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SOME INEQUALITIES FOR SECTOR MATRICES

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To the memory of Leiba Rodman (1949-2015)

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Abstract. Two new inequalities are proved for sector matrices. The first one complements a recent result in [Oper. Matrices, 8 (2014) 1143–1148]; the second one is an analogue of the AM-GM inequality, where the geometric mean for two sector matrices was introduced in [Linear Multilinear Algebra 63 (2015) 296-301]. As an application of the second inequality, we present similar inequalities for singular values or norms.

1. Introduction

By a sector, we mean a region on the complex plane

$$S_\alpha = \{z \in \mathbb{C} : \Re z > 0, |\Im z| \leqslant (\Re z)\tan\alpha\}, \qquad \alpha \in [0, \pi/2).$$

The set of all $n \times n$ complex matrices is denoted by $\mathbb{M}_n$. Recall that the numerical range of an $n \times n$ matrix $M \in \mathbb{M}_n$ is defined by

$$W(M) = \{x^*Mx : x \in \mathbb{C}^n,\ x^*x = 1\}.$$

Sector matrices is a class of matrices whose numerical ranges are contained in S_α (for some fixed α), though the numerical range of a sector matrix may not be a sector. This class of matrices has been the subject of a number of recent papers [3, 4, 5, 6, 8, 9]. We follow up the study by contributing some new inequalities.

Consider $A \in \mathbb{M}_n$ partitioned as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \text{ where } A_{22} \in \mathbb{M}_q,\ q \leqslant \lfloor n/2 \rfloor. \tag{1}$$

Assume that A_{11} is invertible, the Schur complement of A_{11} in A is defined as $A/A_{11} := A_{22} - A_{21}A_{11}^{-1}A_{12}$. It is clear that A is invertible whenever $W(A) \subset S_\alpha$. If $W(A) \subset S_\alpha$, then $W(A_{11}) \subset S_\alpha$, thus A/A_{11} is well defined.

Our starting point is the following singular value inequality

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THEOREM 1. [4, Theorem 1.1] *Let* $A \in \mathbb{M}_n$ *be partitioned as in (1) and* $W(A) \subset S_\alpha$. *Then*

$$\sigma_j(A/A_{11}) \leqslant \sec^2(\alpha)\sigma_j(A_{22}), \qquad j = 1, \ldots, q, \tag{2}$$

where $\sigma_j(\cdot)$ *are the singular values, arranged in descending order.*

For two Hermitian matrices $A, B \in \mathbb{M}_n$, we write $A \geqslant B$ (or $B \leqslant A$) to mean that $A - B$ is positive semidefinite. The absolute value of X is defined as $|X| = (X^*X)^{1/2}$. With this notation, (2) can be equivalently written as

$$|A/A_{11}| \leqslant \sec^2(\alpha)U^*|A_{22}|U \tag{3}$$

for some unitary matrix $U \in \mathbb{M}_q$.

2. An inequality involving Schur complements

The real part (or the Hermitian part) of $A \in \mathbb{M}_n$ is denoted by $\Re A := \frac{A+A^*}{2}$. We present the following result, which says that concerning the real parts of $A/A_{11}, A_{22}$, an analogue of (3) is valid without bringing in a unitary matrix.

THEOREM 2. *Let* $A \in \mathbb{M}_n$ *be partitioned as in (1) and* $W(A) \subset S_\alpha$. *Then*

$$\Re(A/A_{11}) \leqslant \sec^2(\alpha)\Re A_{22}. \tag{4}$$

If $X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$ is invertible, then we also partition X^{-1} comformally as X so that $(X^{-1})_{22}$ means the $(2,2)$ block of X^{-1}. We need two simple lemmas. These lemmas should be well known to experts on matrix analysis, but I include proofs for the convenience of readers.

LEMMA 1. *If* $X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$ *is positive definite, then*

$$(X_{22})^{-1} \leqslant (X^{-1})_{22}.$$

Proof. Note that $(X^{-1})_{22} = (X/X_{11})^{-1} = (X_{22} - X_{21}X_{11}^{-1}X_{12})^{-1} \geqslant (X_{22})^{-1}$. A generalization of this lemma can be found in [7]. $\square$

LEMMA 2. *If* $X \in \mathbb{M}_n$ *has a positive definite real part, then*

$$\Re(X^{-1}) \leqslant (\Re X)^{-1}.$$

Proof. Consider the Cartesian decomposition $X = Y + iZ$. Then

$$\Re(X^{-1}) = (Y + ZY^{-1}Z)^{-1} \leqslant Y^{-1} = (\Re X)^{-1}. \quad \square$$

Proof of Theorem 2. Consider the Cartesian decomposition $A = B + iC$. The condition $W(A) \subset S_\alpha$ implies that $\pm C \leqslant \tan(\alpha)B$ and so

$$\pm B^{-1/2}CB^{-1/2} \leqslant \tan(\alpha).$$

This yields $(B^{-1/2}CB^{-1/2})^2 \leqslant \tan^2(\alpha)$, i.e.,

$$CB^{-1}C \leqslant \tan^2(\alpha)B.$$

In particular,

$$(CB^{-1}C)_{22} \leqslant \tan^2(\alpha)B_{22}. \tag{5}$$

Note that $\sec^2(\alpha) = 1 + \tan^2(\alpha)$, so (5) is equivalent to

$$\cos^2(\alpha)(B + CB^{-1}C)_{22} \leqslant B_{22}. \tag{6}$$

With (6), we can find upper bounds for $(B_{22})^{-1}$,

$$\begin{aligned}
(\Re A_{22})^{-1} = (B_{22})^{-1} &\leqslant \sec^2(\alpha)\Big((B + CB^{-1}C)_{22}\Big)^{-1} \\
&\leqslant \sec^2(\alpha)\Big((B + CB^{-1}C)^{-1}\Big)_{22} \\
&= \sec^2(\alpha)\Big(\Re(A^{-1})\Big)_{22} \\
&= \sec^2(\alpha)\Re(A^{-1})_{22} \\
&= \sec^2(\alpha)\Re((A/A_{11})^{-1}) \\
&\leqslant \sec^2(\alpha)\Big(\Re(A/A_{11})\Big)^{-1},
\end{aligned}$$

in which the second inequality is by Lemma 1 and the third inequality is by Lemma 2. Therefore, $\Re(A/A_{11}) \leqslant \sec^2(\alpha)\Re A_{22}$, as desired. $\square$

REMARK 1. Note that (4) can be written as

$$\Re(\tan^2(\alpha)A_{22} + A_{21}A_{11}^{-1}A_{12}) \geqslant 0.$$

On the other hand, if $W(A) \subset S_\alpha$, then $W(AA^{-1}A^*) = W(A^*) \subset S_\alpha$, which yields $W(A^{-1}) \subset S_\alpha$. As $(A/A_{11})^{-1}$ is a principal submatrix of A^{-1}, we have $W((A/A_{11})^{-1}) \subset S_\alpha$ and so $W(A/A_{11}) \subset S_\alpha$. In particular,

$$\Re(A_{22} - A_{21}A_{11}^{-1}A_{12}) \geqslant 0.$$

However, under the assumption $W(A) \subset S_\alpha$, it is in general not true that

$$\Re(A_{22} + A_{21}A_{11}^{-1}A_{12}) \geqslant 0.$$

3. AM-GM inequalities

The geometric mean of two positive definite matrices $A,B\in\mathbb{M}_n$ is defined by

$$A\sharp B := B^{1/2}(B^{-1/2}AB^{-1/2})^{1/2}B^{1/2}.$$

It is easy to check that the geometric mean $A\sharp B$ is the unique positive definite solution to the Ricatti equation $XB^{-1}X=A$. For more information about matrix geometric mean, we refer to [1, Chapter 4].

Generalizing this, Drury [3] defined the geometric mean for two sector matrices $A,B\in\mathbb{M}_n$ via the formula

$$A\sharp B := \left(\frac{2}{\pi}\int_0^\infty (tA+t^{-1}B)^{-1}\frac{dt}{t}\right)^{-1}, \tag{7}$$

in which we continue to use the standard notation $A\sharp B$ for the geometric mean.

Clearly, from (7), one observes that $A\sharp B=B\sharp A$ and that if $W(A)\subset S_\alpha$ and $W(B)\subset S_\alpha$, then $W(A\sharp B)\subset S_\alpha$. Though not obvious, one could verify that the geometric mean in (7) satisfies (see [3, Theorem 3.4])

(i) $A\sharp B = B^{1/2}(B^{-1/2}AB^{-1/2})^{1/2}B^{1/2}$.

(ii) $A\sharp B$ is a solution to the Ricatti equation $XB^{-1}X=A$. Moreover, if a solution X to the Ricatti equation $XB^{-1}X=A$ has positive definite real part, then $X=A\sharp B$ (see [3, Proposition 3.5]).

The following noncommutative AM-GM inequality is known for positive definite matrices $A,B\in\mathbb{M}_n$ (e.g. [1])

$$A\sharp B \leqslant \frac{A+B}{2}. \tag{8}$$

Is there an analogue for sector matrices? The first thought is whether it holds

$$\Re(A\sharp B) \leqslant \Re\frac{A+B}{2} \tag{9}$$

for sector matrices $A,B\in\mathbb{M}_n$. The answer is no as the following example shows

EXAMPLE 1. Let

$$A=\begin{bmatrix}10 & 3+i\\ 3+i & 2+4i\end{bmatrix}, B=\begin{bmatrix}2-4i & -1-4i\\ -1-4i & 2-i\end{bmatrix}.$$

It is easy to verify that A,B have positive definite real part. Using the Matlab, one computes that $\Re(A\sharp B)=\begin{bmatrix}6.2830 & 2.0747\\ 2.0747 & 3.2251\end{bmatrix}$. However, in this case, $\det\left(\Re\frac{A+B}{2}-\Re(A\sharp B)\right)=-0.8083<0$, violating (9).

The main result of this section is a correct extension of (8). We need a lemma, which can be regarded as a complement of Lemma 2.

LEMMA 3. *If $X \in \mathbb{M}_n$ with $W(A) \subset S_\alpha$, then*

$$\sec^2(\alpha)\Re(X^{-1}) \geqslant (\Re X)^{-1}.$$

Proof. The inequality is implicit in the proof of [4, Theorem 3.1], we omit the details. □

THEOREM 3. *Let $A,B \in \mathbb{M}_n$ be such that $W(A), W(B) \subset S_\alpha$. Then*

$$\Re(A\sharp B) \leqslant \frac{\sec^2(\alpha)}{2}\Re(A+B). \tag{10}$$

Proof. Compute

$$\begin{aligned}
\Re(A\sharp B) &= \Re(A^{-1}\sharp B^{-1})^{-1} \\
&= \Re\frac{2}{\pi}\int_0^\infty (tA^{-1}+t^{-1}B^{-1})^{-1}\frac{dt}{t} \\
&= \frac{2}{\pi}\int_0^\infty \Re(tA^{-1}+t^{-1}B^{-1})^{-1}\frac{dt}{t} \\
&\leqslant \frac{2}{\pi}\int_0^\infty \left(t\Re A^{-1}+t^{-1}\Re B^{-1}\right)^{-1}\frac{dt}{t} \\
&\leqslant \sec^2(\alpha)\frac{2}{\pi}\int_0^\infty \left(t(\Re A)^{-1}+t^{-1}(\Re B)^{-1}\right)^{-1}\frac{dt}{t} \\
&= \sec^2(\alpha)\left((\Re A)^{-1}\sharp(\Re B)^{-1}\right)^{-1} \\
&= \sec^2(\alpha)(\Re A)\sharp(\Re B) \\
&\leqslant \sec^2(\alpha)(\Re A+\Re B) \\
&= \sec^2(\alpha)\Re(A+B),
\end{aligned}$$

in which the first inequality is by Lemma 2 and the second inequality is by Lemma 3, respectively. □

4. Applications

This section presents some implications of Theorem 3. For a Hermitian matrix $X \in \mathbb{M}_n$, $\lambda_j(X)$ means the j-th largest eigenvalue of X. We need an auxiliary result.

LEMMA 4. *Let $A \in \mathbb{M}_n$ be such that $W(A) \subset S_\alpha$. Then*

$$\lambda_j(\Re A) \leqslant \sigma_j(A) \tag{11}$$

$$\leqslant \sec^2(\alpha)\lambda_j(\Re A), \qquad j=1,\dots,n. \tag{12}$$

Proof. The first inequality is due to Fan and Hoffman (see, [2, p. 73]), while the second one was recently proved in [4, Theorem 3.1]. □

THEOREM 4. *Let $A,B \in \mathbb{M}_n$ be such that $W(A), W(B) \subset S_\alpha$. Then*

$$\sigma_j(A \sharp B) \leqslant \frac{\sec^4(\alpha)}{2} \sigma_j(A+B) \tag{13}$$

for $j = 1, \dots, n$.

Proof. Compute

$$\begin{aligned} \sigma_j(A \sharp B) &\leqslant \sec^2(\alpha) \sigma_j(\Re(A \sharp B)) && \text{by (12)} \\ &\leqslant \frac{\sec^4(\alpha)}{2} \sigma_j(\Re(A+B)) && \text{by Theorem 3} \\ &\leqslant \frac{\sec^4(\alpha)}{2} \sigma_j(A+B), && \text{by (11)} \end{aligned}$$

as claimed. □

A norm on the algebra of $\mathbb{M}_n$ is unitarily invariant if $\|X\| = \|UXV\|$ for all unitaries U and V and all $X \in \mathbb{M}_n$.

THEOREM 5. *Let $A,B \in \mathbb{M}_n$ be such that $W(A), W(B) \subset S_\alpha$. Then*

$$\|A \sharp B\| \leqslant \frac{\sec^3(\alpha)}{2} \|A+B\| \tag{14}$$

for any unitarily invariant norm.

Proof. The claimed result follows from the following chain of inequalities

$$\begin{aligned} \|A \sharp B\| &\leqslant \sec(\alpha) \|\Re(A \sharp B)\| \\ &\leqslant \frac{\sec^3(\alpha)}{2} \|\Re(A+B)\| \\ &\leqslant \frac{\sec^3(\alpha)}{2} \|A+B\|. \end{aligned}$$

The argument in each step is the same as in the proof of Theorem 4 except for the first inequality, where we used a result of Zhang [8, Eq.(6)]. □

We finish the paper by proposing the following open problem.

AN OPEN PROBLEM. What is the optimal p in $\sec^p(\alpha)$ that appears in (4), (10), (13) and (14), respectively?

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