factors of automorphy $\widetilde{\zeta}$ and ζ such that $h^0(E_{\widetilde{\zeta}} \otimes \Delta) = h^0(E_\zeta \otimes \Delta) = 0$ where Δ is a line bundle (or divisor) of half-order differentials (see Section 4.4 below) and given a left matrix null/pole divisor $\mathscr{D}$ on M (assumed to have pole and zero order of at most 1 at each point), find a meromorphic matrix function $\widehat{G}$ on $\widehat{M}$ so that (i) $\widehat{G}(Tu) = \widetilde{\zeta}(T)\widehat{G}(u)\zeta(T)^{-1}$ for all $u \in \widehat{M}$ and deck transformations T, and (ii) the null/pole structure of F on M is as prescribed by $\mathscr{D}$.

The interpolation problem considered in [9] was the same with two modifications: (1) rather than specifying the input factor of automorphy ζ (or equivalently, the associated bundle E_ζ), it was only specified that there should be such a bundle and part of the problem was to solve also for this bundle, and (2) the formulation was more concrete: it was assumed that M is the normalizing Riemann surface for an algebraic curve $\mathbf{C} = \{\mu \in \mathbb{P}^w : \mathbf{p}(\mu) = 0\}$ embedded in projective space $\mathbb{P}^2$ and that the given bundle $E_{\widetilde{\zeta}}$ and the bundle to be found E_ζ are presented concretely as kernel bundles associated with determinantal representations of the polynomial $\mathbf{p}^r$ (r equal to the rank of the $E_{\widetilde{\zeta}}$ and E_ζ):

$$E_{\widetilde{\zeta}} = \{((\mu,u) : \mu \in \mathbf{C}, u \in \mathbb{C}^M : (\mu_2\widetilde{\sigma}_1 + \mu_1\widetilde{\sigma}_2 + \mu_0\widetilde{\gamma})u = 0\},$$
$$E_\zeta = \{((\mu,u) : \mu \in \mathbf{C}, u \in \mathbb{C}^M : (\mu_2\sigma_1 + \mu_1\sigma_2 + \mu_0\gamma)u = 0\}$$

where

$$\mathbf{p}(\mu)^r = \det(\mu_2\widetilde{\sigma}_1 + \mu_1\widetilde{\sigma}_2 + \mu_0\widetilde{\gamma}) = \det(\mu_2\sigma_1 + \mu_1\sigma_2 + \mu_0\gamma).$$

The precise connection between the Fourth Interpolation Problem and the variant considered in [9] is explained in some detail in Section 6 of [10]

To compare Problems (IV) and (I), we identify a special case of Problem (IV) which can be related to a special case of Problem (I) as follows. We first note that one point of incompatibility between the two problems is that Problem (IV) is formulated in terms of a left matrix null/pole divisor while Problem (I) is formulated in terms of a right matrix null/pole divisor. However, if $\mathscr{D}$ as in (4.10) and (4.11) is a right matrix null/pole divisor, then $\mathscr{D}' = \{\Upsilon_{q_0} : q_0 \in M_0\}$ with

$$\Upsilon'_{q_0} = \{(C_{\pi_{q_0}}, A_{\pi_{q_0}}), (A_{\zeta_{q_0}}, B_{\zeta_{q_0}}), -S_{q_0})$$

is a left matrix null/pole divisor and if the meromorphic matrix function $\widehat{F}$ has right null/pole structure fitting $\mathscr{D}$ if and only if $G = F^{-1}$ has left null/pole structure fitting $\mathscr{D}'$. Thus we may reformulate Problem (IV) in terms of $F = G^{-1}$ rather than F: then we must have an F with prescribed right null/pole structure prescribed by the right matrix null/pole divisor $\mathscr{D}$ over M_0 which in addition has the relative automorphy property

$$F(Tu) = \zeta(T)\widehat{F}(u)\widetilde{\zeta}(T)^{-1}. \tag{4.37}$$

If we also insist that $\zeta = \widetilde{\zeta} = \zeta_0 \otimes I_r$ for a flat scalar factor of automorphy ζ_0, then the relative automorphy property (4.37) imposed on F just means that F is automorphic: $F(Tu) = F(u)$ for all deck transformations T. Let $D = p_1 + \cdots + p_g$ be a nonspecial divisor and choose ζ_0 to be the flat line bundle so that so that $\zeta_0 \otimes \Delta = \lambda_{D_0}$ where $D_0 = D - p_0$. Then the reformulation of Problem (IV) becomes exactly Problem (I). The one additional point is that the work in [10] was only carried out for the case where the divisor $\mathscr{D}$ has pole and zero order of at most 1 at each point. To compare solutions, for simplicity we shall also insist that poles and zeros are disjoint and of multiplicity 1.

We therefore assume that there are points $\mu_1, \ldots, \mu_N \in M$ (the poles) and $\lambda_1, \ldots, \lambda_N \in M$ (the zeros), all distinct, along with a specified nonzero $1 \times r$ row vector u_j (the left pole vector at μ_j) and a nonzero $r \times 1$ column vector x_j (the right null vector at λ_j) ($j = 1, \dots, N$) so that

$$\Upsilon_{q_0} = \begin{cases} ((x_j, 0), (\emptyset, \emptyset), \emptyset) & \text{if } q_0 = \mu_j, \\ ((\emptyset, \emptyset), (0, u_i), \emptyset) & \text{if } q_0 = \lambda_i, \\ \emptyset & \text{otherwise.} \end{cases} \tag{4.38}$$

Then the solution criterion from [10, Theorem 3.1] (after transcription from right to left formulation as explained above) is: *a (necessarily unique) solution exists if and only if*

$$\Gamma^0 = \left[\Gamma^0_{ij}\right]_{i,j=1}^N = \left[-u_i \left(K(\zeta_0; \mu_i, \lambda_j) \otimes I_r\right) x_j\right]_{i,j=1}^N \tag{4.39}$$

is invertible, together with a linear side-constraint to guarantee that the solution has no poles at the points $p_1, \dots, p_g$. Here $K(\zeta_0; \cdot, \cdot)$ is the Cauchy kernel associated with the flat factor of automorphy ζ_0 (see the appendix Section 4.4 below for a brief introduction to this Cauchy kernel).

We note that the same problem has a solution via statement (1) in Theorem 4.10: *a (necessarily unique) solution exists if and only if the matrix*

$$\Gamma^{D_0}_{\mathscr{D}} = [-\mathrm{res}_{p=\lambda_j} f^{D_0}_{\mu_i}(p) x_i u_j (z_{\lambda_j})^{-1}]_{i,j=1}^N$$

is invertible, together with a linear side-constraint to guarantee that the solution has no poles at the points $p_1, \dots, p_g$. By our assumptions that poles and zeros are distinct, $f^{D_0}_{\mu_i}$ is analytic at λ_j. From the formula (4.76) explained in Theorem 4.14 below, there is a connection between the building-block functions $f^{D_0}_{\mu}$ and the Cauchy kernel, namely:

$$f^{D_0}_{\mu_i}(\lambda_j) = \frac{K(\zeta_0; \lambda_j, \mu_i)}{K(\zeta_0; \lambda_j, p_0)} K(\zeta_0; \mu_i, p_0).$$

Trivially $\mathrm{res}_{p=\lambda_j}(z_{\lambda_j})^{-1} = 1$ (where z_{λ_j} is the local coordinate on M centered at λ_j). Hence the formula for $\Gamma^{D_0}_{\mathscr{D}}$ in this case becomes

$$\Gamma^{D_0}_{\mathscr{D}} = \left[-f^{D_0}_{\mu_i}(\lambda_j) x_i u_j\right]_{i,j=1}^N = \left[-\frac{K(\zeta_0; f\lambda_j, \mu_i)}{K(\zeta_0; \lambda_j, p_0)} K(\zeta_0; \mu_i, p_0) x_i u_j\right]_{i,j=1}^N. \tag{4.40}$$

We note that the matrix $\Gamma^{D_0}_{\mathscr{D}}$ is equal to the matrix Γ^0 (4.39) multiplied on the left and on the right by invertible diagonal matrices, i.e., invertibility of Γ^0 is equivalent to invertibility of $\Gamma^{D_0}_{\mathscr{D}}$. In this way we see directly the equivalence of the solutions of this special interpolation problem as given by Theorem 4.10 and as given by Theorem 3.1 in [10].

REMARK 4.13. (**Abel's theorem.**) Let us specialize the setting of Remark 4.12 even further by assuming that $r = 1$, i.e., we wish to solve for a scalar meromorphic function (or more generally, relatively automorphic function with flat factor of automorphy) on $\widehat{M}$ with prescribed distinct simple zeros $\lambda_1, \dots, \lambda_N$ and prescribed distinct simple poles $\mu_1, \dots, \mu_N$ in M. We therefore assume that the divisor $\mathscr{D}$ is given as $\mathscr{D} = \{\Upsilon_{q_0} : q_0 \in M\}$ where Υ_{q_0} is given as in (4.38) with each u_j and x_i taken to be the complex number 1. When combined with formula (4.74) from the Appendix for the Cauchy kernel, we see that the formula (4.40) for $\Gamma_{\mathscr{D}}^{D_0}$ simplifies further to

$$\Gamma_{\mathscr{D}}^{D_0} = \left[-f_{\mu_i}^{D_0}(\lambda_j)\right]_{i,j=1}^{N} = \left[-\frac{K(\zeta_0;\lambda_j,\mu_i)K(\zeta_0;\mu_i,p_0)}{K(\zeta_0;\lambda_j,p_0)}\right]_{i,j=1}^{N} \tag{4.41}$$
$$= \left[-\frac{1}{\theta(\mathbf{e})}\frac{\theta(\phi(\mu_i)-\phi(\lambda_j)+\mathbf{e})}{E_\Delta(\mu_i,\lambda_j)}\frac{E_\Delta(p_0,\lambda_j)}{\theta(\phi(p_0)-\phi(\lambda_j)+\mathbf{e})}\frac{\theta(\phi(p_0)-\phi(\mu_i)+\mathbf{e})}{E_\Delta(p_0,\mu_i)}\right]$$

where we use the notation

$$\mathbf{e} = a\Omega + b. \tag{4.42}$$

We note that our Theorem 4.10 part (2) gives invertibility of $\Gamma_{\mathscr{D}}^{D_0}$ for some $\mathscr{D}$-admissible divisor D_0 as a sufficient condition for the existence of a flat relatively automorphic solution of the interpolation problem.

On the other hand, it is well known for this scalar version of the problem that, given such a scalar divisor $D = \lambda_1 + \cdots + \lambda_N - \mu_1 - \cdots - \mu_N$, there always exists a function f_D on $\widehat{M}$ with flat factor of automorphy interpolating the data D on M. Such a function can be constructed using the prime form:

$$f(p) = \frac{\prod_{j=1}^{N} E_\Delta(p,\lambda_j)}{\prod_{i=1}^{N} E_\Delta(p,\mu_i)}. \tag{4.43}$$

The construction in the single-valued case where $\sum_{i=1}^{N}\phi(\mu_i) = \sum_{j=1}^{N}\phi(\lambda_j)$ mod the period lattice is carried out in Mumford's book[23, pages 3.209–3.212]; the reader can check that more generally the factor of automorphy in (4.43) is flat as long as the number of prescribed zeros is equal to the number of prescribed poles (counting multiplicities).

This stronger result for the scalar case can be explained as follows. From the last formula for $\Gamma_{\mathscr{D}}^{D_0}$ in (4.41), we see the factorization for $\Gamma_{\mathscr{D}}^{D_0}$:

$$\Gamma_{\mathscr{D}}^{D_0} = -\frac{1}{\theta(\mathbf{e})} \cdot \mathbf{D}_{\boldsymbol{\mu}} \cdot \mathbf{M} \cdot \mathbf{D}_{\boldsymbol{\lambda}} \tag{4.44}$$

where

$$\mathbf{D}_{\boldsymbol{\mu}} = \mathrm{diag}_{1\leqslant i\leqslant N}\left[\frac{\theta(\phi(p_0)-\phi(\mu_i)+\mathbf{e})}{E_\Delta(p_0,\mu_i)}\right],$$
$$\mathbf{M} = \left[\frac{\theta(\phi(\mu_i)-\phi(\lambda_j)+\mathbf{e})}{E_\Delta(\mu_i,\lambda_j)}\right]_{i,j=1}^{N},$$
$$\mathbf{D}_{\boldsymbol{\lambda}} = \mathrm{diag}_{1\leqslant j\leqslant N}\left[\frac{E_\Delta(p_0,\lambda_j)}{\theta(\phi(p_0)-\phi(\lambda_j)+\mathbf{e})}\right].$$

As $\mathbf{D}_{\boldsymbol{\mu}}$ and $\mathbf{D}_{\boldsymbol{\lambda}}$ are invertible diagonal matrices, we see that $\Gamma_{\mathscr{D}}^{D_0}$ is invertible if and only if the middle factor $\mathbf{M}$ is invertible. It turns out the $\det\mathbf{M}$ can be computed explicitly (see Corollary 2.19 page 33 in Fay's book [14]):

$$\det \mathbf{M} = \theta\left(\sum_i \phi(\mu_i) - \sum_j \phi(\lambda_j) + \mathbf{e}\right)\theta(\mathbf{e})^{N-1}\frac{\prod_{i<j} E_\Delta(\mu_i,\mu_j)E_\Delta(\lambda_j,\lambda_i)}{\prod_{i,j} E_\Delta(\mu_i,\lambda_j)}. \tag{4.45}$$

Since $\theta(\mathbf{e}) \neq 0$ and the last factor is automatically nonzero by the assumption that the points $\mu_1,\dots,\mu_N,\lambda_1,\dots,\lambda_N$ are all distinct, we see that the criterion for $\det\Gamma_{\mathscr{D}}^{D_0} \neq 0$ is given by

$$\theta\left(\sum_{i=1}^N \phi(\mu_i) - \sum_{j=1}^N \phi(\lambda_j) + \mathbf{e}\right) \neq 0 \tag{4.46}$$

with $\mathbf{e}$ as in (4.42). By inspection we see that condition (4.46) holds for a generic choice of line bundle ζ. We conclude that in the scalar case, part (2) of Theorem 4.10 can be strengthened to: *the matrix $\Gamma_{\mathscr{D}}^{D_0}$ is invertible for a generic choice of $\mathscr{D}$-admissible divisor D_0 and the Second Interpolation Problem is always solvable.*

Unlike the classical solution in terms of prime forms from [22], our solution of the Second Interpolation Problem for the simple-multiplicity scalar case goes through Weil's characterization of flat bundles and is not constructive. The formula (4.34) used to solve the First Interpolation Problem for this case gives an automorphic (single-valued) meromorphic function interpolating the divisor $\mathscr{D}$ but carrying possible extra poles in $\{p_1,\dots,p_g\}$ along with compensating additional zeros at undetermined points. The side constraint (4.33) removes any extra poles in $\{p_1,\dots,p_g\}$ and thereby generates a single-valued solution of the interpolation problem (i.e., a solution of the First Interpolation Problem). The linear side constraint (4.33), when spelled out for this case, becomes

$$\sum_{i,j=1}^N \left[(\Gamma_{\mathscr{D}}^{D_0})^{-1}\right]_{ij}\frac{K(\zeta_0;p_k,\mu_j)K(\zeta_0;\mu_j,p_0)}{\frac{d}{dp}|_{p=p_k}\{K(\zeta_0;p,p_0)\}} = 0 \text{ for } k=1,\dots,g \tag{4.47}$$

with $\Gamma_{\mathscr{D}}^{D_0}$ as in (4.41).

The entries of $(\Gamma_{\mathscr{D}}^{D_0})^{-1}$ can be spelled out more explicitly as follows. From the factorization (4.44) for $\Gamma_{\mathscr{D}}^{D_0}$, we get the factorization for $(\Gamma_{\mathscr{D}}^{D_0})^{-1}$:

$$(\Gamma_{\mathscr{D}}^{D_0})^{-1} = -\theta(\mathbf{e})\cdot \mathbf{D}_{\boldsymbol{\lambda}}^{-1}\cdot \mathbf{M}^{-1}\cdot \mathbf{D}_{\boldsymbol{\mu}}^{-1}. \tag{4.48}$$

We note that each minor of $\mathbf{M}$, $\mathbf{M}_{ij}$ (the $(N-1)\times(N-1)$ submatrix of $\mathbf{M}$ formed by crossing out row i and column j), has the same form as M; the only adjustment is that the set of poles is one less with μ_i omitted and the set of zeros is one less with λ_j omitted. One can therefore compute the entries $C_{\alpha\beta} = (-1)^{\alpha+\beta}\det\mathbf{M}_{\beta\alpha}$ by another application of Fay's identity; the result is

$$\begin{aligned} C_{\alpha\beta} =& (-1)^{\alpha+\beta}\,\theta\Big(\sum_{j\neq\beta}\phi(\mu_j) - \sum_{i\neq\alpha}\phi(\lambda_i) + \mathbf{e}\Big)\cdot\theta(\mathbf{e})^{N-2} \\ &\cdot\frac{\prod_{i<j:\ i,j\neq\beta} E_\Delta(\mu_i,\mu_j)\prod_{i<j:\ i,j\neq\alpha} E_\Delta(\lambda_j,\lambda_i)}{\prod_{i,j:\ i\neq\alpha,j\neq\beta} E_\Delta(\mu_j,\lambda_i)}. \end{aligned} \tag{4.49}$$

By Cramer's Rule, the entries $(M^{-1})_{\alpha\beta}$ $(1 \leqslant \alpha, \beta \leqslant N)$ of M^{-1} are given by $(\mathbf{M}^{-1})_{\alpha\beta} = \frac{C_{\alpha\beta}}{\det \mathbf{M}}$. We then compute

$$(\mathbf{M}^{-1})_{\alpha\beta} = \frac{C_{\alpha\beta}}{\det \mathbf{M}} = (-1)^{\alpha+\beta} \frac{1}{\theta(\mathbf{e})} \cdot \frac{\theta(\sum_{j\neq\beta}\phi(\mu_j) - \sum_{i\neq\alpha}\phi(\lambda_i) + \mathbf{e})}{\theta(\sum_j \phi(\mu_j) - \sum_i \phi(\lambda_i) + \mathbf{e})} \cdot \frac{\prod_{i<j:\, i,j\neq\beta} E_\Delta(\mu_i,\mu_j)}{\prod_{i<j} E_\Delta(\mu_i,\mu_j)} \cdot \frac{\prod_{i<j:\, i,j\neq\alpha} E_\Delta(\lambda_j,\lambda_i)}{\prod_{i<j} E_\Delta(\lambda_j,\lambda_i)} \cdot \frac{\prod_{i,j} E_\Delta(\mu_j,\lambda_i)}{\prod_{i,j:\, i\neq\alpha, j\neq\beta} E_\Delta(\mu_j,\lambda_i)}.$$

Noting the cancellations in the prime-form terms then compactifies this expression to

$$\begin{aligned}(\mathbf{M}^{-1})_{\alpha\beta} &= (-1)^{\alpha+\beta} \frac{1}{\theta(\mathbf{e})} \cdot \frac{\theta(\sum_{j\neq\beta}\phi(\mu_j) - \sum_{i\neq\alpha}\phi(\lambda_i) + \mathbf{e})}{\theta(\sum_j \phi(\mu_j) - \sum_i \phi(\lambda_i) + \mathbf{e})} \cdot \\ &\quad \cdot \frac{1}{\prod_{i<\beta} E_\Delta(\mu_i,\mu_\beta)\cdot \prod_{\beta<j} E_\Delta(\mu_\beta,\mu_j)} \cdot \frac{1}{\prod_{i<\alpha} E_\Delta(\lambda_\alpha,\lambda_i)\cdot \prod_{\alpha<j} E_\Delta(\lambda_j,\lambda_\alpha)} \cdot \\ &\quad \cdot \prod_i E_\Delta(\mu_\beta,\lambda_i)\cdot \prod_j E_\Delta(\mu_j,\lambda_\alpha) \cdot \frac{1}{E(\mu_\beta,\lambda_\alpha)} \\ &= \frac{1}{\theta(\mathbf{e})} \cdot \frac{\theta(\sum_{j\neq\beta}\phi(\mu_j) - \sum_{i\neq\alpha}\phi(\lambda_i) + \mathbf{e})}{\theta(\sum_j \phi(\mu_j) - \sum_i \phi(\lambda_i) + \mathbf{e})} \cdot \\ &\quad \cdot \frac{\prod_i E_\Delta(\mu_\beta,\lambda_i)\cdot \prod_j E_\Delta(\mu_j,\lambda_\alpha)}{\prod_{j\neq\beta} E_\Delta(\mu_j,\mu_\beta)\cdot \prod_{i\neq\alpha} E_\Delta(\lambda_\alpha,\lambda_i)\cdot E(\mu_\beta,\lambda_\alpha)} \end{aligned} \tag{4.50}$$

where we used the prime-form property $E_\Delta(x,y) = -E_\Delta(y,x)$ in the last step. From the factorization (4.48) we get the still longer formula for $(\Gamma_{\mathscr{D}}^{D_0})^{-1}$:

$$\begin{aligned}[(\Gamma_{\mathscr{D}}^{D_0})^{-1}]_{\alpha\beta} &= -\frac{\theta(\phi(p_0) - \phi(\lambda_\alpha) + \mathbf{e})}{E_\Delta(p_0,\lambda_\alpha)} \cdot \frac{\theta(\sum_{j\neq\beta}\phi(\mu_j) - \sum_{i\neq\alpha}\phi(\lambda_i) + \mathbf{e})}{\theta(\sum_j \phi(\mu_j) - \sum_i \phi(\lambda_i) + \mathbf{e})} \cdot \\ &\quad \cdot \frac{\prod_j E_\Delta(\mu_j,\lambda_\alpha)\cdot \prod_i E_\Delta(\mu_\beta,\lambda_i)}{\prod_{j\neq\beta} E_\Delta(\mu_j,\mu_\beta)\cdot \prod_{i\neq\alpha} E_\Delta(\lambda_i,\lambda_\alpha)} \cdot \frac{1}{E_\Delta(\mu_\beta,\lambda_\alpha)} \cdot \\ &\quad \cdot \frac{E_\Delta(p_0,\mu_\beta)}{\theta(\phi(p_0) - \phi(\mu_\beta) + \mathbf{e})}. \end{aligned} \tag{4.51}$$

On the other hand, Abel's Theorem (see e.g. [13, page 97], [22, pages 145–160], or [21, Chapter 8]) tells us that this scalar null/pole interpolation problem has a solution exactly when

$$\sum_{i=1}^N \phi(\mu_i) = \sum_{j=1}^N \phi(\lambda_j) + m + n\Omega \text{ for some } m,n \in \mathbb{Z}^g \tag{4.52}$$

where ϕ is the Abel-Jacobi map (4.61); indeed, this is just the condition to force the prime-form solution (4.43) to be single-valued. It follows that the side condition (4.47) must be equivalent to the Abel condition (4.53).

Assume that the Abel condition (4.53) holds. For purposes of computation, we can view all functions as being defined on the universal cover of M and choose a new divisor $\widetilde{\lambda}_1 + \cdots \widetilde{\lambda}_N - \widetilde{\mu}_1, \dots, \widetilde{\mu}_N$ sitting above $\lambda_1 + \cdots + \lambda_n - \mu_1 - \cdots - \mu_N$ on the universal cover so that we have the stronger version $\sum_{i=1}^N \phi(\widetilde{\mu}_i) = \sum_{j=1}^N \phi(\widetilde{\lambda}_j)$ of the Abel condition (4.52). In the sequel we shall assume that this normalization has been done so that we can assume the stronger form of the Abel condition:

$$\sum_{i=1}^{N} \phi(\mu_i) = \sum_{j=1}^{N} \phi(\lambda_j). \tag{4.53}$$

Then it is immediate from the Fay criterion (4.46) (in fact, even without the normalization) that $\det \mathbf{M}$ and hence also $\det \Gamma_{\mathscr{D}}^{D_0}$ are nonzero, since in this case

$$\theta(\sum_i \phi(\mu_i) - \sum_j \phi(\lambda_j) + \mathbf{e}) = \theta(\mathbf{e}).$$

Then we have two formulas for the zero-pole interpolant, namely (4.43) and (4.34). If U_0 in (4.34) is chosen to match the value of the first solution at the point p_0, these two formulas must yield the same function. For the simple-multiplicity scalar-valued case which we are discussing here, the formula (4.34) can be made more explicit as follows. Recall that $B_\zeta = \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}$, $C_\pi = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ and the matrix $F^{D_{\zeta_0}}$ has a theta function representation by formula (4.76):

$$F_{A_\pi}^{D_{\zeta_0}}(p) = \mathrm{diag}_{1 \leqslant j \leqslant N} \left[f_{\mu_j}^{D_{\zeta_0}}(p) \right].$$

We then arrive at the conclusion: *if the Abel condition* (4.53) *is satisfied, then there is a nonzero complex number K so that the following identity for all $p \in M \setminus \{\mu_1, \dots, \mu_N\}$:*

$$1 + \sum_{i,j=1}^{N} \left[(\Gamma_{\mathscr{D}}^{D_{\zeta_0}})^{-1} \right]_{ij} f_{\mu_j}^{D_{\zeta_0}}(p) = K \frac{\prod_{j=1}^N E_\Delta(p, \lambda_j)}{\prod_{i=1}^N E_\Delta(p, \mu_i)} \tag{4.54}$$

where the value of the constant K is necessarily given by

$$K = \frac{\prod_{i=1}^N E_\Delta(p_0, \mu_i)}{\prod_{j=1}^N E_\Delta(p_0, \lambda_j)} \tag{4.55}$$

in order that the two sides of this expression agree at the base point p_0. We now give an independent direct computational verification that the identity (4.54) indeed does hold under the assumption that the Abel condition (4.53) holds as follows.

As a first observation, we note that: *to show that* (4.54)–(4.55) *holds, it suffices to show the equality of residues*

$$\begin{aligned} &\mathrm{res}_{p=\mu_\beta} \left(1 + \sum_{i,j=1}^{N} \left[(\Gamma_{\mathscr{D}}^{D_{\zeta_0}})^{-1} \right]_{ij} f_{\mu_j}^{D_{\zeta_0}}(p) \right) \\ &\quad = \mathrm{res}_{p=\mu_\beta} \frac{\prod_{i=1}^N E_\Delta(p_0, \mu_i)}{\prod_{j=1}^N E_\Delta(p_0, \lambda_j)} \frac{\prod_{j=1}^N E_\Delta(p, \lambda_j)}{\prod_{i=1}^N E_\Delta(p, \mu_i)} \end{aligned} \tag{4.56}$$

for each $\beta = 1,\dots,N$. Indeed, suppose that (4.56) for all β and set

$$h(p) = 1 + \sum_{i,j=1}^{N} \left[(\Gamma_{\mathscr{D}}^{D\zeta_0})^{-1}\right]_{ij} f_{\mu_j}^{D\zeta_0}(p) - \frac{\prod_{i=1}^{N} E_\Delta(p_0,\mu_i)}{\prod_{j=1}^{N} E_\Delta(p_0,\lambda_j)} \frac{\prod_{j=1}^{N} E_\Delta(p,\lambda_j)}{\prod_{i=1}^{N} E_\Delta(p,\mu_i)}.$$

Then h is a (single-valued) meromorphic function on M which has a zero at p_0 and only possible poles equal to at most simple poles at $p_1,\dots,p_g$, i.e., h is a holomorphic section of the bundle associated with the divisor $D - \{p_0\}$. Since D by assumption is nonspecial, it follows that $h \equiv 0$, i.e., (4.54)–(4.55) holds. As the right-hand side of (4.54) has no poles at the points $p_1,\dots,p_g$, it follows that the apparent possible poles at $p_1,\dots,p_g$ of the left-hand expression are all removable, from which the side condition (4.47) follows as well.

We now assume that the strong Abel condition (4.53) holds. Our goal is to show the residue equality (4.56).

We first note that the second factor in the formula (4.51) simplifies when (4.53) holds, namely:

$$\frac{\theta(\sum_{j\neq\beta}\phi(\mu_j) - \sum_{i\neq\alpha}\phi(\lambda_i) + \mathbf{e})}{\theta(\sum_j \phi(\mu_j) - \sum_i \phi(\lambda_i) + \mathbf{e})} = \frac{\theta(\phi(\lambda_\alpha) - \phi(\mu_\beta) + \mathbf{e})}{\theta(\mathbf{e})}.$$

Thus the expression (4.51) for $[(\Gamma_{\mathscr{D}}^{D_0})^{-1}]_{\alpha\beta}$ simplifies to

$$\begin{aligned}
[(\Gamma_{\mathscr{D}}^{D_0})^{-1}]_{\alpha\beta} &= -\frac{\theta(\phi(p_0) - \phi(\lambda_\alpha) + \mathbf{e})}{E_\Delta(p_0,\lambda_\alpha)} \cdot \frac{\theta(\phi(\lambda_\alpha) - \phi(\mu_\beta) + \mathbf{e})}{\theta(\mathbf{e})E(\mu_\beta,\lambda_\alpha)} \cdot \\
&\quad \cdot \frac{E_\Delta(p_0,\mu_\beta)}{\theta(\phi(p_0) - \phi(\mu_\beta) + \mathbf{e})} \cdot \frac{\prod_j E_\Delta(\mu_j,\lambda_\alpha)\cdot\prod_i E_\Delta(\mu_\beta,\lambda_i)}{\prod_{j\neq\beta} E_\Delta(\mu_j,\mu_\beta)\cdot\prod_{i\neq\alpha} E_\Delta(\lambda_i,\lambda_\alpha)} \\
&= -f_{\lambda_\alpha}^{D_0}(\mu_\beta) \cdot \frac{\prod_j E_\Delta(\mu_j,\lambda_\alpha)\cdot\prod_i E_\Delta(\mu_\beta,\lambda_i)}{\prod_{j\neq\beta} E_\Delta(\mu_j,\mu_\beta)\cdot\prod_{i\neq\alpha} E_\Delta(\lambda_i,\lambda_\alpha)}.
\end{aligned}$$

where we use the identity (4.77) as well as $E(\mu_\beta,\lambda_\alpha) = -E_\Delta(\lambda_\alpha,\mu_\beta)$ for the last step.

The problem of verifying (4.56) therefore comes down to verifying

$$\sum_{\alpha=1}^{N} [(\Gamma_{\mathscr{D}}^{D_0})^{-1}]_{\alpha\beta} = \frac{\prod_{j=1}^{N} E_\Delta(p_0,\mu_j)}{\prod_{i=1}^{N} E_\Delta(p_0,\lambda_i)} \cdot \frac{\prod_i E_\Delta(\mu_\beta,\lambda_i)}{\prod_{j\neq\beta} E_\Delta(\mu_\beta,\mu_j)} \tag{4.57}$$

for all $\beta = 1,\dots,N$, Let us rewrite the right-hand side of (4.57) as

$$\frac{\prod_{j=1}^{N} E_\Delta(p_0,\mu_j)}{\prod_{i=1}^{N} E_\Delta(p_0,\lambda_i)} \cdot \frac{\prod_i E_\Delta(\mu_\beta,\lambda_i)}{\prod_{j\neq\beta} E_\Delta(\mu_\beta,\mu_j)} = -\frac{\prod_{j=1}^{N} E_\Delta(\mu_j,p_0)}{\prod_{i=1}^{N} E_\Delta(p_0,\lambda_i)} \cdot \frac{\prod_i E_\Delta(\mu_\beta,\lambda_i)}{\prod_{j\neq\beta} E_\Delta(\mu_j,\mu_\beta)}.$$

The left-hand side of (4.57) is given by

$$\sum_{\alpha=1}^{N} [(\Gamma_{\mathscr{D}}^{D_0})^{-1}]_{\alpha\beta} = \sum_{\alpha=1}^{N} -f_{\lambda_\alpha}^{D_0}(\mu_\beta) \cdot \frac{\prod_j E_\Delta(\mu_j,\lambda_\alpha)\cdot\prod_i E_\Delta(\mu_\beta,\lambda_i)}{\prod_{j\neq\beta} E_\Delta(\mu_j,\mu_\beta)\cdot\prod_{i\neq\alpha} E_\Delta(\lambda_i,\lambda_\alpha)}.$$

Cancellation of the common factor $\frac{\prod_i E_\Delta(\mu_\beta,\lambda_i)}{\prod_{j\neq\beta} E_\Delta(\mu_\beta,\mu_j)}$ and writing $f_{\lambda_\alpha}^{D_0}$ in the more explicit form $f_{\lambda_\alpha}^{D-p_0}$ then converts the desired identity (4.57) to

$$\sum_{\alpha=1}^{N} f_{\lambda_\alpha}^{D-p_0}(\mu_\beta)\frac{\prod_j E_\Delta(\mu_j,\lambda_\alpha)}{\prod_{i\neq\alpha} E_\Delta(\lambda_i,\lambda_\alpha)} = \frac{\prod_j E_\Delta(\mu_j,p_0)}{\prod_i E_\Delta(p_0,\lambda_i)}. \tag{4.58}$$

We now view $f_{\lambda_\alpha}^{D-p_0}(\mu_\beta)$ as a function of p_0 for each fixed λ_α and μ_β. As noted in Corollary 4.15 below, when viewed as a (single-valued) meromorphic function on M as a function of p_0, $f_{\lambda_\alpha}^{D-p_0}(\mu_\beta)$ has a simple pole at λ_α with residue equal to -1 together with a zero at μ_β, and other possible poles at most g in number in the zero divisor of $K(\zeta_0;\mu_\beta,\cdot)$ on M. For generic choice of μ_β, this zero divisor is nonspecial. Hence we can argue just as in the proof of the reduction of (4.54)–(4.55) to (4.56) above that to show (4.58), it suffices to verify the equality of residues at $p_0=\lambda_\alpha$ for the two sides of (4.58) for $\alpha=1,\dots,N$. Again making use of the general identity $E_\Delta(x,y)=-E_\Delta(y,x)$ as well as the local development (4.70) for the prime form, one can check that the residues of the left and right hand sides at the simple pole λ_α have the common value

$$-\frac{\prod_j E_\Delta(\mu_j,\lambda_\alpha)}{\prod_{i\neq\alpha} E_\Delta(\lambda_i,\lambda_\alpha)}.$$

for each $\alpha=1,\dots,N$. Putting all the pieces together, we see that we have now verified the identity (4.54), providing an independent proof of the formula (4.34) for the solution of the First Interpolation Problem for this simplest case.

A similar phenomenon occurs already in the genus $g=0$ case. Given specified distinct simple-multiplicity poles $\{\mu_1,\dots,\mu_N\}$ and distinct simple-multiplicity zeros $\{\lambda_1,\dots,\lambda_N\}$ in the complex plane $\mathbb{C}$, it is completely elementary to write down the associated rational zero-pole interpolant

$$f(z)=\frac{\prod_{i=1}^N(z-\lambda_i)}{\prod_{j=1}^N(z-\mu_j)}, \tag{4.59}$$

the genus-0 analogue of the prime-form solution of the problem given in [22]. On the other hand, one can solve the same problem in realization or partial fraction form

$$f(z)=1+\begin{bmatrix}1 & \cdots & 1\end{bmatrix}\begin{bmatrix}(z-\mu_1)^{-1} & & \\ & \ddots & \\ & & (z-\mu_N)^{-1}\end{bmatrix} S^{-1}\begin{bmatrix}1\\ \vdots \\ 1\end{bmatrix} \tag{4.60}$$

where $S=-\left[\frac{1}{\mu_j-\lambda_i}\right]_{i,j=1,\dots,N}$ is the *Cauchy matrix* (see [6, Theorem 4.3.2]), the genus-0 analogue of the formula (4.34). Note that in the genus-0 case, there is no analogue of the side constraint (4.47). The fact that (4.59) and (4.60) agree can be seen via explicit inversion of the Cauchy matrix (see [24] as well as [6, Lemma A.1.5]).

4.4. Appendix: Explicit formulas for building-block functions $f_{kw}^{D_0}$

For completeness we now review here results on theta functions and the prime form; for more complete details we refer to [1, 14, 13, 23]

We assume that we are given a compact Riemann surface M. We let Δ be a line bundle of half-order differentials, i.e., $\Delta \otimes \Delta \cong K_M$ where K_M is the canonical line bundle with local holomorphic sections equal to holomorphic differentials on M. Assume also that we are given a holomorphic complex bundle E of rank r and degree 0 over M such that $h^0(E \otimes \Delta) = 0$. As necessarily $\deg \Delta = g-1$, it follows from the discussion in Section 4.2 that E has property NSF (with test line bundle λ taken to be equal to Δ) and so in particular E is flat with flat factor of automorphy denoted by χ. We let $\pi_1 \colon M \times M \to M$ be the projection map onto the first component and $\pi_2 \colon M \times M \to M$ be the projection map onto the second component. The defining property for the *Cauchy kernel* $K(\chi, \cdot, \cdot)$ is that $K(\chi; \cdot, \cdot)$ be a meromorphic mapping between the vector bundle $\pi_2^* E$ and $\pi_1^* E \otimes \pi_1^* \Delta \otimes \pi_2^* \Delta$ on $M \times M$ which is holomorphic outside of the diagonal $\mathfrak{D} = \{(p,p) \in M \times M \colon p \in M\}$, and that the singularity of $K(\chi; \cdot, \cdot)$ on the diagonal be a simple pole with residue equal to the identity I_r. It is straightforward to show by making use of the assumption that $h^0(E \otimes \Delta) = 0$ that such a Cauchy kernel is unique if it exists. The main result of Section 2 of [10] is that the Cauchy kernel indeed does exists (see also [9] for an alternative derivation in terms of a representation of the Riemann surface as the normalizing Riemann surface for an algebraic curve $\mathbf{C}$ embedded in projective space $\mathbb{P}^2$ with the representation of the bundle E as a kernel bundle (up to a twist) associated with a determinantal representation for the defining polynomial of the curve $\mathbf{C}$).

Let us next restrict to the case where E is taken to be a line bundle. Without loss of generality, we may assume that χ is a flat unitary line bundle (see [14, page 4]). Then in this case there is an explicit formula for $K(\chi; \cdot, \cdot)$ in terms of theta functions which we now describe. We first need to recall some Riemann-surface function theory.

We mark M by fixing a canonical basis $A_1, \dots, A_g, B_1, \dots, B_g$ for the the homology of M. We then also fix a normalized basis for holomorphic differentials on M, where normalized means that $\int_{A_j} \omega_i = \delta_{ij}$. Then the B-period matrix Ω for M is defined as the matrix with j-th column equal to $\begin{bmatrix} \int_{B_j} \omega_1 \\ \vdots \\ \int_{B_j} \omega_g \end{bmatrix}$. The matrix Ω has positive imaginary part and the Jacobian variety of M is defined to be the quotient space $J(M) = \mathbb{C}^g / (\mathbb{Z}^g + \Omega \mathbb{Z}^g)$. We fix a base point $p_0 \in M$ and then define the Abel-Jacobi map $\phi \colon M \to J(M)$ by

$$\phi(p) = \left(\int_{p_0}^{p} \omega_1, \dots, \int_{p_0}^{p} \omega_g \right). \tag{4.61}$$

We write $\theta(z)$ for the theta function associated with the lattice $\mathbb{Z}^g + \Omega \mathbb{Z}^g$ where Ω is the period matrix of M (the multivariable version of the classical theta function already introduced above in (2.2)), namely

$$\theta(z) = \sum_{m \in \mathbb{Z}^g} e^{\pi i \langle \Omega m, m \rangle + 2\pi i \langle z, m \rangle}.$$

We will also need the theta function with characteristic, defined for $a,b \in (\mathbb{R}/\mathbb{Z})^g$ as:

$$\theta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right](z) = \sum_{m\in\mathbb{Z}^g} e^{\pi i\langle\Omega(m+a),m+a\rangle} e^{2\pi i\langle z+b,m+a\rangle}. \tag{4.62}$$

which can be expressed directly in terms of θ as

$$\theta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right](z) = \exp(\pi i\langle\Omega a,a\rangle + 2\pi i\langle z+b,a\rangle)\theta(z+\Omega a+b). \tag{4.63}$$

In particular, if we set $a=b=0$, then $\theta\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right](z) = \theta(z)$. A direct verification shows that $\theta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ has the quasi-periodicity property with respect to the period lattice $\mathbb{Z}^g+\Omega\mathbb{Z}^g$ given by

$$\theta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right](z+m+\Omega n) = \exp(2\pi i\langle m,a\rangle - 2\pi i\langle n,b\rangle - \pi i\langle\Omega n,n\rangle - 2\pi i\langle z,n\rangle)\theta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right](z). \tag{4.64}$$

A useful variant of the prime form (defined below in (4.69)) is given by

$$E_{\mathbf{e}}(p,q) = \theta(\mathbf{e}+\phi(p)-\phi(q))$$

where $\mathbf{e}\in\mathbb{C}^g$, $p,q\in M$ and $\theta(\mathbf{e})=0$. By following the construction in [22, pages 158–160], one can show that, given a divisor

$$D = \lambda_1+\cdots+\lambda_n-\mu_1-\cdots-\mu_n \tag{4.65}$$

of degree zero, for an appropriate choice of $\mathbf{e}$ with $\theta(\mathbf{e})=0$, the function f given by

$$f(p) = \frac{\prod_{i=1}^n E_{\mathbf{e}}(p,\lambda_i)}{\prod_{j=1}^n E_{\mathbf{e}}(p,\mu_i)}$$

is relatively automorphic on $\widehat{M}$ with divisor $\widehat{D}$ having projection down to M equivalent to the prescribed divisor D (4.64) and has flat factor of automorphy holomorphically equivalent to the factor of automorphy χ whose action on the canonical basis for the homology of M is given by

$$\begin{aligned}&\chi(A_\ell) = e^{-2\pi i a_\ell}, \quad \chi(B_\ell) = e^{2\pi i b_\ell} \text{ for } \ell=1,\dots,g \text{ where } \phi(D)=\Omega a+b,\\ &a=(a_1,\dots,a_g) \text{ and } b=(b_1,\dots,b_g)\in\mathbb{R}^g/\mathbb{Z}^g.\end{aligned} \tag{4.66}$$

In this way we may associate a flat factor of automorphy and associated line bundle $L_{\mathbf{a}}$ with a point $\mathbf{a}=\Omega a+b$ (modulo the period lattice $\mathbb{Z}^g+\Omega\mathbb{Z}^g$) in the Jacobian $J(M)$ of M.

An important special case of theta function with characteristic (4.62) is the situation where the characteristic components a,b are taken to be in $(\frac{1}{2}\mathbb{Z}/\mathbb{Z})^g$ (half-order characteristic): for the case where say $a_*,b_*\in(\frac{1}{2}\mathbb{Z}/\mathbb{Z})^g$, the associated theta function with half-order characteristic $\theta\left[\begin{smallmatrix} a_* \\ b_* \end{smallmatrix}\right]$ has the property that its zero set is invariant under the symmetry $z\mapsto -z$:

$$\theta\left[\begin{smallmatrix} a_* \\ b_* \end{smallmatrix}\right](z)=0 \Leftrightarrow \theta\left[\begin{smallmatrix} a_* \\ b_* \end{smallmatrix}\right](-z)=0.$$

This happens in exactly two possible ways: either $\theta\left[\begin{smallmatrix} a_* \\ b_* \end{smallmatrix}\right]$ is even ($\theta\left[\begin{smallmatrix} a_* \\ b_* \end{smallmatrix}\right](-z) = \theta\left[\begin{smallmatrix} a_* \\ b_* \end{smallmatrix}\right](z)$) or $\theta\left[\begin{smallmatrix} a_* \\ b_* \end{smallmatrix}\right]$ is odd ($\theta\left[\begin{smallmatrix} a_* \\ b_* \end{smallmatrix}\right](-z) = -\theta\left[\begin{smallmatrix} a_* \\ b_* \end{smallmatrix}\right](z)$). In the first case (a_*, b_*) is said to be an even half-order characteristic and in the second case (a_*, b_*) is said to be an odd half-order characteristic. An important result is that nonsingular odd half-order characteristics always exist, i.e., an odd half-order characteristic (a_*, b_*) for which in addition the differential $d\theta\left[\begin{smallmatrix} a_* \\ b_* \end{smallmatrix}\right](0)$ is not zero (see Lemma 1 page 3.208 of [23]).

Let us now fix a choice (a_*, b_*) of nonsingular odd half-order characteristic. Since the half-order characteristic (a_*, b_*) is nonsingular, Riemann's zero theorem (see e.g. [22, page 149] or [13, pages 308–309]) implies that $\theta\left[\begin{smallmatrix} a_* \\ b_* \end{smallmatrix}\right](\phi(q) - \phi(\cdot))$ has precisely g zeros $p_{*1}, \dots, p_{*g}$ which are uniquely determined by the equality

$$\phi(p_{*,1}) + \cdots + \phi(p_{*,g}) = -\kappa_0 + \phi(q) + \Omega a_* + b_* \tag{4.67}$$

where κ_0 is Riemann's constant. Since (a_*, b_*) is odd, one of these zeros is q. Hence without loss of generality we may take $p_{*g} = q$ and we are left with

$$\phi(p_{*,1}) + \cdots + \phi(p_{*,g-1}) = -\kappa_0 + \Omega a_* + b_*.$$

As $2(\Omega a_* + b_*) = 0$ in $J(M)$, multiplying this last expression by 2 converts it to

$$2(\phi(p_{*,1}) + \cdots + \phi(p_{*,g-1}) + \mathbf{a}_*) = -2\kappa_0 \tag{4.68}$$

where we set $\mathbf{a}_* = \Omega a_* + b_*$. It is known that a divisor K is the divisor of a meromorphic differential on M if and only if $\phi(K) = -2\kappa_0$ (see [13, page 318]). We let Δ_* be the line bundle associated with the divisor $p_{*1} + \cdots + p_{*g-1}$. We now conclude from the equality (4.68) that

$$(L_{\mathbf{a}_*} \otimes \Delta_*) \otimes (L_{\mathbf{a}_*} \otimes \Delta_*) = K_M$$

where K_M is the line bundle associated with the canonical divisor on M. Thus the bundle $\Delta := L_{\mathbf{a}_*} \otimes \Delta_*$ is a bundle of half-order differentials; we shall henceforth assume that *the bundle of half-order differentials in the definition of the Cauchy kernel arises in this way from a half-order characteristic* (a_*, b_*). The fact that $d\theta\left[\begin{smallmatrix} a_* \\ b_* \end{smallmatrix}\right](0)$ is nonzero also implies that the bundle Δ has only one nonzero holomorphic section (up to a multiplicative constant) (see [1, page 293]). We let $\sqrt{\xi_*(p)}$ denote the choice such that

$$\left(\sqrt{\xi_*(p)}\right)^2 = \sum_{j=1}^{g} \frac{\partial\theta\left[\begin{smallmatrix} a_* \\ b_* \end{smallmatrix}\right]}{\partial z_j}(0)\omega_j(p).$$

We now define the *prime form* $E_\Delta(p,q)$ by

$$E_\Delta(p,q) = \frac{\theta\left[\begin{smallmatrix} a_* \\ b_* \end{smallmatrix}\right](\phi(q) - \phi(p))}{\sqrt{\xi_*(p)}\sqrt{\xi_*(q)}}. \tag{4.69}$$

If we use $\xi_*(p) = \sum_{j=1}^{g} \frac{\partial\theta\left[\begin{smallmatrix} a_* \\ b_* \end{smallmatrix}\right]}{\partial z_j}(0)\omega_j(p)$ as a local coordinate $t = t(p)$, then E_Δ has a local representation at points (p,q) near $p = q$ given by

$$E_\Delta(p,q) = \frac{t(q) - t(p)}{\sqrt{dt(p)}\sqrt{dt(q)}}(1 + O(t(p) - t(q))^2) \tag{4.70}$$

(see formula (26) page 19 of [14], a corrected version of Property 4 page 3.210 of [23]).

Let now χ be a general rank-1 unitary factor of automorphy, with action on the generators $A_1,\dots,A_g,B_1,\dots,B_g$ for the homology basis for M given by

$$\chi(A_\ell)=\exp(-2\pi i a_\ell),\quad \chi(B_\ell)=\exp(2\pi i b_\ell) \tag{4.71}$$

for $\ell=1,\dots,g$ where $a,b\in(\mathbb{R}/\mathbb{Z})^g$. It then can be checked that the Cauchy kernel $K(\chi;\cdot,\cdot)$ is given explicitly by

$$K(\chi;p,q)=\frac{\theta\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right](\phi(q)-\phi(p))}{\theta\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right](0)E_\Delta(q,p)} \tag{4.72}$$

(see [1, Proposition 2.8] for the real Riemann surface setting). Furthermore, the divisor (E_χ) of the bundle E_χ associated with the factor of automorphy χ and the characteristic $\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]$ are related to each other as in (4.66), namely:

$$\phi((E_\chi))=\Omega a+b. \tag{4.73}$$

By using the formula (4.63) for $\theta\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]$, we can express $K(\chi;p,q)$ in terms of the theta function θ itself:

$$K(\chi;p,q)=\frac{\exp(2\pi i a^\top(\phi(q)-\phi(p))\cdot\theta(\phi(q)-\phi(p)+\mathbf{e})}{\theta(\mathbf{e})\cdot E_\Delta(q,p)} \tag{4.74}$$

where $\mathbf{e}=b+\Omega a$ as as in (4.42).

As the following result shows, the Cauchy kernel and the building-block functions $f_{kw}^{D_0}$ used to build the matrices of the form $f_{w,A}^{D_0}$ (4.18) are closely related.

THEOREM 4.14. *Suppose that $D_0=p_1+\cdots+p_g-p_0$ is the divisor on M where the divisor $D=p_1+\cdots+p_g$ is nonspecial. Write the associated line bundle λ_{D_0} in the form*

$$\lambda_{D_0}=\zeta_0\otimes\Delta \tag{4.75}$$

for a degree-0 necessarily flat line bundle ζ_0, where Δ is the bundle of half-order differentials used in the prime form $E_\Delta(\cdot,\cdot)$. Then the canonical function $f_w^{D_0}(p)$ associated with D_0 as introduced in Section 4.2 is given by

$$f_w^{D_0}(p)=\frac{K(\zeta_0;p,w)}{K(\zeta_0;p,p_0)}K(\zeta_0;w,p_0) \tag{4.76}$$

or in terms of theta functions,

$$f_w^{D_0}(p)=\frac{1}{\theta(\mathbf{e})}\frac{\theta(\phi(w)-\phi(p)+\mathbf{e})}{E_\Delta(w,p)}\frac{E_\Delta(p_0,p)}{\theta(\phi(p_0)-\phi(p)+\mathbf{e})}\frac{\theta(\phi(p_0)-\phi(w)+\mathbf{e})}{E_\Delta(p_0,w)}. \tag{4.77}$$

Furthermore, for $k>1$, the function $f_{kw}^{D_0}(p)$ is given by

$$f_{kw}^{D_0}(p)=\frac{1}{(k-1)!}\frac{d^{(k-1)}}{dw^{(k-1)}}f_w^{D_0}(p). \tag{4.78}$$

Proof. Let us set the right hand side of (4.76) equal to $\widetilde{f}_w^{D_0}(p)$. From the defining properties of the Cauchy kernel $K(\zeta_0;\cdot,\cdot)$, we see that $\widetilde{f}_w^{D_0}(p)$ is automorphic as a function of p and has a pole at w with residue 1. Note next that $\widetilde{f}_w^{D_0}$ has potential additional poles at the zeros of $K(\zeta_0;\cdot,p_0)$. These zeros in turn arise as the zeros of $\Theta\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right](\phi(p_0)-\phi(\cdot))$. By Riemann's Theorem again (as in (4.67)), there are g such zeros $\widetilde{p}_1,\dots,\widetilde{p}_g$ determined by

$$\phi(\widetilde{p}_1)+\cdots+\phi(\widetilde{p}_g)=-\kappa_0+\phi(p_0)+\Omega a+b.$$

On the other hand, as a consequence of the general relation (4.73) we have

$$\phi(p_1)+\cdots+\phi(p_g)-\phi(p_0)=-\kappa_0+\Omega a+b.$$

Since $D=p_1+\cdots+p_g$ is nonspecial, the Abel-Jacobi map is injective at this point and we conclude that

$$p_1+\cdots+p_g=\widetilde{p}_1+\cdots+\widetilde{p}_g,$$

i.e., the potential additional poles of $\widetilde{f}_w^{D_0}$ are at the points $p_1,\dots,p_g$. Note in addition that $K(\zeta_0;\cdot,p_0)$ having a pole at p_0 tells us that $\widetilde{f}_w^{D_0}$ has a zero at p_0. Thus $\widetilde{f}_w^{D_0}$ has all the defining properties uniquely determining $f_w^{D_0}$ and we conclude that $\widetilde{f}_w^{D_0}=f_w^{D_0}$ as claimed.

The formula for the higher multiplicity case is checked similarly. □

The following corollaries are of interest for us in the body of the paper.

COROLLARY 4.15. *When the building-block function $f_w^{D_0}(p)=f_w^{D-p_0}(p)$ is considered as a function of p_0 for each fixed w and p, then $f_w^{D_0}(p)=f_w^{D-p_0}(p)$ is a single-valued meromorphic function on M with only pole a simple pole at w having residue there given by*

$$\operatorname{res}_{p_0=w}f_w^{D-p_0}(p)=-1. \tag{4.79}$$

Proof. Use either of formulas (4.76), (4.77) to see that $f_w^{D-p_0}(p)$ as a function of p has a pole of residue 1 at $p=w$ while as a function of p_0 there is a pole at $p_0=w$ of residue -1. In fact one can make the identification

$$f_w^{D-p_0}(p)=-f_w^{D'-p}(p_0).$$

where D is the zero divisor of $K(\zeta_0;\cdot,p_0)$ (or equivalently of $\theta\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right](\phi(p_0)-\phi(\cdot)+\mathbf{e})$) while D' is the zero divisor of $K(\zeta_0;p,\cdot)$ (or equivalently of $\theta\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right](\phi(\cdot)-\phi(p)+\mathbf{e})$). □

COROLLARY 4.16. *The matrix $\Gamma_{\mathscr{D}}^{D_0}$ depends continuously on the support of the divisor D.*

Proof. We fix the base point p_0 and write $D_0=D-p_0$. From the identification (4.75), we have

$$D_0=(\zeta_0)+(\Delta)$$

where (Δ) (the divisor of the bundle Δ of half-order differentials) is fixed. The divisor (ζ_0) of the flat line bundle ζ_0 in turn is specified by

$$\phi((\zeta_0)) = \Omega a + b$$

where $a,b \in \mathbb{R}^g$ are such that the Cauchy kernel $K(\zeta_0; p,q)$ is given by the right-hand side of (4.72). Without loss of generality we can fix all the poles in the divisor (ζ_0) to be also at the base point p_0. From formula (4.62) and the fact that the theta function is infinitely differentiable in its arguments, we see that $\theta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right](\phi(q)-\phi(p))$ is continuous with respect to the parameters a,b. From formulas (4.72) and (4.76) (and more generally (4.78)), we see that $f_{kw}^{D_0}$ is continuous with respect to the parameters a,b. Then by Jacobi inversion, we see that $f_{kw}^{D_0}$ depends continuously on the support of the divisor $((\zeta_0))$, and hence also on the support of the divisor D_0. As we are fixing the base point, we then have continuous dependence on the support of the divisor D. □

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(Received October 27, 2015)

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