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# DILATIONS AND CONSTRAINED ALGEBRAS

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*For our friend Leiba*

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*Abstract.* It is well known that contractive representations of the disk algebra are completely contractive. The Neil algebra $\mathscr{A}$ is the subalgebra of the disk algebra consisting of those functions $f$ for which $f'(0) = 0$. There is a complete isometry from the algebra $R(W)$ of rational functions with poles off of the distinguished variety $W = \{(z,w) : z^2 = w^3,\ |z| < 1\}$ to $\mathscr{A}$. We prove that there are contractive representations of $\mathscr{A}$ which are not completely contractive, and furthermore provide a Kaiser and Varopoulos inspired example of a representation $\pi$ of $R(W)$ whereby $\pi(z)$ and $\pi(w)$ are contractions, yet $\pi$ is not contractive. We also present a characterization of those contractive representations of $R(W)$ that are completely contractive. Finally, we show that by contrast, for the variety $\mathscr{V} = \{(z,w) : z^2 = w^2,\ |z| < 1\}$, all contractive representations of the algebra $R(\mathscr{V})$ of rational functions with poles off $\mathscr{V}$ are completely contractive, and we as well provide a simplified proof of Agler's analogous result over an annulus.

## 1. Introduction

Let $\mathbb{D}$ denote the unit disk in the complex plane and $\overline{\mathbb{D}}$ its closure. The disk algebra, $\mathbb{A}(\mathbb{D})$, is the closure of analytic polynomials in $C(\overline{\mathbb{D}})$, the space of continuous functions on $\overline{\mathbb{D}}$ with the supremum norm. The *Neil algebra* is the subalgebra of the disk algebra given by

$$\mathscr{A} = \{f \in \mathbb{A}(\mathbb{D}) : f'(0) = 0\} = \mathbb{C} + z^2\mathbb{A}(\mathbb{D}).$$

Constrained algebras, of which $\mathscr{A}$ is one of the simplest examples, are of current interest as a venue for function theoretic operator theory, such as Pick interpolation. See for instance [15, 26, 20, 10] and the references therein.

Let $H$ denote a complex Hilbert space and $B(H)$ the bounded linear operators on $H$. A unital representation $\pi : \mathscr{A} \to B(H)$ on $H$ is *contractive* if $\|\pi(f)\| \leqslant \|f\|$ for all $f \in \mathscr{A}$, where $\|f\|$ represents the norm of $f$ as an element of $\mathrm{C}(\overline{\mathbb{D}})$ and $\|\pi(f)\|$ is the operator norm of $\pi(f)$. Unless otherwise indicated, in this article representations are unital and contractive.

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Let $M_n(\mathscr{A})$ denote the $n\times n$ matrices with entries from $\mathscr{A}$. The norm $\|F\|$ of an element $F=(f_{j,\ell})$ in $M_n(\mathscr{A})$ is the supremum of the set $\{\|F(z)\| : z\in\mathbb{D}\}$, where $\|F(z)\|$ is the operator norm of the $n\times n$ matrix $F(z)$. Applying $\pi$ to each entry of $F$,

$$\pi^{(n)}(F)=1_n\otimes\pi(F)=\big(\pi(f_{j,\ell})\big)$$

produces an operator on the Hilbert space $\bigoplus_1^n H$ and $\|\pi^{(n)}(F)\|$ is then its operator norm. The mapping $\pi$ is *completely contractive* if for each $n$ and $F\in M_n(\mathscr{A})$,

$$\|\pi^{(n)}(F)\|\leqslant\|F\|.$$

The following theorem is the first main result of this article.

THEOREM 1.1. *There exists a finite dimensional Hilbert space and a unital contractive representation $\pi:\mathscr{A}\to B(H)$ which is not completely contractive. In fact, there exists a $2\times 2$ matrix rational inner function $F$ (with poles outside of the closed disk) such that $\|F\|\leqslant 1$, but $\|\pi(F)\|>1$.*

Theorem 2.1 gives a necessary and sufficient condition for a unital representation of $\mathscr{A}$ to be completely contractive. An operator $T\in B(H)$ is a *contraction* if it has operator norm less than or equal to one. Since the algebra $\mathscr{A}$ is generated by $z^2$ and $z^3$, a contractive representation $\pi$ of $\mathscr{A}$ is determined by the pair of contractions $X=\pi(z^2)$ and $Y=\pi(z^3)$. In the spirit of the examples of Kaiser and Varopoulos [28] for the polydisk $\mathbb{D}^d$ ($d>2$), Corollary 3.2 asserts the existence of commuting contractions $X$ and $Y$ such that $X^3=Y^2$, but for which the unital representation $\tau$ of $\mathscr{A}$ with $X=\tau(z^2)$ and $Y=\tau(z^3)$ is not contractive.

Given $0<q<1$, let $\mathbb{A}$ denote the annulus $\{z\in\mathbb{C}: q<|z|<1\}$ and $A(\mathbb{A})$ the annulus algebra, consisting of those functions continuous on the closure of $\mathbb{A}$ and analytic in $\mathbb{A}$ in the uniform norm. A well known theorem of Agler [1] says that contractive representations of $A(\mathbb{A})$ are completely contractive. If $W$ is a variety in $\mathbb{C}^2$ which intersects the (topological) boundary of the bidisk $\mathbb{D}^2$ only in the torus $\mathbb{T}^2$, then the set $V=W\cap\mathbb{D}^2$ is called a *distinguished variety*. The annuli (parametrized by $0<q<1$) can be identified with the distinguished varieties determined by

$$z^2=\frac{w^2-t^2}{1-t^2w^2}$$

for $0<t<1$ [27, 12, 13]. The limiting case, $z^2=w^2$ corresponds to two disks intersecting at the origin $(0,0)\in\mathbb{C}^2$. Section 6 contains a streamlined proof of Agler's result which readily extends to show that contractive representations of the algebra associated to the variety $z^2=w^2$ are also completely contractive. See Theorem 6.6 and Corollary 6.11.

The remainder of this introduction places Theorems 1.1 and 2.1 and Corollary 3.2, as well as Theorem 6.6 and Corollary 6.11 in the larger context of rational dilation.

### 1.1. Rational dilation

The Sz.-Nagy dilation theorem states that every contraction operator dilates to a unitary operator. Unitary operators can be characterized in various ways, and in particular, they are normal operators with spectrum contained in the boundary of $\mathbb{D}$; that is, $\mathbb{T}$. A corollary of the Sz.-Nagy dilation theorem is the von Neumann inequality, which implies that $T$ is a contraction if and only if $\|p(T)\| \leqslant \|p\|$ for every polynomial $p$, where $\|p\|$ is the again the norm of $p$ in $C(\overline{\mathbb{D}})$.

More generally, following Arveson [8], given a compact subset $X$ of $\mathbb{C}^d$, let $R(X)$ denote the algebra of rational functions with poles off $X$ with the norm $\|r\|_X$ equal to the supremum of the values of $|r(x)|$ for $x \in X$. The set $X$ is a *spectral set* for the commuting $d$-tuple $T$ of operators on the Hilbert space $H$ if the spectrum of $T$ lies in $X$ and $\|r(T)\| \leqslant \|r\|_X$ for each $r \in R(X)$. If $N$ is also a $d$-tuple of commuting operators with spectrum in $X$ and acting on the Hilbert space $K$, then $T$ *dilates* to $N$ provided there is an isometry $V : H \to K$ such that $r(T) = V^* r(N)V$ for all $r \in R(X)$. The *rational dilation problem* asks: if $X$ is a spectral set for $T$ does $T$ dilate to a tuple $N$ of commuting normal operators with spectrum in the Shilov boundary of $X$ relative to the algebra $R(X)$?

Choosing $X$ to be the closure of a finitely connected domain $D$ in $\mathbb{C}$ with analytic boundary, it turns out the Shilov boundary is the topological boundary and the problem has a positive answer when $X$ is an annulus [1]. On the other hand, for planar domains of higher connectivity, rational dilation fails, at least when the Schottky double is hyperelliptic (a condition which is automatic for triply connected domains – though it is felt that for domains of higher connectivity this requirement is probably an artifact of the proof and rational dilation will likewise fail without this extra condition) [18, 3, 25].

With the choice of $X = \overline{\mathbb{D}}^d$, the rational dilation problem becomes, if $T = (T_1, \dots, T_d)$ is a tuple of commuting operators acting on a Hilbert space $H$ and if

$$\|p(T_1, \dots, T_d)\| \leqslant \|p\|_X$$

for every analytic polynomial $p = p(z_1, \dots, z_d)$ in $d$-variables, does there exist a Hilbert space $K$, an isometry $V : H \to K$, and a commuting tuple $N = (N_1, \dots, N_d)$ of normal operators on $K$ with spectrum in $\mathbb{T}^d$ (the Shilov boundary of $X$) such that $p(T) = V^* p(N)V$ for every polynomial $p$? Andô's theorem implies the result is true for the bidisk $\mathbb{D}^2$. An example due to Parrott implies that rational dilation fails for the polydisk $\mathbb{D}^d$, $d > 2$. Thus as things stand, the rational dilation problem has been settled for the disk, the annulus, hyperelliptic planar domains, and for polydisks.

Arveson [8] gave a profound reformulation of the rational dilation problem in terms of contractive and completely contractive representations. A tuple $T$ acting on the Hilbert space $H$ with spectrum in $X$ determines a unital representation of $\pi_T$ of $R(X)$ on $H$ via $\pi_T(r) = r(T)$ and the condition that $X$ is a spectral set for $T$ is equivalent to the condition that this representation is contractive.

In this context, a representation $\pi$ of $R(X)$ is *completely contractive* if for all $n$ and all $F \in M_n(R(X))$, $\pi^{(n)}(F) := (\pi(F_{i,j}))$ is contractive, the norm of $F$ being given by $\|F\|_\infty = \sup\{\|F(x)\| : x \in X\}$ with $\|F(x)\|$ the operator norm of $F(x)$. Arveson showed that $T$ dilates to a tuple $N$ with spectrum in the (Shilov) boundary of $X$ (with respect to $R(X)$) if and only if $\pi_T$ is completely contractive. Thus the rational dilation problem can be reformulated as: Is every contractive representation of $R(X)$ completely contractive?

Let $W = \{(z,w) \in C^2 : z^2 = w^3\}$. The mapping from $R(W)$ to the Neil algebra $\mathscr{A}$ sending $p(z,w)$ to $p(t^2,t^3)$ is a (complete) isometry. Much of this paper concentrates on studying the connection between contractive and completely contractive representations of $\mathscr{A}$, though the results are readily translated to $R(W)$. Thus, Theorem 1.1 implies that there are contractive representations of $R(W)$ which are not completely contractive.

Note that excluding a cusp at $(0,0)$, $W$ is a manifold, and this cusp makes things just different enough so that $R(W)$ is a tractable though nontrivial algebra on which to study the rational dilation problem. Indeed, many mathematicians have found distinguished varieties to be attractive venues for function theoretic operator theory [26, 5, 6, 4, 21, 29, 20] and in particular, they provide interesting examples when trying to delineate the border between those domains where rational dilation holds and those where it fails. Theorem 6.6 say that on the distinguished variety $\mathscr{V} = \{(z,w) \in \mathbb{D}^2 : z^2 = w^2\}$, every contractive representation of $R(\mathscr{V})$ is completely contractive; that is, rational dilation holds.

While rational dilation fails for the Neil parabola, in Theorem 2.1 we also provide a characterization of the completely contractive representations of $\mathscr{A}$ [14]. However, this positive result is not used to establish Theorem 1.1. Rather the proof of Theorem 1.1 essentially comes down to a cone separation argument. The mechanics of this argument appear in Section 3. The construction of the counterexample and preliminary results are in Section 4. The proof of Theorem 1.1 concludes in Section 5, while the statement and proof of Theorem 2.1 and general facts about representations of $\mathscr{A}$ are the subject of Section 2.

The article concludes with Section 6, which contains a proof of Agler's rational dilation theorem for the annulus that takes advantage of subsequent developments in the theory of matrix-valued functions of positive real part on multiply connected domains. As a limiting case, we prove Theorem 6.6, which shows that rational dilation holds for the algebra $R(\mathscr{V})$, $\mathscr{V} = \{(z,w) \in \mathbb{D}^2 : z^2 = w^2\}$. Corollary 6.11 then gives a reasonably tractable condition to determine if a given representation of $R(\mathscr{V})$ is contractive, and hence completely contractive.

## 2. Representations of $\mathscr{A}$

In this section we characterize the completely contractive representations of $\mathscr{A}$ and consider some examples. The characterization of contractive representations is essentially contained in the paper [16] on test functions for $\mathscr{A}$, and this is described in the next section.

As a (unital) Banach algebra, $\mathscr{A}$ is generated by the functions $z^2$ and $z^3$. It follows that any bounded unital representation is determined by its values on these two functions. If $\pi : \mathscr{A} \to B(H)$ is a bounded representation, $X = \pi(z^2)$ and $Y = \pi(z^3)$, then $X, Y$ are commuting operators which satisfy $X^3 = Y^2$. If we further insist that $\pi$ is contractive, then $X$ and $Y$ are contractions. In summary, every contractive representation $\pi : \mathscr{A} \to B(H)$ determines a pair of commuting contractions $X, Y$ such that $X^3 = Y^2$. However, as we see in Corollary 3.2, not every such pair gives rise to a contractive representation.

The following theorem characterizes the completely contractive representations of $\mathscr{A}$. For Hilbert spaces $H \subseteq K$, let $P_H$ denote the orthogonal projection of $K$ onto $H$ and $|_H$ the inclusion of $H$ into $K$.

THEOREM 2.1. ([14]) *A representation $\pi : \mathscr{A} \to B(H)$ is completely contractive if and only if there is a Hilbert space $K \supset H$ and a unitary operator $U \in B(K)$ such that for all $n \geqslant 0$, $n \neq 1$,*

$$\pi(z^n) = P_H U^n|_H. \tag{1}$$

Theorem 2.1 is a consequence of the Sz.-Nagy dilation theorem together with applications of the Arveson extension and Stinespring dilation theorems. In the case of $\mathbb{A}(\mathbb{D})$, by the Sz.-Nagy dilation theorem every completely contractive representation $\pi : \mathbb{A}(\mathbb{D}) \to B(H)$ is determined by a contraction $T$, with $\pi(z^n) = T^n$, and $T^n = P_H U^n|_H$ for some unitary $U$ and all $n \geqslant 0$. Thus a simple way to construct completely contractive representations of $\mathscr{A}$ is to fix a contraction $T$ and restrict: put $\pi(z^2) = T^2$ and $\pi(z^3) = T^3$. However, in spite of Theorem 2.1 it is *not* the case that every completely contractive representation of $\mathscr{A}$ arises in this way, as we see in Example 2.3 below.

*Proof of Theorem* 2.1. Let $\pi : \mathscr{A} \to B(H)$ be a unital, completely contractive representation. Let $\mathscr{A}^* \subseteq C(\mathbb{T})$ denote the set of complex conjugates of functions in $\mathscr{A}$. Then $\mathscr{A} + \mathscr{A}^*$ is an operator system and $\rho : \mathscr{A} + \mathscr{A}^* \to B(H)$ given by

$$\rho(f + g^*) = \pi(f) + \pi(g)^*$$

is well defined. Since $\pi$ is unital and $\mathscr{A} \cap \mathscr{A}^* = \mathbb{C}1$, $\rho$ is completely positive. By the Arveson extension theorem, $\rho$ extends to a unital, completely positive (ucp) map $\sigma : C(\mathbb{T}) \to B(H)$. By the Stinespring theorem there is a larger Hilbert space $K \supset H$, and a unitary $U \in B(K)$ such that for all $n \geqslant 0$,

$$\sigma(z^n) = P_H U^n|_H.$$

Since $\pi(z^n) = \sigma(z^n)$ for all nonnegative $n \neq 1$, one direction follows.

Conversely, suppose that there is a unitary operator $U \in B(K)$ such that for all $n \geqslant 0$, $n \neq 1$, $\pi(z^n) = P_H U^n|_H$. Then $\tilde{\pi}$ defined as $\tilde{\pi}(z^n) = U^n$, $n \in \mathbb{Z}$ defines a completely contractive representation of $C(\mathbb{T})$. So $\tilde{\pi}$ restricted the operator system $\mathscr{A} \cap \mathscr{A}^*$ is completely positive, as is $\rho$, its compression to $H$, by the Stinespring dilation theorem. Since unital completely positive maps are completely contractive, $\pi = \rho|\mathscr{A}$ is completely contractive. $\square$

REMARK 2.2. In the above proof, obviously $T = P_H U|_H$ is a contraction. However since the restriction of $\sigma$ to $\mathbb{A}(\mathbb{D})$ is not necessarily multiplicative, we cannot conclude that $\pi(z^2) = T^2$ and $\pi(z^3) = T^3$. Indeed the following example illustrates this concretely:

EXAMPLE 2.3. Let $K$ be a separable Hilbert space with orthonormal basis $\{e_j\}_{j\in\mathbb{Z}}$, and let $U$ be the bilateral shift. Let $H \subseteq K$ be defined as $H = e_0 \vee \bigvee_{n=2}^{\infty} e_n$. Then $H$ is invariant for $U^2$ and $U^3$, and so by Theorem 2.1, $\pi$ given by $\pi(z^n) = P_H U^n|_H = U^n|_H$, $n \geqslant 0$, $n \neq 1$, is a completely contractive representation of $\mathscr{A}$.

If it were the case that for some $T \in B(H)$, $T^2 = \pi(z^2)$ and $T^3 = \pi(z^3)$, it would follow that

$$e_3 = U^3 e_0 = \pi(z^3) = \pi(z^2) T e_0.$$

However, $\langle \pi(z^2) e_n, e_3 \rangle = \langle U^2 e_n, e_3 \rangle = 0$ for $n \geqslant 0$, $n \neq 1$, and hence $e_3$ is orthogonal to the range of $\pi(z^2)$. Thus there is no way to define $Te_0$ so that $e_3 = \pi(z^2) T e_0$, and so there can be no such $T$.

EXAMPLE 2.4. If $\pi : \mathscr{A} \to B(H)$ is a unital contractive representation, then the image of the generators $z^2, z^3$ of $\mathscr{A}$ are evidently contractions, $S = \pi(z^2)$ and $T = \pi(z^3)$. Further $S^3 = T^2$. By Andô's Theorem, there exists a pair of commuting unitaries $X$ and $Y$ on a larger Hilbert space $K$ containing $H$ such that

$$S^n T^m = V^* X^n Y^m V,$$

where $V$ is the inclusion of $H$ into $K$. Because $X$ and $Y$ are unitary and commute, $X^* Y = Y X^*$ by the Putnam-Fuglede theorem. The operator $U = X^* Y$ is a contraction, but unfortunately, there is no reason to expect that $U^2 = X$ and $U^3 = Y$ or equivalently, $X^3 = Y^2$. In general then, it will not be the case that $V^* U^{2n+3m} V = S^n T^m = \pi(z^{2n+3m})$. Indeed, Theorems 1.1 and Theorem 2.1 imply that $\pi$ contractive is not a sufficient assumption to guarantee the existence of such a $U$.

It is worth noting that the construction of $U = X^* Y$ via Ando's Theorem did not use the full strength of the contractive hypothesis on $\pi$, but rather only that $S$ and $T$ are commuting contractions with $S^3 = T^2$. Perhaps surprisingly, in view of Corollary 3.2 below, the representation $\pi$ of $\mathscr{A}$ determined by $\pi(z^2) = S$ and $\pi(z^3) = T$ need not even be contractive.

## 3. The set of test functions and its cone

Given $\lambda \in \mathbb{D}$, let

$$\varphi_\lambda(z) = \frac{z - \lambda}{1 - \lambda^* z}, \tag{2}$$

and let

$$\psi_\lambda(z) = z^2 \varphi_\lambda(z) \tag{3}$$

the (up to a unimodular constant) *Blaschke factor* with zero at $\lambda$, times $z^2$. It will be convenient to let

$$\psi_\infty = z^2$$

and at the same time let $\infty$ denote the point at infinity in the one point compactification $\mathbb{D}_\infty$ of the unit disk $\mathbb{D}$. Let

$$\Psi = \{\psi_\lambda : \lambda \in \mathbb{D}_\infty\},$$

with the topology and Borel structure inherited from $\mathbb{D}_\infty$. We refer to this as a set of *test functions*. It has the properties that it separates the points of $\mathbb{D}$ and for all $z \in \mathbb{D}$, $\sup_{\psi \in \Psi} |\psi(z)| < 1$.

Recall that for a set $X$ and $C^*$-algebra $\mathscr{A}$, a function $k : X \times X \to \mathscr{A}$ is called a *kernel*. It is a *positive kernel* if for every finite subset $\{x_1, \dots, x_n\}$ of $X$, $(k(x_i, x_j)) \in M_n(\mathscr{A})$ is positive semidefinite.

Let $M(\Psi)$ be the space of finite Borel measures on the set of test functions. Given a subset $S$ of $\mathbb{D}$, denote by $M^+(S) = \{\mu : S \times S \to M(\Psi)\}$ the collection of positive kernels on $S \times S$ into $M(\Psi)$. Write $\mu_{xy}$ for the value of $\mu$ at the pair $(x, y)$. By $\mu$ being positive, we mean that for all finite sets $\mathscr{G} \subseteq S$ and all Borel sets $\omega \subseteq \Psi$, the matrix

$$(\mu_{x,y}(\omega))_{x,y \in \mathscr{G}} \tag{4}$$

is positive semidefinite. For example, if $\mu$ is identically equal to a fixed positive measure $\nu$, or more generally is of the form $\mu_{xy} = f(x) f(y)^* \nu$ for a fixed positive measure $\nu$ and bounded measurable function $f : \mathbb{C} \to \mathbb{D}$, or more generally still is a finite sum of such terms, then it is positive.

Our starting point is the following result from [16, Theorem 3.8] (stated there for functions of positive real part):

PROPOSITION 3.1. *An analytic function $f$ in the disk belongs to $\mathscr{A}$ and satisfies $\|f\|_\infty \leqslant 1$ if and only if there is a positive kernel $\mu \in M^+(\mathbb{D})$ such that*

$$1 - f(x) f(y)^* = \int_\Psi (1 - \psi(x)\psi(y)^*)\, d\mu_{xy}(\psi). \tag{5}$$

*for all $x, y \in \mathbb{D}$. Furthermore, $\Psi$ is minimal, in the sense that there is no proper closed subset of $E \subseteq \Psi$ such that for any $f$ with $\|f\|_\infty \leqslant 1$, there exists a $\mu \in M^+(\mathbb{D})$ such that*

$$1 - f(x) f(y)^* = \int_E (1 - \psi(x)\psi(y)^*)\, d\mu_{xy}(\psi). \tag{6}$$

For $E \subseteq \Psi$ a closed subset, let $C_{1,E}$ denote the cone consisting of the kernels

$$\left( \int_E (1 - \psi(x)\psi(y)^*)\, d\mu_{x,y}(\psi) \right)_{x,y \in \mathbb{D}}. \tag{7}$$

(Equivalently, we could consider only those $\mu$ such that $\mu_{xy}$ is supported in $E$ for all $x, y$.) In particular, if we choose $E = \{z^2, z^3\}$, it follows from the above proposition that there exists a function $f \in \mathscr{A}$ with $\|f\|_\infty \leqslant 1$ such that $1 - f(x) f(y)^* \notin C_{1,E}$. This yields in our context an analogue of the Kaiser and Varopoulos example for the tridisk:

COROLLARY 3.2. *There exists a pair of commuting contractive matrices $X, Y$ with $X^3 = Y^2$, but such that the representation of $\mathscr{A}$ determined by $\pi(z^2) = X$, $\pi(z^3) = Y$ is not contractive.*

*Proof.* By a standard cone separation argument (as, for example, in the proof of Proposition 3.5), there is a bounded representation $\pi$ of $\mathscr{A}$ (determined by a pair of matrices $X,Y$ with spectrum in $\mathbb{D}$) such that $\|\pi(\psi)\| \leqslant 1$ for each $\psi \in E$ but $\|\pi(f)\| > 1$. In particular, if we take $E$ to be the closed set $\{z^2, z^3\}$, we see that $X = \pi(z^2)$ and $Y = \pi(z^3)$ satisfy the conditions of the corollary. $\square$

Because a cone separation argument is used, it is unfortunately not possible by these means to explicitly construct an example of such a representation.

### 3.1. The matrix cone

To study the action of representations on $M_2(\mathscr{A})$, consider a finite subset $\mathscr{F} \subseteq \mathbb{D}$. As usual, $M_2(\mathbb{C})$ stands for the $2\times 2$ matrices with entries from $\mathbb{C}$. Let $\mathscr{X}_{2,\mathscr{F}}$ denote the set of all kernels $G : \mathscr{F} \times \mathscr{F} \to M_2(\mathbb{C})$ and $\mathscr{L}_{2,\mathscr{F}} \subseteq \mathscr{X}_{\mathscr{F}}$ denote the selfadjoint kernels $F : \mathscr{F} \times \mathscr{F} \mapsto M_2(\mathbb{C})$, in the sense that $F(x,y)^* = F(y,x)$. Finally, write $C_{2,\mathscr{F}}$ for the cone in $\mathscr{L}_{2,\mathscr{F}}$ of elements of the form

$$\left(\textstyle\int_\Psi (1-\psi(x)\psi(y)^*)\, d\mu_{x,y}(\psi)\right)_{x,y\in\mathscr{F}} \tag{8}$$

where $\mu = (\mu_{x,y}) \in M_2^+(\mathscr{F})$ is a kernel taking its values $\mu_{x,y}$ in the $2\times 2$ matrix valued measure on $\Psi$ such that the measure

$$M(\omega) = \left(\mu_{x,y}(\omega)\right)_{x,y} \tag{9}$$

takes positive semidefinite values (in $M_N(M_2(\mathbb{C}))$). Given $f : \mathscr{F} \to \mathbb{C}^2$, the kernel $(f(x)f(y)^*)_{x,y\in\mathscr{F}}$ is called a *square*.

LEMMA 3.3. *The cone $C_{2,\mathscr{F}}$ is closed and contains all squares.*

*Proof.* For $x \in \mathscr{F}$,

$$\sup_{\psi\in\Psi} |\psi(x)| < |x|.$$

Hence as $\mathscr{F}$ is finite, and $\Psi$ is compact, there exists $0 < \kappa \leqslant 1$ such that for all $x \in \mathscr{F}$ and $\psi \in \Psi$

$$1 - \psi(x)\psi(x)^* \geqslant \kappa.$$

Consequently, since $\Gamma$ defined by

$$\Gamma(x,y) = \int_\Psi (1-\psi(x)\psi(y)^*)\, d\mu_{x,y}(\psi)$$

is in $C_{2,\mathscr{F}}$,

$$\frac{1}{\kappa}\Gamma(x,x) \succeq \mu_{x,x}(\Psi),$$

where the inequality is in the sense of the positive semidefinite order on $2\times 2$ matrices.

Now suppose $(\Gamma_n)$ is a sequence from $C_{2,\mathscr{F}}$ converging to some $\Gamma$. For each $n$ there is a measure $\mu^n$ such that $\Gamma_n$ given by

$$\Gamma_n(x,y) = \int_\Psi (1-\psi(x)\psi(y)^*)\, d\mu^n_{x,y}(\psi).$$

Hence there exists a $\tilde{\kappa} > 0$ such that for all $n$ and all $x \in \mathscr{F}$, $\tilde{\kappa} \geqslant \Gamma_n(x,x)$. Consequently, for all $n$ and all $x \in \mathscr{F}$,

$$\tfrac{\tilde{\kappa}}{\kappa} I \succeq \mu^n_{x,x}.$$

By positivity of the $\mu^n$s, it now follows that the measures $\mu^n_{x,y}$ are uniformly bounded. Hence there exists a subsequence $\mu^{n_j}$ and a measure $\mu$ such that $\mu^{n_j}$ converges weak-$*$ to $\mu$, which therefore is positive. We conclude that

$$\Gamma = \int_\Psi (1-\psi(x)\psi(y)^*)\, d\mu_{x,y}(\psi) \in C_{2,\mathscr{F}},$$

establishing the fact that $C_{2,\mathscr{F}}$ is closed.

Now let $f : \mathscr{F} \to \mathbb{C}^2$ be given. Let $\delta$ denote the unit scalar point mass at $z^3$. Then for $\omega \subseteq \Psi$ a Borel subset,

$$\mu_{x,y}(\omega) = f(x)\frac{1}{1-x^3 y^{*3}}\delta(\omega) f(y)^*$$

defines a positive $M_{\mathscr{F}}(\mathbb{C})$-valued measure and

$$\int_\Psi (1-\psi(x)\psi(y)^*)\, d\mu_{x,y}(\psi) = f(x)f(y)^*,$$

showing that $C_{2,\mathscr{F}}$ contains the squares. $\square$

Elaborating on the construction at the end of the last proof, if

$$\nu(\omega) = \big(\nu_{x,y}(\omega)\big)_{x,y\in\mathscr{F}}$$

is positive semidefinite for every Borel subset $\omega$ of $\Psi$, each $\nu_{xy}$ a scalar valued measure, and if $f : \mathscr{F} \to \mathbb{C}^2$, then

$$\mu_{x,y}(\omega) = f(x)\nu_{x,y}(\omega) f(y)^*,$$

defines an $M_{\mathscr{F}}(M_2(\mathbb{C}))$-valued positive measure $\mu$ and

$$\int_\Psi (1-\psi(x)\psi(y)^*)\, d\mu_{x,y}(\psi) \in C_{2,\mathscr{F}}.$$

We therefore have the following from [16] (see also [9]).

PROPOSITION 3.4. *If $g \in \mathscr{A}$ is analytic in a neighborhood of the closure of the disk and if $\|g\|_\infty \leqslant 1$, then $1-g(x)g(y)^* \in C_{1,\mathscr{F}}(1)$. Thus, if $f : \mathscr{F} \to \mathbb{C}^2$, then*

$$f(x)(1-g(x)g(y)^*)f(y)^* \in C_{2,\mathscr{F}}.$$

### 3.2. The cone separation argument

Continue to let $\mathscr{F}$ denote a finite subset of $\mathbb{D}$. Given $F \in M_2(\mathscr{A})$, let $\Sigma_{F,\mathscr{F}}$ denote the kernel

$$\Sigma_{F,\mathscr{F}} = (1 - F(x)F(y)^*)_{x,y\in\mathscr{F}}. \tag{10}$$

Let $I$ denote the ideal of functions in $\mathscr{A}$ which vanish on $\mathscr{F}$. Write $q : \mathscr{A} \to \mathscr{A}/I$ for the canonical projection, which is completely contractive. We use the standard notation $\sigma(T)$ for the spectrum of an operator $T$ on Hilbert space, as well as $F^t$ for the transpose of the matrix function $F$. Thus, $F^t(z) = F(z)^t$. Obviously, when $F \in M_2(\mathscr{A})$, $F^t$ is as well, and $\|F\|_\infty = \|F^t\|_\infty$.

PROPOSITION 3.5. *If $F \in M_2(\mathscr{A})$ and $\|F\| \leqslant 1$, but $\Sigma_{F,\mathscr{F}} \notin C_{2,\mathscr{F}}$, then there exists a Hilbert space $H$ and representation $\tau : \mathscr{A}/I \to B(H)$ such that for all $a \in \mathscr{A}$,*

*(i)* $\sigma(\tau(a)) \subseteq a(\mathscr{F})$;

*(ii)* $\|\tau(q(a))\| \leqslant 1$ *when* $\|a\| \leqslant 1$; *but*

*(iii)* $\|\tau^{(2)}(q(F^t))\| > 1$;

*that is, the representation $\tau \circ q$ is contractive, but not completely contractive.*

*Proof.* The proof proceeds by a cone separation argument: the representation is obtained by applying the GNS construction to a linear functional that separates $\Sigma_{F,\mathscr{F}}$ from $C_{2,\mathscr{F}}$.

The cone $C_{2,\mathscr{F}}$ is closed and by assumption $\Sigma_{F,\mathscr{F}}$ is not in the cone. Hence there is an $\mathbb{R}$-linear functional $\Lambda : \mathscr{L}_\mathscr{F} \to \mathbb{R}$ such that $\Lambda(C_{2,\mathscr{F}}) \geqslant 0$, but $\Lambda(\Sigma_{F,\mathscr{F}}) < 0$. Given $f : \mathscr{F} \to \mathbb{C}^2$ (that is, $f \in (\mathbb{C}^2)^\mathscr{F}$), recall that the square $ff^* := (f(x)f(y)^*)_{x,y\in\mathscr{F}}$ is in the cone and hence $\Lambda(ff^*) \geqslant 0$. Since every element of $\mathscr{X}_\mathscr{F}$ can be expressed uniquely in the form $G = U + iV$ where $U, V \in \mathscr{L}_\mathscr{F}$, there is a unique extension of $\Lambda$ to a $\mathbb{C}$-linear functional $\Lambda : \mathscr{X}_\mathscr{F} \to \mathbb{C}$. With this extended $\Lambda$, let $H$ denote the Hilbert space obtained by giving $(\mathbb{C}^2)^\mathscr{F}$ the (pre)-inner product

$$\langle f, g\rangle = \Lambda(fg^*)$$

and passing to the quotient by the space of null vectors (those $f$ for which $\Lambda(ff^*) = 0$ – since $\mathscr{F}$ is finite, the quotient will be complete).

Define a representation $\rho$ of $\mathscr{A}$ on $H$ by

$$\rho(g)f(x) = g(x)f(x),$$

where the scalar valued $g$ multiplies the vector valued $f$ entrywise.

If $g \in \mathscr{A}$ is analytic in a neighborhood of the closure of the disk and $\|g\|_\infty \leqslant 1$, then by Proposition 3.4, $f(x)(1 - g(x)g(y)^*)f(y) \in C_{2,\mathscr{F}}$. Thus,

$$\langle f, f\rangle - \langle \rho(g)f, \rho(g)f\rangle = \Lambda((f(x)(1 - g(x)g(y)^*)f(y)^*)_{x,y\in\mathscr{F}}) \geqslant 0. \tag{11}$$

Hence, if $\|g\|_\infty \leqslant 1$, then $\|\rho(g)\| \leqslant 1$ and $\rho$ is a contractive representation of $\mathscr{A}$. Moreover, since the definition of $\rho$ depends only on the values of $g$ on $\mathscr{F}$, it passes to a contractive representation $\tau : \mathscr{A}/I \to B(H)$. The restriction of $\mathscr{A}$ to $\mathscr{F}$ separates points of $\mathscr{F}$ (indeed, the elements of $\Psi$ do so), and so it follows that for each $a \in \mathscr{A}$ the eigenvalues of the matrix representing $\tau(a)$ constitute the set $a(\mathscr{F})$. This proves (*i*) and (*ii*).

To prove (*iii*), let $\{e_1, e_2\}$ denote the standard basis for $\mathbb{C}^2$ and let $[e_j] : \mathscr{F} \to \mathbb{C}^2$ be the constant function $[e_j](x) = e_j$. Note that $\{e_i e_j^*\}_{i,j=1}^2$ is a system of $2\times 2$ matrix units. We find

$$\rho^{(2)}(F^t)([e_1]\oplus[e_2]) = \begin{pmatrix} F_{1,1}e_1 + F_{2,1}e_2 \\ F_{1,2}e_1 + F_{2,2}e_2 \end{pmatrix}.$$

Since

$$\begin{aligned}
&(F_{1,1}e_1 + F_{2,1}e_2)(F_{1,1}e_1 + F_{2,1}e_2)^* \\
&= F_{1,1}F_{1,1}^* e_1 e_1^* + F_{2,1}F_{1,1}^* e_2 e_1^* + F_{1,1}F_{2,1}^* e_1 e_2^* + F_{2,1}F_{2,1}^* e_2 e_2^* \\
&= \begin{pmatrix} F_{1,1}F_{1,1}^* & F_{1,1}F_{2,1}^* \\ F_{2,1}F_{1,1}^* & F_{2,1}F_{2,1}^* \end{pmatrix},
\end{aligned}$$

and

$$\begin{aligned}
&(F_{1,2}e_1 + F_{2,2}e_2)(F_{1,2}e_1 + F_{2,2}e_2)^* \\
&= F_{1,2}F_{1,2}^* e_1 e_1^* + F_{2,2}F_{1,2}^* e_2 e_1^* + F_{1,2}F_{2,2}^* e_1 e_2^* + F_{2,2}F_{2,2}^* e_2 e_2^* \\
&= \begin{pmatrix} F_{1,1}F_{1,1}^* & F_{1,1}F_{2,1}^* \\ F_{2,1}F_{1,1}^* & F_{2,1}F_{2,1}^* \end{pmatrix},
\end{aligned}$$

it follows that

$$\begin{aligned}
&\left\langle \rho^{(2)}(F^t)([e_1]\oplus[e_2]), \rho^{(2)}(F^t)([e_1]\oplus[e_2]) \right\rangle \\
&= \Lambda\left( \begin{pmatrix} F_{1,1}F_{1,1}^* + F_{1,2}F_{1,2}^* & F_{1,1}F_{2,1}^* + F_{1,2}F_{2,2}^* \\ F_{2,1}F_{1,1}^* + F_{2,2}F_{1,2}^* & F_{2,1}F_{2,1}^* + F_{2,2}F_{2,2}^* \end{pmatrix} \right) \\
&= \Lambda(FF^*),
\end{aligned}$$

and so

$$\left\langle (I - \rho^{(2)}(F^t)^* \rho^{(2)}(F^t))[e_1]\oplus[e_2], [e_1]\oplus[e_2] \right\rangle < 0.$$

We conclude that $\|\rho(F^t)\| > 1$, and since by assumption $\|F\|_\infty \leqslant 1$, the representation $\rho$ is not $2$-contractive, and thus not completely contractive. $\square$

REMARK 3.6. Though it is not needed in what follows, observe that the converse of the first part of Proposition 3.5 is true: If $T$ is an operator on Hilbert space with spectrum in $\mathscr{F}$, if $\Sigma_{F,\mathscr{F}} \in C_{2,\mathscr{F}}$ and if $\psi(T)$ is contractive for all $\psi \in \Psi$, then $F(T)$ is also contractive.

A proof follows along now standard lines (see, for instance, [17], where the needed theorems are proved for scalar valued functions, though the proofs remain valid in the matrix case). The assumption that $\Sigma_{F,\mathscr{F}} \in C_{2,\mathscr{F}}$ means that $F$ has a $\Psi$-unitary colligation transfer function representation. Since the operator $T$ has spectrum in the finite set $\mathscr{F}$, it determines a representation of $\mathscr{A}$ which sends bounded pointwise convergent sequences in $M_2(\mathscr{A})$ to weak operator topology convergent sequences. Representations of $M_2(\mathscr{A})$ with this property and for which $\psi(T)$ is contractive for all $\psi \in \Psi$, are contractive.

## 4. Construction of the counterexample preliminaries

For $\lambda \in \mathbb{D}\backslash\{0\}$, let

$$\varphi_\lambda = \frac{z-\lambda}{1-\lambda^* z}.$$

Fix distinct points $\lambda_1, \lambda_2 \in \mathbb{D}\backslash\{0\}$. As a shorthand notation, write $\varphi_j$ for $\varphi_{\lambda_j}$. Set

$$\Phi = \begin{pmatrix} \varphi_1 & 0 \\ 0 & 1 \end{pmatrix} U \begin{pmatrix} 1 & 0 \\ 0 & \varphi_2 \end{pmatrix}, \tag{12}$$

where $U$ is a $2\times 2$ unitary matrix with no non-zero entries. To be concrete, choose

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

In particular $\Phi$ is a $2\times 2$ matrix inner function with $\det \Phi(\lambda) = 0$ at precisely the two nonzero points $\lambda_1$ and $\lambda_2$. The function

$$F = z^2 \Phi \tag{13}$$

is in $M_2(\mathscr{A})$ and is a rational inner function, so $\|F\|_\infty = 1$.

Ultimately we will identify a finite set $\mathscr{F}$ and show that $\Sigma_{F,\mathscr{F}} \notin \mathscr{C}_{2,\mathscr{F}}$ and thus, in view of Proposition 3.5 establish Theorem 1.1. In the remainder of this section we collect some needed preliminary lemmas.

LEMMA 4.1. *Given distinct points $\lambda_1, \lambda_2 \in \mathbb{D}\backslash\{0\}$ and a $2\times 2$ unitary matrix $U$, let*

$$\Theta = \begin{pmatrix} \varphi_1 & 0 \\ 0 & 1 \end{pmatrix} U \begin{pmatrix} 1 & 0 \\ 0 & \varphi_2 \end{pmatrix}, \tag{14}$$

*where $\varphi_j = \varphi_{\lambda_j}$. The matrix $U$ is diagonal; that is, there exist unimodular constants $s$ and $t$ such that*

$$\Theta = \begin{pmatrix} s\varphi_1 & 0 \\ 0 & t\varphi_2 \end{pmatrix},$$

*if and only if there exist $2\times 2$ unitaries $V$ and $W$ such that*

$$\Theta = V^* \begin{pmatrix} \varphi_1 & 0 \\ 0 & \varphi_2 \end{pmatrix} W.$$

*Proof.* The forward implication is trivial. For the converse, let $\{e_1, e_2\}$ denote the standard basis for $\mathbb{C}^2$. Evaluating at $\lambda_2$ it follows that $We_2 = \alpha e_2$. Because $W$ is unitary, it now follows that $W$ is diagonal. A similar argument shows that $V$ is diagonal, and the result follows. $\square$

LEMMA 4.2. *Suppose $\mu_{i,j}$ are $2\times 2$ matrix-valued measures on a measure space $(X,\Sigma)$ for $i,j=0,1$. If $\mu_{i,j}(X)=I$ for all $i,j$ and if for each $\omega\in\Sigma$ the $4\times 4$ matrix-valued measure (block $2\times 2$ matrix with $2\times 2$ matrix entries)*

$$\big(\mu_{i,j}(\omega)\big)_{i,j=1}^2$$

*is positive semidefinite, then $\mu_{i,j}=\mu_{1,1}$ for each $i,j=1,2$.*

*Proof.* Fix a unit vector $f\in\mathbb{C}^2$ and let

$$\nu_{i,j}(\omega)=\langle \mu_{i,j}(\omega)f,f\rangle.$$

It follows that $\nu_{i,j}(X)=1$ and for each $\omega\in\Sigma$

$$\gamma(\omega)=\big(\nu_{i,j}(\omega)\big)_{i,j=1}^2$$

is positive semidefinite. On the other hand,

$$\gamma(X)-\gamma(\omega)\geqslant 0$$

and since $\gamma(X)$ is rank one (with a one in each entry), there is a constant $c=c_\omega$ such that

$$\gamma(\omega)=c\gamma(X).$$

Consequently, $\nu_{i,j}(\omega)=\nu_{1,1}(\omega)$. By polarization it follows that $\mu_{i,j}=\mu_{1,1}$ for all $i,j$. $\square$

LEMMA 4.3. *There exist independent vectors $v_1,v_2\in\mathbb{C}^2$ and, for any finite subset $\mathscr{F}$ of the disc, functions $a,b:\mathscr{F}\to\mathbb{C}^2$ in the span of $\{x^2k_{\lambda_1}(x)v_1, x^2k_{\lambda_2}(x)v_2\}$ such that*

$$\frac{I-\Phi(x)\Phi(y)^*}{1-xy^*}=a(x)a(y)^*+b(x)b(y)^*.$$

*Proof.* Let $M_\Phi$ denote the operator of multiplication by $\Phi$ on $H^2_{\mathbb{C}^2}$, the Hardy-Hilbert space of $\mathbb{C}^2$-valued functions on the disk. Because $\Phi$ is unitary-valued on the boundary, $M_\Phi$ is an isometry. In fact, $M_\Phi$ is the product of three isometries in view of Equation (12). The adjoints of the first and third have one dimensional kernels. The middle term is unitary and so its adjoint has no kernel. Thus, the kernel of $M_\Phi^*$ has dimension at most two. It is evident that $k_{\lambda_1}e_1$ is in the kernel of $M_\Phi^*$. Choose a unit vector $v_2$ in $\mathbb{C}^2$ with entries $\alpha$ and $\beta\neq 0$ such that

$$\begin{pmatrix}\alpha\overline{\varphi_{\lambda_1}(\lambda_2)}\\ \beta\end{pmatrix}=Ue_2,$$

with $U$ the unitary appearing in Equation (12). That such a choice of $\alpha$ and $\beta \neq 0$ is possible follows from the assumption that $\lambda_1 \neq \lambda_2$ which ensures that $\varphi_{\lambda_1}(\lambda_2) \neq 0$, and the assumption that $U$ has no non-zero entries, giving $\beta \neq 0$. Further, with this choice of $v_2$ a simple calculation shows that $k_{\lambda_2} v_2$ is also in the kernel of $M_\Phi^*$. Hence, the dimension of the kernel of $M_\Phi^*$ is two. Since $M_\Phi$ is an isometry, $I - M_\Phi M_\Phi^*$ is the projection onto the kernel of $M_\Phi^*$.

Choose an orthonormal basis $\{a,b\}$ for the kernel of $M_\Phi^*$ so that $I - M_\Phi M_\Phi^* = aa^* + bb^*$. It now follows that for vectors $v, w \in \mathbb{C}^2$,

$$\begin{aligned}\left\langle \frac{I - \Phi(x)\Phi(y)^*}{1 - xy^*} v, w \right\rangle &= \left\langle (I - M_\Phi M_\Phi^*) k_y v, k_x w \right\rangle \\ &= \left\langle (aa^* + bb^*) k_y v, k_x w \right\rangle \\ &= \left\langle k_y v, a \right\rangle \langle a, k_x w \rangle + \left\langle k_y v, b \right\rangle \langle b, k_x w \rangle \\ &= \langle (a(x)a(y)^* + b(x)b(y)^*) v, w \rangle. \quad \square\end{aligned}$$

The following is well known.

LEMMA 4.4. *Let $s$ be the Szegő kernel,*

$$s(x,y) = \frac{1}{1 - xy^*}.$$

*If $x_1, \ldots, x_m$ and $y_1, \ldots, y_m$ are two $m$-tuples each of distinct points in the unit disk $\mathbb{D}$, then the matrix*

$$M = \big(s(x_j, y_\ell)\big)_{j,\ell=1}^n$$

*is invertible.*

*Proof.* Suppose $Mc = 0$ where $c$ is the vector with entries $c_1, \ldots, c_m$. Let

$$r(x) = \sum c_\ell s(x, y_\ell) = [(1 - xy_1^*) \cdots (1 - xy_m^*)]^{-1} \sum c_\ell p_\ell(x),$$

for polynomials $p_\ell$ of degree at most $m-1$. Hence $r$ is a rational function with numerator a polynomial $p$ of degree at most $m-1$ and denominator which does not vanish on $\mathbb{D}$. The hypotheses imply that $p(x_j) = 0$ for $j = 1, 2, \ldots, m$. Hence $p$ is identically zero, as then is $r$. Since the kernel functions $\{s(\cdot, t_\ell) : \ell = 1, 2, \ldots, m\}$ form a linearly independent set in $H^2(\mathbb{D})$, it follows that $c = 0$. $\square$

Given a $2 \times 2$ matrix valued measure and a vector $\gamma \in \mathbb{C}^2$, let $\nu_\gamma$ denote the scalar measure defined by $\nu_\gamma(\omega) = \gamma^* \nu(\omega) \gamma$. Note that if $\nu$ is a positive measure (that is, takes positive semidefinite values), then each $\nu_\gamma$ is a positive measure. Let $\Psi_0 = \Psi \setminus \{\psi_\infty\}$.

LEMMA 4.5. *Suppose $\nu$ is a $2 \times 2$ positive matrix-valued measure on $\Psi_0$. For each $\gamma$ the measure $\nu_\gamma$ is a nonnegative linear combination of at most two point masses* *if and only if there exist (possibly not distinct) points* $\mathfrak{z}_1,\mathfrak{z}_2$ *and positive semidefinite matrices* $Q_1$ *and* $Q_2$ *such that*

$$\nu = \sum_{j=1}^{2} \delta_{\mathfrak{z}_j} Q_j,$$

*where* $\delta_{\mathfrak{z}_1}, \delta_{\mathfrak{z}_2}$ *are scalar unit point measures on* $\Psi$ *supported at* $\psi_{\mathfrak{z}_1}, \psi_{\mathfrak{z}_2}$*, respectively.*

*Proof.* If $\nu = \sum_{j=1}^{2} \delta_{\mathfrak{z}_j} Q_j$ with $\mathfrak{z}_1,\mathfrak{z}_2$ and $Q_1, Q_2$ as in the statement of the lemma, then clearly each $\nu_\gamma$ is a nonnegative linear combination of at most two point masses.

For the converse, the $M_2$-valued measure $\nu$, expressed as a $2\times 2$ matrix of scalar measures with respect to the standard orthonormal basis $\{e_1, e_2\}$ of $\mathbb{C}^2$ has the form

$$\nu = \begin{pmatrix} \nu_{11} & \nu_{12} \\ \nu_{21} & \nu_{22} \end{pmatrix}. \tag{15}$$

Since $\nu(\omega)$ is a positive matrix for every measurable set $\omega$, it follows that $\nu_{11}, \nu_{22}$ are positive measures. Moreover for the off-diagonal entries we have $\nu_{21} = \nu_{12}^*$. If $\omega$ is such that $\nu_{11}(\omega) = 0$, then by positivity $\nu_{12}(\omega) = 0$, and similarly if $\nu_{22}(\omega) = 0$. So it follows that $\nu_{12}$ and $\nu_{21}$ are absolutely continuous with respect to both $\nu_{11}$ and $\nu_{22}$. This argument also shows that $\nu_{12}$ and $\nu_{21}$ are supported on the intersection of the supports for $\nu_{11}$ and $\nu_{22}$.

Choosing $\gamma = e_1$, the hypotheses imply there exist $\alpha_1, \alpha_2 \geqslant 0$ and points $\mathfrak{z}_1, \mathfrak{z}_2$ such that

$$\nu_{11} = \sum_{j=1}^{2} \alpha_j \delta_{\mathfrak{z}_j}.$$

Likewise there exist points $\mathfrak{w}_1, \mathfrak{w}_2$ and scalars $\beta_1, \beta_2 \geqslant 0$ such that

$$\nu_{22} = \sum_{j=1}^{2} \beta_j \delta_{\mathfrak{w}_j}.$$

There are several cases to consider. First suppose that the $\{\mathfrak{z}_1,\mathfrak{z}_2\}$ and $\{\mathfrak{w}_1,\mathfrak{w}_2\}$ have no points in common. Then $\nu_{12} = 0 = \nu_{21}$. Also, for $\gamma = e_1 + e_2$, by assumption

$$\nu_\gamma = \nu_{11} + \nu_{22}$$

has support at two points, and so $\mathfrak{z}_1 = \mathfrak{z}_2$ and $\mathfrak{w}_1 = \mathfrak{w}_2$. It follows that the union of the supports of $\nu_{11}$ and $\nu_{22}$ has cardinality at most two, yielding the desired result.

Next suppose that the sets $\{\mathfrak{z}_1,\mathfrak{z}_2\}$ and $\{\mathfrak{w}_1,\mathfrak{w}_2\}$ have one point in common, say $\mathfrak{z}_1 = \mathfrak{w}_1$. In this case $\nu_{12}$ is supported at $\mathfrak{z}_1$ and there is a complex number $s$ so that

$$\nu_{12} = s\delta_{\mathfrak{z}_1}.$$

If $s = 0$, choose $\gamma = e_1 + e_2$, so that $\nu_\gamma = \nu_{11} + \nu_{22}$. Otherwise set $\gamma = e_1 + s^* e_2$, in which case,

$$\nu_\gamma = \nu_{11} + 2|s|^2 \delta_{\mathfrak{z}_1} + |s|^2 \nu_{22}.$$

In either case, $\nu_\gamma$ has support at $\{\mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{w}_2\}$ and only two of these can be distinct.

The remaining case has the sets $\{\mathfrak{z}_1,\mathfrak{z}_2\}$ and $\{\mathfrak{w}_1,\mathfrak{w}_2\}$ equal, and the result is immediate.

Positivity of $\nu$ implies positivity of $Q_1$ and $Q_2$. $\square$

## 5. The proof of Theorem 1.1

Fix a finite set $\mathscr{F}$ containing $0, \lambda_1, \lambda_2$ and consisting of at least six distinct points. This choice of $\mathscr{F}$ along with the prior choices of $\Phi$ and $F$ as in Equations (12) and (13) remain in effect for the rest of this section. Accordingly, let $\Sigma_F = \Sigma_{F,\mathscr{F}}$.

We next prove the following diagonalization result.

THEOREM 5.1. *If $\Sigma_F$ lies in the cone $C_{2,\mathscr{F}}$; that is, there exists an $M_2(\mathbb{C})$-valued positive measure $\mu$ such that*

$$I - F(x)F(y)^* = \int_\Psi (1 - \psi(x)\psi(y)^*)\, d\mu_{x,y}(\psi) \qquad x, y \in \mathscr{F}, \tag{16}$$

*then there exists rank one orthogonal projections $Q_1, Q_2$ summing to $I$, such that, for $x, y \in \mathscr{F}$,*

$$I - F(x)F(y)^* = (1 - x^2 y^{*2} \varphi_1(x)\varphi_1(y)^*)Q_1 + (1 - x^2 y^{*2} \varphi_2(x)\varphi_2(y)^*)Q_2. \tag{17}$$

The proof proceeds by a sequence of lemmas which increasingly restrict the measures $\mu_{x,y}$ in (16).

Assume that $\Sigma_F \in C_{2,\mathscr{F}}$. Multiplying (16) by the Szegő kernel $s(x,y) = (1 - xy^*)^{-1}$ obtains

$$\left( \frac{I - F(x)F(y)^*}{1 - xy^*} \right)_{x,y \in \mathscr{F}} = \left( \int_\Psi \left( \frac{1 - \psi(x)\psi(y)^*}{1 - xy^*} \right) d\mu_{x,y}(\psi) \right)_{x,y \in \mathscr{F}}. \tag{18}$$

Next, since $F$ has the form $x^2\Phi(x)$,

$$\begin{aligned} \frac{I - F(x)F(y)^*}{1 - xy^*} &= \frac{I_2 - x^2 y^{*2} I_2 + x^2 y^{*2} I_2 - x^2 y^{*2} \Phi(x)\Phi(y)^*}{1 - xy^*} \\ &= (1 + xy^*) I_2 + x^2 y^{*2} \left( \frac{I - \Phi(x)\Phi(y)^*}{1 - xy^*} \right). \end{aligned}$$

Similarly, for the test functions $\psi_\lambda(x) = x^2 \varphi_\lambda(x)$ at points $\lambda \in \mathbb{D}$,

$$\frac{1 - \psi_\lambda(x)\psi_\lambda(y)^*}{1 - xy^*} = (1 + xy^*) + x^2 y^{*2} \left( \frac{1 - \varphi_\lambda(x)\varphi_\lambda(y)^*}{1 - xy^*} \right). \tag{19}$$

(Here we take $\varphi_\infty = 1$.) Letting

$$k_\lambda(x) = \frac{\sqrt{1 - |\lambda|^2}}{1 - \lambda^* x}$$

denote the normalized Szegő kernel at $\lambda \in \mathbb{D}$ and using the identity

$$\frac{1 - \varphi_\lambda(x)\varphi_\lambda(y)^*}{1 - xy^*} = k_\lambda(x) k_\lambda(y)^*, \tag{20}$$

for such $\lambda$, equation (19) gives,

$$\frac{1-\psi_\lambda(x)\psi_\lambda(y)^*}{1-xy^*} = (1+xy^*)+x^2y^{*2}k_\lambda(x)k_\lambda(y)^*.$$

For $\lambda=\infty$ (correspondingly, $\psi_\infty(z)=z^2$),

$$\frac{1-\psi_\infty(x)\psi_\infty(y)^*}{1-xy^*} = 1+xy^*.$$

Putting these computations together, we rewrite (18) as

$$\begin{aligned}\frac{I-F(x)F(y)^*}{1-xy^*} &= (1+xy^*)I_2+x^2y^{*2}\left(\frac{I-\Phi(x)\Phi(y)^*}{1-xy^*}\right)\\ &= (1+xy^*)\int_\Psi d\mu_{x,y}(\psi)+x^2y^{*2}\int_{\Psi_0} k_\lambda(x)k_\lambda(y)^*\,d\mu_{x,y}(\psi).\end{aligned} \tag{21}$$

Note that the first integral is over $\Psi$ while the second is just over $\Psi_0=\Psi\backslash\{z^2\}$ since $k_\infty(x)=0$.

Combining Lemma 4.3 with Equation (21) gives functions $a,b:\mathscr{F}\to\mathbb{C}^2$ as in that lemma, with

$$\begin{aligned}&(1+xy^*)I+x^2y^{*2}\left(a(x)a(y)^*+b(x)b(y)^*\right)\\ &\qquad=\int_\Psi(1+xy^*)\,d\mu_{x,y}(\psi)+\int_{\Psi_0} x^2y^{*2}k_\lambda(x)k_\lambda(y)^*\,d\mu_{x,y}(\psi).\end{aligned} \tag{22}$$

The next step will be to remove the $x,y$ dependence in $\mu$. Introducing some notation, let

$$\begin{aligned}\tilde{A}(x,y) &= \int_\Psi d\mu_{x,y}(\psi);\\ R(x,y) &= x^2y^{*2}\left(a(x)a(y)^*+b(x)b(y)^*\right); \quad \text{and}\\ \tilde{R}(x,y) &= (xy^*)^2\int_{\Psi_0} k_\lambda(x)k_\lambda(y)^*\,d\mu_{x,y}(\psi).\end{aligned}$$

Thus, $\tilde{A}$, $R$, and $\tilde{R}$ are all positive kernels on $\mathscr{F}$. With this notation and some rearranging of Equation (22), for $x,y\in\mathscr{F}$,

$$(1+xy^*)(\tilde{A}(x,y)-I) = R(x,y)-\tilde{R}(x,y). \tag{23}$$

Let

$$\mathbb{K}=\{x^2k_{\lambda_1}(x)v_1, x^2k_{\lambda_2}(x)v_2\}, \tag{24}$$

the set of vectors spanning the kernel of $I-M_\Phi M_\Phi^*$ appearing in Lemma 4.3.

LEMMA 5.2. *With the above notations, the assumption that* $\Sigma_F\in C_{2,\mathscr{F}}$ *and for* $x,y\in\mathscr{F}$, *(i) The $M_2(\mathbb{C})$ valued kernel $(\tilde{A}-I)(x,y) = (\tilde{A}(x,y)-I)$ is positive semidefinite;*

*(ii) The $M_2(\mathbb{C})$ valued kernel $R(x,y)-\tilde{R}(x,y)$ is positive semidefinite with rank at most two;*

*(iii) The range of $\tilde{R}$ lies in the range of $R$, which is in the span of $\mathbb{K}$; and*

*(iv) Either*

*(a) The kernel $\tilde{A}-I$ has rank at most one; i.e., there is a function $r:\mathscr{F}\to\mathbb{C}^2$ such that*

$$\tilde{A}(x,y) = I + r(x)r(y)^*, \quad \text{or;} \tag{25}$$

*(b) there exist functions $r,s:\mathscr{F}\to\mathbb{C}^2$ such*

$$\tilde{A}(x,y) = I + r(x)r(y)^* + s(x)s(y)^*,$$

*and a point $\mathfrak{z}\in\mathscr{F}\setminus\{0\}$ such that $r(\mathfrak{z}) = 0 = s(\mathfrak{z})$.*

*Proof.* Since $\psi(0)=0$ for all $\psi\in\Psi$, it follows from (16) that for all $y\in\mathscr{F}$,

$$I = I - F(0)F(y)^* = \int_\Psi (1-\psi(0)\psi(y)^*)d\mu_{0,y}(\psi) = \int d\mu_{0,y}(\lambda) = \tilde{A}(0,y).$$

By positivity we can factor $(A(x,y))_{x,y\neq 0} = C^*C$, and there is a contraction $G$ such that the row $(A(0,y))_{y\neq 0} = (I\cdots I) = GC$. Consequently, $(A(x,y))_{x,y\neq 0} \geqslant (I\cdots I)^t(I\cdots I)$, and so $(i)$ is seen to hold.

That $R-\tilde{R}$ is positive semidefinite follows from $(i)$ and Equation (23). Since $R$ is rank two it must be the case that the rank of $R-\tilde{R}$ is rank at most two, completing the proof of $(ii)$.

By $(ii)$ and Douglas' lemma, the range of $\tilde{R}$ is contained in the range of $R$. By Lemma 4.3, the range of $R$ is spanned by the set $\mathbb{K}$ and $(iii)$ follows.

To prove $(iv)$, first note that in any case Equation (23) and $(ii)$ imply $\tilde{A}-I$ has rank at most two; i.e., there exists $r,s:\mathscr{F}\to\mathbb{C}^2$ such that

$$\tilde{A}-I = r(x)r(y)^* + s(x)s(y)^*.$$

From Equation (23), each of $r, xr, s, xs$ lie in the range of $R$, which equals the span of $\mathbb{K}$. If $r$ is nonzero at two points in $\mathscr{F}$, then $r$ and $xr$ are linearly independent and hence span the range of $R$. In this case, as both $s$ and $xs$ are in the range of $R$ there exists $\alpha_j$ and $\beta_j$ (for $j=1,2$) such that

$$\begin{aligned} s =& \alpha_1 r + \alpha_2 xr \\ xs =& \beta_1 r + \beta_2 xr. \end{aligned}$$

It follows that

$$0 = xs - xs = (\beta_1 + (\beta_2-\alpha_1)x + \alpha_2 x^2)r(x). \tag{26}$$

If $\alpha_2 = 0$, then $s$ is a multiple of $r$ and case $(iv)(a)$ holds. Otherwise, in view of (26), $r$ is zero with the exception of at most two points. Thus $r$ is zero at two points, one of which, say $\mathfrak{z}$, must be different from $0$. Since $s$ must be zero when $r$ is, $s(\mathfrak{z}) = 0$ too and $(iv)(b)$ holds.

The remaining possibility is that both $r$ and $s$ are non-zero at at most one point each, and these points may be distinct. In this situation $r$ and $s$ have at least two common zeros, one of which must be different from $0$ and again $(iv)(b)$ holds. $\square$

LEMMA 5.3. *Under the assumption that $\Sigma_F \in C_{2,\mathscr{F}}$, the $2\times 2$ matrix-valued kernel $\tilde{A}$ is constantly equal to $I$; that is, $\tilde{A}(x,y) = I$ for all $x,y \in \mathscr{F}$.*

*Proof.* In the case that $(iv)(a)$ holds in Lemma 5.2, it (more than) suffices to prove that the $r$ in Equation (25) is $0$. To this end, let $\mathfrak{R}$ denote the range of $R$ which, by Lemma 5.2, is spanned by the set $\mathbb{K}$ appearing in Equation (24). From Equations (23) and (25),

$$\tilde{R} + (1+xy^*)r(x)r(y)^* = R.$$

Thus, $\mathfrak{R}$ contains both $r$ and $xr$; that is, both $r$ and $xr$ are in the span of $\mathbb{K}$. Consequently, there exists $\alpha_j$ and $\beta_j$ $(j = 1,2)$ such that

$$\begin{aligned} r =& x^2 \sum_{j=1}^{2} \alpha_j k_{\lambda_j}(x) v_j \\ xr =& x^2 \sum_{j=1}^{2} \beta_j k_{\lambda_j}(x) v_j. \end{aligned}$$

Hence,

$$0 = xr - xr = x^2 \sum_{j=1}^{2} (\beta_j - x\alpha_j) k_{\lambda_j}(x) v_j. \tag{27}$$

Since the set $\{v_1, v_2\}$ is a basis for $\mathbb{C}^2$ (see Lemma 4.3), it has a dual basis $\{w_1, w_2\}$. Taking the inner product with $w_\ell$ in Equation (27) gives,

$$0 = x^2(\beta_\ell - x\alpha_\ell) k_{\lambda_\ell}(x)$$

for $x \in \mathscr{F}$. Choosing $x = \lambda_\ell$ (which is not zero) implies $\beta_\ell - \lambda_\ell \alpha_\ell = 0$. But then choosing any $x \in \mathscr{F}$ different from both $0$ and $\lambda_j$ (and using $k_{\lambda_j}(x) \neq 0$) implies $\beta_\ell - x\alpha_\ell = 0$. Hence $\alpha_\ell = 0 = \beta_\ell$ and consequently $r(x) = 0$ for all $x$.

Now suppose $(iv)(b)$ in Lemma 5.2 holds. In particular, there exists a point $\mathfrak{z}$ in $\mathscr{F} \setminus \{0\}$ such that $r(\mathfrak{z}) = 0 = s(\mathfrak{z})$. By the same reasoning as in the first part of this proof, there exist $\alpha_j$ and $\beta_j$ such that

$$\begin{aligned} r =& x^2 \sum_{j=1}^{2} \alpha_j k_{\lambda_j}(x) v_j \\ s =& x^2 \sum_{j=1}^{2} \beta_j k_{\lambda_j}(x) v_j. \end{aligned}$$

Taking the inner product with $w_\ell$ and evaluating at $\mathfrak{z}$ yields

$$0 = \alpha_\ell k_{\lambda_\ell}(\mathfrak{z}).$$

Thus $\alpha_\ell = 0$. Likewise, $\beta_\ell = 0$. Thus $r = 0 = s$ and the proof is complete. $\square$

REMARK 5.4. Observe that if it were the case that $v_1 = v_2$ in Equation (27), then it would not be possible to conclude that the $\alpha_j$ and $\beta_j$ are $0$. Indeed, in such a situation, choosing $\beta_j = (-1)^j$ and $\alpha_j = (-1)^j \lambda_j^*$ gives a non-trivial solution. However, the case $v_1 = v_2$ corresponds to a $\Phi$ having the form

$$\Phi = \begin{pmatrix} 1 & 0 \\ 0 & \varphi_{\lambda_1}\varphi_{\lambda_2} \end{pmatrix},$$

which is explicitly ruled out by our choice of $\Phi$ and Lemma 4.1.

LEMMA 5.5. *There exists a $2 \times 2$ matrix valued positive measure $\mu$ on $\Psi$ such that $\mu(\Psi) = I_2$ and*

$$K^\Phi(x,y) := \frac{1 - \Phi(x)\Phi(y)^*}{1 - xy^*} = \int_{\Psi_0} k_\lambda(x) k_\lambda(y)^* \, d\mu(\psi) \tag{28}$$

*for all $x, y \in \mathscr{F} \setminus \{0\}$.*

*Proof.* By Lemma 5.3, $\tilde{A}(x,y) = I$ for all $x, y \in \mathscr{F}$. An examination of the definition of $\tilde{A}$ and application of Lemma 4.2 implies there is a positive measure $\mu$ such that $\mu_{x,y} = \mu$ for all $(x,y)$. Substituting this representation for $\mu_{x,y}$ into (21) after some canceling and rearranging one has,

$$(xy^*)^2 \left( \frac{I - \Phi(x)\Phi(y)^*}{1 - xy^*} \right) = x^2 y^{*2} \int_{\Psi_0} k_\lambda(x) k_\lambda(y)^* \, d\mu(\psi).$$

Dividing by $(xy^*)^2$ (and of course excluding either $x = 0$ or $y = 0$) gives the result. $\square$

Now that $\mu$ has no $x, y$ dependence, the next step is to restrict its support. For this we employ Lemma 4.3. Recall that $\mu$ is a positive $2 \times 2$ matrix-valued measure on $\Psi$. Let $\delta_\infty$ denote point mass at the point $\psi_\infty = z^2$.

LEMMA 5.6. *Under the assumption that $\Sigma_F \in C_{2,\mathscr{F}}$, and with notation as above, there are two points $\mathfrak{z}_1, \mathfrak{z}_2$ in $\mathscr{F}$ such that the measure $\mu$ has the form $\mu = \delta_{\mathfrak{z}_1} Q_1 + \delta_{\mathfrak{z}_2} Q_2 + \delta_\infty P$, where $Q_1, Q_2, P$ are $2 \times 2$ matrices satisfying $0 \leqslant Q_1, Q_2, P \leqslant 1$ and $Q_1 + Q_2 + P = I$, and $\delta_{\mathfrak{z}_1}, \delta_{\mathfrak{z}_2}$ are scalar unit point measures on $\Psi$ supported at $\psi_{\mathfrak{z}_1}, \psi_{\mathfrak{z}_2}$, respectively.*

*Proof.* We first show that the restriction of $\mu$ to $\mathbb{D}$ has support at no more than two points. Accordingly, let $\nu$ denote the restriction of $\mu$ to $\mathbb{D}$.

From Lemma 4.3, for $x,y \in \mathscr{F} \setminus \{0\}$,

$$\frac{I_2 - \Phi(x)\Phi(y)^*}{1 - xy^*} = a(x)a(y)^* + b(x)b(y)^*$$

where $a, b$ are $\mathbb{C}^2$ valued functions on $\mathfrak{F}$. Fix a vector $\gamma$ and define a scalar measure $\nu_\gamma$ on $\Psi$ by $\nu_\gamma(\omega) = \gamma^* \nu(\omega)\gamma$. Note that

$$\begin{aligned}\gamma^* \left(a(x)a(y)^* + b(x)b(y)^*\right)\gamma &= \gamma^* \left(\int_\Psi k_\lambda(x)k_\lambda(y)^* d\mu(\psi)\right)\gamma \\ &= \int_{\Psi_0} k_\lambda(x)k_\lambda(y) d\nu_\gamma(\psi)\end{aligned}$$

is a kernel of rank (at most) two.

Choosing a three point subset $\mathfrak{G} \subseteq \mathscr{F} \setminus \{0\}$ and a nonzero scalar-valued function $c : \mathfrak{G} \to \mathbb{C}$ such that

$$\sum_{x,y \in \mathfrak{G}} c(x)\gamma^* \left(a(x)a(y)^* + b(x)b(y)^*\right)\gamma c(y)^* = 0 \tag{29}$$

gives

$$0 = \int_{\Psi_0} \left| \sum_{x \in \mathfrak{G}} k_\lambda(x)c(x) \right|^2 d\nu_\gamma(\psi), \tag{30}$$

which means that the function $f = \sum_{x \in \mathfrak{G}} k_\lambda(x)c(x)$ vanishes for $\nu_\gamma$-a.e. on $\Psi_0$. The function $f$ is a linear combination of at most three Szegő kernels, and hence can vanish at at most two points in $\mathbb{D}$. It follows that $\nu_\gamma$ is supported at at most two points in $\mathbb{D}$. An application of Lemma 4.5 now implies that there exist points $\mathfrak{z}_1, \mathfrak{z}_2$ and positive semidefinite matrices $Q_1, Q_2$ such that

$$\nu = \sum_{j=1}^{2} \delta_{\mathfrak{z}_j} Q_j.$$

Letting $P = \mu(\{\infty\})$, it follows that $\mu$ has the promised form,

$$\mu = \delta_{\mathfrak{z}_1} Q_1 + \delta_{\mathfrak{z}_2} Q_2 + \delta_\infty P.$$

Finally, because $\mu$ has total mass equal to the identity,

$$I = \mu(\Psi) = Q_1 + Q_2 + P. \quad \square$$

To eliminate $P$ and show that the $Q_i$ are orthogonal, rank one projections, return to Equation (28) and rearrange it once again. Recall the identity of Equation (20), multiply through by $1 - xy^*$ and use Lemma 5.6. We have by the description of $\mu$ from the previous lemma, for $x, y \in \mathscr{F} \setminus \{0\}$,

$$1 - \Phi(x)\Phi(y)^* = (1 - \varphi_{\mathfrak{z}_1}(x)\varphi_{\mathfrak{z}_1}(y)^*)Q_1 + (1 - \varphi_{\mathfrak{z}_2}(x)\varphi_{\mathfrak{z}_2}(y)^*)Q_2,$$

where $\psi_{\mathfrak{z}_1}, \psi_{\mathfrak{z}_2}$ are the support points of the measure $\mu$. Using the fact that $Q_1+Q_2+P=I$, we obtain, for $x,y \in \mathscr{F}\setminus\{0\}$,

$$\Phi(x)\Phi(y)^* = \varphi_{\mathfrak{z}_1}(x)\varphi_{\mathfrak{z}_1}(y)^*Q_1 + \varphi_{\mathfrak{z}_2}(x)\varphi_{\mathfrak{z}_2}(y)^*Q_2 + P. \tag{31}$$

LEMMA 5.7. *Let $\Phi$ be as above. In the representation (31),*

*(i)* $\{\mathfrak{z}_1,\mathfrak{z}_2\} = \{\lambda_1,\lambda_2\}$*;*

*(ii)* $P=0$*; and*

*(iii)* $Q_1,Q_2$ *are rank one projections summing to* $I$ *(and hence mutually orthogonal).*

*Proof.* By the identity (31)

$$\Phi(\lambda_1)\Phi(\lambda_1)^* = |\varphi_{\mathfrak{z}_1}(\lambda_1)|^2Q_1 + |\varphi_{\mathfrak{z}_2}(\lambda_1)|^2Q_2 + P,$$

and since $\det\Phi(\lambda_1)=0$, both sides of this equation have rank at most one. It follows that at least one of $\varphi_{\mathfrak{z}_1}, \varphi_{\mathfrak{z}_2}$ (and hence exactly one, since the $\lambda_j$ are distinct) must have a zero at $\lambda_1$, since otherwise the three positive matrices $Q_1,Q_2,P$ would all be scalar multiples of the same rank one matrix, which violates $Q_1+Q_2+P=I$. Similarly for $\lambda_2$, so (*i*) is proved. Further, without loss of generality, it can be assumed that $\mathfrak{z}_j=\lambda_j$ for $j=1,2$.

It follows from evaluating at the $\lambda_j$ that each of $Q_1,Q_2,P$ has rank at most one. In particular we have for $k,j=1,2$, $k\neq j$,

$$\Phi(\lambda_j)\Phi(\lambda_j)^* = |\varphi_k(\lambda_j)|^2Q_k + P. \tag{32}$$

This means that $\operatorname{ran}P \subseteq \operatorname{ran}Q_1 \cap \operatorname{ran}Q_2$. On the other hand, if $\operatorname{ran}Q_1\cap\operatorname{ran}Q_2 \neq \{0\}$, we have $\operatorname{ran}Q_1 \subseteq \operatorname{ran}Q_2$ or vice versa, which again contradicts $Q_1+Q_2+P=1$. Thus $\operatorname{ran}Q_1 \vee \operatorname{ran}Q_2 = \mathbb{C}^2$, and so $P=0$, which is (*ii*). Since $Q_2 = 1-Q_1$, if $f\in\ker Q_1$, then $Q_2f=f$. However, $Q_2$ is a rank one contraction, so it must be a projection, and then the same follows for $Q_1$. Thus we have (*iii*). □

*Proof of Theorem* 5.1. Since $F(x)=x^2\Phi(x)$, Theorem 5.1 is now immediate from Lemma 5.7. □

### 5.1. The proof of Theorem 1.1

The proof of Theorem 1.1 concludes in this subsection. Recall that we are assuming that $F(z)=z^2\Phi(z)$, where $\Phi$ is as in (12).

Suppose that $\Sigma_F \in C_{2,\mathscr{F}}$. From Equation (31) and Lemma 5.7, for $x,y\in\mathscr{F}\setminus\{0\}$,

$$\Phi(x)\Phi(y)^* = \sum_{j=1}^{2}\varphi_j(x)\varphi_j(y)^*Q_j. \tag{33}$$

Since the $Q_j$ are rank one projections which sum to $I$, there exists an orthonormal basis $\{\gamma_1, \gamma_2\}$ such that

$$Q_j = \gamma_j \gamma_j^*.$$

Let $V$ be the unitary matrix with columns $\gamma_j$, and let

$$G(z) = V \begin{pmatrix} \varphi_1(z) & 0 \\ 0 & \varphi_2(z) \end{pmatrix}.$$

Observe $\Phi(x)\Phi(y)^* = G(x)G(y)^*$ for $x, y \in \mathscr{F} \setminus \{0\}$.

Fix $\zeta \in \mathscr{F} \setminus \{0, \lambda_1, \lambda_2\}$. Then $\Phi(\zeta)$ is invertible and further $\Phi(\zeta)\Phi(\zeta)^* = G(\zeta)G(\zeta)^*$. Hence by Douglas' Lemma, there is a unitary $W$ such that $\Phi(\zeta) = G(\zeta)W^*$. Consequently,

$$0 = \Phi(\zeta)\Phi(y)^* - G(\zeta)G(y)^* = G(\zeta)(\Phi(y)W - G(y))^*,$$

and therefore $\Phi(y)W = G(y)$, for $y \in \mathscr{F} \setminus \{0\}$. Returning to the definition of $G$, we arrive at the conclusion that, for $x \in \mathscr{F} \setminus \{0\}$,

$$\Phi(x) = U \begin{pmatrix} \varphi_1(x) & 0 \\ 0 & \varphi_2(x) \end{pmatrix} W^*. \tag{34}$$

Now $\Phi$ and $G$ are both rational matrix inner functions of degree at most two. Since $\mathscr{F} \setminus \{0\}$ contains at least five points it is a set of uniqueness for rational functions of degree at most two, and hence (34) must hold on all of $\mathbb{D}$. It now follows that, on all of $\mathbb{D}$,

$$\Phi = U \begin{pmatrix} \varphi_1 & 0 \\ 0 & \varphi_2 \end{pmatrix} W^*.$$

By Lemma 4.1,

$$\Phi = \begin{pmatrix} s\varphi_1 & 0 \\ 0 & t\varphi_2 \end{pmatrix}$$

for unimodular constants $s$ and $t$, contrary to our choice of $\Phi$ in (12). We conclude that $\Sigma_F \notin C_{2,\mathscr{F}}$, and so by Proposition 3.5, there exists a contractive representation of $\mathscr{A}$ which is not completely contractive.

## 6. Rational dilation for the annulus and the variety $z^2 = w^2$

This section provides a proof of rational dilation for the annulus along the lines of [22], but with a major simplification suggested by Agler [2] (see also [3]). Direct appeal to the systematic study of the extreme rays of functions of positive real part on a multiply connected domain found in [16, 10] and [11] (see also [19, 18]) also significantly streamlines the argument. This proof for the annulus, with minor modifications indicated in Subsection 6.5, also establishes rational dilation for the distinguished variety defined by $z^2 = w^2$.

### 6.1. A Naimark dilation Theorem

The following version of the Naimark dilation Theorem will be used to reduce the extreme rays of functions of positive real part on an annulus to a much smaller collection.

THEOREM 6.1. *Fix positive integers $m,n$ and suppose that $A_1,\dots,A_m;B_1,\dots,B_m$ are rank one positive semidefinite $n\times n$ matrices. If*

$$\sum A_j = I = \sum B_\ell,$$

*then there exists an isometry $V:\mathbb{C}^n\to\mathbb{C}^m$ and $m\times m$ matrices $P_1,\dots,P_m;Q_1\dots,Q_m$ such that*

*(i) Each of $P_1,\dots,P_m;Q_1,\dots Q_m$ are rank one projections;*

*(ii)*

$$\sum P_j = I = \sum Q_\ell;$$

*(iii) and*

$$A_j = V^*P_jV,\;\; B_\ell = V^*Q_\ell V.$$

*Proof.* Since for each $j$, $A_j$ is rank one and positive semidefinite, there exists $a_j\in\mathbb{C}^n$ such that

$$A_j = a_ja_j^*.$$

Let $V$ denote the matrix whose $j$-th row is $a_j^*$ (the $1\times n$) matrix. It follows that $V$ is an $m\times n$ matrix and moreover,

$$V^*V = \sum a_ja_j^* = I.$$

Thus $V$ is an isometry. Let $P_j = e_je_j^*$, where $\{e_1,\dots,e_n\}$ is the standard orthonormal basis for $\mathbb{C}^n$ and note that

$$V^*P_jV = a_ja_j^* = A_j.$$

The analogous construction with $B_j = b_jb_j^*$ produces an isometry $W:\mathbb{R}^n\to\mathbb{R}^m$ such that

$$W^*P_\ell W = B_\ell.$$

Since $V$ and $W$ are isometries, the mapping $U:\operatorname{range}(V)\to\operatorname{range}(W)$ defined by $UVx = Wx$ is a unitary mapping. Since the codimensions of the range of $V$ and the range of $W$ are the same, $U$ can be extended to a unitary mapping on $\mathbb{C}^m$. Let $Q_\ell = U^*P_\ell U$. Then each $Q_\ell$ is a rank one projection, $\sum Q_\ell = I$ and

$$V^*Q_\ell V = V^*U^*P_\ell UV = W^*P_\ell W = B_\ell. \quad \square$$

### 6.2. Extremal functions of positive real part

As a special case of the results in [10, 11], the matrix-valued functions of positive real part on an annulus are characterized.

Fix $0 < q < 1$ and let $\mathbb{A}$ denote the annulus,

$$\mathbb{A} = \{z \in \mathbb{C} : q < |z| < 1\},$$

with its boundary components

$$\partial_0 = \{|z| = 1\}, \;\; \partial_1 = \{|z| = q\}.$$

Let $M_n$ denote the $n \times n$ matrices. An analytic function $F : \mathbb{A} \to M_n$ whose real part

$$\operatorname{Re} F(z) = \frac{F(z) + F(z)^*}{2}$$

takes positive definite values in $\mathbb{A}$ has an $n \times n$ matrix-valued measure $\mu_F$ on $\partial = \partial_0 \cup \partial_1$ for its boundary values. On the other hand, a positive semidefinite $n \times n$ matrix-valued measure $\mu$ on $\partial$ is the boundary values of a matrix-valued harmonic function $H$ on $\mathbb{A}$. Moreover, $H$ is the real part of analytic function if and only if

$$\mu(\partial_0) = \mu(\partial_1).$$

By compressing to the range of $\mu(\partial_0)$, it can be assumed that $\mu(\partial_0)$ has full rank.

Let $\Gamma_n$ denote the set of positive semidefinite $n \times n$ matrix-valued measures $\mu$ on $\partial$ such that $\mu(\partial_0) = I = \mu(\partial_1)$. The results of [10] and [11] imply that the extreme points of the set $\Gamma_n$ have the form,

$$\mu = \sum_{j=1}^{m} A_j \delta_{\alpha_j} + \sum_{\ell=1}^{m} B_\ell \delta_{\beta_\ell}, \tag{35}$$

where for any $j, \ell$, each $A_j$ and $B_\ell$ is a rank one, positive semidefinite $n \times n$ matrix, $\alpha_j \in \partial_1$ and $\beta_\ell \in \partial_0$ and

$$\sum A_j = I = \sum B_\ell.$$

Repetition is allowed in the sets of points $\{\alpha_j\}$ and $\{\beta_\ell\}$ to allow for attaching arbitrary positive semidefinite matrices to a point on $\partial$ and $A_j$ or $B_\ell$ may the zero matrix, so as to ensure that $\{A_j\}$ and $\{B_\ell\}$ have the same cardinality. It should be noted that not every measure of the form in Equation (35) is an extreme point (a characterization is given in [10, 11]).

For a $\mu \in \Gamma_n$, let $F_\mu$ denote a corresponding analytic function of positive real part. Thus, the real part of $F_\mu$ is the harmonic function whose boundary values are $\mu$. Such an $F$ is not unique, but any two differ by a matrix $C$ which is skew self-adjoint, $C^* = -C$. Note that the real part of $F$ is zero except at the $m$ points (counting multiplicity) in the support of $\mu$ on each of the components of $\partial$.

An operator $T$ has $\mathbb{A}$ as a spectral set if $\sigma(T) \subseteq \mathbb{A}$ and $\|f(T)\| \leqslant 1$ for each analytic function $f : \mathbb{A} \to \mathbb{D}$. Here $\mathbb{D}$ is the unit disc, $\{z \in \mathbb{C} : |z| < 1\}$ and $\sigma(T)$ is the spectrum of $T$. The following proposition is a consequence of the results of [10, 11].

THEOREM 6.2. *Let $T$ be a operator on the Hilbert space $H$ with $\sigma(T) \subseteq \mathbb{A}$, and suppose $\mathbb{A}$ is a spectral set for $T$. Then there exists a normal operator $N$ acting on a Hilbert space $K$ with $\sigma(N) \subseteq \partial$ and an isometry $V : H \to K$ such that $r(T) = V^* r(N) V$ for all rational functions $r$ with poles off the closure of $\mathbb{A}$ if and only if*

$$\frac{F_\mu(T) + F_\mu(T)^*}{2} \succeq 0$$

*for each $n$ and each $\mu$ as in Equation (35).*

REMARK 6.3. The first equivalent condition of the theorem says that $T$ has a rational dilation to a normal operator with spectrum in the boundary of $\mathbb{A}$.

### 6.3. Matrix extreme functions of positive real part

There is a particularly nice subset of the extreme points of $\Gamma_n$ from which all the extreme points of $\Gamma_n$ can be recovered in a canonical fashion.

Let $\mathscr{E}_n$ denote those elements $\nu$ of $\Gamma_n$ of the form,

$$\nu = \sum_{j=1}^{n} A_j \delta_{\alpha_j} + \sum_{\ell=1}^{n} B_\ell \delta_{\beta_\ell}.$$

In particular, each $A_j$ $B_\ell$ is a rank one projection with $\sum A_j = \sum B_\ell = I$.

LEMMA 6.4. *Let*

$$G = (F_\nu - I)(F_\nu + I)^{-1}.$$

*For each $n \times n$ unitary matrix $U$ the function*

$$\det(I - G(z)U)$$

*has precisely $n$ zeros on each boundary component of $\partial$.*

*Proof.* Because $F$ has positive real part, $G$ is contractive-valued in $\mathbb{A}$. Further, as the real part of $F$ is $0$, except on a finite subset of $\partial$, the function $G$ is unitary-valued on a cofinite subset of $\partial$ and hence extends by the reflection principle to a function analytic in a neighborhood of the closure of $\mathbb{A}$.

On the other hand,

$$F_\nu = (I + G)(I - G)^{-1}.$$

Thus the real part of $F_\nu$ is zero at $z$ unless $1$ is in the spectrum of $G(z)$. Hence, $\det(I - G(z))$ has exactly $n$ zeros (counting multiplicity) on each boundary component of $\mathbb{A}$.

Let $\mathscr{U}$ denote the collection of $n \times n$ unitary matrices, and let

$$\mathscr{U}_k = \{U \in \mathscr{U} : \det(I - G(z)U) \text{ has } k \text{ zeros on } \partial_0\}.$$

Note that as the zeros of $\det(I - G(z)U)$ can only occur on the boundary, this number of zeros is stable with respect to small perturbations of $U$. Thus, $\mathscr{U}_k$ is open. But $\mathscr{U} = \cup \mathscr{U}_k$ is compact, and hence this union is finite. Since $\mathscr{U}_n$ is not empty and $\mathscr{U}$ is connected, it follows that $\mathscr{U} = \mathscr{U}_n$. $\square$

### 6.4. Rational dilation on $\mathbb{A}$

THEOREM 6.5. ([1, 22]) *If the operator $T$ has the annulus as a spectral set, then $T$ has a normal dilation to an operator with spectrum in the boundary of $\mathbb{A}$.*

*Proof.* It suffices to verify the second of the equivalent conditions in Theorem 6.2. Accordingly, let such an $F_\mu$ be given. By Theorem 6.1, there is an $m$, an isometry $V: \mathbb{C}^n \to \mathbb{C}^m$ and rank one projections $P_j$ and $Q_\ell$ as described in that theorem so that

$$V^* P_j V = A_j, \quad V^* Q_\ell V = B_\ell.$$

Consider the measure

$$\nu = \sum_{j=1}^m P_j \delta_{\alpha_j} + \sum_{j=1}^m Q_\ell \delta_{\beta_\ell}.$$

Because the $\sum P_j = \sum Q_\ell = I$, it follows that there is an $m \times m$ matrix-valued analytic function $G$ of positive real part whose boundary values are the measure $\mu$. Further, since

$$V^* \nu V = \mu,$$

if $\operatorname{Re} G(T) \succeq 0$, then also $\operatorname{Re} F(T) \succeq 0$. Hence it suffices to prove $\operatorname{Re} G(T) \succeq 0$ under the assumption that $\mathbb{A}$ is a spectral set for $T$.

Let

$$G = (F_\nu - I)(F_\nu + I)^{-1}.$$

Then $G$ is a contractive analytic function in $\mathbb{A}$. By Lemma 6.4, with $U = G(1)^*$ and $\tilde{G} = GU$, the function $\det(I - G(z)U)$ has exactly $n$ zeros on each of $\partial_0$ and $\partial_1$. Let

$$\tilde{F} = (I + \tilde{G})(I - \tilde{G})^{-1}.$$

Thus $\tilde{F}$ has positive real part and moreover its boundary values determine a measure $\tilde{\mu}$ with support at exactly $n$ points (counting multiplicity) on each boundary component. On the other hand, the choice of $U$ implies that

$$\tilde{\mu} = \sum_{j=1}^n \tilde{A}_j \delta_{\gamma_j} + \tilde{B} \delta_1,$$

where each $\tilde{A}_j$ is a rank one, positive semidefinite $n \times n$ matrix and

$$\tilde{B} = \sum \tilde{A}_j \succ 0.$$

Thus,

$$\operatorname{Re} \tilde{F} = \sum_{j=1}^n \tilde{A}_j (\delta_{\gamma_j} + \delta_1)$$

on $\partial \mathbb{A}$.

Consider now the scalar measures $\delta_{\gamma_j} + \delta_1$. Each of these measure puts unit mass on each boundary component $\partial_0, \partial_1$, and hence for each $j$ there is a holomorphic function $\psi_j$ of positive real part in $\mathbb{A}$ such that

$$\operatorname{Re}\psi_j = \delta_{\gamma_j} + \delta_1$$

on $\partial\mathbb{A}$. By the assumption that $\mathbb{A}$ is a spectral set for $T$, we have $\operatorname{Re}\psi_j(T) \geqslant 0$ for each $j$.

Now let

$$\Gamma = \sum \psi_j \tilde{A}_j.$$

It follows that $\Gamma$ and $\tilde{F}$ agree up to a skew symmetric matrix. Thus,

$$\operatorname{Re}\tilde{F}(T) = \operatorname{Re}\Gamma(T) = \sum(\operatorname{Re}\psi_j(T)) \otimes \tilde{A}_j \succeq 0.$$

Hence,

$$\|\tilde{G}(T)\| \leqslant 1;$$

and thus

$$\|G(T)\| \leqslant 1;$$

and so finally,

$$\operatorname{Re}F(T) \succeq 0,$$

and the proof is complete. $\square$

### 6.5. The variety $z^2 = w^2$

In this section we consider the hypo-Dirichlet algebra $\mathscr{A}(\mathscr{V})$ of functions which are analytic on the variety $\mathscr{V} = \{(z,w) \in \mathbb{D}^2 : z^2 = w^2\}$ in the bidisk and continuous on its boundary $\partial\mathscr{V}$. The boundary of $\mathscr{V}$ consists of the two disjoint circles $z = w$, and $z = -w$ with $|z| = |w| = 1$. The proof of rational dilation for the annulus given above may be straightforwardly modified to prove the following theorem:

THEOREM 6.6. *Every contractive homomorphism $\pi : \mathscr{A}(\mathscr{V}) \to B(H)$ is completely contractive.*

The algebra $\mathscr{A}(\mathscr{V})$ is in some sense a limiting case of the annulus algebras. Indeed for a fixed real number $0 < t < 1$, the variety in $\mathbb{D}^2$ defined by

$$z^2 = \frac{w^2 - t^2}{1 - t^2 w^2} \tag{36}$$

is an open Riemann surface which is topologically an annulus [27], and in fact by varying $t$ every annulus $\mathbb{A}_q$ is conformally equivalent to one of these [12, 13]. The variety $\mathscr{V}$ is of course the limiting case $t \to 0$.

To get started we record some basic facts about $\mathscr{V}$ and $\mathscr{A}(\mathscr{V})$. To fix some notation: $\mathscr{V}$ is the union of the two sheets

$$\mathscr{V}_+ = \{(z,w) : z = w\}, \quad \mathscr{V}_- = \{(z,w) : z = -w\}, \tag{37}$$

which can each be identified with unit disk $\mathbb{D}$ via the parametrization $\psi_+(t) = (t,t)$ and $\psi_-(t) = (t,-t)$ respectively. The sheets $\mathcal{V}_\pm$ intersect only at the origin, and the boundary of $\mathcal{V} \cap \mathbb{D}^2$ is the disjoint union of the circles $\partial\mathcal{V}_+$, and $\partial\mathcal{V}_-$. We equip each of these circles with normalized Lebesgue measure (that is, the push-forward of Lebesgue measure under the maps $\psi_\pm$). We also recall that, by definition, a (scalar or matrix valued) function $F$ is holomorphic on the variety $\mathcal{V}$ if and only if for each point $(z,w) \in \mathcal{V}$, there is a neighborhood $\Omega$ of this point in $\mathbb{C}^2$ such that $F$ extends to be holomorphic in $\Omega$. $\mathscr{A}(\mathcal{V})$ is then the algebra of functions holomorphic on $\mathcal{V}$ and continuous on $\mathcal{V} \cup \partial\mathcal{V}$, equipped with the supremum norm, which we denote $\|f\|_{\mathcal{V}}$.

Given any function $F$ on $\mathcal{V}$, we let $F_\pm$ denote its restrictions to the disks $\mathcal{V}_\pm$. In particular, if $F$ is holomorphic on $\mathcal{V}$, then $H_\pm(t) := F_\pm(\psi_\pm(t))$ are holomorphic functions on the disk, and $H_+(0) = H_-(0)$. The converse is also true:

LEMMA 6.7. *Given any pair of holomorphic functions $H_\pm : \mathbb{D} \to M_n(\mathbb{C})$ with $H_+(0) = H_-(0)$, there exists a holomorphic function $F : \mathcal{V} \to M_n(\mathbb{C})$ such that $F_\pm \circ \psi_\pm = H_\pm$.*

*Proof.* It suffices to assume $H_\pm(0) = 0$, in which case the function

$$F(z,w) = (1-(z-w))H_+\left(\frac{z+w}{2}\right) + (1-(z+w))H_-\left(\frac{z-w}{2}\right) \tag{38}$$

is holomorphic in $\mathbb{D}^2$ and restricts to $H_\pm$ on $\mathcal{V}_\pm$. $\square$

Since polynomials are dense in the disk algebra $\mathscr{A}(\mathbb{D})$, an immediate consequence is that polynomials in $z,w$ are dense in $\mathscr{A}(\mathcal{V})$. It is also evident that $\|F\|_{\mathcal{V}} = \max(\|H_+\|_\infty, \|H_-\|_\infty)$, and that $\mathscr{A}(\mathcal{V})$ is a uniform algebra with Shilov boundary $\partial\mathcal{V}$.

PROPOSITION 6.8. *Let $\mu$ be a finite, nonnegative $n \times n$ matrix valued measure on $\partial\mathcal{V}$. Then there is a function $F \in M_n(Hol(\mathcal{V}))$ such that $\mu = Re\ F$ on $\partial V$ if and only if $\mu(\partial\mathcal{V}_+) = \mu(\partial\mathcal{V}_-)$.*

*Proof.* This is more or less immediate from the foregoing description of the holomorphic functions on $\mathcal{V}$; indeed the necessity of the condition $\mu(\partial\mathcal{V}_+) = \mu(\partial\mathcal{V}_-)$ is evident since by restricting to each disk $\mu(\partial\mathcal{V}_\pm) = F_\pm(0)$. Conversely, suppose this constraint holds. Let $\mu_\pm$ denote the restriction of $\mu$ to the respective boundary circles. On each of the disks $\mathcal{V}_\pm$ there is a holomorphic function $H_\pm$, real-valued at the origin, such that $\mathrm{Re}\, H_\pm = \mu_\pm$ on $\partial\mathcal{V}_\pm$, and we have $H_+(0) = H_-(0) = \mu(\partial\mathcal{V}_\pm)$. Thus by the lemma, each $H_\pm$ is the restriction to $\mathcal{V}_\pm$ of the same function $F$, holomorphic on $\mathcal{V}$. $\square$

One immediate consequence of Proposition 6.8 is that a continuous real-valued function $u$ on $\partial\mathcal{V}$ is the real part of the boundary values of a holomorphic function on $\mathcal{V}$ if and only if $\int_{\mathcal{V}_+} u\, dm = \int_{\mathcal{V}_-} u\, dm$. Using this fact and the density of polynomials in $\mathscr{A}(\mathcal{V})$, it follows that, viewing $\mathscr{A}(\mathcal{V})$ as a subalgebra of $C(\partial\mathcal{V})$, the closure of $\mathrm{Re}\,\mathscr{A}(\mathcal{V})$ in $C_\mathbb{R}(\partial\mathcal{V})$ is equal to the pre-annihilator of the measure $m_+ - m_-$

on $\partial\mathscr{V}$ (here $m_\pm$ is Lebesgue measure on $\partial\mathscr{V}_\pm$). Thus the closure of $\operatorname{Re}\mathscr{A}(\mathscr{V})$ in $C_{\mathbb{R}}(\partial\mathscr{V})$ has codimension 1 (in particular $\mathscr{A}(\mathscr{V})$ is a hypo-Dirichlet algebra on $\partial\mathscr{V}$, as claimed). Note that the codimension is also 1 in the case of the annulus.

Consider the collection of $n\times n$ matrix-valued holomorphic functions on $\mathscr{V}$ with positive real part, normalized to $F(0)=I_n$, and let $\Gamma_n$ denote the extreme points of this set. Using Proposition 6.8, the results of [10, 11] are again applicable and exactly as in the case of the annulus, every extreme point of $\Gamma_n$ has the form $F_\mu$ for some finitely supported $\mu$ of the form

$$\mu=\sum_{j=1}^{m}A_j\delta_{\alpha_j}+\sum_{j=1}^{m}B_j\delta_{\beta_j} \tag{39}$$

with $\{\alpha_1,\dots\alpha_m\},\{\beta_1,\dots\beta_m\}$ subsets of $\partial\mathscr{V}_\pm$ respectively, and $\sum A_j=\sum B_j=I$ (though again not every such $F_\mu$ is an extreme point).

We say that $\overline{\mathscr{V}}$ is a *spectral set* for the pair of commuting operators $S,T$ if the joint spectrum of $S,T$ lies in $\overline{\mathscr{V}}$ and, for every polynomial $p(z,w)$, we have

$$\|p(S,T)\|\leqslant\|p\|_{\mathscr{V}}. \tag{40}$$

Note that this condition forces $S^2=T^2$, since $p(z,w)=z^2-w^2$ vanishes on $\mathscr{V}$.

We say that the pair $(S,T)$ acting on the Hilbert space $H$ has a *normal $\partial\mathscr{V}$ dilation* if there exists a pair of commuting normal operators $U,V$ acting on a Hilbert space $K$ with spectrum in $\partial\mathscr{V}$ and an isometry $\iota: H\to K$ such that $p(S,T)=\iota^*p(U,V)\iota$ for all polynomials $p$. By the definition of $\mathscr{V}$ and the spectral theorem, the commuting normal pairs with spectrum in $\partial\mathscr{V}$ are precisely the pairs of unitary operators $U,V$ satisfying $U^2=V^2$.

PROPOSITION 6.9. *Let $S,T$ be a pair of commuting operators with joint spectrum in $\mathscr{V}$ and suppose $\overline{\mathscr{V}}$ is a spectral set for $S,T$. Then $(S,T)$ has a normal $\partial\mathscr{V}$ dilation if and only if*

$$F_\mu(S,T)+F_\mu(S,T)^*\succeq 0 \tag{41}$$

*for all $\mu$ as in (39).*

*Proof.* If a dilation exists, then (41) holds by the spectral theorem. Conversely, suppose (41) holds. Then by the Choquet integral arguments of [10, 11], and the fact that the joint spectral radius of $S,T$ is strictly less than 1, we have that $F(S,T)+F(S,T)^*\succeq 0$ for all matrix-valued functions $F$ on $\mathscr{V}$ with positive real part. In particular, if $P$ is a matrix-valued polynomial with $\|P\|_{\mathscr{V}}<1$, then $F=(I+P)(I-P)^{-1}$ has positive real part, so $F(S,T)+F(S,T)^*\succeq 0$ and thus $\|P(S,T)\|\leqslant 1$. This says that the map $p\to p(S,T)$ is completely contractive. $\square$

*Proof of Theorem 6.6.* Let $\pi$ be a contractive representation of $\mathscr{A}(\mathscr{V})$, then $\pi$ is determined by a pair of commuting contractions $S,T$ satisfying $S^2=T^2$. To prove that $\pi$ is completely contractive, observe that we may replace $S,T$ by $rS,rT$ for $r<1$. (Note that $r^2S^2=r^2T^2$ so $rS,rT$ still determine a homomorphism $\pi_r$ of $\mathscr{A}(\mathscr{V})$.) Indeed, since the map $p(z,w)\to p(rz,rw)$ is completely contractive on $\mathscr{A}(V)$, if the maps $\pi_r$ are completely contractive then so is $\pi$. So, now that $\|S\|,\|T\|<1$, it suffices to verify the condition of Proposition 6.9. But the proof now reduces to one essentially identical to the proof given for the annulus above; the boundary components $\partial\mathscr{V}_\pm$ playing the roles of $\partial_0,\partial_1$. The only modification is to Lemma 6.4. In particular when we speak of extending $G$ to a neighborhood of a boundary point it should be understood that this is a neighborhood in the union of the planes $z\pm w=0$ (the full variety $z^2=w^2$ in $\mathbb{C}^2$); all zero-counting is done here. The proof of Theorem 6.5 then goes through unchanged. $\square$

The question still remains as to which pairs $S,T$ with $S^2=T^2$ have $\mathscr{V}$ as a spectral set. It is evident that $S$ and $T$ must be contractions, but this alone is not sufficient. As in the case of the annulus, there is a one-parameter family of conditions that must be checked. The following is Theorem 9.4 of [7]. For completeness, we include a proof.

THEOREM 6.10. ([7]) *Let $S,T$ be commuting operators with $S^2=T^2$. Then $\mathscr{V}$ is a spectral set for $S,T$ if and only if*

$$\|\lambda S+(1-\lambda)T\|\leqslant 1 \tag{42}$$

*for every complex number $\lambda$ lying on the circle $|\lambda-\frac{1}{2}|=\frac{1}{2}$.*

*Proof.* For the $\lambda$ described in the theorem one may check that the functions

$$\lambda z+(1-\lambda)w \tag{43}$$

are bounded by $1$ on $\mathscr{V}$, so the condition is necessary.

Conversely, using again the Choquet integral arguments of [10, 11], for $\mathscr{V}$ to be a spectral set it suffices to check that $\operatorname{Re}F_\mu(S,T)\succeq 0$ for every extreme point $F_\mu$ of the set of functions of positive real part on $\mathscr{V}$ (normalized to $F(0,0)=1$). From [10, 11] we also know that the $\mu$ representing these functions are precisely those that put a single unit point mass on each boundary component. By the description of $\operatorname{Hol}\mathscr{V}$ in Lemma 6.7, these are the functions whose restrictions satisfy

$$F_+(t)=\frac{1+\alpha t}{1-\alpha t},\quad F_-(t)=\frac{1+\beta t}{1-\beta t} \tag{44}$$

for unimodular constants $\alpha,\beta$. Taking Cayley transforms $f_\pm=(F_\pm-1)(F_\pm+1)^{-1}$ we get simply the functions

$$f_+(t)=\alpha t, f_-(t)=\beta t, \tag{45}$$

and we require $\|f(S,T)\|\leqslant 1$ for all $\alpha,\beta$ where $f$ is any function on $\mathbb{D}^2$ with $f|_{\mathscr{V}_\pm}=f_\pm$. Multiplying $f$ by $\alpha^*$, we may assume $\alpha=1$, and now it is straightforward to check that, putting $\lambda=\frac{1+\beta}{2}$, the functions

$$f(z,w)=\lambda z+(1-\lambda)w \tag{46}$$

do the job. $\square$

Combining Theorems 6.6 and 6.10 we have:

COROLLARY 6.11. *Let $S,T$ be commuting operators on Hilbert space with $S^2 = T^2$. Then $S,T$ dilate to a commuting pair of unitaries $U,V$ satisfying $U^2 = V^2$ if and only if*

$$\|\lambda S + (1-\lambda)T\| \leqslant 1 \tag{47}$$

*for every complex number $\lambda$ on the circle $|\lambda - \frac{1}{2}| = \frac{1}{2}$.*

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