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MINKOWSKI PRODUCT OF CONVEX SETS AND PRODUCT NUMERICAL RANGE

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In memory of Leiba Rodman

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Abstract. Let K_1,K_2 be two compact convex sets in $\mathbf{C}$. Their Minkowski product is the set $K_1K_2 = \{ab : a \in K_1, b \in K_2\}$. We show that the set K_1K_2 is star-shaped if K_1 is a line segment or a circular disk. Examples for K_1 and K_2 are given so that K_1 and K_2 are triangles (including interior) and K_1K_2 is not star-shaped. This gives a negative answer to a conjecture by Puchala et. al concerning the product numerical range in the study of quantum information science. Additional results and open problems are presented.

1. Introduction

Let K_1,K_2 be compact convex sets in $\mathbf{C}$. We study the Minkowski product of the sets defined and denoted by

$$K_1K_2 = \{ab : a \in K_1, b \in K_2\}.$$

This topic arises naturally in many branches of research. For example, in numerical analysis, computations are subject to errors caused by the precision of the machines and round-off errors. Sometimes measurement errors in the raw data may also affect the accuracy. So, when two real numbers a and b are multiplied, the actual answer may actually be the product of numbers in two intervals containing a and b; when two complex numbers a and b are multiplied, the actual answer may actually be the product of numbers from two regions in the complex plane. The study of the product set also has applications in computer-aided design, reflection and refraction of wavefronts in geometrical optics, stability characterization of multi-parameter control systems, and the shape analysis and procedural generation of two-dimensional domains. For more discussion about these topics, see [3] and the references therein. Another application comes from the study of quantum information science. For a complex $n \times n$ matrix A, its numerical range is defined and denoted by

$$W(A) = \{x^*Ax : x \in \mathbf{C}^n, x^*x = 1\}.$$

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The numerical range of a matrix is always a compact convex set and carries a lot of information about the matrix, e.g., see [5].

Denote by $X \otimes Y$ the Kronecker product of two matrices or vectors. Then the decomposable numerical range of $T \in M_m \otimes M_n \equiv M_{mn}$ is defined by

$$W^{\otimes}(T) = \{(x^* \otimes y^*)T(x \otimes y) : x \in \mathbf{C}^m, y \in \mathbf{C}^n, x^*x = y^*y = 1\},$$

which is a subset of $W(T)$. In the context of quantum information science, this set corresponds to the collection of $\langle T, P \otimes Q \rangle$, where $P \in M_m, Q \in M_n$ are pure states (i.e., rank one orthogonal projections). In particular, if $T = A \otimes B$ with $(A,B) \in M_m \times M_n$, then

$$W^{\otimes}(A \otimes B) = \{(x^* \otimes y^*)(A \otimes B)(x \otimes y) : x \in \mathbf{C}^m, y \in \mathbf{C}^n, x^*x = y^*y = 1\} = W(A)W(B).$$

So, the set $W^{\otimes}(A \otimes B)$ is just the Minkowski product of the two compact convex sets $W(A)$ and $W(B)$. In particular, the following was proved in [8]. (Their proofs concern the product numerical range that can be easily adapted to general compact convex sets.)

PROPOSITION 1.1. *Suppose K_1, K_2 are compact convex sets in $\mathbf{C}$.*
(a) *The set K_1K_2 is simply connected.*
(b) *If $0 \in K_1 \cup K_2$, then K_1K_2 is star-shaped with 0 as a star center.*

It was conjectured in [8] that the set K_1K_2 is always star-shaped. In this paper, we will show that the conjecture is not true in general (Section 3.1). The proof depends on a detailed analysis of the product sets of two closed line segments (Section 2). Then we obtain some conditions under which the product set of two convex polygons is star-shaped (Sections 3.2). Furthermore, we show that K_1K_2 is star-shaped for any compact convex set K_2 if K_1 is a closed line segment or a closed circular disk in Sections 4 and 5. Some additional results and open problems are mentioned in Section 6. In particular, in Theorem 6.2, we will improve the following result, which is a consequence of the simply connectedness of K_1K_2 [8, Proposition 1].

PROPOSITION 1.2. *Suppose K_1, K_2 are compact convex sets in $\mathbf{C}$ and $p \in K_1K_2$. Then K_1K_2 is star-shaped with p as a star center if and only if K_1K_2 contains the line segment joining p to ab for any $a \in \partial K_1$ and $b \in \partial K_2$. Here, ∂K denotes the boundary of K.*

In our discussion, we denote by $K(z_1, z_2, \ldots, z_m)$ the convex hull of the set $\{z_1, \ldots, z_m\} \subseteq \mathbf{C}$. In particular, $K(z_1, z_2)$ is the line segment in $\mathbf{C}$ joining z_1, z_2. Also, if $K_1 = \{\alpha\}$, we write $K_1K_2 = \alpha K_2$.

2. The product set of two segments

We first give a complete description of the set K_1K_2 when $K_1 = K(\alpha_1, \alpha_2)$ and $K_2 = K(\beta_1, \beta_2)$ are two line segments. McAllister has plotted some examples in [7] but the analysis is not complete. In the context of product numerical range, it is known, see for example, [6, Theorem 4.3], that $W(T)$ is a line segment if and only if T is normal with collinear eigenvalues. In such a case, $W(T) = W(T_0)$ for a normal matrix $T_0 \in M_2$ having the two endpoints of $W(T)$ as its eigenvalues. Thus, the study of K_1K_2 when K_1, K_2 are closed line segments corresponds to the study of $W^{\otimes}(A \otimes B) = W(A)W(B)$ for $A \in M_m, B \in M_n$ with special structure, and $W^{\otimes}(A \otimes B) = W^{\otimes}(A_0 \otimes B_0)$ for some normal matrices $A_0, B_0 \in M_2$. We have the following result.

THEOREM 2.1. *Let $K_1 = K(\alpha_1, \alpha_2)$ and $K_2 = K(\beta_1, \beta_2)$ be two line segments in* $\mathbf{C}$. *Then K_1K_2 is a star-shaped subset of $K(\alpha_1\beta_1, \alpha_1\beta_2, \alpha_2\beta_1, \alpha_2\beta_2)$.*

In general, $K(\alpha_1, \ldots, \alpha_n)K(\beta_1, \ldots, \beta_m) \subseteq K(\alpha_1\beta_1, \alpha_1\beta_2, \ldots, \alpha_i\beta_j, \ldots, \alpha_n\beta_m)$ because

$$\left(\sum_i p_i\alpha_i\right)\left(\sum_j q_j\beta_j\right) = \left(\sum_{i,j} p_iq_j\alpha_i\beta_j\right)$$

and $\sum_i p_i = 1$ and $\sum_j q_j = 1$ imply that $\sum_{i,j} p_iq_j = 1$. The key point of Theorem 2.1 is the star-shapedness of the product of two line segments in $\mathbf{C}$.

We will give a complete description of the set K_1K_2 in the following. If one or both of the line segments K_1, K_2 lie(s) in a line passing through origin, the description is relatively easy as shown in the following.

PROPOSITION 2.2. *Let $K_1 = K(\alpha_1, \alpha_2)$ and $K_2 = K(\beta_1, \beta_2)$ be two line segments in* $\mathbf{C}$.

1. *If both $K(0, \alpha_1, \alpha_2)$ and $K(0, \beta_1, \beta_2)$ are line segments, then K_1K_2 is the line segment*
$$K(\alpha_1\beta_1, \alpha_1\beta_2, \alpha_2\beta_1, \alpha_2\beta_2).$$
2. *Suppose $K(0, \alpha_1, \alpha_2)$ is a line segment and $K(0, \beta_1, \beta_2)$ is not.*

   (2.a) *If $0 \in K(\alpha_1, \alpha_2)$, then $K_1K_2 = K(0, \alpha_1\beta_1, \alpha_1\beta_2) \cup K(0, \alpha_2\beta_1, \alpha_2\beta_2)$ is the union of two triangles (one of them may degenerate to $\{0\}$) meeting at 0, which is the star center of K_1K_2.*

   (2.b) *If $0 \notin K(\alpha_1, \alpha_2)$ then $K_1K_2 = K(\alpha_1\beta_1, \alpha_1\beta_2, \alpha_2\beta_1, \alpha_2\beta_2)$.*

Proof. 1. There exist α, $\beta, a_1, a_2, b_1, b_2 \in \mathbf{R}$ such that $K_1 = \{re^{i\alpha} : a_1 \leqslant r \leqslant b_1\}$ and $K_2 = \{re^{i\beta} : a_2 \leqslant r \leqslant b_2\}$. So, we have

$$K_1K_2 = \{re^{i(\alpha+\beta)} : a_3 \leqslant r \leqslant b_3\} \text{ for some } a_3,\ b_3 \in \mathbf{R}.$$

(2.a) Evidently, $K_1K_2 = K(0, \alpha_1)K_2 \cup K(0, \alpha_2)K_2$ and $K(0, \alpha_i)K_2 \subseteq K(0, \alpha_i\beta_1, \alpha_i\beta_2)$ for $i = 1,\ 2$. We are going to show that $K(0, \alpha_i)K(\beta_1, \beta_2) = K(0, \alpha_i\beta_1, \alpha_i\beta_2)$ for $i = 1,\ 2$.

Clearly, $0 \in K(0, \alpha_i)K(\beta_1, \beta_2)$. If $x \in K(0, \alpha_i\beta_1, \alpha_i\beta_2) \setminus \{0\}$, then there exist $s,\ t \geqslant 0$ with $0 < s + t \leqslant 1$ such that $x = s\alpha_i\beta_1 + t\alpha_i\beta_2$. Therefore, $x = ab$, where

$$a = (s+t)\alpha_i \in K(0, \alpha_i) \quad \text{and} \quad b = \frac{s}{s+t}\beta_1 + \frac{t}{s+t}\beta_2 \in K(\beta_1, \beta_2)$$

Thus $K(0,\alpha_i)K(\beta_1,\beta_2) = K(0,\alpha_i\beta_1,\alpha_i\beta_2)$ and $K_1K_2 = K(0,\alpha_2\beta_1,\alpha_2\beta_2) \cup K(0,\alpha_1\beta_1,\alpha_1\beta_2)$.

(2.b) Let $x \in K(\alpha_1\beta_1,\alpha_1\beta_2,\alpha_2\beta_1,\alpha_2\beta_2)$. Then $x = s\alpha_1\beta_1 + t\alpha_1\beta_2 + u\alpha_2\beta_1 + v\alpha_2\beta_2$ for some $s,\ t,\ u,\ v \geqslant 0$ with $s+t+u+v = 1$. Since $0 \notin K(\alpha_1,\alpha_2)$, $\alpha_2 = k\alpha_1$ for some $k > 0$, then $x = (p\alpha_1 + (1-p)\alpha_2)(q\beta_1 + (1-q)\beta_2)$, where

$$p = s+t, \quad q = \frac{s+uk}{s+t+k(u+v)} \in [0,1]. \quad \square$$

The situation is more involved if neither $K(0,\alpha_1,\alpha_2)$ nor $K(0,\beta_1,\beta_2)$ is a line segment. To describe the shape of K_1K_2 in such a case, we put the two segments in a certain "canonical" position. More specifically, the next proposition shows that we can find α_0 and $\beta_0 \in \mathbf{C}$ such that $\alpha_0^{-1}K_1$ and $\beta_0^{-1}K_2$ lie in the vertical line $\{z \in \mathbf{C} : \Re(z) = 1\}$.

PROPOSITION 2.3. *Let $K_1 = K(\alpha_1,\alpha_2)$ and $K_2 = K(\beta_1,\beta_2)$ be two line segments in $\mathbf{C}$ such that neither $K(0,\alpha_1,\alpha_2)$ nor $K(0,\beta_1,\beta_2)$ is a line segment. Let*

$$\alpha_0 = \frac{\alpha_1\overline{\alpha_2} - \alpha_2\overline{\alpha_1}}{2(\overline{\alpha_2} - \overline{\alpha_1})} \ \ \textit{and} \ \ \beta_0 = \frac{\beta_1\overline{\beta_2} - \beta_2\overline{\beta_1}}{2(\overline{\beta_2} - \overline{\beta_1})} \tag{1}$$

Then α_0 (respectively, β_0) is the point on the line passing through α_1 and α_2 (respectively, β_1 and β_2) closest to 0. We have

$$\frac{\alpha_1}{\alpha_0} = 1 + a_1 i, \quad \frac{\alpha_2}{\alpha_0} = 1 + a_2 i, \quad \frac{\beta_1}{\beta_0} = 1 + b_1 i, \quad \frac{\beta_2}{\beta_0} = 1 + b_2 i \tag{2}$$

for some a_1, a_2, b_1 and $b_2 \in \mathbf{R}$.

Proof. The line passing through α_1 and α_2 is given by the parametric equation $r(t) = \alpha_1 + t(\alpha_1 - \alpha_2)$, $t \in \mathbf{R}$. α_0 in (1) is obtained by minimizing $|r(t)|^2$. Similarly, we have β_0. By direct calculation we have (2) with

$$a_1 = \frac{\alpha_1\overline{\alpha_2} + \alpha_2\overline{\alpha_1} - 2|\alpha_1|^2}{i(\alpha_1\overline{\alpha_2} - \alpha_2\overline{\alpha_1})}, \qquad a_2 = \frac{\alpha_1\overline{\alpha_2} + \alpha_2\overline{\alpha_1} - 2|\alpha_2|^2}{i(\alpha_2\overline{\alpha_1} - \alpha_1\overline{\alpha_2})},$$

$$b_1 = \frac{\beta_1\overline{\beta_2} + \beta_2\overline{\beta_1} - 2|\beta_1|^2}{i(\beta_1\overline{\beta_2} - \beta_2\overline{\beta_1})}, \qquad b_2 = \frac{\beta_1\overline{\beta_2} + \beta_2\overline{\beta_1} - 2|\beta_2|^2}{i(\beta_2\overline{\beta_1} - \beta_1\overline{\beta_2})}. \quad \square$$

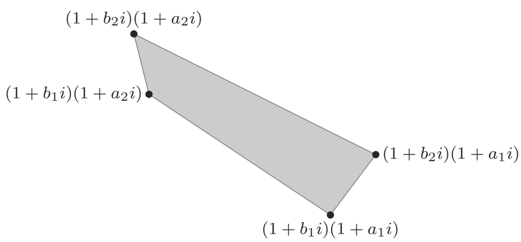
We can now describe K_1K_2 for two line segments $K_1 = K(\alpha_1,\alpha_2)$ and $K_2 = K(\beta_1,\beta_2)$ in the "canonical" position. Because $K(\alpha_1,\alpha_2)K(\beta_1,\beta_2)$ is a simply connected set, we focus on the description of the boundary and the set of star centers of K_1K_2 in the following.

THEOREM 2.4. *Let $K_1 = K(\alpha_1, \alpha_2)$ and $K_2 = K(\beta_1, \beta_2)$ with $\alpha_1 = 1 + ia_1$, $\alpha_2 = 1 + ia_2$, $\beta_1 = 1 + ib_1$, $\beta_2 = 1 + ib_2$ such that $a_1 < a_2$ and $b_1 < b_2$. Assume $a_1 \leqslant b_1$; otherwise, interchange the roles of K_1 and K_2. Then one of the following holds.*

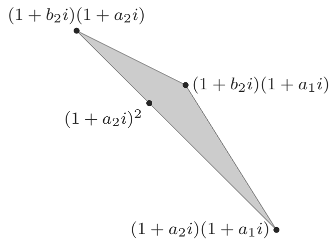
(a) *$a_1 < a_2 \leqslant b_1 < b_2$. Then K_1K_2 is the convex quadrilateral $K(\alpha_1\beta_1, \alpha_1\beta_2, \alpha_2\beta_1, \alpha_2\beta_2)$, which will degenerate to the triangle $K(\alpha_1\beta_1, a_1\beta_2, \alpha_2\beta_2)$ if $a_2 = b_1$; see Figure* 1 (a) and (a.i).

(b) *$a_1 \leqslant b_1 < a_2 \leqslant b_2$. Then $K_1K_2 \subseteq K(\alpha_1\beta_1, \alpha_1\beta_2, \alpha_2\beta_2)$, and the boundary of K_1K_2 consists of the line segments $K(\alpha_2^2, \alpha_2\beta_2)$, $K(\alpha_2\beta_2, \alpha_1\beta_2)$, $K(\alpha_1\beta_2, \alpha_1\beta_1)$, $K(\alpha_1\beta_1, \beta_1^2)$, and the curve $\mathbf{E} = \{(1+si)^2 : s \in [b_1, a_2]\} \subseteq \mathbf{C}$. Here, $K(\alpha_2^2, \alpha_2\beta_2)$ lies on the tangent line of the curve $\mathbf{E}$ at α_2^2, and $K(\beta_1^2, \alpha_1\beta_1)$ lies on the tangent line of the curve $\mathbf{E}$ at β_1^2. The set of star centers equals $K(\alpha_1, \beta_1)K(\alpha_2, \beta_2)$, which may be a quadrilateral, a line or a point; see Figure* 1 (b), (b.i), (b.ii), b(iii).

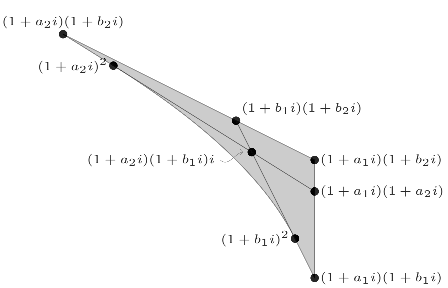
(c) *Suppose $a_1 < b_1 < b_2 < a_2$. Then the boundary of K_1K_2 consists of the line segments $K(\beta_2^2, \alpha_2\beta_2)$, $K(\alpha_2\beta_2, \alpha_2\beta_1)$, $K(\alpha_2\beta_1, \beta_1\beta_2)$, $K(\beta_1\beta_2, \alpha_1\beta_2)$, $K(\alpha_1\beta_2, \alpha_1\beta_1)$, $K(\beta_1^2, \alpha_1\beta_1)$ and the curve segment $\{(1+si)^2 : s \in [b_1, b_2]\} \subseteq \mathbf{C}$. Here, $K(\beta_2^2, \alpha_2\beta_2)$ lies on the tangent line of the curve $\mathbf{C}$ at β_2^2, and $K(\beta_1^2, \alpha_1\beta_1)$ lies on the tangent line of the curve $\mathbf{C}$ at β_1^2. The unique star center is $\beta_1\beta_2$; see Figure* 1 (c).



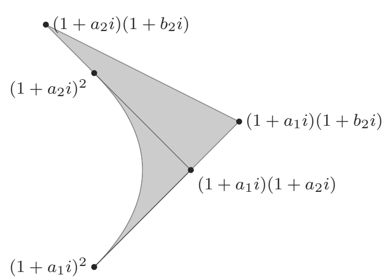
(a) $a_1 < a_2 < b_1 < b_2$,



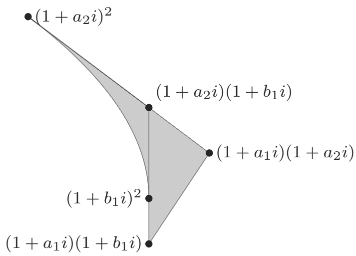
(a.i) $a_1 < a_2 = b_1 < b_2$.



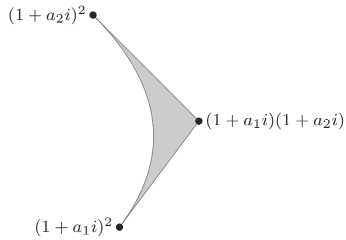
(b) $a_1 < b_1 < a_2 < b_2$,



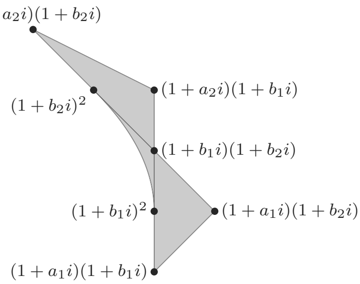
(b.i) $a_1 = b_1 < a_2 < b_2$,



(b.ii) $a_1 < b_1 < a_2 = b_2$,



(b.iii) $a_1 = b_1 < b_2 = a_2$.



(c) $a_1 < b_1 < b_2 < a_2$.

Figure 1. *The set* $K(1+a_1i, 1+a_2i)K(1+b_1i, 1+b_2i)$ *described in Theorem* 2.4.

To prove Theorem 2.4, we need the following lemma that treat some special cases of the theorem. It turns out that these special cases are the building blocks for the general case.

LEMMA 2.5. *Let* $a_1 < a_2 \leqslant b_1 < b_2$. *Then*

(a) $K(1+a_1i, 1+a_2i)K(1+b_1i, 1+b_2i)$ *is the quadrilateral (or triangle if* $a_2 = b_1$ *),*

$$\mathbf{K} = K\Big((1+a_1i)(1+b_1i), (1+a_1i)(1+b_2i), (1+a_2i)(1+b_1i), (1+a_2i)(1+b_2i)\Big).$$

(b) $K(1+a_1i, 1+a_2i)K(1+a_1i, 1+a_2i)$ *is the simply connected region bounded by the line segments*

$$\mathbf{L}_1 = K\Big((1+a_1i)^2, (1+a_1i)(1+a_2i)\Big), \quad \mathbf{L}_2 = K\Big((1+a_2i)^2, (1+a_1i)(1+a_2i)\Big),$$

and the curve $\mathbf{E} = \{(1+si)^2 \ : \ s \in [a_1, a_2]\}$. *The set* $\mathbf{L}_1$ *is a segment of the tangent line of* $\mathbf{E}$ *at* $(1+a_1i)^2$*, and* $\mathbf{L}_2$ *is a segment of the tangent line of* $\mathbf{E}$ *at* $(1+a_2i)^2$.

Proof. (a) Suppose $\alpha_j = 1 + a_ji$ and $\beta_j = 1 + b_ji$ for $j = 1, 2$ are such that $a_1 < a_2 \leqslant b_1 < b_2$. Let $K_1 = K(\alpha_1, \alpha_2)$ and $K_2 = K(\beta_1, \beta_2)$. It suffices to show that the union of the line segments

$$\ell_1 = \beta_2 K_1, \quad \ell_2 = \beta_1 K_1, \quad \ell_3 = \alpha_2 K_2, \quad \ell_4 = \alpha_1 K_2$$

forms the boundary of the quadrilateral (or triangle) $\mathbf{K}$, that is, the union is a simple closed curve. By simply connectedness and the fact that K_1K_2 is a subset of $\mathbf{K}$, we get the desired conclusion. For the convenience of discussion, we will identify $x+iy \in \mathbf{C}$ with $(x,y) \in \mathbf{R}^2$ and $(x,y,0) \in \mathbf{R}^3$. Note that since $\arg(\alpha_1\beta_1) < \arg(\alpha_2\beta_1), \arg(\alpha_1\beta_2) < \arg(\alpha_2\beta_2)$, it suffices to show that $\alpha_1\beta_2$ and $\alpha_2\beta_1$ are on opposite sides of the line ℓ passing through $\alpha_1\beta_1$ and $\alpha_2\beta_2$. This is true if and only if the cross product $(\alpha_2\beta_1-\alpha_2\beta_2)\times(\alpha_1\beta_1-\alpha_2\beta_2)$ and $(\alpha_1\beta_2-\alpha_2\beta_2)\times(\alpha_1\beta_1-\alpha_2\beta_2)$ are pointing in opposite directions, that is

$$\det\begin{bmatrix}\Re(\alpha_2\beta_1-\alpha_2\beta_2) & \Re(\alpha_1\beta_1-\alpha_2\beta_2)\\ \Im(\alpha_2\beta_1-\alpha_2\beta_2) & \Im(\alpha_1\beta_1-\alpha_2\beta_2)\end{bmatrix}\cdot\det\begin{bmatrix}\Re(\alpha_1\beta_2-\alpha_2\beta_2) & \Re(\alpha_1\beta_1-\alpha_2\beta_2)\\ \Im(\alpha_1\beta_2-\alpha_2\beta_2) & \Im(\alpha_1\beta_1-\alpha_2\beta_2)\end{bmatrix}\leqslant 0$$

The expression on the left hand side is

$$[(b_1-b_2)(a_2-a_1)(a_2-b_1)]\cdot[(b_1-b_2)(a_2-a_1)(b_2-a_1)]$$
$$=(b_1-b_2)^2(a_2-a_1)^2(a_2-b_1)(b_2-a_1)$$

Since $a_2 \leqslant b_1$ and $b_2 > a_1$, then we are done.

To prove (b), first note that $\mathbf{L}_1, \mathbf{L}_2$ and $\mathbf{E}$ are clearly in K_1K_1. Direct calculation shows that $\mathbf{L}_1$ with equation $x = 1-a_1(y-a_1)$ and $\mathbf{L}_2$ with equation $x = 1-a_2(y-a_2)$ are tangent to the parabola $\mathbf{E}$ with equation $x = 1-\frac{y^2}{4}$ at the points $(1-a_1^2,\ 2a_1)$ and $(1-a_2^2,\ 2a_2)$ respectively.

Since K_1K_1 is simply connected, the region

$$\mathbf{S} = \left\{x+iy : 1-\frac{y^2}{4} \leqslant x \leqslant 1-a_1(y-a_1),\ 1-a_2(y-a_2)\right\}, \qquad (3)$$

which is the region enclosed by $\mathbf{L}_1, \mathbf{L}_2$ and $\mathbf{E}$ is a subset of K_1K_1. Now, suppose $x+iy \in K_1K_1$. Then there exist r and s with $a_1 \leqslant r, s \leqslant a_2$ such that

$$x+iy = (1+ir)(1+is) = 1-rs+i(r+s).$$

Note that

$$x = 1-rs \geqslant 1-\frac{1}{4}(r+s)^2 = 1-\frac{y^2}{4}$$

always holds. Also, if $a \leqslant t \leqslant b$, then $(a+b-t)t \geqslant ab$. Since

$$a_1 \leqslant r \leqslant s+r-a_1 \text{ and } s+r-a_2 \leqslant r \leqslant a_2,$$

we have $rs \geqslant a_1(s+r-a_1),\ a_2(s+r-a_2)$. Hence,

$$x = 1-rs \leqslant 1-a_1(r+s-a_1) = 1-a_1(y-a_1), \quad \text{and}$$
$$x = 1-rs \leqslant 1-a_2(r+s-a_2) = 1-a_2(y-a_2).$$

This shows that K_1K_1 lies inside $\mathbf{S}$. Thus $K_1K_1 = \mathbf{S}$. $\square$

Proof of Theorem 2.4. Suppose $K_1 = K(1+ia_1, 1+ia_2)$ and $K_2 = K(1+ib_1, 1+ib_2)$ such that $a_1 \leqslant a_2, b_1 \leqslant b_2$. We show that if K_1K_2 can be written as the union of subsets of the form in Lemma 2.5. In fact, if $[a_1,a_2] \cap [b_1,b_2] = [c_1,c_2]$, then

$$K_1K_2 = (\alpha_0\beta_0)\left[(AC)\cup(AB)\cup(CC)\cup(CB)\right],$$

where $C = K(1+c_1i, 1+c_2i)$, $B = K(1+b_1i, 1+b_2i)\setminus C$ and $A = K(1+a_1i, 1+a_2i)\setminus C$. By Lemma 2.5, we get the conclusion. □

By Theorem 2.4, we have the following corollary giving information about the star center of the product of two line segments without putting them in the "canonical" position.

COROLLARY 2.6. *Let $K_1 = K(\alpha_1,\alpha_2)$ an $K_2 = K(\beta_1,\beta_2)$, where $\alpha_1,\alpha_2,\beta_1,\beta_2 \in \mathbb{C}$ such that $\arg(\alpha_1) < \arg(\alpha_2) < \arg(\alpha_1)+\pi$ and $\arg(\beta_1) < \arg(\beta_2) < \arg(\beta_1)+\pi$. Then K_1K_2 is star-shaped and one of the following holds.*

(a) *There exists $\xi \in \mathbb{C}$ such that $\xi K_1 \subseteq K_2$. Equivalently, the segments $K(\alpha_1\beta_1,\alpha_1\beta_2)$ and $K(\alpha_2\beta_1,\alpha_2\beta_2)$ intersect at $\xi\alpha_1\alpha_2$. In this case, $\xi\alpha_1\alpha_2$ is the unique star-center of K_1K_2.*

(b) *There exists $\xi \in \mathbb{C}$ such that $\xi K_2 \subseteq K_1$. Equivalently, the segments $K(\alpha_1\beta_1,\alpha_2\beta_1)$ and $K(\alpha_1\beta_2,\alpha_2\beta_2)$ intersect at $\xi\beta_1\beta_2$. In this case, $\xi\beta_1\beta_2$ is the unique star-center of K_1K_2.*

(c) *Condition* (a) *and* (b) *do not hold, and every point in $K(\beta_1\alpha_2,\beta_2\alpha_1)$ is a star center of K_1K_2*

3. The product set of two convex polygons

In this section, we study the product set of two convex polygons (including interior). It is known that for every convex polygon K_1 with vertexes $\mu_1,\ldots,\mu_n$, then $K_1 = W(T)$ for $T = \mathrm{diag}(\mu_1,\ldots,\mu_n) \in M_n$. In Section 3.1, we will show that the product set of two convex polygons may not be star-shaped. In particular, we have a product set of two triangles that are not star-shaped. This gives a negative answer to the conjecture in [8].

3.1. Products of polygons that are not star-shaped

In this subsection, we show that there are examples of convex sets K_1 and K_2 such that K_1K_2 is not star-shaped. The first example has the form $K_1 = K_2 = K(\alpha_1,\bar{\alpha}_1,\alpha_2)$, where $\alpha_2 \notin \mathbf{R}$. One can regard $K_1 = W(T)$ with $T = \mathrm{diag}(\alpha_1,\bar{\alpha}_1,\alpha_2) \in M_3$ so that the set $W^{\otimes}(T\otimes T) = W(T)W(T)$ is not star-shaped. We can construct another example of the form $K_1 = K_2 = K(\alpha_1,\bar{\alpha}_1,\alpha_2,\bar{\alpha}_2)$, which is symmetric about the real axis, such that K_1K_2 is not star-shaped. One can regard $K_1 = W(A)$ for a real normal matrix $A \in M_4$ with eigenvalues $\alpha_1,\bar{\alpha}_1,\alpha_2,\bar{\alpha}_2$ so that $W^{\otimes}(A\otimes A)$ is not star-shaped.

EXAMPLE 3.1. Let $K_1 = K(e^{i\frac{\pi}{3}}, e^{-i\frac{\pi}{3}}, 0.95e^{i\frac{\pi}{4}})$. Then K_1K_1 is not star-shaped.

Proof. Let $\alpha_1 = e^{i\frac{\pi}{3}}$ and $\alpha_2 = 0.95e^{i\frac{\pi}{4}}$, $K_1 = K(\alpha_1, \overline{\alpha_1}, \alpha_2)$. Then $1 = \alpha_1\overline{\alpha_1}$, $0.95^2 i = \alpha_2^2 \in K_1K_1$. We are going to show that **a)** if s is a star center of K_1K_1, then $s = 1$ and **b)** $(1-t) + t0.95^2 i \notin K_1K_1$ for all $t \in (0,1)$.

Let S be a closed and bounded subset of $\mathbf{C}$, with $0 \notin S$. Suppose $t \in \mathbf{R}$ and $S \cap \{re^{it} : r > 0\} \neq \emptyset$. Let $\rho_0^S(t) = \min\{r > 0 : re^{it} \in S\}$ and $\rho_1^S(t) = \max\{r > 0 : re^{it} \in S\}$.

Let $L_1 = K(\alpha_1, \overline{\alpha_1})$, $S_1 = K_1K_1$ and $S_2 = L_1L_1$. Since $\rho_0^{K_1}(\theta) = \rho_0^{L_1}(\theta)$ for $-\dfrac{\pi}{3} \leqslant \theta \leqslant \dfrac{\pi}{3}$, it follows that $\rho_0^{S_1}(\theta) = \rho_0^{S_2}(\theta)$ for $-\dfrac{2\pi}{3} \leqslant \theta \leqslant \dfrac{2\pi}{3}$.

Note that $x + iy \in S_2 \Leftrightarrow 4(x+iy) \in (2L_1)(2L_1)$. Then, applying Lemma 2.5 (b) to $2L_1 = K(1 - i\sqrt{3}, 1 + i\sqrt{3})$, we have

$$S_2 = \{x + iy : 1 - 4y^2 \leqslant 4x \leqslant 1 - \sqrt{3}(4y - \sqrt{3}),\ 1 + \sqrt{3}(4y + \sqrt{3})\}$$

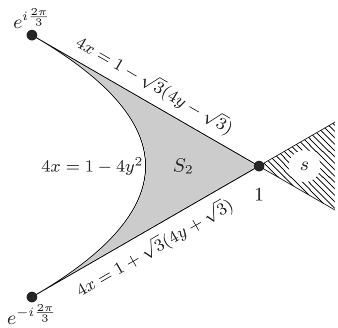


Figure 2. *Plot of* $S_2 = L_1L_1$

a) Note that $\left\{\rho_0^{S_1}(\theta) : \theta \in [-2\pi/3, 2\pi/3]\right\} = \left\{\rho_0^{S_2}(\theta) : \theta \in [-2\pi/3, 2\pi/3]\right\} = \{z^2 : z \in L_1\}$. This means that the curve $\{z^2 : z \in L_1\}$ is a boundary curve of S_2. By Proposition 1.2, if s were a star-center of S_2, then the segment $K(s, z^2)$ must be in S_2 for any $z \in L_1$.

If $s = x + iy$ is a star center of S_1, then we must have

$$4x \geqslant 1 - \sqrt{3}(4y - \sqrt{3}), \quad 1 + \sqrt{3}(4y + \sqrt{3}) \Rightarrow x \geqslant 1$$

Since $|z| \leqslant 1$ for all $z \in S_1$, we have $s = 1$.

b) Let $L_2 = K(\alpha_1, \alpha_2)$, $L_3 = K(\bar{\alpha}_1, \alpha_2)$. Then the boundary of the simply connected set $S_1 = K_1K_1$ is a subset of $\cup_{1 \leqslant i \leqslant j \leqslant 3} L_iL_j$.

Suppose $0 < \theta < \frac{\pi}{2}$ and $\rho_1^{S_1}(t) = r$. Then $re^{i\theta} \in L_2L_3 \cup L_3L_3$. Direct calculation using Lemma 2.5 and Proposition 2.3 shows that $\rho_1^{L_2L_3}(\theta),\ \rho_1^{L_3L_3}(\theta) < \rho_1^{K(1,\alpha_2^2)}(\theta)$; see the following figures.

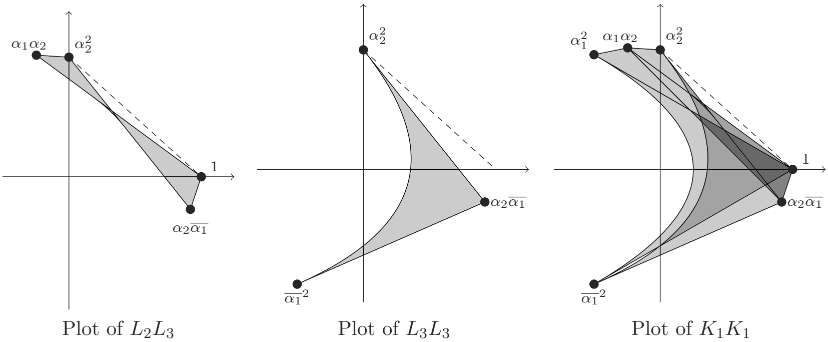


Figure 3.

We conclude that K_1K_1 is not star-shaped. □

Next, we modify Example 3.1 to Example 3.2 so that $\bar{K}_1 = K_1(\alpha_1,\alpha_2,\bar{\alpha}_1,\bar{\alpha}_2)$ with $\alpha_1 = e^{i\frac{\pi}{3}}$ and $\alpha_2 = 0.95e^{i\frac{\pi}{4}}$. In this case, one can regard $K_1 = W(A)$ for some real symmetric $A \in M_4$. The product set K_1K_2 will be larger than the product set considered in Example 3.1. Never-the-less, we can analyze the product of the sets L_iL_j for $i,j = 1,2,3,4$, where $L_1 = K(\alpha_1,\bar{\alpha}_1)$, $L_2 = K(\alpha_1,\alpha_2)$, $L_3 = K(\alpha_2,\bar{\alpha}_2)$, $L_4 = K(\bar{\alpha}_2,\bar{\alpha}_1)$ so that $\cup_{1\leqslant i\leqslant j\leqslant 4}L_iL_j$ contains the boundary of the simply connected set K_1K_1. Again one can show that the part of the boundary $\{z^2 : z \in K(\alpha_1,\bar{\alpha}_1)\}$ of L_1L_1 is also part of the boundary of K_1K_1 so that $1 = \alpha_1\bar{\alpha}_1 \in K_1K_1$ is the only possible candidate to serve as a star-center for K_1K_1. However, none of the set L_iL_j contains the set $\{t+(1-t)0.95^2i : 0 < t < 1/3\}$. Thus, the line segment joining 1 and $\alpha_2^2 = 0.95^2i$ is not in K_1K_1. Hence, 1 is not the star center of K_1K_1, and K_1K_1 is not star-shaped.

EXAMPLE 3.2. Let $K_1 = K(e^{i\frac{\pi}{3}},e^{-i\frac{\pi}{3}},0.95e^{i\frac{\pi}{4}},0.95e^{-i\frac{\pi}{4}})$. Then K_1 is symmetric about the x-axis but $P = K_1K_1$ is not star-shaped.

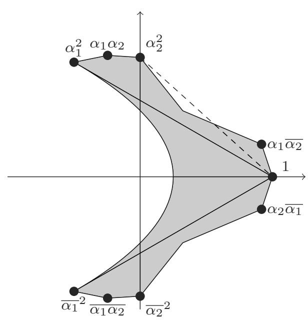


Figure 4. *The set $P = K_1K_1$ in Example 3.2 does not contain the segment $K(1,\alpha_2^2)$.*

3.2. A necessary and sufficient condition

In the following result, we establish a necessary and sufficient condition for the product of two polygons to be a star-shaped set.

THEOREM 3.3. *Let* $K_1 = K(\alpha_1,\ldots,\alpha_n)$ *and* $K_2 = K(\beta_1,\ldots,\beta_m)$. *Then* K_1K_2 *is star-shaped if and only if there is* $p \in K_1K_2$ *such that* $K(p,\alpha_i\beta_j) \subseteq K_1K_2$ *for all* $1 \leqslant i \leqslant n$ *and* $1 \leqslant j \leqslant m$.

Proof. Assume that $K_1 = K(\alpha_1,\ldots,\alpha_n)$ and $K_2 = K(\beta_1,\ldots,\beta_m)$. From Proposition 1.1 (a), we only need to prove that given any $1 \leqslant i_1, i_2 \leqslant n$ and $1 \leqslant j_1, j_2 \leqslant m$, $K(p,q) \subseteq K_1K_2$ for all $q \in K(\alpha_{i_1},\alpha_{i_2})K(\beta_{j_1},\beta_{j_2})$. Without loss of generality, we may assume that for $r = 1,\ 2,\ i_r = j_r = r,\ \alpha_r = 1+ia_r$ and $\beta_r = 1+ib_r$ satisfy one of the conditions (a), (b) or (c) in Theorem 2.4.

Since $K(p,\alpha_r\beta_t) \subseteq K_1K_2$ for $r,t = 1,2$, by the fact that K_1K_2 is simply connected, we see that

$$\mathbf{K} = K(p,\alpha_1\beta_1,\alpha_1\beta_2)\cup K(p,\alpha_2\beta_1,\alpha_2\beta_2)\cup K(p,\alpha_1\beta_1,\alpha_2\beta_1)\cup K(p,\alpha_1\beta_2,\alpha_2\beta_2) \subseteq K_1K_2.$$

If $K(\alpha_1,\alpha_2)K(\beta_1,\beta_2)$ is convex, then $K(p,q) \subseteq \mathbf{K}$ for all $q \in K(\alpha_1,\alpha_2)K(\beta_1,\beta_2)$.

If $K(\alpha_1,\alpha_2)K(\beta_1,\beta_2)$ is not convex, then a_1,a_2,b_1 and b_2 satisfy conditions (b) or (c) in Theorem 2.4. Let $[a_1,a_2]\cap[b_1,b_2] = [c_1,c_2]$, $C = K(1+c_1i,1+c_2i)$, $B = K(1+b_1i,1+b_2i)\setminus C$ and $A = K(1+a_1i,1+a_2i)\setminus C$. Since $K_1K_2 = (AC)\cup(AB)\cup(CC)\cup(CB)$, and previous argument shows that $K(p,q) \subseteq K_1K_2$ for all $q \in (AC)\cup(AB)\cup(CB)$, it remains to show that $K(p,q) \subseteq K_1K_2$ for all $q \in \partial(CC)$. Let

$$\mathbf{V} = (1+c_1i)K(1+c_1i,1+c_2i)\cup(1+c_2i)K(1+c_1i,1+c_2i)$$

and

$$\mathbf{U} = \{(1+si)^2 : s \in (c_1,c_2)\}.$$

Note that $\partial(CC) = \mathbf{V}\cup\mathbf{U}$ and $\mathbf{V} \subseteq K(\alpha_1\beta_1,\alpha_1\beta_2)\cup K(\alpha_2\beta_1,\alpha_2\beta_2)\cup K(\alpha_1\beta_1,\alpha_2\beta_1)\cup K(\alpha_1\beta_2,\alpha_2\beta_2)$. So it remains to show that $K(p,q) \subseteq K_1K_2$ for all $q \in \mathbf{E}^o = \{(1+si)^2 : s \in (c_1,c_2)\}$.

Suppose $q \in \mathbf{E}^o$. Let $\mathbf{L}$ be the tangent line to $\mathbf{E}^o$ at q and $\mathbf{H}$ the open half plane determined by $\mathbf{L}$ and contains 0.

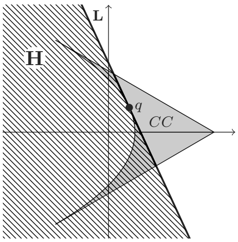


Figure 5.

Consider the following three cases:

Case 1. If $p \in \mathbf{H}$, then there exists $t > 1$ such that $s = p + t(q-p) \in \mathbf{V}$. Therefore, $K(p,q) \subseteq K(p,s) \subseteq K_1K_2$.

Case 2. If $p \in (\mathbf{C} \setminus \mathbf{H}) \cap (CC)$, then $K(p,q) \subseteq (CC) \subseteq K_1K_2$ because $(\mathbf{C} \setminus \mathbf{H}) \cap (CC)$ is a triangular region containing q.

Case 3. If $p \in \mathbf{C} \setminus (\mathbf{H} \cup (CC))$, then there exists $0 < t < 1$ such that $s = p + t(q-p) \in \mathbf{V}$. Therefore, $K(p,q) = K(p,s) \cup K(s,q) \subseteq K_1K_2$. $\square$

We have the following consequence of Theorem 3.3.

COROLLARY 3.4. *Let K_1 be a triangular region with $K_1 = \overline{K}_1$. Then $K_1 = K(r,a,\overline{a})$ for some $r \in \mathbf{R}$ and $a \in \mathbf{C}$. The product set $P = K_1K_1$ is a star-shaped set with $|a|^2$ as a star center.*

Proof. By Theorem 3.3, it suffices to show that $K(|a|^2, q) \in P$ for all $q \in \{r^2, ra, r\overline{a}, a^2, \overline{a}^2\}$.

1. For $0 \leqslant t \leqslant 1$, let $f(t) = (tr + (1-t)a)(tr + (1-t)\overline{a}) \subseteq P$. Since $f(0) = |a|^2$ and $f(1) = r^2$, we have $K(|a|^2, r^2) \subseteq P$.
2. $K(|a|^2, ra) = aK(\overline{a}, r) \subseteq P$.
3. $K(|a|^2, r(\overline{a}) = \overline{a}K(a,r) \subseteq P$.
4. $K(|a|^2, a^2) = aK(\overline{a}, a) \subseteq P$.
5. $K(|a|^2, \overline{a}^2) = \overline{a}K(a, \overline{a}) \subseteq P$. $\square$

Suppose $A \in M_n$ is a real matrix. Then $W(A)$ is symmetric about the real axis. By Corollary 3.4, if $A \in M_3$ is a real normal matrix, then $W(A)W(A)$ is star-shaped. In fact, if A is Hermitian, then $W(A)W(A)$ is convex; otherwise, $|a|^2$ is a star center, where $a, \bar{a}$ are the complex eigenvalues of A.

4. A line and a convex set

In this section, we consider the product of a line segment and a convex set. In the context of numerical range, we consider $W(A)W(B)$, where A is a normal matrix with collinear eigenvalues, and B is a general matrix.

THEOREM 4.1. *Let $K_1 = K(\alpha, \beta)$ for some $\alpha, \beta \in \mathbb{C}$ and K_2 be a compact convex sets in $\mathbb{C}$. Then K_1K_2 is star-shaped.*

We begin with the following easy cases.

PROPOSITION 4.2. *Suppose that $K_1 = K(\alpha, \beta)$ is a line segment and that K_2 is a (not necessarily compact) convex set.*

(1) *If* $0 \in K_1 \cup K_2$*, then* K_1K_2 *is star-shaped with 0 as a star center.*

(2) *If there is a nonzero* $\xi_1 \in \mathbf{C}$ *such that* $\xi_1 K_1 \subseteq (0,\infty)$*, then* K_1K_2 *is convex.*

(3) *If there is a nonzero* $\xi_1 \in \mathbf{C}$ *such that* $\xi_1 K_1 \subseteq K_2$*, then* K_1K_2 *is star-shaped with* $\xi_1\alpha\beta$ *as a star center.*

Proof. (1) It follows from Proposition 1.1 (b).
(2) We may assume that $\xi_1 = 1$. Then $K_1K_2 = \cup_{\alpha \leqslant t \leqslant \beta} tK_2$ is convex.
(3) Assume $\xi_1 = 1$. For every $p \in K_1$ and $q \in K_2$, we will show that

$$K(\alpha\beta, pq) \subseteq K(\alpha,\beta)K(\alpha,\beta,q) \subseteq K_1K_2.$$

To this end, note that

$$K(\alpha\beta,\alpha^2) = \alpha K(\alpha,\beta), \qquad K(\alpha\beta,\beta^2) = \beta K(\alpha,\beta),$$

$$K(\alpha\beta,\alpha q) = \alpha K(\beta,q), \qquad K(\alpha\beta,\beta q) = \beta K(\alpha,q).$$

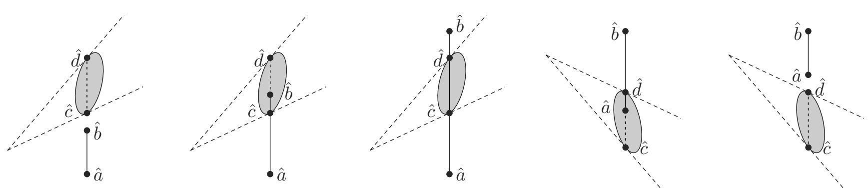
So, we have $K(\alpha\beta,v) \in K(\alpha,\beta)K(\alpha,\beta,q)$ for any $v \in \{\alpha^2,\alpha\beta,\alpha q,\beta^2,\beta q\}$, which is the set of the product of vertexes of $K(\alpha,\beta)$ and $K(\alpha,\beta,q)$. By Theorem 3.3, $K(\alpha,\beta)K(\alpha,\beta,q)$ is star-shaped with $\alpha\beta$ as a star center. Thus, $K(\alpha\beta,pq) \subseteq K(\alpha,\beta)K(\alpha,\beta,q) \subseteq K_1K_2$.

If $\xi_1 \neq 1$, then $(\xi_1\alpha)(\xi_1\beta)$ is a star center of $(\xi_1K_1)K_2 = \xi_1K_1K_2$ by the above argument. Thus, $\xi_1(\alpha\beta)$ is a star center of K_1K_2. □

From now on, we will focus on convex sets K_1 and K_2 that do not satisfy the hypotheses in Proposition 4.2 (1) – (3). In particular, we may find ξ_1 and ξ_2 so that $\xi_1K_1 = K(\hat{a},\hat{b})$ and ξ_2K_2 is a compact convex set containing $\hat{c},\hat{d}$ and lying in the cone

$$\mathscr{C} = \{t_1\hat{c} + t_2\hat{d} : t_1,t_2 \geqslant 0\},$$

where $\hat{a} = 1+ia, \hat{b} = 1+ib, \hat{c} = 1+ic, \hat{d} = 1+id$ with $a \leqslant b, c \leqslant d$. There could be five different configurations of the two sets ξ_1K_1 and ξ_2K_2 as illustrated in Figure 6. (Here, we assume that Proposition 4.2 (3) does not hold so that we do not have the case $c \leqslant a < b \leqslant d$.) If K_1,K_2 are put in these "canonical" positions, we can describe the star centers of K_1K_2 in the next theorem.



(a) $a < b \leq c < d$ (b) $a \leq c \leq b \leq d$ (c) $a \leq c < d \leq b$ (d) $c \leq a \leq d \leq b$ (e) $c < d \leq a < b$

Figure 6. *The above figures illustrate the canonical representations of a line segment* $K_1 = K(a,b)$ *and a convex set* K_2 *described in Theorem 4.3.*

THEOREM 4.3. *Let $\hat{a} = 1 + ia, \hat{b} = 1 + ib, \hat{c} = 1 + ic, \hat{d} = 1 + id$ with $a \leqslant b, c \leqslant d$. Suppose $K_1 = K(\hat{a}, \hat{b})$ and K_2 be a compact convex set containing $\hat{c}, \hat{d}$ and lying in the cone*

$$\mathscr{C} = \{t_1\hat{c} + t_2\hat{d} : t_1, t_2 \geqslant 0\}$$

such that the hypotheses of Proposition 4.2 (1) – (3) *do not hold. Then K_1K_2 is star-shaped and one of the following holds.*

(a) *If $a \leqslant b \leqslant c \leqslant d$, then $\hat{b}\hat{c}$ is a star center.*
(b) *If $a \leqslant c \leqslant b \leqslant d$, then $\hat{b}\hat{c}$ is a star center.*
(c) *If $a \leqslant c \leqslant d \leqslant b$, then $\hat{c}\hat{d}$ is a star center.*
(d) *If $c \leqslant a \leqslant d \leqslant b$, then $\hat{a}\hat{d}$ is a star center.*
(e) *If $c \leqslant d \leqslant a \leqslant b$, then $\hat{a}\hat{d}$ is a star center.*

We need some lemmas to prove Theorem 4.3.

LEMMA 4.4. *Suppose $C = 1 + i\tan\theta_C$, $D = 1 + i\tan\theta_D$ and $P = re^{i\theta_P}$ with $r > 0$, $-\dfrac{\pi}{2} < \theta_C < \theta_P < \theta_D < \dfrac{\pi}{2}$. Let*

$$\frac{-i(P-C)}{|P-C|} = e^{i\theta_1} \quad \text{and} \quad \frac{i(P-D)}{|P-D|} = e^{i\theta_2} \quad \text{with} \quad -\frac{\pi}{2} < \theta_1,\ \theta_2 < \frac{\pi}{2}.$$

Then there exists ξ_1, ξ_2 such that $\xi_1 C = 1 + i\tan(\theta_C - \theta_1)$ and $\xi_1 P = 1 + i\tan(\theta_P - \theta_1)$, $\xi_2 D = 1 + i\tan(\theta_D - \theta_2)$ and $\xi_2 P = 1 + i\tan(\theta_P - \theta_2)$.

Consequently, we have

1. *If $\Re(P) \leqslant 1$, then $\theta_2 \leqslant 0 \leqslant \theta_1$ and $\theta_C - \theta_1 \leqslant \theta_P - \theta_1 \leqslant \theta_P \leqslant \theta_P - \theta_2 \leqslant \theta_D \leqslant \theta_D - \theta_2$.*

2. *If $\Re(P) \geqslant 1$, then $\theta_1 \leqslant 0 \leqslant \theta_2$ and $\theta_C \leqslant \theta_C - \theta_1 \leqslant \theta_P - \theta_1$ and $\theta_P - \theta_2 \leqslant \theta_D - \theta_2 \leqslant \theta_D$.*

Proof. First consider C and P. Then θ_1 is the angle from $\overrightarrow{CD}$ to $\overrightarrow{CP}$. Then the result follows from simple geometry.

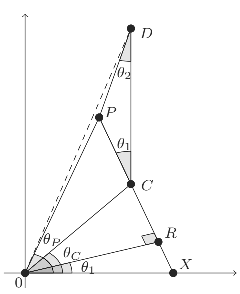


Figure 7.

On one also can calculate directly with $\xi_1 = \dfrac{\cos\theta_C}{\cos(\theta_C - \theta_1)} e^{-i\theta_1}$.

For the second statement, apply the above result on $\overline{D}$ and $\overline{P}$, the complex conjugate of D and P. $\square$

LEMMA 4.5. *Suppose $a \leqslant c \leqslant d$, $p = t_1(1+ic)+t_2(1+id)$ is nonzero for some $t_1,t_2 \geqslant 0$, $K_1 = K(1+ia,1+id)$, and $K_2 = K(1+ic,1+id,p)$. Then K_1K_2 is star-shaped with $(1+ic)(1+id)$ as a star center.*

Proof. Let $\hat{a} = 1+ia$, $\hat{c} = 1+ic$, $\hat{d} = 1+id$. By Theorem 3.3, it suffices to show that $K(\hat{c}\hat{d},uv) \subseteq K_1K_2$ for each pair of elements (u,v) in $\{\hat{a},\hat{d}\} \times \{\hat{c},\hat{d},p\}$. If $u = \hat{d}$, then $K(\hat{c}\hat{d},\hat{d}v) = \hat{d}K(\hat{c},v) \subseteq K_1K_2$. Similarly, if $u = \hat{c}$, then $K(\hat{c}\hat{d},\hat{c}v) = \hat{c}K(\hat{d},v) \subseteq K_1K_2$. Thus, the only nontrivial case is when $(u,v) = (\hat{a},p)$.

By continuity, we may assume that $t_1,\ t_2 > 0$. We consider two cases.

Case 1. Suppose $\Re(p) \leqslant 1$. Then by Lemma 4.4 and Theorem 2.4, $K(\hat{a},\hat{c})K(p,\hat{d})$ is convex. So

$$K(\hat{c}\hat{d},\hat{a}p) \subseteq K(\hat{a},\hat{c})K(p,\hat{d}) \subseteq K_1K_2\,.$$

Case 2. Suppose $\Re(p) > 1$. By Lemma 4.4, there exists α_0 such that $\alpha_0\hat{c} = 1+c_1i$ and $\alpha_0 p = 1+p_1 i$ such that $c_1 > c$. By Theorem 2.4, if $p_1 \geqslant d$, then $\hat{c}\hat{d}$ is a star center of $K(\hat{a},\hat{d})K(\hat{c},p)$. If $p_1 < d$, then $K(\hat{a}\hat{c},\hat{d}\hat{c})$ intersects $K(\hat{a}p,\hat{d}p)$ and $\hat{c}\hat{d}$ lies inside the triangle with vertices $\hat{a}p$, $\hat{d}p$, $\hat{a}\hat{d}$ (see Figure 8). Thus, $K(\hat{c}\hat{d},\hat{a}p)$ is in the interior of the region enclosed by $K(\hat{d}p,\hat{c}\hat{d}) \cup K(\hat{c}\hat{d},\hat{a}\hat{d}) \cup K(\hat{a}\hat{d},\hat{a}p) \cup K(\hat{a}p,\hat{c}\hat{a}) \subseteq K_1K_2$.

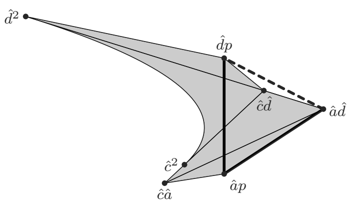


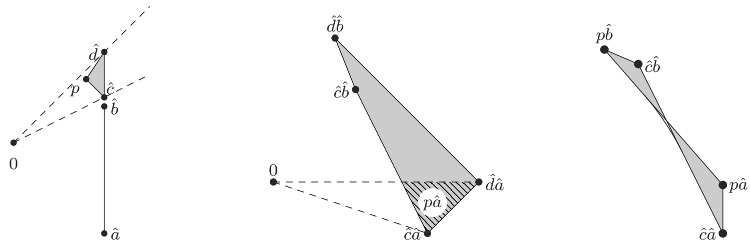
Figure 8.

In both cases, we have $K(\hat{c}\hat{d},\hat{a}p) \subseteq K_1K_2$. $\square$

LEMMA 4.6. *Suppose $a < b \leqslant c < d$, $p = t_1(1+ic)+t_2(1+id)$ is nonzero for some $t_1,t_2 \geqslant 0$ and $K_1 = K(1+ia,1+ib)$, and $K_2 = K(1+ic,1+id,p)$. Assume also that there is no $\xi \in \mathbf{C}$ such that $K_1 \subseteq \xi K_2$. Then K_1K_2 is star-shaped and $(1+bi)(1+ci)$ is a star center.*

Proof. Let $\hat{a} = 1+ia$, $\hat{c} = 1+ic$, $\hat{d} = 1+id$. Similar to the previous lemma, it is enough to show that $K(\hat{b}\hat{c},\hat{a}p) \subseteq K_1K_2$ for any $p = t_1\hat{c}+t_2\hat{d}$ such that $t_1,t_2 \geqslant 0$.

Let $\xi \in \mathbf{C}$ such that $\xi K(\hat{c},p)$ is a vertical line segment with real part 1. If $\xi K(\hat{c},p) \not\subseteq K(\hat{a},\hat{b})$, then by Corollary 2.6, $\hat{b}\hat{c}$ is a star-center of $K_1K(\hat{c},p)$ and hence $K(\hat{b}\hat{c},\hat{a}p) \subseteq K_1K_2$. Otherwise, we have $\xi K(\hat{c},p) \subseteq K(\hat{a},\hat{b})$ and $K_1K(\hat{c},p)$ is as shown in Figure 9(c) in the figure below. This will only happen if $\Re(p) < 1$. Since $\hat{a}p = t_1(\hat{c}\hat{a}) + t_2\hat{d}\hat{a}$ for some $t_1, t_2 \geqslant 0$ such that $t_1 + t_2 < 1$, then $\hat{a}p \in K(0, \hat{c}\hat{a}, \hat{d}\hat{a})$ and $\hat{b}p \in K(0, \hat{c}\hat{b}, \hat{d}\hat{b})$. Note also that 0 and $p\hat{a}$ are separated by the line segment $K(\hat{c}\hat{b}, \hat{c}\hat{a})$. Hence, $p\hat{a}$ is in the quadrilateral $K_1K(\hat{c}, \hat{d})$ and therefore $K(\hat{a}p, \hat{c}\hat{b}) \subseteq K_1K_2$. This finishes the proof that $\hat{c}\hat{b}$ is a star center for K_1K_2.



(a) $K_1 = K(\hat{a}, \hat{b})$ and $K(\hat{c}, \hat{d}, p)$ **(b)** $K_1K(\hat{c}, \hat{d})$ **(c)** $K_1K(\hat{c}, p)$

Figure 9.

□

Proof of Theorem 4.3. Note that (d) follows from (b) by considering $\bar{K}_1\bar{K}_2$. Similarly, (e) follows from (a). Thus, we only need to prove (a)-(c).

To prove that s is a star center of K_1K_2, we show that for any $p \in K_2$, s is a star center of $K_1K(\hat{c}, \hat{d}, p)$. To accomplish this, it is enough to show that $K(s, uv) \subseteq K_1K_2$ for all pairs $(u, v) \in \{\hat{b}, \hat{a}\} \times \{\hat{c}, \hat{d}, p\}$ by Theorem 3.3, where $p = t_1\hat{c} + t_2\hat{d}$ for some $t_1, t_2 \geqslant 0$.

For (a), the conclusion follows directly from Lemma 4.6.

To prove (c), the only nontrivial cases to consider are when $(u, v) = (\hat{a}, p)$ or $(u, v) = (\hat{b}, p)$. By Lemma 4.5, $K(\hat{c}\hat{d}, \hat{a}p) \subseteq K(\hat{a}, \hat{d})K(\hat{c}, \hat{d}, p) \subseteq K_1K_2$. By Lemma 4.5 again, the product $\overline{K(\hat{b}, \hat{c})}\ \overline{K(\hat{c}, \hat{d}, \hat{p})}$, has $\overline{\hat{c}\hat{d}}$ as a star center. Thus, $\hat{c}\hat{d}$ is a star center of $K(\hat{b}, \hat{c})K(\hat{c}, \hat{d}, \hat{p})$ and thus $K(\hat{c}\hat{d}, \hat{b}p) \subseteq K(\hat{b}, \hat{c})K(\hat{c}, \hat{d}, \hat{p}) \subseteq K_1K_2$.

To prove (b), it is enough to show that $K(\hat{c}\hat{b}, \hat{a}p) \subseteq K_1K_2$ for all $p \in K_2$. We consider two cases,

1. Suppose $p = t_1\hat{d} + t_2\hat{b}$ for some $t_1, t_2 \geqslant 0$. Then by Lemma 4.6, $\hat{b}\hat{c}$ is a star-center of $K(\hat{a}, \hat{c})K(\hat{b}, \hat{d}, p)$. Thus $K(\hat{b}\hat{c}, \hat{a}p) \subseteq K(\hat{a}, \hat{c})K(\hat{b}, \hat{d}, p) \subseteq K_1K_2$.

2. Suppose $p = t_1\hat{b} + t_2\hat{c}$ for some $t_1, t_2 \geqslant 0$. Then by Lemma 4.5, $\hat{b}\hat{c}$ is a star-center of $K(\hat{a}, \hat{b})K(\hat{b}, \hat{c}, p)$. Thus $K(\hat{b}\hat{c}, \hat{a}p) \subseteq K(\hat{a}, \hat{b})K(\hat{b}, \hat{c}, p) \subseteq K_1K_2$.

In both cases, $\hat{b}\hat{c}$ is a star-center for K_1K_2. □

It is clear that Theorem 4.1 follows from Proposition 4.2 and Theorem 4.3.

5. A circular disk and a closed set

It is known that the product of two circular disks is star-shaped [3, 4, 7, 8]. In this section, we will prove some unexpected results that if K_1 is a circular disk, then for many closed sets K_2, the product set is star-shaped. We will use $D(\mu,R)$ to denote the closed disk with center $\mu \in \mathbf{C}$ and radius $R \geqslant 0$.

Note that if $0 \in K_1$, then for every non-empty set K_2, K_1K_2 is star-shaped with 0 as star center. Suppose $0 \notin K_1$, we can always scale K_1 so that it is a circular disk centered at 1 with radius $r < 1$.

We have the following results showing that the product set of a circular disk and another set would be star-shaped under some very general conditions. We begin with the following observation.

LEMMA 5.1. *Suppose $r \in (0,1]$ and $b \in D(1,r)$. Then the product $D(1,r)\{b\}$ is a disk containing $1-r^2$.*

Proof. Let $b \in D(1,r)$. Then $bD(1,r) = D(b,|b|r)$.

$$\begin{aligned} |b-(1-r^2)|^2 &= (b-(1-r^2))(\overline{b}-(1-r^2)) \\ &= |b|^2-(b+\overline{b})(1-r^2)+(1-r^2)^2 \\ &= |b|^2r^2-(1-r^2)(-|b|^2+(b+\overline{b})-(1-r^2)) \\ &= |b|^2r^2-(1-r^2)(r^2-(b-1)(\overline{b}-1)) \\ &\leqslant |b|^2r^2 \qquad\qquad \text{because } |b-1| \leqslant r \leqslant 1. \quad \square \end{aligned}$$

From the above simple proposition, we get the following.

THEOREM 5.2. *Suppose $K_1 = D(\mu,R)$ does not contain 0. For every nonempty subset S of K_1, the product set K_1S is star shaped with star center $\mu^2(1-r^2)$, where $r = |\mu^{-1}R|$.*

In the numerical range context, for every circular disk K_1, there is $A \in M_2$ such that $A-(\mathrm{tr}A)I/2$ is nilpotent and $W(A) = K_1$. Moreover, $B \in M_n$ satisfies $W(B) \subseteq W(A)$ if and only if B admits a dilation of the form $I \otimes A$; see [1, 2]. By Theorem 5.2, if $A \in M_2$ such that $(A-\mathrm{tr}AI)/2$ is nilpotent, then $W(A)W(B)$ is star-shaped for any $B \in M_n$ satisfying $W(B) \subseteq W(A)$.

Next, we have the following.

THEOREM 5.3. *Suppose $r \in (0,1]$ and $b \in \mathbf{C}$ with $\Re(b) \geqslant 1$. Then the product $K(1,b)D(1,r)$ is star-shaped with 1 as star center.*

Proof. Suppose $b = 1+Re^{i\theta}$ with $R \geqslant 0$ and $-\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$. Let $c \in K(1,b)$. Then $c = 1+sRe^{i\theta}$ for some $0 \leqslant s \leqslant 1$. $cK_1 = D(c,|c|r)$. Therefore, $K(1,b)D(1,r) =$

$\cup\{D(c,|c|r) : c \in K(1,b)\}$. Let $z \in K(1,b)D(1,r)$. Then $|z-(1+sRe^{i\theta})| \leqslant |1+sRe^{i\theta}|r$ for some $0 \leqslant s \leqslant 1$. Let $0 \leqslant t \leqslant 1$. We have

$$\begin{aligned}
&|tz+(1-t)-(1+tsRe^{i\theta})|^2 \\
&= |t(z-(1+sRe^{i\theta}))|^2 \\
&\leqslant t^2|1+sRe^{i\theta}|^2 r^2 \\
&= \left((t+tsR\cos\theta)^2+(tsR\sin\theta)^2\right) r^2 \\
&= \left((1+tsR\cos\theta)^2+(tsR\sin\theta)^2-(1-t)(1+t+2tsR\cos\theta)\right) r^2 \\
&\leqslant \left((1+tsR\cos\theta)^2+(tsR\sin\theta)^2\right) r^2 \\
&= |1+tsRe^{i\theta}|^2 r^2.
\end{aligned}$$

Therefore, $tz+(1-t) \in D(1+tsRe^{i\theta}, |1+tsRe^{i\theta}|r) \subseteq K(1,b)D(1,r)$. □

THEOREM 5.4. *Suppose S is a star-shaped subset of $\mathbf{C}$ with star center s such that $|s| \leqslant |z|$ for every $z \in S$. Then $D(a,r)S$ is star-shaped for every circular disk $D(a,r)$. In particular, if S is convex, then $D(a,r)S$ is star-shaped for every circular disk $D(a,r)$.*

Proof. If either S or $D(a,r)$ contains 0, the result holds. So we may assume that $0 \notin S \cup D(a,r)$.

We may assume that $s = 1$ and $D(a,r) = D(1,r)$ with $0 \leqslant r \leqslant 1$. Then for every $z \in S$, $z = 1+Re^{i\theta}$ for some $-\dfrac{\pi}{2} \leqslant \theta \leqslant \dfrac{\pi}{2}$. By Theorem 5.3, the product $K(1,z)D(1,r)$ is star shaped with star center 1. Hence, $SD(1,r)$ is also star shaped with star center 1. □

Apart from the nice results above, there are some limitations about the star-shapedness of the product set of a circular disk and another set in $\mathbf{C}$ as shown in the following.

EXAMPLE 5.5. Let $S = K(1,2e^{i\frac{11\pi}{12}}) \cup K(1,2e^{-i\frac{11\pi}{12}})$. Then S is star-shaped with 1 as star center. Let $D(1,\frac{1}{2})$ be the disk centered at 1 with radius $\frac{1}{2}$. Then the product set $SD(1,\frac{1}{2})$ is not simply connected.

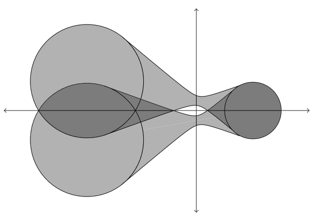


Figure 10. *The product set $(K(1,2e^{i\frac{11\pi}{12}}) \cup K(1,2e^{-i\frac{11\pi}{12}})) \cdot D(1,\frac{1}{2})$ is not simply connected.*

6. Additional results and further research

We have to assume compactness in most of our results. One may wonder what happen if we relax this assumption. The following example shows that without the end points, the product of two line segments may not be star-shaped.

EXAMPLE 6.1. Let $K_1 = K_2$ be the line segment joining $1+i$ and $1-i$ without the end points. Then K_1K_2 has no star center.

Verification. Note that the closure of K_1K_2 equals $S = K(1+i,1-i)K(1+i,1-i)$ has a unique star-center 2. The set K_1K_2 is obtained from S by removing the line segments $K(2,2i)$ and $K(2,-2i)$. The only point in the closure can reach all the points in K_1K_2 is 2, but it is not in K_1K_2. So, K_1K_2 is not star-shaped. □

Recall that an extreme point of a compact convex set $S \subseteq \mathbf{C}$ is an element in S that cannot be written as the mid-point of two different elements in S. If S is a polygon (with interior) then its vertexes are the extreme points. We can extend Theorem 3.3 to the following.

THEOREM 6.2. *Let $K_1, K_2 \subseteq \mathbf{C}$ be compact convex sets. Then K_1K_2 is star-shaped if and only if there is $p \in K_1K_2$ such that $K(p,ab) \subseteq K_1K_2$ for any extreme points $a \in K_1$ and $b \in K_2$.*

Proof. If K_1K_2 is star-shaped, then a star center $p \in K_1K_2$ satisfies $K(p,c) \subseteq K_1K_2$ for any $c \in K_1K_2$. Now, suppose there is $p \in K_1K_2$ satisfying $K(p,ab) \subseteq K_1K_2$ for any extreme points $a \in K_1$ and $b \in K_2$. Let $\mu = \mu_1\mu_2$ with $\mu_1 \in K_1, \mu_2 \in K_2$. By the Caretheodory theorem $\mu_1 \in K(a_1,a_2,a_3)$ and $\mu_2 \in K(b_1,b_2,b_3)$ for some extreme points $a_1,a_2,a_3 \in K_1$ and $b_1,b_2,b_3 \in K_2$. (Some of the a_i's may be the same, and also some of the b_i's may be the same.) Suppose $p = p_1p_2$ with $p_1 \in K_1$ and $p_2 \in K_2$. Then $p_1 \in K(a_4,a_5,a_6)$ and $p_2 \in K(b_4,b_5,b_6)$ for some extreme points $a_4,a_5,a_6 \in K_1$ and $b_4,b_5,b_6 \in K_2$. By Theorem 3.3, $K(p,\mu_1\mu_2) \subseteq K(a_1,\ldots,a_6)K(b_1,\ldots,b_6) \subseteq K_1K_2$. Thus, p is a star center of K_1K_2 □

Another observation is the following extension of Proposition 1.1(b). Note that we do not need to impose compactness conditions on K_1 or K_2.

PROPOSITION 6.3. *Suppose $K_1 \subseteq \mathbf{C}$ is star-shaped with 0 as a star center. Then for any non-empty subset $K_2 \subseteq \mathbf{C}$, the set K_1K_2 is star-shaped with 0 as a star center.*

Proof. Let $p = p_1p_2 \in K_1K_2$ with $p_1 \in K_1, p_2 \in K_2$. Then $K(0,p) = K(0,p_1)\{p_2\} \subseteq K_1K_2$. □

There are other interesting questions deserve further research. We mention a few of them in the following.

P1 Find necessary and sufficient conditions on K_1 and K_2 so that K_1K_2 is convex or star-shaped.

In the context of numerical range if $A \in M_2$, then $W(A)$ is an elliptical disk. So, it is also of interest to study the following.

P2 Let K_1, K_2 be two elliptical disks. Determine conditions on K_1, K_2 so that K_1K_2 is star-shaped or convex.

One may also consider the following.

P3 Characterize those elliptical disks K_1 such that K_1K_2 is star-shaped for all compact convex set K_2.

More generally, one may consider the following.

P4 Characterize those compact convex sets K_1 such that K_1K_2 is convex or star-shaped for any compact convex set K_2.

In connection to Problem P4, we have shown that if K_1 is a close line segment or a close circular disk, then K_1K_2 is star-shaped for any compact convex set K_2. These results are are also connected to Problem P3 because a line segment and a circular disk can be viewed as elliptical disks.

It is also interesting to study the Minkowski product of s (convex) sets $K_1, \ldots, K_s$. The study will be more challenging. As pointed out in [8], the set $K_1 \cdots K_s$ may not be simply connected in general. Nevertheless, our results in Section 5 and Proposition 6.2 imply the following.

PROPOSITION 6.4. *Suppose* $K_1, \ldots, K_s \subseteq \mathbf{C}$.

1. *If any one of the sets* $K_1, \ldots, K_s$ *is star-shaped with 0 as a star center, then* $K_1 \cdots K_s$ *is star-shaped with 0 as a star center.*

2. *Suppose there is a nonzero number* μ_1 *such that* $\mu_1 K_1$ *is a circular disk center at 1 with radius* $r < 1$.

(2.a) *If there is* $\mu \in \mathbf{C}$ *such that* $\mu K_2 \cdots K_s \subseteq \mu_1 K_1$ *for* $i = 2, \ldots, s$, *then* $K_1 \cdots K_r$ *is star-shaped with* $(\mu_1\mu)^{-1}(1 - r^2)$ *as a star center.*

(2.b) *If there is* $\mu \in \mathbf{C}$ *such that* $\mu K_2 \cdots K_s \subseteq \{z \in \mathbf{C} : \Re(z) \geqslant 1\}$ *for* $i = 2, \ldots, s$, *then* $K_1 \cdots K_r$ *is star-shaped with* $(\mu_1\mu)^{-1}$ *as a star center.*

It is also interesting to study the following problem.

P5 Characterize those compact (convex) sets K such that K^2 is convex or star-shaped.

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