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SOME GRÜSS' TYPE INEQUALITIES FOR TRACE OF OPERATORS IN HILBERT SPACES

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Abstract. Some inequalities of Grüss' type for trace of operators in Hilbert spaces, under suitable assumptions for the involved operators, are given.

1. Introduction

In 1935, G. Grüss [31] proved the following integral inequality which gives an approximation of the integral mean of the product in terms of the product of the integrals means as follows:

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(x)g(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx\right| \\ &\leqslant\frac{1}{4}(\Phi-\phi)(\Gamma-\gamma),\end{aligned}\tag{1.1}$$

where f, $g:[a,b]\to\mathbb{R}$ are integrable on $[a,b]$ and satisfy the condition

$$\phi\leqslant f(x)\leqslant\Phi,\ \gamma\leqslant g(x)\leqslant\Gamma \tag{1.2}$$

for each $x\in[a,b]$, where ϕ,Φ,γ,Γ are given real constants.

Moreover, the constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller one.

In 1950, M. Biernacki, H. Pidek and C. Ryll-Nardjewski [38, Chapter X] established the following discrete version of Grüss' inequality:

Let $a=(a_1,\dots,a_n)$, $b=(b_1,\dots,b_n)$ be two $n-$tuples of real numbers such that $r\leqslant a_i\leqslant R$ and $s\leqslant b_i\leqslant S$ for $i=1,\dots,n$. Then one has

$$\left|\frac{1}{n}\sum_{i=1}^n a_ib_i-\frac{1}{n}\sum_{i=1}^n a_i\cdot\frac{1}{n}\sum_{i=1}^n b_i\right|\leqslant\frac{1}{n}\left[\frac{n}{2}\right]\left(1-\frac{1}{n}\left[\frac{n}{2}\right]\right)(R-r)(S-s), \tag{1.3}$$

where $[x]$ denotes the integer part of x, $x\in\mathbb{R}$.

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For a simple proof of (1.1) as well as for some other integral inequalities of Grüss type, see Chapter X of the book [38].

For other related results see the papers [1]–[3], [8]–[10], [11]–[13], [17]–[24], [29], [40], [50] and the references therein.

In [18], in order to generalize the above result in abstract structures the author has proved the following Grüss' type inequality in real or complex inner product spaces.

THEOREM 1. (Dragomir, 1999, [18]) *Let $(H,\langle .,.\rangle)$ be an inner product space over $\mathbb{K}\,(\mathbb{K}=\mathbb{R},\mathbb{C})$ and $e\in H, \|e\|=1$. If $\varphi,\ \gamma,\ \Phi,\ \Gamma$ are real or complex numbers and x, y are vectors in H such that the conditions*

$$\operatorname{Re}\langle \Phi e-x, x-\varphi e\rangle \geqslant 0 \text{ and } \operatorname{Re}\langle \Gamma e-y, y-\gamma e\rangle \geqslant 0 \tag{1.4}$$

hold, then we have the inequality

$$|\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle| \leqslant \frac{1}{4}|\Phi-\varphi||\Gamma-\gamma|. \tag{1.5}$$

The constant $\frac{1}{4}$ is best possible in the sense that it can not be replaced by a smaller constant.

For other results of this type, see the recent monograph [21] and the references therein.

Let A be a selfadjoint linear operator on a complex Hilbert space $(H;\langle .,.\rangle)$. The *Gelfand map* establishes a $*$-isometrically isomorphism Φ between the set $C(\operatorname{Sp}(A))$ of all *continuous functions* defined on the *spectrum* of A, denoted $\operatorname{Sp}(A)$, and the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows:

For any $f,\ g\in C(\operatorname{Sp}(A))$ and any $\alpha,\ \beta\in\mathbb{C}$ we have

(i) $\Phi(\alpha f+\beta g)=\alpha\Phi(f)+\beta\Phi(g)$;

(ii) $\Phi(fg)=\Phi(f)\Phi(g)$ and $\Phi\left(\bar{f}\right)=\Phi(f)^*$;

(iii) $\|\Phi(f)\|=\|f\|:=\sup_{t\in\operatorname{Sp}(A)}|f(t)|$;

(iv) $\Phi(f_0)=1_H$ and $\Phi(f_1)=A$, where $f_0(t)=1$ and $f_1(t)=t$, for $t\in\operatorname{Sp}(A)$.

With this notation we define

$$f(A):=\Phi(f) \text{ for all } f\in C(\operatorname{Sp}(A))$$

and we call it the *continuous functional calculus* for a selfadjoint operator A.

If A is a selfadjoint operator and f is a real valued continuous function on $\operatorname{Sp}(A)$, then $f(t)\geqslant 0$ for any $t\in\operatorname{Sp}(A)$ implies that $f(A)\geqslant 0$, i.e. $f(A)$ is a positive operator on H. Moreover, if both f and g are real valued functions on $\operatorname{Sp}(A)$ then the following important property holds:

$$f(t)\geqslant g(t) \text{ for any } t\in\operatorname{Sp}(A) \text{ implies that } f(A)\geqslant g(A) \tag{P}$$

in the operator order of $B(H)$.

In the recent paper [26], we obtained amongst other the following refinement of the Grüss inequality:

THEOREM 2. (Dragomir, 2009, [26]) *Let A be a selfadjoint operator on the Hilbert space $(H;\langle .,. \rangle)$ and assume that* $\operatorname{Sp}(A) \subseteq [m,M]$ *for some scalars $m < M$. If f and g are continuous on $[m,M]$ and* $\gamma := \min_{t\in[m,M]} f(t)$ *and* $\Gamma := \max_{t\in[m,M]} f(t)$ *then*

$$|\langle f(A)g(A)x,x\rangle - \langle f(A)x,x\rangle \langle g(A)x,x\rangle| \leqslant \frac{1}{2}(\Gamma-\gamma)\left[\|g(A)x\|^2 - \langle g(A)x,x\rangle^2\right]^{1/2} \leqslant \frac{1}{4}(\Gamma-\gamma)(\Delta-\delta) \tag{1.6}$$

for each $x \in H$ with $\|x\| = 1$, where $\delta := \min_{t\in[m,M]} g(t)$ *and* $\Delta := \max_{t\in[m,M]} g(t)$.

In order to state some Grüss' type inequalities for trace operators on complex Hilbert spaces we need some preparations as follows.

2. Some facts on trace of operators

Let $(H,\langle\cdot,\cdot\rangle)$ be a complex Hilbert space and $\{e_i\}_{i\in I}$ an *orthonormal basis* of H. We say that $A \in \mathscr{B}(H)$ is a *Hilbert-Schmidt operator* if

$$\sum_{i\in I}\|Ae_i\|^2 < \infty. \tag{2.1}$$

It is well know that, if $\{e_i\}_{i\in I}$ and $\{f_j\}_{j\in J}$ are orthonormal bases for H and $A \in \mathscr{B}(H)$ then

$$\sum_{i\in I}\|Ae_i\|^2 = \sum_{j\in I}\|Af_j\|^2 = \sum_{j\in I}\|A^*f_j\|^2 \tag{2.2}$$

showing that the definition (2.1) is independent of the orthonormal basis and A is a Hilbert-Schmidt operator iff A^* is a Hilbert-Schmidt operator.

Let $\mathscr{B}_2(H)$ the set of Hilbert-Schmidt operators in $\mathscr{B}(H)$. For $A \in \mathscr{B}_2(H)$ we define

$$\|A\|_2 := \left(\sum_{i\in I}\|Ae_i\|^2\right)^{1/2} \tag{2.3}$$

for $\{e_i\}_{i\in I}$ an orthonormal basis of H. This definition does not depend on the choice of the orthonormal basis.

Using the triangle inequality in $l^2(I)$, one checks that $\mathscr{B}_2(H)$ is a *vector space* and that $\|\cdot\|_2$ is a norm on $\mathscr{B}_2(H)$, which is usually called in the literature as the *Hilbert-Schmidt norm*.

Denote *the modulus* of an operator $A \in \mathscr{B}(H)$ by $|A| := (A^*A)^{1/2}$.

Because $\||A|x\| = \|Ax\|$ for all $x \in H$, A is Hilbert-Schmidt iff $|A|$ is Hilbert-Schmidt and $\|A\|_2 = \||A|\|_2$. From (2.2) we have that if $A \in \mathscr{B}_2(H)$, then $A^* \in \mathscr{B}_2(H)$ and $\|A\|_2 = \|A^*\|_2$.

The following theorem collects some of the most important properties of Hilbert-Schmidt operators:

THEOREM 3. *We have*

(i) $(\mathscr{B}_2(H), \|\cdot\|_2)$ *is a Hilbert space with inner product*

$$\langle A,B\rangle_2 := \sum_{i\in I} \langle Ae_i, Be_i\rangle = \sum_{i\in I} \langle B^*Ae_i, e_i\rangle \tag{2.4}$$

and the definition does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$*;*

(ii) We have the inequalities

$$\|A\| \leqslant \|A\|_2 \tag{2.5}$$

for any $A \in \mathscr{B}_2(H)$ *and*

$$\|AT\|_2, \|TA\|_2 \leqslant \|T\| \|A\|_2 \tag{2.6}$$

for any $A \in \mathscr{B}_2(H)$ *and* $T \in \mathscr{B}(H)$*;*

(iii) $\mathscr{B}_2(H)$ *is an operator ideal in* $\mathscr{B}(H)$*, i.e.*

$$\mathscr{B}(H)\mathscr{B}_2(H)\mathscr{B}(H) \subseteq \mathscr{B}_2(H);$$

(iv) $\mathscr{B}_{fin}(H)$*, the space of operators of finite rank, is a dense subspace of* $\mathscr{B}_2(H)$*;*

(v) $\mathscr{B}_2(H) \subseteq \mathscr{K}(H)$*, where* $\mathscr{K}(H)$ *denotes the algebra of compact operators on* H*.*

If $\{e_i\}_{i\in I}$ an orthonormal basis of H, we say that $A \in \mathscr{B}(H)$ is *trace class* if

$$\|A\|_1 := \sum_{i\in I} \langle |A| e_i, e_i\rangle < \infty. \tag{2.7}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathscr{B}_1(H)$ the set of trace class operators in $\mathscr{B}(H)$.

The following proposition holds:

PROPOSITION 1. *If* $A \in \mathscr{B}(H)$*, then the following are equivalent:*

(i) $A \in \mathscr{B}_1(H)$*;*

(ii) $|A|^{1/2} \in \mathscr{B}_2(H)$*;*

(ii) A (or $|A|$*) is the product of two elements of* $\mathscr{B}_2(H)$*.*

The following properties are also well known:

THEOREM 4. *With the above notations:*

(i) We have

$$\|A\|_1 = \|A^*\|_1 \text{ and } \|A\|_2 \leqslant \|A\|_1 \tag{2.8}$$

for any $A \in \mathscr{B}_1(H)$*;*

(ii) $\mathscr{B}_1(H)$ *is an operator ideal in* $\mathscr{B}(H)$*, i.e.*

$$\mathscr{B}(H)\mathscr{B}_1(H)\mathscr{B}(H) \subseteq \mathscr{B}_1(H);$$

(iii) We have

$$\mathscr{B}_2(H)\mathscr{B}_2(H) = \mathscr{B}_1(H);$$

(iv) We have

$$\|A\|_1 = \sup\{\langle A,B\rangle_2 \mid B \in \mathscr{B}_2(H),\ \|B\| \leqslant 1\};$$

(v) $(\mathscr{B}_1(H), \|\cdot\|_1)$ *is a Banach space.*
(iv) We have the following isometric isomorphisms

$$\mathscr{B}_1(H) \cong K(H)^* \text{ and } \mathscr{B}_1(H)^* \cong \mathscr{B}(H),$$

where $K(H)^*$ *is the dual space of* $K(H)$ *and* $\mathscr{B}_1(H)^*$ *is the dual space of* $\mathscr{B}_1(H)$.

We define the *trace* of a trace class operator $A \in \mathscr{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i\in I} \langle Ae_i, e_i\rangle, \tag{2.9}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (2.9) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

THEOREM 5. *We have*
(i) If $A \in \mathscr{B}_1(H)$ *then* $A^* \in \mathscr{B}_1(H)$ *and*

$$\operatorname{tr}(A^*) = \overline{\operatorname{tr}(A)}; \tag{2.10}$$

(ii) If $A \in \mathscr{B}_1(H)$ *and* $T \in \mathscr{B}(H)$, *then* $AT,\ TA \in \mathscr{B}_1(H)$ *and*

$$\operatorname{tr}(AT) = \operatorname{tr}(TA) \text{ and } |\operatorname{tr}(AT)| \leqslant \|A\|_1 \|T\|; \tag{2.11}$$

(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on* $\mathscr{B}_1(H)$ *with* $\|\operatorname{tr}\| = 1$;
(iv) If $A,\ B \in \mathscr{B}_2(H)$ *then* $AB,\ BA \in \mathscr{B}_1(H)$ *and* $\operatorname{tr}(AB) = \operatorname{tr}(BA)$;
(v) $\mathscr{B}_{fin}(H)$ *is a dense subspace of* $\mathscr{B}_1(H)$.

Utilising the trace notation we obviously have that

$$\langle A,B\rangle_2 = \operatorname{tr}(B^*A) = \operatorname{tr}(AB^*) \text{ and } \|A\|_2^2 = \operatorname{tr}(A^*A) = \operatorname{tr}\left(|A|^2\right)$$

for any $A,\ B \in \mathscr{B}_2(H)$.

The following Hölder's type inequality has been obtained by Ruskai in [42]

$$|\operatorname{tr}(AB)| \leqslant \operatorname{tr}(|AB|) \leqslant \left[\operatorname{tr}\left(|A|^{1/\alpha}\right)\right]^{\alpha} \left[\operatorname{tr}\left(|B|^{1/(1-\alpha)}\right)\right]^{1-\alpha} \tag{2.12}$$

where $\alpha \in (0,1)$ and $A,B \in \mathscr{B}(H)$ with $|A|^{1/\alpha}, |B|^{1/(1-\alpha)} \in \mathscr{B}_1(H)$.

In particular, for $\alpha = \frac{1}{2}$ we get the Schwarz inequality

$$|\operatorname{tr}(AB)| \leqslant \operatorname{tr}(|AB|) \leqslant \left[\operatorname{tr}\left(|A|^2\right)\right]^{1/2} \left[\operatorname{tr}\left(|B|^2\right)\right]^{1/2} \tag{2.13}$$

with $A,\ B \in \mathscr{B}_2(H)$.

For the theory of trace functionals and their applications the reader is referred to [45].

For some classical trace inequalities see [14], [16], [39] and [49], which are continuations of the work of Bellman [5]. For related works the reader can refer to [4], [6], [14], [30], [33], [34], [36], [43] and [46].

3. Some Grüss' type trace inequalities

We denote by $\mathscr{B}_1^+(H) := \{P : P \in \mathscr{B}_1(H) \text{ and } P \geqslant 0\}$.

We have the following result:

THEOREM 6. *For any $A,\ C \in \mathscr{B}(H)$ and $P \in \mathscr{B}_1^+(H) \setminus \{0\}$ we have the inequality*

$$\begin{aligned} &\left| \frac{\operatorname{tr}(PAC)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} \right| \\ &\leqslant \inf_{\lambda \in \mathbb{C}} \|A - \lambda \cdot 1_H\| \frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left(\left| \left(C - \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} 1_H \right) P \right| \right) \\ &\leqslant \inf_{\lambda \in \mathbb{C}} \|A - \lambda \cdot 1_H\| \left[\frac{\operatorname{tr}\left(P|C|^2\right)}{\operatorname{tr}(P)} - \left| \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} \right|^2 \right]^{1/2}, \end{aligned} \tag{3.1}$$

where $\|\cdot\|$ is the operator norm.

Proof. We observe that, for any $\lambda \in \mathbb{C}$ we have

$$\begin{aligned} &\frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left[P(A - \lambda 1_H) \left(C - \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} 1_H \right) \right] \\ &= \frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left[PA \left(C - \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} 1_H \right) \right] \\ &- \frac{\lambda}{\operatorname{tr}(P)} \operatorname{tr}\left[P \left(C - \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} 1_H \right) \right] \\ &= \frac{\operatorname{tr}(PAC)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}. \end{aligned} \tag{3.2}$$

Taking the modulus in (3.2) and utilizing the properties of the trace, we have

$$\begin{aligned}
&\left|\frac{\operatorname{tr}(PAC)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right| \qquad (3.3)\\
&=\frac{1}{\operatorname{tr}(P)}\left|\operatorname{tr}\left[P(A-\lambda 1_H)\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)\right]\right|\\
&=\frac{1}{\operatorname{tr}(P)}\left|\operatorname{tr}\left[(A-\lambda 1_H)\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right]\right|\\
&\leqslant \|A-\lambda 1_H\|\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right)
\end{aligned}$$

for any $\lambda \in \mathbb{C}$, where for the last inequality we used the inequality (2.11).

Utilising Schwarz's inequality (2.13) we also have

$$\begin{aligned}
&\operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right) \qquad (3.4)\\
&=\operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P^{1/2}P^{1/2}\right|\right)\\
&\leqslant\left[\operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P^{1/2}\right|^2\right)\right]^{1/2}[\operatorname{tr}(P)]^{1/2}.
\end{aligned}$$

Observe that

$$\begin{aligned}
&\operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P^{1/2}\right|^2\right) \qquad (3.5)\\
&=\operatorname{tr}\left(\left(\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P^{1/2}\right)^*\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P^{1/2}\right)\\
&=\operatorname{tr}\left(P^{1/2}\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)^*\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P^{1/2}\right)\\
&=\operatorname{tr}\left(\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)^*\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right)\\
&=\operatorname{tr}\left(\left(C^*-\frac{\overline{\operatorname{tr}(PC)}}{\operatorname{tr}(P)}1_H\right)\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right)\\
&=\operatorname{tr}\left[\left(|C|^2-\frac{\overline{\operatorname{tr}(PC)}}{\operatorname{tr}(P)}C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}C^*+\left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^2 1_H\right)P\right]\\
&=\left(\frac{\operatorname{tr}\left(|C|^2P\right)}{\operatorname{tr}(P)}-\left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^2\right)\operatorname{tr}(P).
\end{aligned}$$

By (3.4) and (3.5) we get

$$\operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right) \leqslant \left(\frac{\operatorname{tr}\left(|C|^2 P\right)}{\operatorname{tr}(P)} - \left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^2\right)^{1/2} \operatorname{tr}(P)$$

and by (3.3) we have

$$\begin{aligned} &\left|\frac{\operatorname{tr}(PAC)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right| \\ &\leqslant \|A-\lambda\cdot 1_H\| \frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right) \\ &\leqslant \|A-\lambda\cdot 1_H\| \left(\frac{\operatorname{tr}\left(|C|^2 P\right)}{\operatorname{tr}(P)} - \left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^2\right)^{1/2} \end{aligned} \tag{3.6}$$

for any $\lambda \in \mathbb{C}$.

Taking the infimum over $\lambda \in \mathbb{C}$ in (3.6) we get the desired result (3.1). □

COROLLARY 1. *For any $C \in \mathscr{B}(H)$ and $P \in \mathscr{B}_1^+(H)\setminus\{0\}$ we have the inequality*

$$\begin{aligned} 0 &\leqslant \frac{\operatorname{tr}\left(P|C|^2\right)}{\operatorname{tr}(P)} - \left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^2 \\ &\leqslant \inf_{\mu\in\mathbb{C}} \|C-\mu\cdot 1_H\| \frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right) \\ &\leqslant \inf_{\mu\in\mathbb{C}} \|C-\mu\cdot 1_H\| \left[\frac{\operatorname{tr}\left(P|C|^2\right)}{\operatorname{tr}(P)} - \left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^2\right]^{1/2}. \end{aligned} \tag{3.7}$$

In particular, we have

$$0 \leqslant \frac{\operatorname{tr}\left(P|C|^2\right)}{\operatorname{tr}(P)} - \left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^2 \leqslant \inf_{\mu\in\mathbb{C}} \|C-\mu\cdot 1_H\|^2. \tag{3.8}$$

Proof. If we take in (3.1) $A = C^*$ then we get

$$\begin{aligned} &\left|\frac{\operatorname{tr}(PC^*C)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PC^*)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right| \\ &\leqslant \inf_{\lambda\in\mathbb{C}} \|C^*-\lambda\cdot 1_H\| \frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right) \\ &\leqslant \inf_{\lambda\in\mathbb{C}} \|C^*-\lambda\cdot 1_H\| \left[\frac{\operatorname{tr}\left(P|C|^2\right)}{\operatorname{tr}(P)} - \left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^2\right]^{1/2}, \end{aligned}$$

which is clearly equivalent to (3.7).

The inequality (3.8) follows from the inequality between the second and fourth term in (3.7). □

COROLLARY 2. *For any $C \in \mathscr{B}(H)$ and $P \in \mathscr{B}_1^+(H) \setminus \{0\}$ we have the inequality*

$$\left| \frac{\operatorname{tr}(PC^2)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} \right)^2 \right| \tag{3.9}$$
$$\leqslant \inf_{\lambda \in \mathbb{C}} \|C - \lambda \cdot 1_H\| \frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left(\left| \left(C - \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} 1_H \right) P \right| \right)$$
$$\leqslant \inf_{\lambda \in \mathbb{C}} \|C - \lambda \cdot 1_H\| \left[\frac{\operatorname{tr}\left(P|C|^2\right)}{\operatorname{tr}(P)} - \left| \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} \right|^2 \right]^{1/2}.$$

Following [27], for the complex numbers α, β and the bounded linear operator T we define the following transform

$$\mathscr{C}_{\alpha,\beta}(T) := (T^* - \overline{\alpha} I)(\beta I - T),$$

where by T^* we denote the adjoint of T.

We list some properties of the transform $\mathscr{C}_{\alpha,\beta}(\cdot)$ that are useful in the following:

(i) For any $\alpha, \beta \in \mathbb{C}$ and $T \in B(H)$ we have:

$$\mathscr{C}_{\alpha,\beta}(I) = (1 - \overline{\alpha})(\beta - 1) I, \qquad \mathscr{C}_{\alpha,\alpha}(T) = -(\alpha I - T)^*(\alpha I - T),$$

$$\mathscr{C}_{\alpha,\beta}(\gamma T) = |\gamma|^2 \mathscr{C}_{\frac{\alpha}{\gamma}, \frac{\beta}{\gamma}}(T) \quad \text{for each } \gamma \in \mathbb{C} \setminus \{0\},$$

$$\left[\mathscr{C}_{\alpha,\beta}(T)\right]^* = \mathscr{C}_{\beta,\alpha}(T)$$

and

$$\mathscr{C}_{\overline{\beta},\overline{\alpha}}(T^*) - \mathscr{C}_{\alpha,\beta}(T) = T^*T - TT^*.$$

(ii) The operator $T \in B(H)$ is normal if and only if $\mathscr{C}_{\overline{\beta},\overline{\alpha}}(T^*) = \mathscr{C}_{\alpha,\beta}(T)$ for each $\alpha, \beta \in \mathbb{C}$.

We recall that a bounded linear operator T on the complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$ is called *accretive* if $\operatorname{Re}\langle Ty, y \rangle \geqslant 0$ for any $y \in H$.

Utilizing the following identity

$$\operatorname{Re}\left\langle \mathscr{C}_{\alpha,\beta}(T)x, x \right\rangle = \operatorname{Re}\left\langle \mathscr{C}_{\beta,\alpha}(T)x, x \right\rangle \tag{3.10}$$
$$= \frac{1}{4}|\beta - \alpha|^2 - \left\| \left(T - \frac{\alpha + \beta}{2} I \right) x \right\|^2$$

that holds for any scalars α, β and any vector $x \in H$ with $\|x\| = 1$ we can give a simple characterization result that is useful in the following:

LEMMA 1. *For $\alpha,\beta\in\mathbb{C}$ and $T\in B(H)$ the following statements are equivalent:*

(i) The transform $\mathscr{C}_{\alpha,\beta}(T)$ (or, equivalently $\mathscr{C}_{\beta,\alpha}(T)$) is accretive;

(ii) The transform $\mathscr{C}_{\overline{\alpha},\overline{\beta}}(T^*)$ $\left(\text{or, equivalently } \mathscr{C}_{\overline{\beta},\bar{\alpha}}(T^*)\right)$ is accretive;

(iii) We have the norm inequality

$$\left\|T-\frac{\alpha+\beta}{2}\cdot I\right\|\leqslant\frac{1}{2}|\beta-\alpha| \tag{3.11}$$

or, equivalently,

$$\left\|T^*-\frac{\bar{\alpha}+\bar{\beta}}{2}\cdot I\right\|\leqslant\frac{1}{2}|\beta-\alpha|. \tag{3.12}$$

REMARK 1. In order to give examples of operators $T\in B(H)$ and numbers $\alpha,\beta\in\mathbb{C}$ such that the transform $\mathscr{C}_{\alpha,\beta}(T)$ is accretive, it suffices to select a bounded linear operator S and the complex numbers z,w with the property that $\|S-zI\|\leqslant|w|$ and, by choosing $T=S$, $\alpha=\frac{1}{2}(z+w)$ and $\beta=\frac{1}{2}(z-w)$ we observe that T satisfies (3.11), i.e., $\mathscr{C}_{\alpha,\beta}(T)$ is accretive.

COROLLARY 3. *Let $\alpha,\beta\in\mathbb{C}$ and $A\in B(H)$ such that the transform $\mathscr{C}_{\alpha,\beta}(A)$ is accretive, or, equivalently*

$$\left\|A-\frac{\alpha+\beta}{2}\cdot I\right\|\leqslant\frac{1}{2}|\beta-\alpha|.$$

For any $C\in\mathscr{B}(H)$ and $P\in\mathscr{B}_1^+(H)\setminus\{0\}$ we have the inequality

$$\begin{aligned}
&\left|\frac{\operatorname{tr}(PAC)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right| \\
&\leqslant\frac{1}{2}|\beta-\alpha|\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right) \\
&\leqslant\frac{1}{2}|\beta-\alpha|\left[\frac{\operatorname{tr}\left(P|C|^2\right)}{\operatorname{tr}(P)}-\left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^2\right]^{1/2}.
\end{aligned} \tag{3.13}$$

In particular, if $C\in\mathscr{B}(H)$ is such that $\mathscr{C}_{\alpha,\beta}(C)$ is accretive, then

$$\begin{aligned}
0&\leqslant\frac{\operatorname{tr}\left(P|C|^2\right)}{\operatorname{tr}(P)}-\left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^2 \\
&\leqslant\frac{1}{2}|\beta-\alpha|\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right) \\
&\leqslant\frac{1}{2}|\beta-\alpha|\left[\frac{\operatorname{tr}\left(P|C|^2\right)}{\operatorname{tr}(P)}-\left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^2\right]^{1/2}\leqslant\frac{1}{4}|\beta-\alpha|^2.
\end{aligned} \tag{3.14}$$

Also

$$\left|\frac{\operatorname{tr}\left(PC^{2}\right)}{\operatorname{tr}(P)}-\left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^{2}\right| \tag{3.15}$$
$$\leqslant \frac{1}{2}|\beta-\alpha| \frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} 1_{H}\right) P\right|\right)$$
$$\leqslant \frac{1}{2}|\beta-\alpha|\left[\frac{\operatorname{tr}\left(P|C|^{2}\right)}{\operatorname{tr}(P)}-\left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^{2}\right]^{1/2} \leqslant \frac{1}{4}|\beta-\alpha|^{2}.$$

We have the following Grüss type inequality:

COROLLARY 4. *Let α, β, γ, $\delta \in \mathbb{C}$ and A, $C \in B(H)$ such that the transforms $\mathscr{C}_{\alpha,\beta}(A)$ and $\mathscr{C}_{\gamma,\delta}(C)$ are accretive. Then for any $P \in \mathscr{B}_1^{+}(H) \setminus \{0\}$ we have the inequality*

$$\left|\frac{\operatorname{tr}(PAC)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right| \leqslant \frac{1}{4}|\beta-\alpha||\gamma-\delta|. \tag{3.16}$$

REMARK 2. In the case when A is a selfadjoint operator and $m1_H \leqslant A \leqslant M1_H$ for some real numbers $m<M$, then

$$\left|A-\frac{m+M}{2} 1_{H}\right| \leqslant \frac{1}{2}(M-m) 1_{H},$$

which implies that

$$\left\|A-\frac{m+M}{2} 1_{H}\right\| \leqslant \frac{1}{2}(M-m).$$

Then by (3.13) we have

$$\left|\frac{\operatorname{tr}(PAC)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right| \tag{3.17}$$
$$\leqslant \frac{1}{2}(M-m) \frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} 1_{H}\right) P\right|\right)$$
$$\leqslant \frac{1}{2}(M-m)\left[\frac{\operatorname{tr}\left(P|C|^{2}\right)}{\operatorname{tr}(P)}-\left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^{2}\right]^{1/2},$$

for any $P \in \mathscr{B}_1^{+}(H) \setminus \{0\}$ and $C \in \mathscr{B}(H)$.

If C is a selfadjoint operator and $k1_H \leqslant C \leqslant K1_H$ for some real numbers $k<K$,

then

$$0 \leqslant \frac{\operatorname{tr}\left(PC^2\right)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^2 \tag{3.18}$$
$$\leqslant \frac{1}{2}(K-k)\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(C - \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right)$$
$$\leqslant \frac{1}{2}(K-k)\left[\frac{\operatorname{tr}\left(PC^2\right)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^2\right]^{1/2} \leqslant \frac{1}{4}(K-k)^2,$$

for any $P \in \mathscr{B}_1^+(H) \setminus \{0\}$.

We have the following Grüss type inequality

$$\left|\frac{\operatorname{tr}(PAC)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right| \leqslant \frac{1}{4}(M-m)(K-k) \tag{3.19}$$

provided that $m1_H \leqslant A \leqslant M1_H$ and $k1_H \leqslant C \leqslant K1_H$.

Let $\mathscr{M}_n(\mathbb{C})$ be the space of all square matrices of order n with complex elements and $A \in \mathscr{M}_n(\mathbb{C})$ be a Hermitian matrix such that $\operatorname{Sp}(A) \subseteq [m,M]$ for some scalars m, M with $m < M$. Then for any $C \in \mathscr{M}_n(\mathbb{C})$ we have

$$\left|\frac{\operatorname{tr}(AC)}{n} - \frac{\operatorname{tr}(A)}{n}\frac{\operatorname{tr}(C)}{n}\right| \leqslant \frac{1}{2}(M-m)\frac{1}{n}\operatorname{tr}\left(\left|C - \frac{\operatorname{tr}(C)}{n}I_n\right|\right) \tag{3.20}$$
$$\leqslant \frac{1}{2}(M-m)\left[\frac{\operatorname{tr}\left(|C|^2\right)}{n} - \left|\frac{\operatorname{tr}(C)}{n}\right|^2\right]^{1/2},$$

where I_n is the identity matrix in $\mathscr{M}_n(\mathbb{C})$.

If C is a Hermitian matrix such that $\operatorname{Sp}(C) \subseteq [k,K]$ for some scalars k, K with $k < K$, then

$$0 \leqslant \frac{\operatorname{tr}\left(C^2\right)}{n} - \left(\frac{\operatorname{tr}(C)}{n}\right)^2 \leqslant \frac{1}{2}(K-k)\frac{1}{n}\operatorname{tr}\left(\left|C - \frac{\operatorname{tr}(C)}{n}I_n\right|\right) \tag{3.21}$$
$$\leqslant \frac{1}{2}(K-k)\left[\frac{\operatorname{tr}\left(C^2\right)}{n} - \left(\frac{\operatorname{tr}(C)}{n}\right)^2\right]^{1/2} \leqslant \frac{1}{4}(K-k)^2.$$

In the case when the operator A is a function of selfadjoint operators we have the following result as well.

THEOREM 7. *Let S be a selfadjoint operator with* $\operatorname{Sp}(S) \subseteq [m,M]$ *and* $f:[m,M] \to \mathbb{C}$ *a continuous function of bounded variation on* $[m,M]$. *For any* $C \in \mathscr{B}(H)$ *and*

$P \in \mathscr{B}_1^+(H) \setminus \{0\}$ we have the inequality

$$\begin{aligned} &\left| \frac{\operatorname{tr}(Pf(S)C)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(Pf(S))}{\operatorname{tr}(P)} \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} \right| \\ &\leqslant \frac{1}{2} \bigvee_m^M (f) \frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left(\left| \left(C - \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} 1_H \right) P \right| \right) \\ &\leqslant \frac{1}{2} \bigvee_m^M (f) \left[\frac{\operatorname{tr}\left(P|C|^2\right)}{\operatorname{tr}(P)} - \left| \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} \right|^2 \right]^{1/2}, \end{aligned} \tag{3.22}$$

where $\bigvee_m^M (f)$ is the total variation of f on the interval.

If the function $f : [m,M] \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$ on $[m,M]$, i.e.

$$|f(t) - f(s)| \leqslant L|t-s|$$

for any $t,s \in [m,M]$, then

$$\begin{aligned} &\left| \frac{\operatorname{tr}(Pf(S)C)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(Pf(S))}{\operatorname{tr}(P)} \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} \right| \\ &\leqslant L \left\| S - \frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)} 1_H \right\| \frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left(\left| \left(C - \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} 1_H \right) P \right| \right) \\ &\leqslant L \left\| S - \frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)} 1_H \right\| \left[\frac{\operatorname{tr}\left(P|C|^2\right)}{\operatorname{tr}(P)} - \left| \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} \right|^2 \right]^{1/2} \end{aligned} \tag{3.23}$$

for any $C \in \mathscr{B}(H)$ and $P \in \mathscr{B}_1^+(H) \setminus \{0\}$.

Proof. From the inequality (3.3) we have

$$\begin{aligned} &\left| \frac{\operatorname{tr}(Pf(S)C)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(Pf(S))}{\operatorname{tr}(P)} \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} \right| \\ &\leqslant \|f(S) - \lambda 1_H\| \frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left(\left| \left(C - \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} 1_H \right) P \right| \right) \end{aligned} \tag{3.24}$$

for any $\lambda \in \mathbb{C}$.

From (3.24) we get

$$\begin{aligned} &\left| \frac{\operatorname{tr}(Pf(S)C)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(Pf(S))}{\operatorname{tr}(P)} \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} \right| \\ &\leqslant \left\| f(S) - \frac{f(m)+f(M)}{2} 1_H \right\| \frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left(\left| \left(C - \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} 1_H \right) P \right| \right). \end{aligned} \tag{3.25}$$

Since f is of bounded variation on $[m,M]$, then we have

$$\begin{aligned}\left|f(t)-\frac{f(m)+f(M)}{2}\right| &= \left|\frac{f(t)-f(m)+f(t)-f(M)}{2}\right| \\ &\leqslant \frac{1}{2}\left[|f(t)-f(m)|+|f(M)-f(t)|\right] \\ &\leqslant \frac{1}{2}\bigvee_{m}^{M}(f)\end{aligned} \tag{3.26}$$

for any $t\in[m,M]$.

From (3.26) we get in the order $\mathscr{B}(H)$ that

$$\left|f(S)-\frac{f(m)+f(M)}{2}1_H\right| \leqslant \frac{1}{2}\bigvee_{m}^{M}(f)\,1_H,$$

which implies that

$$\left\|f(S)-\frac{f(m)+f(M)}{2}1_H\right\| \leqslant \frac{1}{2}\bigvee_{m}^{M}(f)\,1_H. \tag{3.27}$$

Making use of (3.25) and (3.27) we get the first inequality (3.22). The second part is obvious.

From (3.24) we have

$$\begin{aligned}&\left|\frac{\operatorname{tr}(Pf(S)C)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(Pf(S))}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right| \\ &\leqslant \left\|f(S)-f\left(\frac{\operatorname{tr}(SP)}{\operatorname{tr}(P)}\right)1_H\right\| \frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right)\end{aligned} \tag{3.28}$$

any $C\in\mathscr{B}(H)$ and $P\in\mathscr{B}_1^+(H)\setminus\{0\}$.

Since

$$|f(t)-f(s)|\leqslant L|t-s|$$

for any $t,s\in[m,M]$, then we have in the order $\mathscr{B}(H)$ that

$$|f(S)-f(s)1_H|\leqslant L|S-s1_H|$$

for any $s\in[m,M]$. In particular, we have

$$\left|f(S)-f\left(\frac{\operatorname{tr}(SP)}{\operatorname{tr}(P)}\right)1_H\right|\leqslant L\left|S-\frac{\operatorname{tr}(SP)}{\operatorname{tr}(P)}1_H\right|,$$

which implies that

$$\left\|f(S)-f\left(\frac{\operatorname{tr}(SP)}{\operatorname{tr}(P)}\right)1_H\right\|\leqslant L\left\|S-\frac{\operatorname{tr}(SP)}{\operatorname{tr}(P)}1_H\right\|$$

and by (3.28) we get the first inequality in (3.23).

The second part is obvious. □

REMARK 3. If we take $f(t)=t$ in (3.22), then we get the inequality (3.17) while from (3.23) we obtain

$$\begin{aligned}
&\left|\frac{\operatorname{tr}(PSC)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right| \qquad (3.29)\\
&\leqslant \left\|S-\frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}1_H\right\|\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right)\\
&\leqslant \left\|S-\frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}1_H\right\|\left[\frac{\operatorname{tr}\left(P|C|^2\right)}{\operatorname{tr}(P)}-\left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^2\right]^{1/2}
\end{aligned}$$

for any $C\in\mathscr{B}(H)$ and $P\in\mathscr{B}_1^+(H)\setminus\{0\}$.

The case of selfadjoint operators C is as follows:

COROLLARY 5. *Let S be a selfadjoint operator with $\operatorname{Sp}(S)\subseteq[m,M]$ and $f:[m,M]\to\mathbb{C}$ a continuous function of bounded variation on $[m,M]$. If C is selfadjoint with $\operatorname{Sp}(C)\subseteq[n,N]$ for some real numbers $n<N$ and $P\in\mathscr{B}_1^+(H)\setminus\{0\}$, then we have the inequality*

$$\begin{aligned}
&\left|\frac{\operatorname{tr}(Pf(S)C)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(Pf(S))}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right| \qquad (3.30)\\
&\leqslant \frac{1}{2}\bigvee_m^M(f)\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right)\\
&\leqslant \frac{1}{2}\bigvee_m^M(f)\left[\frac{\operatorname{tr}\left(PC^2\right)}{\operatorname{tr}(P)}-\left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^2\right]^{1/2}\leqslant\frac{1}{4}(M-n)\bigvee_m^M(f).
\end{aligned}$$

If the function $f:[m,M]\to\mathbb{C}$ is Lipschitzian with the constant $L>0$ on $[m,M]$, then

$$\begin{aligned}
&\left|\frac{\operatorname{tr}(Pf(S)C)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(Pf(S))}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right| \qquad (3.31)\\
&\leqslant L\left\|S-\frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}1_H\right\|\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right)\\
&\leqslant L\left\|S-\frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}1_H\right\|\left[\frac{\operatorname{tr}\left(PC^2\right)}{\operatorname{tr}(P)}-\left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^2\right]^{1/2}\\
&\leqslant \frac{1}{2}(M-n)L\left\|S-\frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}1_H\right\|.
\end{aligned}$$

4. Some examples

If we write the inequality (3.22) for the function $f:[m,M]\subset[0,\infty)\to[0,\infty)$, $f(t)=t^r$, $r>0$, then we get

$$\begin{aligned}&\left|\frac{\operatorname{tr}(PS^rC)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(PS^r)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right| \\ &\leqslant \frac{1}{2}(M^r-m^r)\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right) \\ &\leqslant \frac{1}{2}(M^r-m^r)\left[\frac{\operatorname{tr}\left(P|C|^2\right)}{\operatorname{tr}(P)}-\left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^2\right]^{1/2},\end{aligned} \tag{4.1}$$

while from (3.23) we have

$$\begin{aligned}&\left|\frac{\operatorname{tr}(PS^rC)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(PS^r)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right| \\ &\leqslant \Delta_r\left\|S-\frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}\right\|\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right) \\ &\leqslant \Delta_r\left\|S-\frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}\right\|\left[\frac{\operatorname{tr}\left(P|C|^2\right)}{\operatorname{tr}(P)}-\left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^2\right]^{1/2}\end{aligned} \tag{4.2}$$

for any S a selfadjoint operator with $\operatorname{Sp}(S)\subseteq[m,M]$, any $C\in\mathscr{B}(H)$ and $P\in\mathscr{B}_1^+(H)\setminus\{0\}$, where

$$\Delta_r:=\begin{cases} rM^{r-1} \text{ if } r\geqslant 1, \\ rm^{r-1} \text{ if } r\in(0,1).\end{cases}$$

If C is selfadjoint with $\operatorname{Sp}(C)\subseteq[n,N]$ for some real numbers $n<N$ then from (4.1) and (4.2) we get the power inequalities

$$\left|\frac{\operatorname{tr}(PS^rC)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(PS^r)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|\leqslant\frac{1}{4}(M^r-m^r)(N-n) \tag{4.3}$$

and

$$\left|\frac{\operatorname{tr}(PS^rC)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(PS^r)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|\leqslant\frac{1}{2}\Delta_r(N-n)\left\|S-\frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}\right\|. \tag{4.4}$$

If we write the inequality (3.22) for the function $f:[m,M]\subset(0,\infty)\to\mathbb{R}$, $f(t)=$

$\ln t$, then we have

$$\begin{aligned} & \left| \frac{\operatorname{tr}(CP\ln S)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(P\ln S)}{\operatorname{tr}(P)} \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} \right| \\ & \leqslant \frac{1}{2} \ln\left(\frac{M}{m}\right) \frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left(\left| \left(C - \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} 1_H \right) P \right| \right) \\ & \leqslant \frac{1}{2} \ln\left(\frac{M}{m}\right) \left[\frac{\operatorname{tr}\left(P|C|^2\right)}{\operatorname{tr}(P)} - \left| \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} \right|^2 \right]^{1/2}, \end{aligned} \tag{4.5}$$

while from (3.23) we have

$$\begin{aligned} & \left| \frac{\operatorname{tr}(CP\ln S)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(P\ln S)}{\operatorname{tr}(P)} \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} \right| \\ & \leqslant \frac{1}{m} \left\| S - \frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)} \right\| \frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left(\left| \left(C - \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} 1_H \right) P \right| \right) \\ & \leqslant \frac{1}{m} \left\| S - \frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)} \right\| \left[\frac{\operatorname{tr}\left(P|C|^2\right)}{\operatorname{tr}(P)} - \left| \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} \right|^2 \right]^{1/2}, \end{aligned} \tag{4.6}$$

for any S a selfadjoint operator with $\operatorname{Sp}(S) \subseteq [m,M]$, any $C \in \mathscr{B}(H)$ and $P \in \mathscr{B}_1^+(H) \setminus \{0\}$.

If C is selfadjoint with $\operatorname{Sp}(C) \subseteq [n,N]$ for some real numbers $n < N$ then from (4.5) and (4.6) we have

$$\left| \frac{\operatorname{tr}(CP\ln S)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(P\ln S)}{\operatorname{tr}(P)} \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} \right| \leqslant \frac{1}{4} (N-n) \ln\left(\frac{M}{m}\right) \tag{4.7}$$

and

$$\left| \frac{\operatorname{tr}(CP\ln S)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(P\ln S)}{\operatorname{tr}(P)} \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} \right| \leqslant \frac{N-n}{2m} \left\| S - \frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)} \right\|. \tag{4.8}$$

If we write the inequality (3.22) for the function $f : [m,M] \subset (0,\infty) \to \mathbb{R}$, $f(t) = t^{-1}$, then we have

$$\begin{aligned} & \left| \frac{\operatorname{tr}\left(PS^{-1}C\right)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}\left(PS^{-1}\right)}{\operatorname{tr}(P)} \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} \right| \\ & \leqslant \frac{1}{2} \frac{M-m}{mM} \frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left(\left| \left(C - \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} 1_H \right) P \right| \right) \\ & \leqslant \frac{1}{2} \frac{M-m}{mM} \left[\frac{\operatorname{tr}\left(P|C|^2\right)}{\operatorname{tr}(P)} - \left| \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} \right|^2 \right]^{1/2}, \end{aligned} \tag{4.9}$$

while from (3.23) we have

$$\begin{aligned} &\left|\frac{\operatorname{tr}\left(PS^{-1}C\right)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}\left(PS^{-1}\right)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right| \\ &\leqslant \frac{1}{m^2}\left\|S-\frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}\right\|\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right) \\ &\leqslant \frac{1}{m^2}\left\|S-\frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}\right\|\left[\frac{\operatorname{tr}\left(P|C|^2\right)}{\operatorname{tr}(P)}-\left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^2\right]^{1/2} \end{aligned} \tag{4.10}$$

for any S a selfadjoint operator with $\operatorname{Sp}(S)\subseteq[m,M]$, any $C\in\mathscr{B}(H)$ and $P\in\mathscr{B}_1^+(H)\setminus\{0\}$.

If C is selfadjoint with $\operatorname{Sp}(C)\subseteq[n,N]$ for some real numbers $n<N$ then from (4.9) and (4.10) we have

$$\left|\frac{\operatorname{tr}\left(PS^{-1}C\right)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}\left(PS^{-1}\right)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|\leqslant\frac{1}{4}\frac{M-m}{mM}(N-n) \tag{4.11}$$

and

$$\left|\frac{\operatorname{tr}\left(PS^{-1}C\right)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}\left(PS^{-1}\right)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|\leqslant\frac{N-n}{2m^2}\left\|S-\frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}\right\|. \tag{4.12}$$

Now, if we take $C=S$ in (4.1), then we get

$$\begin{aligned} 0 &\leqslant \frac{\operatorname{tr}\left(PS^{r+1}\right)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(PS^r)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)} \\ &\leqslant \frac{1}{2}(M^r-m^r)\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(S-\frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}1_H\right)P\right|\right) \\ &\leqslant \frac{1}{2}(M^r-m^r)\left[\frac{\operatorname{tr}\left(PS^2\right)}{\operatorname{tr}(P)}-\left(\frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}\right)^2\right]^{1/2} \\ &\leqslant \frac{1}{4}(M^r-m^r)(M-m), \end{aligned} \tag{4.13}$$

for any S a selfadjoint operator with $\operatorname{Sp}(S)\subseteq[m,M]\subset[0,\infty)$ and any $P\in\mathscr{B}_1^+(H)\setminus\{0\}$.

Also, if we take $C = S$ in (4.5), then we obtain

$$\begin{aligned} 0 &\leqslant \frac{\operatorname{tr}(PS\ln S)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(P\ln S)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)} \\ &\leqslant \frac{1}{2}\ln\left(\frac{M}{m}\right)\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(S - \frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}1_H\right)P\right|\right) \\ &\leqslant \frac{1}{2}\ln\left(\frac{M}{m}\right)\left[\frac{\operatorname{tr}\left(PS^2\right)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}\right)^2\right]^{1/2} \\ &\leqslant \frac{1}{4}(M-m)\ln\left(\frac{M}{m}\right) \end{aligned} \tag{4.14}$$

for any S a selfadjoint operator with $\operatorname{Sp}(S) \subseteq [m,M] \subset (0,\infty)$ and any $P \in \mathscr{B}_1^+(H) \setminus \{0\}$.

Finally, if we take $C = S$ in (4.9), then we get

$$\begin{aligned} 0 &\leqslant \frac{\operatorname{tr}\left(PS^{-1}\right)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)} - 1 \\ &\leqslant \frac{1}{2}\frac{M-m}{mM}\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(S - \frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}1_H\right)P\right|\right) \\ &\leqslant \frac{1}{2}\frac{M-m}{mM}\left[\frac{\operatorname{tr}\left(PS^2\right)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}\right)^2\right]^{1/2} \\ &\leqslant \frac{1}{4}\frac{(M-m)^2}{mM}, \end{aligned} \tag{4.15}$$

for any S a selfadjoint operator with $\operatorname{Sp}(S) \subseteq [m,M] \subset (0,\infty)$ and any $P \in \mathscr{B}_1^+(H) \setminus \{0\}$.

From the first and last terms in (4.15) we get the Kantorovich type inequality

$$1 \leqslant \frac{\operatorname{tr}\left(PS^{-1}\right)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)} \leqslant \frac{1}{4}\frac{(M+m)^2}{mM}.$$

We notice that, the positivity of the first terms in (4.13), (4.14) and (4.15) follows from the Čebyšev's type trace inequality obtained in [28].

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