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INVERSE PROBLEMS FOR SELF–ADJOINT DIRAC SYSTEMS: EXPLICIT SOLUTIONS AND STABILITY OF THE PROCEDURE

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To the memory of Leiba Rodman,
a wonderful mathematician and an admirable person

(*Communicated by F. Gesztesy*)

Abstract. A procedure to recover explicitly self-adjoint matrix Dirac systems on the semi-axis (with both discrete and continuous components of spectrum) from rational Weyl functions is considered. Its stability is proved. GBDT version of Bäcklund-Darboux transformation and various important results on Riccati equations are used for this purpose.

1. Introduction

Self-adjoint Dirac system has the form

$$\frac{d}{dx}y(x,z) = \mathrm{i}(zj + jV(x))y(x,z), \qquad x \geqslant 0, \tag{1.1}$$

where

$$j = \begin{bmatrix} I_{m_1} & 0 \\ 0 & -I_{m_2} \end{bmatrix}, \quad V = \begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix}, \quad m_1 + m_2 =: m, \tag{1.2}$$

I_{m_k} is the $m_k \times m_k$ identity matrix and $v(x)$ is an $m_1 \times m_2$ matrix function. We assume that the *potential* v is locally summable (i.e., summable on all the finite intervals $[0,l]$).

The inverse spectral problem for a general-type self-adjoint Dirac system, and closely related problem to recover a Dirac system from its Weyl–Titchmarsh (Weyl) function, has been actively studied since 1950's [28, 32], and many interesting results were published in recent years (see, e.g., [2, 3, 6, 7, 10, 24, 35, 43] and various references therein). When speaking about inverse spectral problems, we mean (in particular) the recovery of systems from their Weyl functions. Inverse spectral problems (in the

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mentioned above sense) are also solved [12, 14, 37, 39, 43] for general-type skew-self-adjoint and discrete Dirac systems.

Procedures to solve these inverse problems are nonlinear and usually unstable. However, stability plays an essential role in theory and applications, and special cases where such stability can be proved are important. In particular, we could mention the paper [34] on the evolution Schrödinger equation and the paper [24], where stability was proved for a class of scalar ($m_1 = m_2 = 1$) Dirac systems (on an interval) with discrete and d-separated spectral data. We consider the case of explicit solutions of inverse problems (i.e., the case of rational Weyl functions), where one can apply procedures, which are different from the procedures for the general-type case.

Thus, we prove stability in solving inverse problems for matrix Dirac systems on the semi-axis with both discrete and continuous components of spectrum. Riccati equations play an essential role in the explicit solving of inverse problems, and so the classical works on Riccati equations by Leiba Rodman and coauthors are actively used in this paper.

Explicit solutions of inverse spectral problems can be obtained either by applying procedures for general-type systems to some specific spectral data (e.g., to rational Weyl or scattering functions) or by using several procedures tailored to specific solutions. The first (general-type) approach was used, for instance, in [1, 17, 42] and [2, Sect. 6]. The second approach includes the Crum-Krein method [8, 29], commutation methods [9, 18, 19, 27, 46] and some versions of Bäcklund-Darboux transformation. Here we use our GBDT version of the Bäcklund-Darboux transformation (see [38, 40, 43] and references therein), see also [4, 5, 33, 47] and references therein on various versions of Bäcklund-Darboux transformations.

In the next section, Preliminaries, we present some basic notions from system theory and formulate several results on Weyl functions. We also present the GBDT procedure to solve the inverse problem for systems (1.1) (more precisely, to recover self-adjoint Dirac systems from Weyl functions). Section 3 is dedicated to the proof of stability of this procedure.

As usual, $\mathbb{R}$ stands for the real axis, $\mathbb{C}$ stands for the complex plane, $\mathbb{C}_+$ is the open upper half-plane $\{z : \Im(z) > 0\}$, $\mathbb{C}_-$ is the open lower half-plane $\{z : \Im(z) < 0\}$, and the notation $\mathrm{diag}\{d_1, \ldots\}$ stands for the diagonal (or block diagonal) matrix with the entries $d_1, \ldots$ on the main diagonal. By $\|A\|$ and by $\sigma(A)$, we denote the l^2-induced norm and the spectrum, respectively, of some matrix A. We say that the matrix X is positive (positive definite) and write $X > 0$ if X is Hermitian (i.e., $X = X^*$) and all the eigenvalues of X are positive.

2. Preliminaries

2.1. Rational functions

Recall that a rational matrix function is called *strictly proper* if it tends to zero at infinity. It is well-known [26, 30] that such an $m_2 \times m_1$ matrix function φ can be represented in the form

$$\varphi(z) = \mathscr{C}(zI_n - \mathscr{A})^{-1}\mathscr{B}, \tag{2.1}$$

where $\mathscr{A}$ is a square matrix of some order n, and the matrices $\mathscr{B}$ and $\mathscr{C}$ are of sizes $n\times m_1$ and $m_2\times n$, respectively. The representation (2.1) is called a *realization* of φ, and the realization (2.1) is said to be *minimal* if n is minimal among all possible realizations of φ. This minimal n is called the *McMillan degree* of φ. The realization (2.1) of φ is minimal if and only if

$$\operatorname{span}\bigcup_{k=0}^{n-1}\operatorname{Im}\mathscr{A}^k\mathscr{B}=\mathbb{C}^n, \quad \operatorname{span}\bigcup_{k=0}^{n-1}\operatorname{Im}(\mathscr{A}^*)^k\mathscr{C}^*=\mathbb{C}^n, \quad n=\operatorname{ord}(\mathscr{A}), \tag{2.2}$$

where Im stands for image and $\operatorname{ord}(\mathscr{A})$ stands for the order of $\mathscr{A}$. If for a pair of matrices $\{\mathscr{A},\mathscr{B}\}$ the first equality in (2.2) holds, then the pair $\{\mathscr{A},\mathscr{B}\}$ is called *controllable*. If the second equality in (2.2) is fulfilled, then the pair $\{\mathscr{C},\mathscr{A}\}$ is said to be *observable*.

Now, let matrix functions φ be contractive, that is, let $\varphi^*\varphi\leqslant I_{m_1}$ (or, equivalently, $\varphi\varphi^*\leqslant I_{m_2}$) hold. From [30, Theorems 21.1.3, 21.2.1] (see also [13, p. 191]), the next proposition easily follows.

PROPOSITION 2.1. *Assume that φ is a strictly proper rational matrix function, which is contractive on $\mathbb{R}$ and has no poles in $\mathbb{C}_+$, and let the realization* (2.1) *be its minimal realization. Then, there is a positive solution $X>0$ of the Riccati equation*

$$X\mathscr{B}\mathscr{B}^*X+\mathrm{i}(\mathscr{A}^*X-X\mathscr{A})+\mathscr{C}^*\mathscr{C}=0. \tag{2.3}$$

Clearly, under conditions of Proposition 2.1, $\varphi(z)$ is contractive on $\mathbb{C}_+\cup\mathbb{R}$.

In the case of the discrete and continuous skew-self-adjoint Dirac systems, we obtain [21, 25, 11] the Riccati equation with minus before $\mathscr{B}\mathscr{B}^*$:

$$X\mathscr{C}^*\mathscr{C}X+\mathrm{i}(\mathscr{A}X-X\mathscr{A}^*)-\mathscr{B}\mathscr{B}^*=0. \tag{2.4}$$

This case is dealt with in a different way (based on some stability results from [31]) in our next paper [15].

2.2. System (1.1): Weyl function and inverse problem

Recall that $Y(x,z)$ is the normalized by $Y(0,z)=I_m$ fundamental solution of Dirac system (1.1), where j and V have the forms (1.2).

DEFINITION 2.2. An $m_2\times m_1$ matrix function $\varphi(z)$ $(z\in\mathbb{C}_+)$ such that

$$\int_0^\infty \begin{bmatrix} I_{m_1} & \varphi(z)^* \end{bmatrix} Y(x,z)^*Y(x,z)\begin{bmatrix} I_{m_1} \\ \varphi(z)\end{bmatrix} dx<\infty \tag{2.5}$$

is called a Weyl function of the Dirac system (1.1) on $[0,\infty)$.

REMARK 2.3. According to [13, Sect. 2] and [43, Sect. 2.2], the Weyl function $\varphi(z)$ of the Dirac system (1.1) always exists and is unique. Moreover, $\varphi(z)$ is holomorphic and contractive in $\mathbb{C}_+$.

If φ is rational, it can be extended (from $\mathbb{C}_+$) on $\mathbb{R}$ and $\mathbb{C}_-$ in a natural way. Each potential v corresponding to a strictly proper rational Weyl function is generated by a fixed value $n \in \mathbb{N}$ and by a quadruple of matrices, namely, by two $n \times n$ matrices α and $S_0 > 0$ and by $n \times m_k$ matrices ϑ_k $(k = 1, 2)$ such that the matrix identity

$$\alpha S_0 - S_0 \alpha^* = \mathrm{i}(\vartheta_1 \vartheta_1^* - \vartheta_2 \vartheta_2^*) \tag{2.6}$$

holds. Such potentials v have the form

$$v(x) = -2\mathrm{i}\vartheta_1^* \mathrm{e}^{\mathrm{i}x\alpha^*} S(x)^{-1} \mathrm{e}^{\mathrm{i}x\alpha} \vartheta_2, \tag{2.7}$$

$$S(x) = S_0 + \int_0^x \Lambda(t)\Lambda(t)^* dt, \quad \Lambda(x) = \begin{bmatrix} \mathrm{e}^{-\mathrm{i}x\alpha}\vartheta_1 & \mathrm{e}^{\mathrm{i}x\alpha}\vartheta_2 \end{bmatrix}. \tag{2.8}$$

DEFINITION 2.4. [20, 13] The potentials v generated by the quadruples $\{\alpha, S_0, \vartheta_1, \vartheta_2\}$ (where $S_0 > 0$ and (2.6) holds) via equalities (2.7) and (2.8), are called pseudo-exponential potentials.

THEOREM 2.5. [13] *Let Dirac system with a pseudo-exponential potential v be given on $[0, \infty)$ and let v be generated by the quadruple $\{\alpha, S_0, \vartheta_1, \vartheta_2\}$. Then the Weyl function φ of this system has the form*

$$\varphi(z) = -\mathrm{i}\vartheta_2^* S_0^{-1}(zI_n - \theta)^{-1}\vartheta_1, \quad \theta = \alpha - \mathrm{i}\vartheta_1\vartheta_1^* S_0^{-1}. \tag{2.9}$$

The following theorem (i.e., [13, Theorem 3.4]) presents a procedure for explicit solution of the inverse problem, which is basic for this paper. (See also [23, Theorem 5.2] for the $m_1 = m_2$ case.)

THEOREM 2.6. *Let $\varphi(z)$ be a strictly proper rational matrix function, which is contractive on $\mathbb{R}$ and has no poles in $\mathbb{C}_+$. Assume that* (2.1) *is its minimal realization and that $X > 0$ is a solution of* (2.3).

Then $\varphi(z)$ is the Weyl function of the Dirac system (1.1)*, the potential v of which has the form* (2.7), (2.8)*, where the quadruple $\{\alpha, S_0, \vartheta_1, \vartheta_2\}$ is given (in terms of $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ and X) by the relations*

$$\alpha = \mathscr{A} + \mathrm{i}\mathscr{B}\mathscr{B}^* X, \quad S_0 = X^{-1}, \quad \vartheta_1 = \mathscr{B}, \quad \vartheta_2 = -\mathrm{i}X^{-1}\mathscr{C}^*. \tag{2.10}$$

In particular, the identity (2.6) easily follows from (2.3) and (2.10). The uniqueness of our explicit solution of the inverse problem is immediate from a much more general uniqueness result.

PROPOSITION 2.7. [41] *The solution of the inverse problem to recover system* (1.1) *from its Weyl function is unique in the class of Dirac systems with the locally square summable potentials.*

REMARK 2.8. We note that there are many quadruples generating the same pseudo-exponential potential. The quadruples, which are recovered using (2.10), have an important additional property: controllability of the pair $\{\alpha, \vartheta_1\}$. This property is immediate from the controllability of the pair $\{\mathscr{A}, \mathscr{B}\}$.

Furthermore, the matrices $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ in the minimal realizations (2.1) of φ are unique up to *basis* (similarity) transformations:

$$\widehat{\mathscr{A}} = \mathscr{T}^{-1}\mathscr{A}\mathscr{T}, \quad \widehat{\mathscr{C}} = \mathscr{C}\mathscr{T}, \quad \widehat{\mathscr{B}} = \mathscr{T}^{-1}\mathscr{B}, \tag{2.11}$$

where $\mathscr{T}$ are invertible $m \times m$ matrices. Choosing the realization of φ with $\widehat{\mathscr{A}}$, $\widehat{\mathscr{B}}$ and $\widehat{\mathscr{C}}$ instead of $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$, and adding the symbol "$\widehat{\ }$" in the notations of the corresponding matrices α, ϑ_i and X, we derive

$$\widehat{\alpha} = \mathscr{T}^{-1}\alpha\mathscr{T}, \quad \widehat{\vartheta}_i = \mathscr{T}^{-1}\vartheta_i \quad (i = 1,2), \quad \widehat{X} = \mathscr{T}^* X \mathscr{T}. \tag{2.12}$$

Setting $\mathscr{T} = X^{-1/2}U^*$, where U is unitary, we have $\widehat{X} = I_m$. Hence, (2.6) takes the form $\widehat{\alpha} - \widehat{\alpha}^* = \mathrm{i}(\widehat{\vartheta}_1\widehat{\vartheta}_1^* - \widehat{\vartheta}_2\widehat{\vartheta}_2^*)$. Moreover, for the case $m_1 = m_2 = p$, it was shown in [22] that U may be chosen in such a way that we have the block representations:

$$\widehat{\beta} := \widehat{\alpha} - \mathrm{i}\widehat{\vartheta}_1(\widehat{\vartheta}_1 + \widehat{\vartheta}_2)^* = \begin{bmatrix} \widetilde{\beta} & 0 \\ 0 & \zeta \end{bmatrix}, \quad \widehat{\vartheta}_1 = \begin{bmatrix} \widetilde{\vartheta}_1 \\ \omega \end{bmatrix}, \quad \widehat{\vartheta}_2 = \begin{bmatrix} \widetilde{\vartheta}_2 \\ -\omega \end{bmatrix}, \tag{2.13}$$

where

$$\zeta = \zeta^* = \mathrm{diag}\{t_1 I_{n_1}, t_2 I_{n_2}, \ldots, t_k I_{n_k}\}, \quad \sigma(\widetilde{\beta}) \in \mathbb{C}_-, \tag{2.14}$$

and ω is some $\widetilde{n} \times p$ matrix, $\widetilde{n} := \sum_{i=1}^k n_k$.

Now, introduce the Dirac operator H associated with the differential expression

$$(\mathscr{H} f)(x) = \left(-\mathrm{i}j\frac{d}{dx} - V(x)\right) f(x), \tag{2.15}$$

the domain of which consists of all absolutely continuous functions f from $L^2_m(0,\infty)$, such that $\mathscr{H} f \in L^2_m(0,\infty)$ and the initial condition

$$\begin{bmatrix} I_p & -I_p \end{bmatrix} f(0) = 0$$

holds. Using (2.13), it is shown in [22] (see also [23, Sect. 2]) that the real eigenvalues of H are concentrated at the points t_k and have multiplicities n_k, whereas the continuous spectrum of H is described by $\widetilde{\beta}$, $\widetilde{\vartheta}_1$ and $\widetilde{\vartheta}_2$. Namely, the spectral density ρ of H has the form

$$\rho(t) = g(t)^* g(t), \quad g(t) := I_p - \mathrm{i}(\widetilde{\vartheta}_1 + \widetilde{\vartheta}_2)^*(tI_{n-\widetilde{n}} - \widetilde{\beta})^{-1}\widetilde{\vartheta}_1. \tag{2.16}$$

In view of the mentioned above connections between the quadruple $\{\alpha, S_0, \vartheta_1, \vartheta_2\}$ and the corresponding Weyl and spectral functions, we can consider this quadruple as the spectral data.

3. Stability of explicit solutions

We see that the procedure (given in Theorem 2.6) of solving inverse problem consists of two steps. Namely, the first step is the solving of the Riccati equation (2.3) and the second step is the recovery of the potential v generated by the quadruple introduced in (2.10). Let us start with the stability in solving the Riccati equation.

1. The following lemma is a stronger statement than Proposition 2.1. (Theorem 7.4.2 from [30] is used for its proof in addition to the Theorems 21.1.3 and 21.2.1 from [30], which yield Proposition 2.1.)

LEMMA 3.1. [16] *Assume that a strictly proper rational $m_2 \times m_1$ matrix function $\varphi(z)$ is contractive on $\mathbb{R}$, and that (2.1) is its minimal realization.*

Then there is a unique Hermitian solution X of the Riccati equation (2.3) such that the relation

$$\sigma(\mathscr{A} + \mathrm{i}\mathscr{B}\mathscr{B}^* X) \subset (\mathbb{C}_- \cup \mathbb{R}) \tag{3.1}$$

holds. This solution X is always invertible. It is also positive if and only if $\varphi(z)$ is contractive in $\mathbb{C}_+$.

Further, in our procedure to recover the potential v from the Weyl function φ, we shall look for this particular solution X of (2.3). More precisely, we start with the strictly proper rational $m_2 \times m_1$ matrix function $\varphi(z)$, which is contractive on $\mathbb{R}$ and has no poles in $\mathbb{C}_+$. Hence, $\varphi(z)$ is contractive in $\mathbb{C}_+$, and so, according to Lemma 3.1, we have $X > 0$. By $\mathscr{G}_n$ we denote the class of triples $\{\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}}, \widetilde{\mathscr{C}}\}$ which determine minimal realizations $\widetilde{\varphi}(z) = \widetilde{\mathscr{C}}(zI_n - \widetilde{\mathscr{A}})^{-1}\widetilde{\mathscr{B}}$ of $m_2 \times m_1$ matrix functions $\widetilde{\varphi}(z)$ contractive on $\mathbb{R} \cup \mathbb{C}_+$. First, we consider the stability in recovery of X from $\{\mathscr{A}, \mathscr{B}, \mathscr{C}\} \in \mathscr{G}_n$.

DEFINITION 3.2. The recovery of $X > 0$, satisfying (3.1), from the minimal realization (2.1) (where $\{\mathscr{A}, \mathscr{B}, \mathscr{C}\} \in \mathscr{G}_n$) of $\varphi(z)$ is called stable if for any $\varepsilon > 0$ there is $\delta > 0$ such that for each $\{\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}}, \widetilde{\mathscr{C}}\}$ satisfying conditions

$$\{\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}}, \widetilde{\mathscr{C}}\} \in \mathscr{G}_n, \quad \|\mathscr{A} - \widetilde{\mathscr{A}}\| + \|\mathscr{B} - \widetilde{\mathscr{B}}\| + \|\mathscr{C} - \widetilde{\mathscr{C}}\| < \delta \tag{3.2}$$

there is a solution $\widetilde{X} = \widetilde{X}^*$ of the equation $\widetilde{X}\widetilde{\mathscr{B}}\widetilde{\mathscr{B}}^*\widetilde{X} + \mathrm{i}(\widetilde{\mathscr{A}}^*\widetilde{X} - \widetilde{X}\widetilde{\mathscr{A}}) + \widetilde{\mathscr{C}}^*\widetilde{\mathscr{C}} = 0$ in the neighborhood $\|X - \widetilde{X}\| < \varepsilon$ of X.

The stability of solutions X of an important class of Riccati equations was shown in [36, Theorem 4.4] for a somewhat wider class of perturbations than described in our definition and we shall use this theorem in order to prove our first stability statement.

THEOREM 3.3. *The recovery of $X > 0$, satisfying (3.1), from the minimal realization (2.1) of $\varphi(z)$ (with $\{\mathscr{A}, \mathscr{B}, \mathscr{C}\} \in \mathscr{G}_n$) is stable.*

Proof. Assuming that a minimal realization (2.1) of $\varphi(z)$ is given (that is, matrices $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ are given), we consider equation (2.3). Putting

$$A_0 = -\mathrm{i}(\mathscr{A} + cI_n) \quad (c \in \mathbb{R}), \quad C_0 = -\mathscr{C}^*\mathscr{C}, \quad D_0 = \mathscr{B}\mathscr{B}^*, \tag{3.3}$$

we see that the equation (2.3) coincides with the Riccati equation

$$XD_0X + XA_0 + A_0^*X - C_0 = 0$$

considered in [36, Subsection 4.2].

Now, we deal with the conditions (i)–(iv) (on the coefficients A_0, C_0, D_0) from [36, Subsection 4.2]. (Only perturbations satisfying these conditions are allowed in [36, Subsection 4.2] and we will show that the conditions (i)–(iv) are satisfied in the case $\{\mathscr{A},\mathscr{B},\mathscr{C}\} \in \mathscr{G}_n$.) Equalities (3.3) and the fact that the pair $\{\mathscr{A},\mathscr{B}\}$ is controllable imply that conditions (i) and (ii) in [36, Subsection 4.2] are fulfilled. In a similar way we derive that conditions (i) and (ii) are fulfilled for the Riccati equations $\widetilde{X}\widetilde{D}_0\widetilde{X} + \widetilde{X}\widetilde{A}_0 + \widetilde{A}_0^*\widetilde{X} - \widetilde{C}_0 = 0$ corresponding to all the triples $\{\widetilde{\mathscr{A}},\widetilde{\mathscr{B}},\widetilde{\mathscr{C}}\} \in \mathscr{G}_n$. For sufficiently large values of $|c|$, the requirement (iii) that the matrix

$$H = \begin{bmatrix} -C_0 & A_0^* \\ A_0 & D_0 \end{bmatrix} \tag{3.4}$$

satisfies the condition $\det H \neq 0$ and that signature of H equals zero is also fulfilled. Clearly, c may be chosen so that (iii) is valid if we substitute the triple $\{\mathscr{A},\mathscr{B},\mathscr{C}\}$ with any triple $\{\widetilde{\mathscr{A}},\widetilde{\mathscr{B}},\widetilde{\mathscr{C}}\} \in \mathscr{G}_n$ in some small neighborhood of $\{\mathscr{A},\mathscr{B},\mathscr{C}\}$. Finally, according to Lemma 3.1, there are hermitian solutions of equations $\widetilde{X}\widetilde{D}_0\widetilde{X} + \widetilde{X}\widetilde{A}_0 + \widetilde{A}_0^*\widetilde{X} - \widetilde{C}_0 = 0$, that is, condition (iv) holds. Since conditions (i)–(iv) from [36, Subsection 4.2] are fulfilled, the stability with respect to perturbations in the class $\mathscr{G}_n$ will follow from the stability in the sense of [36, Theorem 4.4].

Again, using Lemma 3.1, we choose the solution $X > 0$ of (2.3) satisfying (3.1). It is immediate that one of the equivalent statements from [36, Theorem 4.4] is valid for our X. That is, according to (3.1) and (3.3), the equality $\Im(\lambda) = 0$ holds for each λ from the set

$$\begin{aligned}\sigma\big(\mathrm{i}(A_0 + D_0X)\big) \cap \sigma\big(-\mathrm{i}(A_0^* + XD_0)\big) \\ = \sigma\big(\mathscr{A} + \mathrm{i}\mathscr{B}\mathscr{B}^*X + cI_n\big) \cap \sigma\big((\mathscr{A} + \mathrm{i}\mathscr{B}\mathscr{B}^*X + cI_n)^*\big),\end{aligned} \tag{3.5}$$

and so the statement (d) from [36, Theorem 4.4] holds. Therefore, by virtue of [36, Theorem 4.4], X is a stable and isolated solution of (2.3). □

2. Now, we will show that small perturbations of the quadruple $\{\alpha, S_0, \vartheta_1, \vartheta_2\}$ result in small perturbations of the corresponding potential v. We note that we consider only perturbations which do not change m_1, m_2 and n.

DEFINITION 3.4. The quadruple $\{\alpha, S_0, \vartheta_1, \vartheta_2\}$ is called *admissible* if $S_0 > 0$ and (2.6) holds, and it is called *spectral* if it is admissible, the pair $\{\alpha, \vartheta_1\}$ is controllable and

$$\sigma(\alpha) \subset (\mathbb{R} \cup \mathbb{C}_-). \tag{3.6}$$

REMARK 3.5. Theorem 2.5 and Remark 2.3 show that the Weyl function corresponding to any pseudo-exponential potential is rational and contractive. Then, Theorem 2.6, Proposition 2.7 and Lemma 3.1 imply that this potential (uniquely recovered from the Weyl function) is generated, in particular, by a spectral quadruple. In other words, each pseudo-exponential potential is generated by some spectral quadruple.

THEOREM 3.6. *Let a spectral quadruple $\{\alpha, S_0, \vartheta_1, \vartheta_2\}$ be given. Then, for any $\varepsilon > 0$ there is $\delta > 0$ such that each pseudo-exponential potential $\widetilde{v}$ generated by an admissible quadruple $\{\widetilde{\alpha}, \widetilde{S}_0, \widetilde{\vartheta}_1, \widetilde{\vartheta}_2\}$ satisfying condition*

$$\|\alpha - \widetilde{\alpha}\| + \|S_0 - \widetilde{S}_0\| + \|\vartheta_1 - \widetilde{\vartheta}_1\| + \|\vartheta_2 - \widetilde{\vartheta}_2\| < \delta$$

belongs to the ε-neighborhood of v generated by $\{\alpha, S_0, \vartheta_1, \vartheta_2\}$, that is,

$$\sup_{x \in [0,\infty)} \|v(x) - \widetilde{v}(x)\| < \varepsilon. \tag{3.7}$$

In order to prove the theorem above, we generalize (for the case when m_1 does not necessarily equal m_2 and S_0 does not necessarily equal I_n) some results from [23] on asymptotics of

$$Q(x) := S_0 + 2\int_0^x \mathrm{e}^{2\mathrm{i}t\alpha} \vartheta_2 \vartheta_2^* \mathrm{e}^{-2\mathrm{i}t\alpha^*} dt. \tag{3.8}$$

LEMMA 3.7. *The following relations are valid for a spectral quadruple $\{\alpha, S_0, \vartheta_1, \vartheta_2\}$:*

$$\lim_{x\to\infty} Q(x)^{-1} = 0, \quad \lim_{x\to\infty} \|Q(x)^{-1} \mathrm{e}^{2\mathrm{i}x\alpha} \vartheta_2\| = 0. \tag{3.9}$$

Proof. The proof uses some steps from the proof of [23, Proposition 3.3]. Since $Q(x)$ is increasing and is bounded below by $S_0 > 0$, there is a limit

$$\kappa_Q := \lim_{x\to\infty} Q(x)^{-1}.$$

Next, we prove that $\kappa_Q = 0$. From the definition (3.8) and identity (2.6) we derive

$$\begin{aligned}\alpha Q(x) - Q(x)\alpha^* &= \alpha S_0 - S_0 \alpha^* - \mathrm{i}\left(\mathrm{e}^{2\mathrm{i}x\alpha} \vartheta_2 \vartheta_2^* \mathrm{e}^{-2\mathrm{i}x\alpha^*} - \vartheta_2 \vartheta_2^*\right) \\ &= \mathrm{i}\vartheta_1 \vartheta_1^* - \mathrm{i}\mathrm{e}^{2\mathrm{i}x\alpha} \vartheta_2 \vartheta_2^* \mathrm{e}^{-2\mathrm{i}x\alpha^*}.\end{aligned} \tag{3.10}$$

Multiplying (from both sides) the left-hand side and right-hand side of (3.10) by $Q(x)^{-1}$, we obtain

$$Q(x)^{-1}\alpha - \alpha^* Q(x)^{-1} - \mathrm{i}Q(x)^{-1}\vartheta_1\vartheta_1^* Q(x)^{-1} = -\mathrm{i}Q(x)^{-1}\mathrm{e}^{2\mathrm{i}x\alpha}\vartheta_2\vartheta_2^*\mathrm{e}^{-2\mathrm{i}x\alpha^*}Q(x)^{-1}. \tag{3.11}$$

Passing in (3.11) to the limit, we see that

$$\lim_{x\to\infty} Q(x)^{-1}\mathrm{e}^{2\mathrm{i}x\alpha}\vartheta_2\vartheta_2^*\mathrm{e}^{-2\mathrm{i}x\alpha^*}Q(x)^{-1} = \mathrm{i}(\kappa_Q\alpha - \alpha^*\kappa_Q - \mathrm{i}\kappa_Q\vartheta_1\vartheta_1^*\kappa_Q). \tag{3.12}$$

On the other hand, formula (3.8) yields

$$\frac{d}{dx}Q(x)^{-1} = -2Q(x)^{-1}\mathrm{e}^{2\mathrm{i}x\alpha}\vartheta_2\vartheta_2^*\mathrm{e}^{-2\mathrm{i}x\alpha^*}Q(x)^{-1},$$

and so we have

$$\int_0^\infty Q(t)^{-1}\mathrm{e}^{2\mathrm{i}t\alpha}\vartheta_2\vartheta_2^*\mathrm{e}^{-2\mathrm{i}t\alpha^*}Q(t)^{-1}dt = \frac{1}{2}(S_0^{-1} - \kappa_Q) < \infty. \tag{3.13}$$

Taking into account (3.13) and the fact that there exists a limit of the expression integrated in (3.13) (see (3.12)), we derive that this limit equals zero. That is, we rewrite (3.12) in the form

$$\kappa_Q\alpha - \alpha^*\kappa_Q - \mathrm{i}\kappa_Q\vartheta_1\vartheta_1^*\kappa_Q = 0. \tag{3.14}$$

Moreover, since the left-hand side in (3.12) tends to zero, the second equality in (3.9) is already proved.

Recall that the first equality in (3.9) is equivalent to $\kappa_Q = 0$. Now, we prove $\kappa_Q = 0$ by negation. For this, we rewrite (3.14) in the form $\alpha^*\kappa_Q = \kappa_Q(\alpha - \mathrm{i}\vartheta_1\vartheta_1^*\kappa_Q)$, which implies that the range of κ_Q is an invariant subspace of α^*. Thus, assuming $\kappa_Q \neq 0$, we obtain that there is an eigenvector $\kappa_Q g$ of α^*: $\alpha^*\kappa_Q g = c\kappa_Q g, \quad \kappa_Q g \neq 0, \quad g \in \mathbb{C}^n$.

Finally, consider the expression $\mathrm{i}g^*(\kappa_Q\alpha - \alpha^*\kappa_Q)g$. In view of (3.6), for the eigenvalue c of α^* we have $\Im(c) \geqslant 0$, and so

$$\mathrm{i}g^*(\kappa_Q\alpha - \alpha^*\kappa_Q)g = \mathrm{i}(\overline{c} - c)g^*\kappa_Q g \geqslant 0.$$

On the other hand, we have $\vartheta_1^*\kappa_Q g \neq 0$ because the pair $\{\alpha, \vartheta_1\}$ is controllable. Hence, the inequality $\mathrm{i}g^*(\kappa_Q\alpha - \alpha^*\kappa_Q)g = -g^*\kappa_Q\vartheta_1\vartheta_1^*\kappa_Q g < 0$ follows from (3.14). We arrive at a contradiction, that is, $\kappa_Q = 0$. □

In the case of admissible quadruples $\{\alpha, S_0, \vartheta_1, \vartheta_2\}$, the matrix identity

$$\alpha S(x) - S(x)\alpha^* = \mathrm{i}\Lambda(x)j\Lambda(x)^* \tag{3.15}$$

(see [13, formula (3.6)]) coincides with (2.6) at $x = 0$ and easily follows from (2.6) and (2.8) for $x > 0$. In other words, α, $S(x)$ and $\Lambda(x)$ form an S-node (and, moreover, the so called Darboux matrix function corresponding to $v(x)$ coincides with the transfer matrix function [44, 45, 43] in Lev Sakhnovich sense). Using (2.8), (3.8) and (3.15), we derive

$$Q'(x) = \left(\mathrm{e}^{\mathrm{i}x\alpha}S(x)\mathrm{e}^{-\mathrm{i}x\alpha^*}\right)', \quad Q(0) = S(0) \quad \left(Q' := \frac{d}{dx}Q\right),$$

and so the following equality is valid:

$$Q(x) = \mathrm{e}^{\mathrm{i}x\alpha}S(x)\mathrm{e}^{-\mathrm{i}x\alpha^*}. \tag{3.16}$$

Proof of Theorem 3.6. Now, we consider a pseudo-exponential potential v generated by the spectral quadruple $\{\alpha, S_0, \vartheta_1, \vartheta_2\}$ and pseudo-exponential potentials $\widetilde{v}$

generated by admissible quadruples $\{\widetilde{\alpha},\widetilde{S}_0,\widetilde{\vartheta}_1,\widetilde{\vartheta}_2\}$ belonging to a neighborhood of $\{\alpha,S_0,\vartheta_1,\vartheta_2\}$. The matrix function Q corresponding to $\{\widetilde{\alpha},\widetilde{S}_0,\widetilde{\vartheta}_1,\widetilde{\vartheta}_2\}$ is denoted by $\widetilde{Q}$. In view of (2.7) and (3.16), we have:

$$v(x) = -2i\vartheta_1^* Q(x)^{-1} e^{2ix\alpha}\vartheta_2, \quad \widetilde{v}(x) = -2i\widetilde{\vartheta}_1^*\widetilde{Q}(x)^{-1} e^{2ix\widetilde{\alpha}}\widetilde{\vartheta}_2. \tag{3.17}$$

It is immediate from the proof of Lemma 3.7 that (3.11) holds for admissible quadruples as well. That is, we may rewrite (3.11) for $\{\widetilde{\alpha},\widetilde{S}_0,\widetilde{\vartheta}_1,\widetilde{\vartheta}_2\}$:

$$\begin{aligned}&\widetilde{Q}(x)^{-1}\widetilde{\alpha} - \widetilde{\alpha}^*\widetilde{Q}(x)^{-1} - i\widetilde{Q}(x)^{-1}\widetilde{\vartheta}_1\widetilde{\vartheta}_1^*\widetilde{Q}(x)^{-1}\\ &= -i\widetilde{Q}(x)^{-1}e^{2ix\widetilde{\alpha}}\widetilde{\vartheta}_2\widetilde{\vartheta}_2^* e^{-2ix\widetilde{\alpha}^*}\widetilde{Q}(x)^{-1}.\end{aligned} \tag{3.18}$$

Since Q and $\widetilde{Q}$ are monotonic and the first relation in (3.9) is valid, we may choose $x_0 > 0$ and some neighborhood of $\{\alpha,S_0,\vartheta_1,\vartheta_2\}$ so that $Q(x)$ and $\widetilde{Q}(x)$ are large enough for $x \geqslant x_0$, and so the left-hand sides of (3.11) and (3.18) are small enough. Hence, the right-hand sides of (3.11) and (3.18) are also small enough. Therefore, taking into account (3.17), we see that for any $\varepsilon > 0$ there are $x_0 > 0$ and $\delta_1 > 0$ such that the inequality

$$\sup_{x\in[x_0,\infty)} \|v(x)-\widetilde{v}(x)\| < \varepsilon \tag{3.19}$$

holds for each admissible quadruple $\{\widetilde{\alpha},\widetilde{S}_0,\widetilde{\vartheta}_1,\widetilde{\vartheta}_2\}$ in the δ_1-neighborhood (i.e., the neighborhood $\|\alpha-\widetilde{\alpha}\|+\|S_0-\widetilde{S}_0\|+\|\vartheta_1-\widetilde{\vartheta}_1\|+\|\vartheta_2-\widetilde{\vartheta}_2\|<\delta_1$) of $\{\alpha,S_0,\vartheta_1,\vartheta_2\}$.

It easily follows from the definition of Q and $\widetilde{Q}$ and from (3.17) that there is some δ_2-neighborhood of $\{\alpha,S_0,\vartheta_1,\vartheta_2\}$, where we have

$$\sup_{x\in[0,x_0)} \|v(x)-\widetilde{v}(x)\| < \varepsilon. \tag{3.20}$$

Clearly, inequalities (3.19) and (3.20) yield (3.7) (for $\delta = \min(\delta_1,\delta_2)$). □

REMARK 3.8. It is immediate from the second relation in (3.9), the first relation in (3.17) and Remark 3.5 that all pseudo-exponential potentials tend to zero at infinity.

3. We already discussed at the beginning of this section that the procedure of solving inverse problem in Theorem 2.6 consists of solving Riccati equation (2.3) and the following recovery of the potential v generated by the quadruple $\{\alpha,S_0,\vartheta_1,\vartheta_2\}$ introduced in (2.10). In view of Lemma 3.1 we may choose the solution $X>0$ of (2.3) which satisfies (3.1). According to Theorem 3.3 the recovery of this solution is stable. Theorem 3.6 shows that the recovery of v from the quadruple introduced in (2.10) is stable as well. Thus, we obtain the next statement on the stability of the procedure of solving inverse problem.

THEOREM 3.9. *The procedure (given in Theorem 2.6) to uniquely recover the pseudo-exponential potential v of Dirac system (1.1) from a minimal realization of the Weyl function (i.e., of some strictly proper rational $m_2\times m_1$ matrix function, which is contractive in $\mathbb{C}_+$) is stable once we agree to choose such a positive solution X of the Riccati equation (2.3) that (3.1) holds.*

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