Operators and Matrices
Volume 10, Number 4 (2016), 1009–1042 doi:10.7153/oam-10-57

THE ELLIS–GOHBERG INVERSE PROBLEM FOR MATRIX–VALUED WIENER FUNCTIONS ON THE LINE

M. A. KAASHOEK AND F. VAN SCHAGEN

In memory of Leiba Rodman

(*Communicated by I. M. Spitkovsky*)

Abstract. This paper deals with the Ellis-Gohberg inverse problem for matrix-valued Wiener functions on the line, instead of on the circle, as was done in [4] for scalar functions and in [14] for matrix-valued functions. The problem is reduced to a linear finite matrix equation of which the right hand side is described explicitly in terms of one of the given functions. The results obtained parallel and extend those derived in [14] for Wiener functions on the circle. Special attention is paid to the case when the given functions are Fourier transforms of functions of finite support. In the final section the results are specified further for the case when the given functions are rational matrix functions.

1. Introduction

In the present paper we deal with an inverse problem for orthogonal functions related to the Nehari problem. These orthogonal functions have been introduced by R. L. Ellis and I. Gohberg in [4] where a continuous infinite analogue of Krein's theorem [15] is proved. The discrete analogue of the inverse problem we shall be dealing with was solved for the scalar case in [4] and for various classes of matrix-valued functions in [14].

To state the problem we need some notation. Throughout $\mathbb{C}^{r\times s}$ denotes the linear space of all $r\times s$ matrices with complex entries and $L^1(\mathbb{R})^{r\times s}$ denotes the space of all $r\times s$ matrices of which the entries are Lebesgue integrable functions on the real line $\mathbb{R}$. For $f\in L^1(\mathbb{R})^{r\times s}$ put

$$(\mathscr{F}f)(\lambda)=\int_{-\infty}^{\infty}e^{-i\lambda t}f(t)\,dt \quad \text{and} \quad (Jf)(\lambda)=f(-\lambda) \quad (\lambda\in\mathbb{R}).$$

Thus $\mathscr{F}$ is the classical Fourier transform and J is just a flip over operator. In what follows we write $\mathscr{F}'$ for the operator $J\mathscr{F}$. Since $\mathscr{F}$ is the classical Fourier transform, we have

$$\mathscr{F}'(f\star g)=(\mathscr{F}'f)(\mathscr{F}'g), \quad f\in L^1(\mathbb{R})^{r\times\ell},\ g\in L^1(\mathbb{R})^{\ell\times m}, \tag{1.1}$$

Mathematics subject classification (2010): 47A56, 47B35, 47A48, 93B15, 15A06.
Keywords and phrases: Inverse problem, Wiener function on the line, Hankel operator, Wiener-Hopf integral operator, convolution operator on a finite interval, bi-inner rational matrix function, realization.

where $f \star g$ is the convolution product of f and g.

Next we define the Wiener space $\mathscr{W}^{r\times s}$, as follows:

$$\mathscr{W}^{r\times s} = \mathscr{W}^{r\times s}_{-,0} \dot{+} \mathscr{W}^{r\times s}_{d} \dot{+} \mathscr{W}^{r\times s}_{+,0}.$$

Here $\mathscr{W}^{r\times s}_{+,0} = \mathscr{F}' L^1(\mathbb{R}_+)^{r\times s}$ and $\mathscr{W}^{r\times s}_{-,0} = \mathscr{F}' L^1(\mathbb{R}_-)^{r\times s}$, where $\mathbb{R}_+ = [0,\infty)$ and $\mathbb{R}_- = (-\infty, 0]$, and $\mathscr{W}^{r\times s}_{d}$ is the space consisting of all constant $r\times s$ matrix functions on $\mathbb{R}$. It follows that $\varphi \in \mathscr{W}^{r\times s}$ decomposes in a unique way as $\varphi = \varphi_{+,0} + \varphi_d + \varphi_{-,0}$, where $\varphi_{\pm,0} \in \mathscr{F}' L^1(\mathbb{R}_\pm)^{r\times s}$ and φ_d is a constant $r\times s$ matrix function. Finally, for $\varphi \in \mathscr{W}^{r\times s}$ we define φ^* by $\varphi^*(\lambda) = \varphi(\lambda)^*$ for $\lambda \in \mathbb{R}$.

The main problem. Let $\alpha \in e + \mathscr{W}^{p\times p}_{+,0}$ and $\gamma \in \mathscr{W}^{q\times p}_{-,0}$. Here e is the $p\times p$ matrix function identically equal to the $p\times p$ identity matrix. Thus

$$\alpha(\lambda) = I_p + \int_0^\infty e^{i\lambda t} a(t)\,dt, \quad \text{where } a \in L^1(\mathbb{R}_+)^{p\times p}, \tag{1.2}$$

$$\gamma(\lambda) = \int_{-\infty}^0 e^{i\lambda t} c(t)\,dt \quad \text{where } c \in L^1(\mathbb{R}_-)^{q\times p}. \tag{1.3}$$

We say that $g \in \mathscr{W}^{p\times q}_{+,0}$ is a solution to the *EG inverse problem associated with* α *and* γ if the following two inclusions are satisfied:

$$\alpha + g\gamma - e \in \mathscr{W}^{p\times p}_{-,0} \quad \text{and} \quad g^*\alpha + \gamma \in \mathscr{W}^{q\times p}_{+,0}. \tag{1.4}$$

To understand better the conditions in (1.4) we take inverse Fourier transforms in (1.4). As a first step write $g = \mathscr{F}' k$, where $k \in L^1(\mathbb{R}_+)^{p\times q}$, put $k^*(t) = k(-t)^*$ for $t \leqslant 0$, and next consider the Hankel operators G and G_* defined by

$$G : L^1(\mathbb{R}_-)^q \to L^1(\mathbb{R}_+)^p, \quad (Gf)(t) = \int_{-\infty}^0 k(t-s)f(s)\,ds, \quad t \geqslant 0; \tag{1.5}$$

$$G_* : L^1(\mathbb{R}_+)^p \to L^1(\mathbb{R}_-)^q, \ (G_*h)(t) = \int_0^\infty k^*(t-s)h(s)\,ds, \ t \leqslant 0. \tag{1.6}$$

Here $L^1(\mathbb{R}_\pm)^r = L^1(\mathbb{R}_\pm)^{r\times 1}$. Using these Hankel operators, the definitions of α and γ in (1.2) and (1.3), respectively, and taking inverse Fourier transforms in (1.4) one sees that the inclusions in (1.4) are equivalent to

$$\begin{bmatrix} I & G \\ G_* & I \end{bmatrix} \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ -k^* \end{bmatrix}. \tag{1.7}$$

Thus the EG inverse problem associated with α and γ can be reformulated as follows. Given $a \in L^1(\mathbb{R}_+)^{p\times p}$ and $c \in L^1(\mathbb{R}_-)^{q\times p}$, find $k \in L^1(\mathbb{R}_+)^{p\times q}$ such that the identity (1.7) is satisfied, where G and G_* are the Hankel operators defined by (1.5) and (1.6), respectively.

The following lemma presents a necessary condition for the EG inverse problem to be solvable.

LEMMA 1.1. *If the EG inverse problem associated with* α *and* γ *has a solution, then the following condition is satisfied:*

$$\text{(C1)} \qquad \alpha(\lambda)^*\alpha(\lambda)-\gamma(\lambda)^*\gamma(\lambda)=I_p \text{ for each } \lambda\in\mathbb{R}.$$

The above result is due to R. L. Ellis and I. Gohberg, see formula (2.5) in [5, Section 12.2]. The analogous identity for functions on the unit circle is older; see [4, formula (1.7)] which deals with scalar functions and [6, formula (2.5)] for matrix-valued functions. For various generalizations, including an abstract version in a band method setting, we refer to [3], [12], and [13]. For the sake of completeness we present the proof.

Proof. Let $g\in\mathscr{W}_{+,0}^{p\times q}$ be a solution of the EG inverse problem. Thus g satisfies (1.4) which tells us that $\alpha+g\gamma=e+\xi_1$ and $\alpha^*g+\gamma^*=\xi_2$ where ξ_1 and ξ_2 are both in $\mathscr{W}_{-,0}^{p\times p}$. It follows that

$$\begin{aligned}\alpha^*\alpha-\gamma^*\gamma&=\alpha^*(\alpha+g\gamma)-(\alpha^*g+\gamma^*)\gamma=\alpha^*(e+\xi_1)-\xi_2\gamma=\\&=e+(\alpha^*-e)+\alpha^*\xi_1-\xi_2\gamma\in e+\mathscr{W}_{-,0}^{p\times p}.\end{aligned}$$

Since $(\alpha^*\alpha-\gamma^*\gamma)^*=\alpha^*\alpha-\gamma^*\gamma$, we obtain

$$\alpha^*\alpha-\gamma^*\gamma\in(e+\mathscr{W}_{-,0}^{p\times p})\cap(e+\mathscr{W}_{+,0}^{p\times p}).$$

Therefore $\alpha^*\alpha-\gamma^*\gamma=e$, and (C1) is proved. □

Already in the scalar case simple examples show that condition (C1) is not sufficient. In fact, as we shall see (Theorem 4.1 in Section 4 below) in the scalar case the EG inverse problem associated with α and γ is solvable if and only if condition (C1) is satisfied and the functions α and γ^* have no common zero in the open upper half plane. This result is the continuous analogue of Theorem 4.1 in [4]. For the square matrix case, i.e., when $p=q$, we prove the following continuous analogue of Theorem 1.1 in [14].

THEOREM 1.2. *Assume* $p=q$. *The EG inverse problem associated with* α *and* γ *is solvable and the solution is unique if and only if condition* (C1) *is satisfied and the functions* $\det\alpha$ *and* $\det\gamma^*$ *have no common zero in the open upper half plane.*

The latter result is not true without the additional uniqueness requirement, that is, it may happen that the EG inverse problem associated with α and γ is solvable while $\det\alpha$ and $\det\gamma^*$ have common zeros. In that case the number of solutions is infinite; see for instance Example 3.8.

Contents. The paper consists of six sections and an appendix, the first section being the present introduction. Section 2 presents a number of auxiliary results using the fact that condition (C1) in Lemma 1.1 above implies that $\alpha(\lambda)^*\alpha(\lambda)$ is positive definite for each $\lambda\in\mathbb{R}$ and is equal to the $p\times p$ identity matrix for $\lambda=\infty$. Our main results are presented in Sections 3, 4, and 5. In Section 3 we reduce the EG inverse problem to a linear matrix equation, we present formulas for all solutions whenever the problem is solvable, and we prove the results referred to in the previous paragraph, including Theorem 1.2. Section 4 deals with uniqueness of the solution mainly for the scalar case. In Section 5 we assume that the functions a and c appearing in (1.2) and (1.3) have finite support, and we show that in that case condition (C1) is necessary and sufficient for the EG inverse problem to be solvable. Moreover, if the functions a and c appearing in (1.2) and (1.3) have finite support and condition (C1) is satisfied, then there exists precisely one solution $g = \mathscr{F}'k$ to the EG-inverse problem such that k has finite support. In Section 6, using elements of mathematical system theory, the results of Section 3 are specified further for the case when α and γ are rational matrix functions. The appendix Section A presents three auxiliary results that are used in proofs appearing in Sections 2, 3, and 5.

Notation and terminology. By $\mathbb{C}_+$ we denote the open upper half plane. A linear operator from $\mathbb{C}^s$ to $\mathbb{C}^r$ is identified with its matrix with respect to the standard bases of $\mathbb{C}^s$ and $\mathbb{C}^r$. For a matrix $A \in \mathbb{C}^{r\times r}$ the set of eigenvalues of A is denoted by $\sigma(A)$. Standard terminology from elementary mathematical systems theory, like realization, minimal realization, controllability and observability, is used without further explanation. For these and related items, see, e.g., [1, Sections 2.1, 2.2 and 8.1], [2, Chapters 4, 5 and 7], [11, Sections 3.1, 3.2 and 3.3] and/or [20, Section 6.5]). Given a and c as in (1.2) and (1.3) the functions a^* and c^* are defined by $a^*(t) = a(-t)^*$, $t \leqslant 0$ and $c^*(t) = c(-t)^*$, $t \geqslant 0$. Thus $a^* \in L^1(\mathbb{R}_-)^{p\times p}$ and $c^* \in L^1(\mathbb{R}_+)^{p\times q}$. Moreover, for $\lambda \in \mathbb{R}$ we have

$$\alpha^*(\lambda) = I_p + \int_{-\infty}^{0} e^{i\lambda t} a^*(t)\,dt \quad \text{and} \quad \gamma^*(\lambda) = \int_0^{\infty} e^{i\lambda t} c^*(t)\,dt, \quad \lambda \in \mathbb{R}.$$

By definition T_α and T_{α^*} are the Wiener-Hopf operators acting on $L^1(\mathbb{R}_+)^p$ defined by α and α^*, respectively, that is, for each $f \in L^1(\mathbb{R}_+)^p$ we have

$$(T_\alpha f)(t) = f(t) + \int_0^t a(t-s)f(s)\,ds, \quad 0 \leqslant t < \infty, \tag{1.8}$$

$$(T_{\alpha^*} f)(t) = f(t) + \int_t^{\infty} a^*(t-s)f(s)\,ds, \quad 0 \leqslant t < \infty. \tag{1.9}$$

2. Preliminaries

Let $\alpha \in e + \mathscr{W}_{+,0}^{p\times p}$, and assume that $\det \alpha(\lambda) \neq 0$ for each $\lambda \in \mathbb{R}$, or equivalently, assume that

$$w(\lambda) = \alpha^*(\lambda)\alpha(\lambda) > 0, \quad \lambda \in \mathbb{R}. \tag{2.1}$$

Note that the positivity condition (2.1) is automatically fulfilled when there exists $\gamma \in \mathscr{W}_{-,0}^{q\times p}$ such that condition (C1) is satisfied. Indeed, in that case

$$w(\lambda) = \alpha^*(\lambda)\alpha(\lambda) = I_p + \gamma^*(\lambda)\gamma(\lambda) \geqslant I_p, \quad \lambda \in \mathbb{R}. \tag{2.2}$$

Let w be given by (2.1). Note that $w \in \mathscr{W}^{p\times p}$, and the value of w at infinity is equal to the $p \times p$ identity matrix I_p. Since $w(\lambda)$ is positive definite for each real λ

and at infinity, w admits a spectral factorization (see, e.g., [9, Corollary XXX.10.3]), that is, $w = w_{\rm sp}^* w_{\rm sp}$, where

$$w_{\rm sp} \in \mathscr{W}_+^{p\times p} := \mathscr{W}_{+,0}^{p\times p} \dot{+} \mathscr{W}_d^{p\times p} \quad \text{and} \quad \det w_{\rm sp}(\lambda) \neq 0 \quad (\lambda \in \overline{\mathbb{C}_+}). \tag{2.3}$$

Without loss of generality we may assume that $w_{\rm sp}(\infty) = I_p$. In that case $w_{\rm sp}$ is uniquely determined by α. If the normalization condition $w_{\rm sp}(\infty) = I_p$ is satisfied, we call $w_{\rm sp}$ the *right spectral factor* of w.

In the sequel, with some abuse of terminology, a matrix function Θ is called *bi-inner in* $\mathscr{W}_+^{p\times p}$ if $\Theta \in \mathscr{W}_+^{p\times p}$, the matrix $\Theta(\lambda)$ is unitary for each $\lambda \in \mathbb{R}$, and $\Theta(\infty) = I_p$. In particular,

$$\Theta(\lambda) = I_p + \int_0^\infty e^{i\lambda t}\theta(t)\,\mathrm{d}t\,, \text{ where } \theta \in L^1(\mathbb{R}_+)^{p\times p} \text{ and } \lambda \in \mathbb{R}. \tag{2.4}$$

The next lemma shows that such a function always is rational.

LEMMA 2.1. *If a matrix function is bi-inner in $\mathscr{W}_+^{p\times p}$, then the function is rational.*

Proof. Assume Θ is a matrix function which is bi-inner in $\mathscr{W}_{+,0}^{p\times p}$. Let T_Θ be the Wiener-Hopf operator on $L^2(\mathbb{R}_+)^p$ defined by Θ, that is

$$(T_\Theta f)(t) = f(t) + \int_0^t \theta(t-s)f(s)\,\mathrm{d}s\,, \quad 0 \leqslant t < \infty,$$

where θ is given by (2.4). Since $\Theta(\lambda)$ is unitary for each $\lambda \in \mathbb{R}$ and $\Theta(\infty) = I_p$, it follows (see, e.g., [9, Theorem XXX10.2]) that T_Θ is a Fredholm operator. In particular, $\operatorname{Im} T_\Theta$, the range of T_Θ, is closed and $\operatorname{codim} \operatorname{Im} T_\Theta$ is finite. Furthermore,

$$T_\Theta^* T_\Theta = T_{\Theta^*} T_\Theta = T_{\Theta^*\Theta} = I_{L^2(\mathbb{R}_+)^p}.$$

Thus T_Θ is an isometry with $\operatorname{codim} \operatorname{Im} T_\Theta$ finite. But then it follows that the operator $I_{L^2(\mathbb{R}_+)^p} - T_\Theta T_\Theta^*$ is a finite rank orthogonal projection. On the other hand (see [8, Section XII.2]) we have $I_{L^2(\mathbb{R}_+)^p} - T_\Theta T_\Theta^* = H_\Theta H_\Theta^*$, where H_Θ is the Hankel operator on $L^2(\mathbb{R}_+)^p$ defined by Θ, that is,

$$(H_\Theta f)(t) = \int_0^\infty \theta(t+s)f(s)\,\mathrm{d}s\,, \quad 0 \leqslant t < \infty \quad (f \in L^2(\mathbb{R}_+)^p). \tag{2.5}$$

Here, as before, Θ is defined by (2.4). It follows that $\operatorname{rank} H_\Theta < \infty$. But then we can apply [19, Lemma 8.12] to conclude that Θ is a rational function. $\square$

PROPOSITION 2.2. *Let $\alpha \in e + \mathscr{W}_{+,0}^{p\times p}$, and assume that condition (2.1) is satisfied. Then $\alpha = \Theta w_{\rm sp}$, where $w_{\rm sp}$ is the right spectral factor of w in (2.1) and the function Θ is bi-inner in $\mathscr{W}_+^{p\times p}$ and is uniquely determined by α. In particular, Θ is rational.*

Proof. Since $w_{\rm sp}$ is the right spectral factor of w, we have

$$\alpha^*(\lambda)\alpha(\lambda) = w(\lambda) = w_{\rm sp}^*(\lambda)w_{\rm sp}(\lambda), \quad \lambda \in \mathbb{R}.$$

It follows that $w_{\rm sp}^{-*}(\lambda)\alpha^*(\lambda)\alpha(\lambda)w_{\rm sp}^{-1}(\lambda) = I_p$ for $\lambda \in \mathbb{R}$. Now let Θ be the $p\times p$ matrix function on $\mathbb{R}$ defined by $\Theta(\lambda) = \alpha(\lambda)w_{\rm sp}^{-1}(\lambda)$. Then $\Theta \in \mathscr{W}_+^{p\times p}$ and $\alpha = \Theta w_{\rm sp}$. Furthermore, $\Theta^*(\lambda)\Theta(\lambda) = I_p$ on $\mathbb{R}$ and $\Theta(\infty) = I_p$. Hence Θ is bi-inner in $\mathscr{W}_+^{p\times p}$. The uniqueness of Θ follows from the identity $\Theta = \alpha w_{\rm sp}^{-1}$ and the fact that $w_{\rm sp}$ is uniquely determined by α because of the normalization condition $w_{\rm sp}(\infty) = I_p$. Finally, by Lemma 2.1, since Θ is bi-inner in $\mathscr{W}_+^{p\times p}$, the function Θ is rational. $\square$

Let $\gamma \in \mathscr{W}_{-,0}^{q\times p}$. Then there are many functions $\alpha \in e + \mathscr{W}_{+,0}^{p\times p}$ such that condition (C1) is satisfied. In fact, Proposition 2.2 provides a recipe to construct all these α which yields the following corollary.

COROLLARY 2.3. *Given $\gamma \in \mathscr{W}_{-,0}^{q\times p}$, put $w = e + \gamma^*\gamma$, and let $w_{\rm sp}$ be the right spectral factor of w. Then $\alpha \in e + \mathscr{W}_{+,0}^{p\times p}$ satisfies condition* (C1) *with the given γ if and only if $\alpha = \Theta w_{\rm sp}$, where Θ is any function bi-inner in $\mathscr{W}_+^{p\times p}$. In particular, if γ is rational, then any $\alpha \in e + \mathscr{W}_{+,0}^{p\times p}$ satisfying condition* (C1) *with the given γ is rational too.*

Proof. Assume that α satisfies condition (C1). Then by identity (2.2) the function $w_{\rm sp}$ is the spectral factor of w as in (2.1). Hence Proposition 2.2 tells us that there exists a bi-inner function Θ such that $\alpha = \Theta w_{\rm sp}$. The proof of the converse is straight forward.

Furthermore, if γ is rational, then w is rational and hence the spectral factor is rational too (see for instance Proposition 6.1 below). Therefore $\alpha = \Theta w_{\rm sp}$ is also rational. $\square$

Let $\alpha \in e + \mathscr{W}_{+,0}^{p\times p}$, and assume that condition (2.1) is satisfied. As in Proposition 2.2 write α as $\alpha = \Theta w_{\rm sp}$, where $w_{\rm sp}$ is the right spectral factor of w in (2.1). We shall refer to Θ as the *bi-inner function in $\mathscr{W}_+^{p\times p}$ associated to α*. Since Θ is rational and $\Theta(\infty) = I_p$, we may assume that Θ is given by a minimal realization

$$\Theta(\lambda) = I_p + C(\lambda I_n - A)^{-1}B. \tag{2.6}$$

The minimality of the realization and the fact that $\Theta \in \mathscr{W}_+^{p\times p}$ imply that the $n\times n$ matrix A has all its eigenvalues in the open lower half plane $\mathbb{C}_-$.

PROPOSITION 2.4. *Let $\alpha \in e + \mathscr{W}_{+,0}^{p\times p}$, and assume that condition* (2.1) *is satisfied. Then $\psi \in \mathscr{W}_{+,0}^{p\times q}$ and $\alpha^*\psi \in \mathscr{W}_{-,0}^{p\times q}$ if and only if*

$$\psi(\lambda) = C(\lambda I_n - A)^{-1}X \quad \textit{for some unique } X \in \mathbb{C}^{n\times q}. \tag{2.7}$$

Here A and C are the matrices A and C appearing in the minimal realization (2.6) of the rational bi-inner function Θ. Furthermore, X is given by

$$X = \frac{1}{2\pi i}\int_\Gamma (\mu I_n - A)^{-1}B(\Theta^*\psi)(\mu)\,\mathrm{d}\mu\,, \tag{2.8}$$

where Γ is a Cauchy contour in $\mathbb{C}_-$ around the eigenvalues of A.

For the definition of a Cauchy contour see [8, page 6]. This definition includes that a Cauchy contour is positively oriented.

Proof. First notice that $\alpha^*\psi = w_{\rm sp}^*\Theta^*\psi$. Since $w_{\rm sp}^*$ and $w_{\rm sp}^{-*}$ both belong to $\mathscr{W}_-^{p\times p}$ we see that $\alpha^*\psi \in \mathscr{W}_{-,0}^{p\times q}$ if and only if $\Theta^*\psi \in \mathscr{W}_{-,0}^{p\times q}$. Hence it suffices to prove the proposition for Θ in place of α.

Assume $\psi \in \mathscr{W}_{+,0}^{p\times q}$ and $\Theta^*\psi \in \mathscr{W}_{-,0}^{p\times q}$. Put $\rho = \Theta^*\psi$. Then $\rho \in \mathscr{W}_{-,0}^{p\times q}$ and $\Theta\rho = \psi \in \mathscr{W}_{+,0}^{p\times q}$. Using the realization (2.6) we see that

$$\psi(\lambda) = \Theta(\lambda)\rho(\lambda) = \rho(\lambda) + C(\lambda I_n - A)^{-1}B\rho(\lambda), \quad \lambda \in \mathbb{R}. \tag{2.9}$$

Put $\varphi(\lambda) = (\lambda I_n - A)^{-1}B\rho(\lambda)$. Since $\rho \in \mathscr{W}_{-,0}^{p\times q}$ and $\psi \in \mathscr{W}_{+,0}^{p\times q}$, the identity (2.9) tells us that

$$\psi = (C\varphi)_{+,0} = C\varphi_{+,0}.$$

Next, applying Lemma A.1 with this choice of ρ and φ, we obtain (2.7) with X being given by (2.8).

The uniqueness statement in (2.7) follows from the fact that the pair $\{C, A\}$ is observable which follows from the fact that the realization (2.6) is minimal.

To prove the converse statement, assume that ψ is given by (2.7). Recall that the eigenvalues of A are in $\mathbb{C}_-$. Thus indeed $\psi \in \mathscr{W}_{+,0}^{p\times q}$. Using $\Theta^* = \Theta^{-1}$ and the realization (2.6), we see that

$$\begin{aligned}\Theta^*(\lambda) = \Theta(\lambda)^{-1} = I_p - C(\lambda I_n - A^\times)^{-1}B,&\\ \text{where } A^\times = A - BC \text{ and } \lambda \in \mathbb{R}.&\end{aligned} \tag{2.10}$$

Since the realization (2.6) is minimal, the same holds true for the realization in (2.10). But then, using $\Theta^* \in \mathscr{W}_{-,0}^{p\times p}$, we may conclude that the eigenvalues of $A^\times$ are in $\mathbb{C}_+$. Next, using a classical product formula (see, e.g., Theorem 2.4 in [1]), we obtain

$$\begin{aligned}(\Theta^*\psi)(\lambda) &= \big(I_p - C(\lambda I_n - A^\times)^{-1}B\big)\,C(\lambda I_n - A)^{-1}X\\ &= C(\lambda I_n - A^\times)^{-1}X,\end{aligned}$$

and the fact that the eigenvalues of $A^\times$ are in $\mathbb{C}_+$ implies $\Theta^*\psi \in \mathscr{W}_{-,0}^{p\times p}$ as desired. □

REMARK 2.5. Let T_{α^*} be the Wiener-Hopf operator on $L^1(\mathbb{R}_+)^p$ defined by α^* (see (1.9)). The first part of Proposition 2.4 is equivalent to the statement that

$$\operatorname{Ker} T_{\alpha^*} = \{f \mid f(t) = Ce^{-itA}x, \quad x \in \mathbb{C}^n \quad (t \geqslant 0)\}.$$

To see this, notice that $T_{\alpha^*} f = 0$ if and only if $(\alpha^* \mathscr{F}'(f))_+ = 0$. With $\psi = \mathscr{F}'(f)$ this means that $\psi \in \mathscr{W}_{+,0}^{p\times 1}$ and $\alpha^* \psi \in \mathscr{W}_{-,0}^{p\times 1}$. So according to Proposition 2.4 this is equivalent to $\psi(\lambda) = C(\lambda I_n - A)^{-1}x$ for some $x \in \mathbb{C}^n$. But $\psi = \mathscr{F}'(f)$ has this form if and only if $f(t) = Ce^{-itA}x$ for some $x \in \mathbb{C}^n$.

PROPOSITION 2.6. *Let $\alpha \in e + \mathscr{W}_{+,0}^{p\times p}$, and assume that condition* (2.1) *is satisfied. Let A, B and C be the matrices appearing in the minimal realization* (2.6) *of the rational bi-inner function Θ. Then*

$$\left(\alpha^{-*}\right)_{+,0}(\lambda) = C(\lambda I_n - A)^{-1}\tilde{B}, \tag{2.11}$$

with $\tilde{B} \in \mathbb{C}^{n\times p}$ given by

$$\tilde{B} = \frac{1}{2\pi i}\int_\Gamma (\mu I_n - A)^{-1} B w_{\rm sp}^{-*}(\mu)\, \mathrm{d}\mu\,, \tag{2.12}$$

where Γ is a Cauchy contour in $\mathbb{C}_-$ around the eigenvalues of A. Moreover, the realization (2.11) *is minimal.*

Proof. Let $\varphi(\lambda) = (\lambda I_n - A)^{-1} B w_{\rm sp}^{-*}(\lambda)$ for λ in the closed lower half plane. We apply Lemma A.1 with $k = m = p$ and $\rho = w_{\rm sp}^{-*}$. Since $w_{\rm sp}^{-*} \in \mathscr{W}_-^{p\times p}$, the same holds true for ρ. Thus $\varphi_{+,0}(\lambda) = (\lambda I_n - A)^{-1}\tilde{B}$ with $\tilde{B}$ given by (2.12). Furthermore,

$$\alpha^{-*}(\lambda) = \Theta(\lambda) w_{\rm sp}^{-*}(\lambda) = w_{\rm sp}^{-*}(\lambda) + C(\lambda I_n - A)^{-1} B w_{\rm sp}^{-*}(\lambda),$$

and hence $(\alpha^{-*})_{+,0}(\lambda) = C\varphi_{+,0}(\lambda)$, which proves (2.11).

It remains to show that the realization (2.11) is minimal. The pair (C, A) is observable. So we only have to prove that the pair $(A, \tilde{B})$ is controllable. Let $\lambda_\circ$ be an eigenvalue of A, and let x be such that $x^* A = \lambda_\circ x^*$ and $x^* \tilde{B} = 0$. According to the so-called Hautus test (see [20, Lemma 3.3.7], [11, Theorem 3.2.3]), in order to prove that $(A, \tilde{B})$ is controllable, it is sufficient to prove that $x = 0$. Using formula (2.11) we obtain

$$\begin{aligned} 0 = x^*\tilde{B} &= \frac{1}{2\pi i}\int_\Gamma x^*(\mu I_n - A)^{-1} B w_{\rm sp}^{-*}(\mu)\, \mathrm{d}\mu = \frac{1}{2\pi i}\int_\Gamma x^*(\mu - \lambda_\circ)^{-1} B w_{\rm sp}^{-*}(\mu)\, \mathrm{d}\mu \\ &= x^* B \left(\int_\Gamma (\mu - \lambda_\circ)^{-1} w_{\rm sp}^{-*}(\mu)\, \mathrm{d}\mu \right) = x^* B w_{\rm sp}^{-*}(\lambda_\circ) \end{aligned}$$

But $w_{\rm sp}^{-*}(\lambda_\circ)$ is an invertible matrix. We conclude that $x^* B = 0$. Since $x^* A = \lambda_\circ x^*$ and the pair (A, B) is controllable, it follows (using [20, Lemma 3.3.7] again) that $x = 0$. □

Minimal realizations of $(\alpha^{-1})_{-,0}$. Let $\alpha \in e + \mathscr{W}_{+,0}^{p\times p}$, and assume that condition (2.1) is satisfied. From Proposition 2.6 we know that the function $(\alpha^{-1})_{-,0}$ is a strictly proper rational matrix function. In what follows we shall assume that $(\alpha^{-1})_{-,0}$ is given by the minimal realization:

$$(\alpha^{-1})_{-,0}(\lambda) = C_+(\lambda I_{n_+} - A_+)^{-1} B_+. \tag{2.13}$$

In that case we can apply the dual version of [10, Theorem A.5.3], in particular, the second part of this theorem, to show that the pair of matrices (A_+, B_+) is a left null pair of α with respect to $\mathbb{C}_+$. By the dual version of [10, Theorem A.5.1] the latter property is equivalent to (A_+, B_+) satisfying the following three conditions:

(a) A_+ is a square matrix which has all its eigenvalues in $\mathbb{C}_+$ and the order n_+ of A_+ is equal to the sum of the multiplicities of the zeros of $\det\alpha(\lambda)$ in $\mathbb{C}_+$;

(b) B_+ is a matrix of size $n_+ \times p$ and the pair (A_+, B_+) is controllable;

(c) $(\lambda I_{n_+} - A_+)^{-1} B_+ \alpha(\lambda)$ is analytic in $\mathbb{C}_+$.

Taking adjoints in (2.11) we see that a minimal realization of $(\alpha^{-1})_{-,0}$ is also given by

$$(\alpha^{-1})_{-,0}(\lambda) = \tilde{B}^*(\lambda I_n - A^*)^{-1} C^*. \tag{2.14}$$

Here A and C are the matrices appearing in the minimal realization of Θ in (2.6), and $\tilde{B}$ is defined by (2.12). Since the right hand sides of (2.14) and (2.13) are minimal realizations of the same function, these realizations are similar. It follows that $n = n_+$ and there exists an invertible $n \times n$ matrix S such that

$$A_+ = S^{-1} A^* S, \quad C_+ = \tilde{B}^* S, \quad B_+ = S^{-1} C^*. \tag{2.15}$$

These remarks will be useful in the next section.

3. First main results

In this section $\alpha \in e + \mathscr{W}_{+,0}^{p\times p}$ and $\gamma \in \mathscr{W}_{-,0}^{q\times p}$, and we assume that condition (C1) is satisfied. From the results of the previous section we know that $\det\alpha(\lambda)$ is nonzero for $\lambda \in \mathbb{R}$ and at infinity, and the function $(\alpha^{-1})_{-,0}$ is rational. Furthermore, we assume that $(\alpha^{-1})_{-,0}$ is given by the minimal realization

$$(\alpha^{-1})_{-,0}(\lambda) = C_+(\lambda I_{n_+} - A_+)^{-1} B_+. \tag{3.1}$$

Note that n_+ can be zero. In that case $(\alpha^{-1})_{-,0}$ is identically zero, and the latter happens if and only if $\det\alpha$ has no zeros in $\mathbb{C}_+$. In fact, see item (a) in the one but last paragraph of the previous section, the integer n_+ is equal to the number of zeros (multiplicities taken into account) of $\det\alpha$ in $\mathbb{C}_+$. We shall prove the following theorems.

THEOREM 3.1. *Let $\alpha \in e + \mathscr{W}_{+,0}^{p\times p}$ and $\gamma \in \mathscr{W}_{-,0}^{q\times p}$, and assume that condition* (C1) *is satisfied. If* $\det\alpha$ *has no zeros in $\mathbb{C}_+$, then the EG inverse problem associated with α and γ is uniquely solvable, and the unique solution $\tilde{g}$ is given by $\tilde{g} = -(\alpha^{-*}\gamma^*)_+$.*

THEOREM 3.2. *Let $\alpha \in e + \mathscr{W}_{+,0}^{p\times p}$ and $\gamma \in \mathscr{W}_{-,0}^{q\times p}$, and assume that condition* (C1) *is satisfied. Assume that* $\det\alpha$ *has zeros in $\mathbb{C}_+$, and let $\left(\alpha^{-1}\right)_{-,0}$ be given by the* *minimal realization* (3.1)*. Then the EG inverse problem associated with* α *and* γ *has a solution if and only if there exists a* $q\times n_+$ *matrix* Y *such that*

$$\frac{1}{2\pi i}\int_\Gamma \gamma^*(\lambda)Y(\lambda I_{n_+}-A_+)^{-1}\,\mathrm{d}\lambda = -C_+. \tag{3.2}$$

Here Γ *is a Cauchy contour in* $\mathbb{C}_+$ *such that all the eigenvalues of* A_+ *are in the inner domain of* Γ*. Furthermore, in that case all solutions to the EG inverse problem associated with* α *and* γ *are given by*

$$g(\lambda) = -(\alpha^{-*}\gamma^*)_+(\lambda) + B_+^*(\lambda I_{n_+}-A_+^*)^{-1}Y^*, \quad \Im\lambda \geqslant 0, \tag{3.3}$$

where Y *is an arbitrary* $q\times n_+$ *matrix satisfying* (3.2).

In the sequel we denote by J the linear map from $\mathbb{C}^{q\times n_+}$ into $\mathbb{C}^{p\times n_+}$ defined by

$$J(X) = \frac{1}{2\pi i}\int_\Gamma \gamma^*(\lambda)X(\lambda I_{n_+}-A_+)^{-1}\,\mathrm{d}\lambda\,, \quad X\in\mathbb{C}^{q\times n_+}, \tag{3.4}$$

where Γ is a Cauchy contour in $\mathbb{C}_+$ around the eigenvalues of A_+.

THEOREM 3.3. *Let* $\alpha\in e+\mathscr{W}_{+,0}^{p\times p}$ *and* $\gamma\in\mathscr{W}_{-,0}^{q\times p}$*, and assume that condition* (C1) *is satisfied. Assume that* $\det\alpha$ *has zeros in* $\mathbb{C}_+$*, and let* $\left(\alpha^{-1}\right)_{-,0}$ *be given by the minimal realization* (3.1)*. If, in addition,* $p=q$*, then the following conditions are equivalent:*

(i) *the EG inverse problem associated with* α *and* γ *is uniquely solvable;*

(ii) *the transformation* J *defined by* (3.4) *is invertible;*

(iii) $\det\alpha$ *and* $\det\gamma^*$ *have no common zero in* $\mathbb{C}_+$.

In order to prove the above theorems we first prove the following lemma.

LEMMA 3.4. *Let* $\alpha\in e+\mathscr{W}_{+,0}^{p\times p}$ *and* $\gamma\in\mathscr{W}_{-,0}^{q\times p}$*, and assume that condition* (C1) *is satisfied. Then all* $g\in\mathscr{W}_{+,0}^{p\times q}$ *satisfying the second inclusion in* (1.4) *are given by*

$$g = -(\alpha^{-*}\gamma^*)_+ + \psi, \text{ where } \psi\in\mathscr{W}_{+,0}^{p\times q} \text{ and } \alpha^*\psi\in\mathscr{W}_{-,0}^{p\times q}. \tag{3.5}$$

Furthermore, in that case

$$(\alpha+g\gamma-e)_+ = (\alpha^{-*})_{+,0} + (\psi\gamma)_{+,0}, \tag{3.6}$$

$$\big(\alpha^*(\alpha+g\gamma-e)_+\big)_+ = 0, \tag{3.7}$$

and g *satifies the first condition in* (1.4) *if and only if*

$$(\psi\gamma)_{+,0} = -(\alpha^{-*})_{+,0}. \tag{3.8}$$

Proof. We split the proof into three parts.

PART 1. In this part we show that all $g \in \mathscr{W}_{+,0}^{p\times q}$ satisfying the second inclusion in (1.4) are given by (3.5). Put $\tilde{g} = -(\alpha^{-*}\gamma^*)_+$. Note that $\tilde{g} \in \mathscr{W}_{+,0}^{p\times q}$. Furthermore, $\tilde{g}^* = -\left(\gamma\alpha^{-1}\right)_- = -\gamma\alpha^{-1} + \left(\gamma\alpha^{-1}\right)_{+,0}$. But then

$$\tilde{g}^*\alpha + \gamma = \left(\gamma\alpha^{-1}\right)_{+,0}\alpha \in \mathscr{W}_{+,0}^{q\times p}\mathscr{W}_{+}^{p\times p} = \mathscr{W}_{+,0}^{q\times p}.$$

Thus the second inclusion in (1.4) holds with $\tilde{g}$ in place of g.

Next, assume that $g \in \mathscr{W}_{+,0}^{p\times q}$ satisfies the second inclusion in (1.4). We claim that (3.5) holds with $\psi = g - \tilde{g}$. Clearly $\psi \in \mathscr{W}_{+,0}^{p\times q}$. Furthermore,

$$\alpha^*\psi = \alpha^*(g-\tilde{g}) = (\alpha^* g + \gamma^*) - (\alpha^*\tilde{g} + \gamma^*) \in \mathscr{W}_{-.0}^{p\times q}.$$

Hence $\alpha^*\psi \in \mathscr{W}_{-,0}^{p\times q}$, and (3.5) is proved.

Conversely, let g be given by (3.5). Thus $g = \tilde{g} + \psi$, with $\psi \in \mathscr{W}_{+,0}^{q\times p}$ and $\alpha^*\psi \in \mathscr{W}_{-,0}^{p\times q}$. Since both $\tilde{g}$ and ψ belong to $\mathscr{W}_{+,0}^{q\times p}$, the same holds true for g. From the first part of the proof we know that $\tilde{g}^*\alpha + \gamma \in \mathscr{W}_{+,0}^{q\times p}$. Thus

$$\begin{aligned} g^*\alpha + \gamma &= g^*\alpha - \tilde{g}^*\alpha + (\tilde{g}^* a + \gamma) = (\alpha^*\psi)^* + (\tilde{g}^* a + \gamma) \\ &\in \mathscr{W}_{+,0}^{q\times p} + \mathscr{W}_{+,0}^{q\times p} = \mathscr{W}_{+,0}^{q\times p}. \end{aligned}$$

Hence g satisfies the second inclusion in (1.4).

PART 2. In this part $g \in \mathscr{W}_{+,0}^{q\times p}$, and we assume that g satisfies the second inclusion in (1.4). Thus g is given by (3.5). As in the first part, $\tilde{g} = -(\alpha^{-*}\gamma^*)_+$. Note that

$$\begin{aligned} \alpha + \tilde{g}\gamma - e &= \alpha - (\alpha^{-*}\gamma^*)_+\gamma - e \\ &= \alpha - (\alpha^{-*}\gamma^*)\gamma + (\alpha^{-*}\gamma^*)_{-,0}\gamma - e \\ &= \alpha - \alpha^{-*}(\gamma^*\gamma) + (\alpha^{-*}\gamma^*)_{-,0}\gamma - e \\ &= \alpha - \alpha^{-*}(\alpha^*\alpha - e) + (\alpha^{-*}\gamma^*)_{-,0}\gamma - e \\ &= \alpha^{-*} - e + (\alpha^{-*}\gamma^*)_{-,0}\gamma, \end{aligned}$$

and that $(\alpha^{-*}\gamma^*)_{-,0}\gamma \in \mathscr{W}_{-,0}^{p\times p}$. Hence

$$(\alpha + \tilde{g}\gamma - e)_+ = (\alpha^{-*} - e)_+ = (\alpha^{-*})_{+,0}.$$

Furthermore, using $\alpha + g\gamma - e = (\alpha + \tilde{g}\gamma - e) + (g - \tilde{g})\gamma$ and $\psi = g - \tilde{g}$ we see that

$$(\alpha + g\gamma - e)_+ = (\alpha^{-*})_{+,0} + (\psi\gamma)_+ = (\alpha^{-*})_{+,0} + (\psi\gamma)_{+,0}.$$

Recall that g satisfies the first inclusion in (1.4) if and only if $(\alpha + g\gamma - e)_+ = 0$. According the preceding formula the latter happens if and only if (3.8) holds.

PART 3. It remains to prove (3.7). Put $\varphi = (\alpha + g\gamma - e)_+$. We use equality (3.6) in

$$\begin{aligned}\alpha^*\varphi &= \alpha^*\left(\alpha^{-*} - e - (\alpha^{-*})_{-,0}\right) + \alpha^*\left(\psi\gamma - (\psi\gamma)_{-,0}\right)\\ &= (e - \alpha^*) - \alpha^*(\alpha^{-*})_{-,0} + (\alpha^*\psi)\gamma - \alpha^*(\psi\gamma)_{-,0}.\end{aligned}$$

Now notice that all four terms in the right hand side of the above expression are in $\mathscr{W}_{-,0}^{p\times p}$. We conclude that $(\alpha^*\varphi)_+ = 0$. $\square$

Proof of Theorem 3.1. In PART 1 of the proof of Lemma 3.4 we proved that $\tilde{g}$ satisfies the second conclusion in (1.4). In PART 2 we showed that

$$\alpha + \tilde{g}\gamma - e = \alpha^{-*} - e + \left(\alpha^{-*}\gamma^*\right)_{-,0}\gamma.$$

Since $\det\alpha$ has no zero in $\mathbb{C}_+$, we know that $(\alpha^{-*} - e)_+ = 0$, and since $(\alpha^{-*}\gamma^*)_{-,0}\gamma \in \mathscr{W}_{-,0}^{p\times p}$, we conclude that $\alpha + \tilde{g}\gamma - e \in \mathscr{W}_{-,0}^{p\times p}$, i.e., also the first inclusion in (1.4) holds. It remains to show that $\tilde{g}$ is the only solution. All solutions are of the form $\tilde{g} + \psi$ with $\psi \in \mathscr{W}_{+,0}^{p\times q}$ such that $\alpha^*\psi \in \mathscr{W}_{-}^{p\times q}$. Since $\alpha^{-*} \in \mathscr{W}_{-}^{p\times p}$, we see that $\psi = \alpha^{-*}(\alpha^*\psi) \in \mathscr{W}_{-}^{p\times q}$, and therefore $\psi = 0$. $\square$

Proof of Theorem 3.2. We split the proof into two parts. Throughout condition (C1) is satisfied.

PART 1. In this first part g is a solution to the EG inverse problem associated with α and γ. Since condition (C1) is satisfied, we know from Lemma 3.4 that $g = -(\alpha^{-*}\gamma^*)_+ + \psi$ for some $\psi \in \mathscr{W}_{+,0}^{p\times q}$ such that $\alpha^*\psi \in \mathscr{W}_{-,0}^{p\times q}$. But then we can apply Proposition 2.4 to show that $\psi(\lambda) = C(\lambda I_n - A)^{-1}X$ for some $X \in \mathbb{C}^{n\times q}$. Here A and C are the matrices appearing in the minimal realization (2.6). Hence, using Proposition 2.6, the matrices A and C also appear in the minimal realization (2.11) of $(\alpha^{-*})_{+,0}$. But then we know from the final paragraph of the previous section that the pair of matrices (C, A) is similar to the pair (B_+^*, A_+^*). More precisely, (2.15) tells us that with S as in (2.15), we have

$$A = S^{-*}A_+^*S^*, \quad C = B_+^*S^*. \tag{3.9}$$

It follows that

$$\psi(\lambda) = C(\lambda I_n - A)^{-1}X = B_+^*(\lambda I_{n_+} - A_+^*)^{-1}Y^*, \tag{3.10}$$

where $Y = X^*S$. Thus (3.3) is satisfied with $Y = X^*S$.

So far we only used that g satisfies the second inclusion in (1.4). But, by assumption, g also satisfies the first inclusion in (1.4), and thus, using the final part of Lemma 3.4, we see that $(\psi\gamma)_{+,0} = -(\alpha^{-*})_{+,0}$. Taking adjoints and using (2.14) we obtain

$$(\gamma^*\psi^*)_{-,0}(\lambda) = -C_+(\lambda I_{n_+} - A_+)^{-1}B_+, \quad \Im\lambda \leqslant 0. \tag{3.11}$$

On the other hand, by taking adjoints in (3.10), we know that ψ^* is given by $\psi^*(\lambda) = Y(\lambda I_{n_+} - A_+)^{-1}B_+$. We proceed by applying Lemma A.2 with $\rho = \gamma^*$ and $\varphi(\lambda) = \gamma^*(\lambda)Y(\lambda I_{n_+} - A_+)^{-1}$. This yields

$$(\gamma^*\psi^*)_{-,0}(\lambda) = (\varphi_{-,0})(\lambda)B_+ = X(\lambda I_{n_+} - A_+)^{-1}B_+, \quad \Im\lambda \leqslant 0,$$

where

$$X = \frac{1}{2\pi i}\int_\Gamma \gamma^*(\lambda)Y(\lambda I_{n_+} - A_+)^{-1}\,\mathrm{d}\lambda\,. \tag{3.12}$$

Comparing (3.11) and (3.12) we see that

$$X(\lambda I_{n_+} - A_+)^{-1}B_+ = -C_+(\lambda I_{n_+} - A_+)^{-1}B_+, \quad \Im\lambda \leqslant 0.$$

The fact that the pair (A_+, B_+) is controllable implies that $X = -C_+$, and hence there exists a $q \times n_+$ matrix Y such that (3.2) holds.

PART 2. It remains to prove the reverse implication. Let Y be a $q \times n_+$ matrix such that (3.2) holds. Define g by (3.3), where Y is given by (3.2). Put

$$\psi(\lambda) = B_+^*(\lambda I_n - A_+^*)^{-1}Y^*, \quad \Im\lambda \geqslant 0. \tag{3.13}$$

Then $\psi \in \mathscr{W}_{+,0}^{p\times q}$. Using the similarity in (3.9) and Proposition 2.4 we see that $\alpha^*\psi \in \mathscr{W}_{-,0}^{p\times q}$. Thus by the first part of Lemma 3.4 the function g satisfies the second inclusion in (1.4).

According to the final part of Lemma 3.4, in order to show that g satisfies the first inclusion (1.4) it suffices to show that (3.8) holds. But this follows by applying Lemma A.2 and using (3.2). Indeed, put $\rho = \gamma^*$, and let φ be defined by $\varphi(\lambda) = \gamma^*(\lambda)Y(\lambda I_{n_+} - A_+)^{-1}$, where Y is given by (3.2). Then, by Lemma A.2 and using (2.14), we have

$$\begin{aligned}(\gamma^*\psi^*)_{-,0}(\lambda) &= (\varphi_{-.0})(\lambda)B_+ = -C_+(\lambda I_{n_+} - A_+)^{-1}B_+ \\ &= -(\alpha^{-1})_{-,0}(\lambda).\end{aligned}$$

Taking adjoints yields (3.8). Hence g also satisfies the first inclusion (1.4), and therefore g is a solution of the EG inverse problem. □

We proceed with proving Theorem 3.3. Since $p = q$, the transformation J defined by (3.4) is invertible if and only if J is surjective or if and only if J is injective. Using this observation, the equivalence of items (i) and (ii) in Theorem 3.3 immediately follows from Theorem 3.2. The main difficulty is to prove the equivalence of items (ii) and (iii). To do this we first prove two auxiliary results.

In what follows $\eta \in \mathscr{W}_+^{p\times q}$ and A is an $n \times n$ matrix whose eigenvalues are all in $\mathbb{C}_+$. We shall deal with the linear map J_η defined by

$$\begin{aligned}&J_\eta : \mathbb{C}^{q\times n} \to \mathbb{C}^{p\times n}, \\ &J_\eta(X) = \frac{1}{2\pi i}\int_\Gamma \eta(\lambda)X(\lambda I_n - A)^{-1}\,d\lambda, \quad X \in \mathbb{C}^{q\times n}.\end{aligned} \tag{3.14}$$

Here Γ is a Cauchy contour in $\mathbb{C}_+$ around $\sigma(A)$. We are interested in the equation $J_\eta(X) = Y$ where $Y \in \mathbb{C}^{p\times n}$.

For $1 \leqslant i \leqslant p$ and $1 \leqslant j \leqslant q$ let $\eta_{ij}(\lambda)$ be the (ij)-th entry of the matrix function $\eta(\lambda)$, where $\Im\lambda \geqslant 0$. In other words

$$\eta(\lambda) = \begin{bmatrix} \eta_{11}(\lambda) & \eta_{12}(\lambda) & \cdots & \eta_{1q}(\lambda) \\ \eta_{21}(\lambda) & \eta_{22}(\lambda) & \cdots & \eta_{2q}(\lambda) \\ \vdots & \vdots & \vdots & \vdots \\ \eta_{p1}(\lambda) & \eta_{p2}(\lambda) & \cdots & \eta_{pq}(\lambda) \end{bmatrix}, \quad \Im\lambda \geqslant 0. \tag{3.15}$$

Let $X \in \mathbb{C}^{q\times n}$ and $Y \in \mathbb{C}^{p\times n}$, and let X_i be the i-th row of X and let Y_j the j-th row of Y, $1 \leqslant i \leqslant q$ and $1 \leqslant j \leqslant p$. Thus we partition X and Y as follows

$$X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_q \end{bmatrix}, \; Y = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_p \end{bmatrix}, \quad \text{where } X_i, Y_j : \mathbb{C}^n \to \mathbb{C}.$$

Note that the function η_{ij} is analytic on $\mathbb{C}_+$. Since $\sigma(A) \subset \mathbb{C}_+$, we can use the classical functional calculus (see, e.g., in [8, Section I.3]), to define the $n\times n$ matrix $\eta_{ij}(A)$ for $1 \leqslant i \leqslant p$ and $1 \leqslant j \leqslant q$.

LEMMA 3.5. *We claim that $J_\eta(X) = Y$ if and only if*

$$\begin{bmatrix} X_1 & X_2 & \cdots & X_q \end{bmatrix} \begin{bmatrix} \eta_{11}(A) & \eta_{21}(A) & \cdots & \eta_{p1}(A) \\ \eta_{12}(A) & \eta_{22}(A) & \cdots & \gamma_{p2}(A) \\ \vdots & \vdots & \ddots & \vdots \\ \eta_{1q}(A) & \eta_{2q}(A) & \cdots & \eta_{pq}(A) \end{bmatrix} = \begin{bmatrix} Y_1 & Y_2 & \cdots & Y_p \end{bmatrix}. \tag{3.16}$$

Proof. Let $J_\eta(X) = Y$. Then (3.14) and (3.15) tell us that for $\lambda \in \mathbb{C}_+$ we have

$$Y_j = \frac{1}{2\pi i}\int_\Gamma \left(\sum_{\nu=1}^{q} \eta_{j\nu}(\lambda) X_\nu \right) (\lambda I_n - A)^{-1}\, d\lambda, \quad j = 1, 2, \cdots, p.$$

But $\eta_{j\nu}(\lambda)$ is a scalar function for each $1 \leqslant j \leqslant p$, $1 \leqslant \nu \leqslant q$. Hence

$$\eta_{j\nu}(\lambda) X_\nu (\lambda I_n - A)^{-1} = X_\nu \eta_{j\nu}(\lambda)(\lambda I_n - A)^{-1}, \; 1 \leqslant j \leqslant p, \; 1 \leqslant \nu \leqslant q.$$

Using the functional calculus it follows that

$$Y_j = \left(\sum_{\nu=1}^{q} X_\nu \left(\frac{1}{2\pi i}\int_\Gamma \eta_{j\nu}(\lambda)(\lambda I_n - A)^{-1}\, d\lambda \right) \right) = \sum_{\nu=1}^{q} X_\nu \eta_{j\nu}(A), \quad j = 1, 2, \cdots, p.$$

This proves (3.16). The converse is proved in a similar way reversing the direction of the arguments. □

Next, assume that $p = q$, and put $\Delta(\lambda) = \det \eta(\lambda)$. Thus

$$\Delta(\lambda) = \sum_{\sigma} \varepsilon(\sigma)\eta_{1\,\sigma(1)}(\lambda)\eta_{2\,\sigma(2)}(\lambda)\cdots\eta_{p\,\sigma(p)}(\lambda), \quad \Im\lambda \geqslant 0. \tag{3.17}$$

Here the sum is taken over all permutations σ of $\{1,2,\ldots,p\}$ and $\varepsilon(\sigma)$ is the sign of the permutation σ. Note that the scalar function $\Delta(\lambda)$ is analytic in $\mathbb{C}_+$, and by the functional calculus

$$\Delta(A) = \sum_{\sigma} \varepsilon(\sigma)\eta_{1\,\sigma(1)}(A)\eta_{2\,\sigma(2)}(A)\cdots\eta_{p\,\sigma(p)}(A). \tag{3.18}$$

LEMMA 3.6. *Assume that $p = q$. Then the linear map J_η is one-to-one and surjective if and only if* $\det \eta$ *has no zero on $\sigma(A)$.*

Proof. Put

$$\Upsilon(A) = \begin{bmatrix} \eta_{11}(A) & \eta_{21}(A) & \cdots & \eta_{p1}(A) \\ \eta_{12}(A) & \eta_{22}(A) & \cdots & \eta_{p2}(A) \\ \vdots & \vdots & \ddots & \vdots \\ \eta_{1p}(A) & \eta_{2p}(A) & \cdots & \eta_{pp}(A) \end{bmatrix}. \tag{3.19}$$

Using Lemma 3.5 we see that J_η is one-to-one and surjective if and only if the matrix $\Upsilon(A)$ is non-singular. Notice that $\Upsilon(A)$ is a $p \times p$ matrix with entries from the commutative ring of matrices of the form $\eta(A)$ with $\eta \in \mathscr{W}_+^{p\times p}$. Hence $\Upsilon(A)$ is non-singular over this ring if and only if its determinant over this ring, $\det \Upsilon(A)$, given by (see Proposition 4 on page 331 of [18])

$$\det \Upsilon(A) = \sum_{\sigma} \varepsilon(\sigma)\eta_{1\,\sigma(1)}(A)\eta_{2\,\sigma(2)}(A)\cdots\eta_{p\,\sigma(p)}(A),$$

is invertible in the ring, i.e., is an invertible matrix. But (3.18) tells us that $\det \Upsilon(A) = \Delta(A)$ and we see that $\Upsilon(A)$ is invertible as a matrix with entries from $\mathbb{C}$ if and only if the matrix $\Delta(A)$ is non-singular.

It remains to show that $\Delta(A)$ is non-singular if and only if Δ has no zero on $\sigma(A)$. This follows from the spectral mapping theorem (see, e.g., [8, Theorem I.3.3]). Indeed, since $\Delta(\lambda) = \det \eta(\lambda)$ by definition, the spectral mapping theorem tells us that $\sigma\big(\Delta(A)\big)$ is equal to $\Delta\big(\sigma(A)\big)$. Thus $0 \in \sigma\big(\Delta(A)\big)$ if and only if Δ has a zero on $\sigma(A)$. This proves the lemma. □

Proof of Theorem 3.3. We already mentioned (see the paragraph after the proof of Theorem 3.2) that items (i) and (ii) directly follow from Theorem 3.2. To prove the equivalence of (ii) and (iii) we apply Lemma 3.6 with $\eta = \gamma^*$ and $A = A_+$. Note that $\gamma^* \in \mathscr{W}_+^{p\times p}$ (recall that we assume that $p = q$) and the eigenvalues of A_+ are in $\mathbb{C}_+$. In this case $J = J_\eta$ and Lemma 3.6 shows that J is invertible if and only if $\det\gamma^*$ has no zero on $\sigma(A_+)$. Recall that A_+ is the state matrix appearing in the minimal realization (2.14). According to item (a) in the one but last paragraph of Section 2 this implies that the eigenvalues of A_+ are the zeros of $\det\alpha$ in $\mathbb{C}_+$. We proved the equivalence of (ii) and (iii). $\square$

Next we present an example that will play a role in later sections too (see e.g., Examples 4.2 and 6.2).

EXAMPLE 3.7. Let $\lambda_0 \in \mathbb{C}_+$, and let M be a $q\times p$ matrix. Put

$$\gamma(\lambda) = \frac{1}{\lambda-\lambda_0}M. \tag{3.20}$$

Clearly $\gamma \in \mathscr{W}_{-,0}^{q\times p}$. Choose $\alpha \in e + \mathscr{W}_{+,0}^{p\times p}$ such that condition (C1) is satisfied with the given γ. As we know from Corollary 2.3 there are many such α and they are rational. For the chosen α and γ the map J reduces to

$$\begin{aligned} J(X) &= M^*X\left(\frac{1}{2\pi i}\int_\Gamma \left(\lambda-\overline{\lambda_0}\right)^{-1}(\lambda I_{n_+}-A_+)^{-1}\,\mathrm{d}\lambda\right)\\ &= M^*X(A_+-\overline{\lambda_0}I_{n_+})^{-1}. \end{aligned} \tag{3.21}$$

Thus in this case the EG inverse problem associated with α and γ is solvable if and only if the matrix equation $M^*X = C_+(\overline{\lambda_0}I_{n_+}-A_+)$ has a solution. Here A_+ and C_+ are determined by the minimal realization (3.1). Since the eigenvalues of A_+ are in $\mathbb{C}_+$, and $\overline{\lambda_0}\in\mathbb{C}_-$, the matrix $\overline{\lambda_0}I_{n_+}-A_+$ is invertible. Multiplying both sides of $M^*X = C_+(\overline{\lambda_0}I_{n_+}-A_+)$ from the right by $(\overline{\lambda_0}I_{n_+}-A_+)^{-1}$ and taking adjoints we conclude that the EG inverse problem associated with α and γ is solvable if and only if the equation $Y^*M = C_+^*$ has a solution Y or, equivalently, if and only if $\operatorname{Ker} M \subset \operatorname{Ker} C_+^*$. In that case all solutions to the EG inverse problem associated with α and γ are given by

$$g(\lambda) = -(\alpha^{-*}\gamma^*)_+(\lambda) + B_+^*(\lambda I_{n_+}-A_+^*)^{-1}X, \quad \Im\lambda \geqslant 0,$$

where X is an arbitrary $n_+\times q$ matrix such that $XM = \left(\lambda_0 I_{n_+}-A_+^*\right)C_+^*$.

Theorem 3.3 states that for the case when $p=q$ the EG inverse problem associated with α and γ has a unique solution if and only if $\det\alpha$ and $\det\gamma^*$ have no common zero. If $\det\alpha$ and $\det\gamma^*$ do have a common zero, then it can happen that the EG inverse problem has no solution at all or infinitely many solutions. In fact, in the scalar case, there is no solution if α and γ^* have a common zero (see Theorem 4.1). On the other hand, Example 3.8 below provides a 2×2 matrix case with infinitely many solutions.

EXAMPLE 3.8. Let $\alpha \in e+\mathscr{W}_{+,0}^{2\times 2}$ and $\gamma \in \mathscr{W}_{-,0}^{q\times p}$ be given by

$$\alpha(\lambda) = \begin{bmatrix} \frac{\lambda-i\sqrt{2}}{\lambda+i} & 0\\ 0 & 1\end{bmatrix}, \quad \gamma(\lambda) = \frac{1}{\lambda-i}\begin{bmatrix} 1 & 0\\ 0 & 0\end{bmatrix}.$$

For this choice of α and γ condition (C1) is satisfied, and $i\sqrt{2}$ is a common zero of $\det\alpha$ and $\det\gamma^*$. Note that γ is of the form (3.20) with $M = \mathrm{diag}\,[1,0]$. Furthermore, $(\alpha^{-1})_{+,0}$ is given by the minimal realization

$$(\alpha^{-1})_{+,0}(\lambda) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} (\lambda - i\sqrt{2})^{-1} \begin{bmatrix} i(1+\sqrt{2}) & 0 \end{bmatrix}.$$

Since $\mathrm{Ker}\,\mathrm{diag}\,[1,0] = \mathrm{Ker}\,\begin{bmatrix} 1 & 0 \end{bmatrix}$, the result of Example 3.7 above shows that the EG inverse problem has infinite many solutions which are given by

$$g(\lambda) = \frac{1}{\lambda + i\sqrt{2}} \begin{bmatrix} \frac{1+\sqrt{2}}{-1+\sqrt{2}} - \frac{\lambda - i}{\lambda + i}\, x \\ 0 & 0 \end{bmatrix}, \text{ where } x \in \mathbb{C} \text{ is a free parameter.}$$

The latter formula can also be checked by direct computation.

Lemma 3.5 with $\eta = \gamma^*$ allows us to rewrite equation (3.2) as

$$\begin{bmatrix} Y_1 & Y_2 & \cdots & Y_q \end{bmatrix} \begin{bmatrix} (\gamma_{11})^*(A_+) & \cdots & (\gamma_{1p})^*(A_+) \\ \vdots & & \vdots \\ (\gamma_{q1})^*(A_+) & \cdots & (\gamma_{qp})^*(A_+) \end{bmatrix} = -\begin{bmatrix} C_{+,1} & \cdots & C_{+,p} \end{bmatrix}. \tag{3.22}$$

Here γ_{ij} is the (i,j)-th entry of the $q \times p$ matrix γ. Identity (3.22) yields the following corollary to Theorem 3.2.

COROLLARY 3.9. *With α, γ, A_+ and C_+ as in* Theorem 3.2, *the EG inverse problem associated with α and γ has a solution if and only if equation* (3.22) *has a solution.*

Let us return to Example 3.7 and specify equation (3.22) for this case. This yields:

$$\begin{bmatrix} Y_1 & Y_2 & \cdots & Y_q \end{bmatrix} \begin{bmatrix} \overline{m_{11}}(A_+ - \overline{\lambda_0} I_{n_+})^{-1} & \cdots & \overline{m_{1p}}(A_+ - \overline{\lambda_0} I_{n_+})^{-1} \\ \vdots & & \vdots \\ \overline{m_{q1}}(A_+ - \overline{\lambda_0} I_{n_+})^{-1} & \cdots & \overline{m_{qp}}(A_+ - \overline{\lambda_0} I_{n_+})^{-1} \end{bmatrix} = -\begin{bmatrix} C_{+,1} & \cdots & C_{+,p} \end{bmatrix}. \tag{3.23}$$

Here $\overline{m_{ij}}$ denotes the complex conjugate of the (i,j)-th entry m_{ij} of the matrix M. Using elementary matrix multiplication rules one sees that the matrix equation (3.23) is equivalent to the equation $M^*Y(A_+ - \overline{\lambda_0} I_{n_+})^{-1} = -C_+$ appearing in Example 3.7. In other words, the latter equation can be viewed as a special case of equation (3.22).

REMARK 3.10. The discrete analogue of the EG inverse problem treated in this paper is considered in [14]. The discrete counterpart of Theorem 3.2 is [14, Theorem 4.5]. However Theorem 3.2 is more explicit than the corresponding result in [14].

REMARK 3.11. It would be interesting to find necessary and sufficient conditions for equation (3.2) to be solvable in terms of the root functions of α and γ^* corresponding to their zeros in $\mathbb{C}_+$. For a number of special cases we have such conditions. See Theorem 3.3 in the present section and Theorems 4.1, 4.3, and 5.1 below.

4. More about uniqueness

In this section we present two additional results about uniqueness of solutions. We first consider the case when the functions α and γ are scalar, that is, $p = q = 1$. The following theorem is the continuous analogue of Theorem 4.1 in [4].

THEOREM 4.1. *Assume $p = q = 1$. Then the EG inverse problem associated with α and γ has a solution if and only if the following two conditions are satisfied:*

(C1) $\alpha(\lambda)^*\alpha(\lambda) - \gamma(\lambda)^*\gamma(\lambda) = 1$ *for each* $\lambda \in \mathbb{R}$*;*

(C2) *the functions α and γ^* have no common zero in $\mathbb{C}_+$.*

Moreover, in that case the EG inverse problem is uniquely solvable and the unique solution is given by

$$g(\lambda) = -\left(\alpha^{-*}\gamma^*\right)_+(\lambda) + B_+^*\left(\lambda I_{n_+} - A_+^*\right)^{-1}\gamma(A_+^*)^{-1}C_+^*, \tag{4.1}$$

where A_+, B_+ and C_+ are defined by (2.14)*.*

Proof. We already know that condition (C1) is a necessary condition for the EG inverse problem associated with α and γ to be solvable. Therefore, in what follows we assume that (C1) is satisfied, and hence we can freely use the notations introduced in the previous section.

In the case that α and γ are scalar functions we know from Theorem 3.3 that (C2) implies that the EG inverse problem is solvable and that the solution is unique.

Next we prove that the condition (C2) also implies that this solution is given by (4.1). In the present case with γ a scalar function, the map J defined by (3.4) can be considered as a linear operator on the finite dimensional space $\mathbb{C}^{n_+}$. Furthermore, since $\sigma(A_+) \subset \mathbb{C}_+$ and γ^* is analytic on $\mathbb{C}_+$, the functional calculus (see [8, Section I.3]) yields

$$J(Y) = Y\frac{1}{2\pi i}\int_\Gamma \gamma^*(\lambda)(\lambda I_{n_+} - A_+)^{-1}\,\mathrm{d}\lambda = Y\gamma^*(A_+), \qquad Y^* \in \mathbb{C}^{n_+}. \tag{4.2}$$

The eigenvalues of A_+ coincide with the zeros of α in $\mathbb{C}_+$. So condition (C2) is satisfied if and only if γ^* has no zero on $\sigma(A_+)$. But then, using the spectral mapping theorem (see [8, Theorem I.3.3]), we conclude that

$$\text{(C2) is satisfied} \iff \gamma^*(A_+) \text{ is invertible.} \tag{4.3}$$

In that case the solution $Y_\circ$ of $J(Y) = C_+$, i.e. the equation (3.2), is $Y_\circ = C_+\gamma^*(A_+)^{-1}$ and hence $Y_\circ^* = (\gamma^*(A_+))^{-*}C_+^* = \gamma(A_+^*)^{-1}C_+^*$. Formula (4.1) now follows from (3.3).

We proved that the conditions (C1) and (C2) are sufficient for solvability of the EG inverse problem associated with α and γ. Now assume that EG inverse problem associated with α and γ is solvable. We already know that this implies that condition (C1) is satisfied and it remains to show that condition (C2) is satisfied. From Theorem

3.2 we know that there exists a Y such that $Y\gamma(A_+^*) = J(Y) = -C_+$. Using the functional calculus we know that that $YA_+^k\gamma^*(A_+) = Y\gamma^*(A_+)A_+^k$ for each positive integer k. It follows that

$$YA_+^k\gamma^*(A_+) = Y\gamma^*(A_+)A_+^k = -C_+A_+^k, \quad k = 0,1,2,\dots. \tag{4.4}$$

Since the realization in (2.14) is minimal, the pair (C_+, A_+) is observable. Hence $\gamma^*(A_+)$ is injective and we conclude that the square matrix $\gamma^*(A_+)$ is invertible. But then the equivalence in (4.3) tells us that (C2) is satisfied. This completes the proof. □

EXAMPLE 4.2. We illustrate Theorem 4.1 with the following choice of α and γ:

$$\alpha(\lambda) = \frac{\lambda - i\sqrt{2}}{\lambda + i} \quad \text{and} \quad \gamma(\lambda) = \frac{1}{\lambda - i}. \tag{4.5}$$

Condition (C1) is satisfied. Note that α has precisely one zero in $\mathbb{C}_+$, namely $\lambda = i\sqrt{2}$. On the other hand, γ has no zeros. It follows that condition (C2) in Theorem 4.1 is also satisfied. Hence for α and γ in (4.5) the EG inverse problem has a unique solution. We use formula (4.1) to compute this solution. One verifies that

$$(\alpha^{-1})_{-,0}(\lambda) = \frac{i(1+\sqrt{2})}{\lambda - i\sqrt{2}} = C_+(\lambda - A_+)^{-1}B_+,$$

with

$$A_+ = i\sqrt{2}, \quad B_+ = 1, \quad C_+ = i(1+\sqrt{2}).$$

Note that $\gamma(A_+^*)^{-1} = -i(1+\sqrt{2})$. Hence we have

$$B_+^*(\lambda - A_+^*)^{-1}\gamma(A_+^*)^{-1}C_+^* = \frac{-(1+\sqrt{2})^2}{\lambda + i\sqrt{2}}.$$

Furthermore

$$-(\alpha^{-*}\gamma^*)_+(\lambda) = \frac{-(\lambda - i)}{(\lambda + i\sqrt{2})(\lambda + i)}.$$

Adding the last two equalities shows that the unique solution g of the EG inverse problem is given by

$$\begin{aligned} g(\lambda) &= -(\alpha^{-*}\gamma^*)_+(\lambda) + B_+^*(\lambda - A_+^*)^{-1}\gamma(A_+^*)^{-1}C_+^* \\ &= -\frac{2(1+\sqrt{2})(\lambda\sqrt{2}+i)}{(\lambda+i)(\lambda+i\sqrt{2})}. \end{aligned} \tag{4.6}$$

Given this formula one can also verify directly that (1.4) is indeed satisfied. On the other hand, it is not straightforward to guess that (4.6) gives the solution of the EG inverse problem. Note that the present example can be viewed as a specification of Example 3.7 in a scalar setting, namely, $M = [1]$ and $\lambda_0 = i$.

We conclude this section with a somewhat stronger version of Theorem 4.1 which does not require $p=q=1$ but only $p=q$ and which coincides with Theorem 4.1 when $p=q=1$.

THEOREM 4.3. *Let $p=q$. Partition C_+ as follows:*

$$C_+ = \begin{bmatrix} C_{+,1} \\ C_{+,2} \\ \vdots \\ C_{+,p} \end{bmatrix}, \text{ where } C_{+,j} : \mathbb{C}^{n_+} \to \mathbb{C}, \quad j = 1,2,\dots,p.$$

Assume that the pair $\big(\mathrm{row}\,(C_{+,1},\dots,C_{+,p}), \mathrm{diag}_p\,(A_+,\dots,A_+)\big)$ is observable. Then the EG inverse problem associated with α and γ has a solution if and only if $\det\alpha$ *and* $\det\gamma^*$ *have no common zero, and in that case the solution is unique.*

Proof. First assume that the EG inverse problem associated with α and γ has a solution. According to Corollary 3.9 this implies that the equation (3.22) with $q=p$ has a solution. Since A_+ commutes with all $(\gamma_{ij})^*(A_+)$, we get that

$$\begin{bmatrix} Y_1 A_+^k & \cdots & Y_p A_+^k \end{bmatrix} \begin{bmatrix} (\gamma_{11})^*(A_+) & \cdots & (\gamma_{1p})^*(A_+) \\ \vdots & & \vdots \\ (\gamma_{p1})^*(A_+) & \cdots & (\gamma_{pp})^*(A_+) \end{bmatrix} = -\begin{bmatrix} C_{+,1}A_+^k & \cdots & C_{+,p}A_+^k \end{bmatrix} \tag{4.7}$$

for any k. So if $\Upsilon(A_+)x=0$, where $\Upsilon(A_+)$ is the second matrix in (4.7), then

$$\begin{bmatrix} C_{+,1}A_+^k & \cdots & C_{+,p}A_+^k \end{bmatrix} x = 0, \qquad k = 0,1,2,\dots.$$

Since we assumed that the pair $\big(\mathrm{row}\,(C_{+,1},\dots,C_{+,p}), \mathrm{diag}_p\,(A_+,\dots,A_+)\big)$ is observable, it follows that $x=0$. We proved that $\Upsilon(A_+)$ is injective and, being square, is non-singular. Again using Lemma 3.5 we conclude that the equation (3.2) has a unique solution, and hence the EG inverse problem is uniquely solvable. Theorem 3.3 gives that this is equivalent to $\det\alpha$ and $\det\gamma^*$ having no common zero.

Conversely, assume that $\det\alpha$ and $\det\gamma^*$ have no common zero. Then it follows from Theorem 3.3 that the EG inverse problem is solvable and that the solution is unique. □

5. The case when a and c have finite support

A function $f \in L^1(\mathbb{R})^{r\times s}$ is said to have *finite support* if there exists real numbers $\tau_1 < \tau_2$ such that $f(t) = 0$ for all $t \notin [\tau_1, \tau_2]$. In this case we also say that the *support of f belongs to* the interval $[\tau_1, \tau_2]$, and we write $\mathrm{supp}\, f \subset [\tau_1, \tau_2]$. The following theorem is the main result of this section. We view this theorem as the natural continuous analogue of [14, Theorem 3.3] with the role of polynomials taken over by functions of the form $\mathscr{F}' f$ with f a function with finite support.

THEOREM 5.1. *Let α and γ be given by* (1.2) *and* (1.3)*, respectively, and assume that the corresponding functions a and c have finite support. Then the EG inverse problem associated with α and γ is solvable if and only if the condition* (C1) *is satisfied. Moreover, in that case there exists precisely one solution $g = \mathscr{F}'k$ to the EG inverse problem such that k has finite support.*

To prove the above theorem we need the following lemma and an additional lemma.

LEMMA 5.2. *Let α be as in* (1.2)*, and assume that a has finite support. Then $f \in \operatorname{Ker} T_{\alpha^*}$ and f has finite support implies $f = 0$.*

Proof. Assume that $\operatorname{supp} a$ and $\operatorname{supp} f$ are both contained in the finite interval $[0,\tau]$ for some $\tau > 0$. Consider the direct sum decomposition

$$L^1(\mathbb{R}_+)^p = L^1([0,\tau])^p \dot{+} L^1([\tau,\infty))^p. \tag{5.1}$$

Write T_{α^*} as a 2×2 block operator matrix relative to the decomposition (5.1), as follows

$$T_{\alpha^*} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} L^1([0,\tau])^p \\ L^1([\tau,\infty))^p \end{bmatrix} \to \begin{bmatrix} L^1([0,\tau])^p \\ L^1([\tau,\infty))^p \end{bmatrix}.$$

We claim that $C = 0$ and A is invertible.

Take $h \in L^1([0,\tau])^p$. Note that $\operatorname{supp} a^* \subset [-\tau, 0]$. It follows that the matrix $a^*(t-s) = 0$ whenever $0 \leqslant s \leqslant t$. But then $h \in L^1([0,\tau])^p$ implies that $(T_{\alpha^*}h)(t) = 0$ for $t > \tau$, and hence $C = 0$. Next, setting $a(t) = 0$ for $t < 0$ or, equivalently, setting $a^*(t) = 0$ for $t > 0$, we see that the operator A is given by

$$(Ah)(t) = h(t) + \int_0^\tau a^*(t-s)h(s)\,\mathrm{d}s, \quad 0 \leqslant t \leqslant \tau. \tag{5.2}$$

Thus we can apply Lemma A.3 with $K = A$ to show that A is invertible.

Finally, let $f \in \operatorname{Ker} T_{\alpha^*}$. Since $\operatorname{supp} f \subset [0,\tau]$, it follows that relative to the direct sum decomposition (5.1) the function f can be written as $f = \begin{bmatrix} f_0 & 0 \end{bmatrix}^{\mathrm{T}}$. But then $T_{\alpha^*} f = 0$ and $C = 0$ imply that $Af_0 = 0$ because

$$\begin{bmatrix} Af_0 \\ 0 \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} \begin{bmatrix} f_0 \\ 0 \end{bmatrix} = T_{\alpha^*} f = 0.$$

Since A is invertible, it follows that $f_0 = 0$, and thus $f = 0$ as desired. □

Proof of Theorem 5.1. We split the proof into four parts.

PART 1. Choose $\tau > 0$ such that $\operatorname{supp} a \subset [0,\tau]$ and $\operatorname{supp} c \subset [-\tau, 0]$. As in the proof of Lemma 5.2 above, we shall use the direct sum decomposition (5.1). From the proof of Lemma 5.2 we know that T_{α^*} partitions as

$$T_{\alpha^*} = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} : \begin{bmatrix} L^1([0,\tau])^p \\ L^1([\tau,\infty))^p \end{bmatrix} \to \begin{bmatrix} L^1([0,\tau])^p \\ L^1([\tau,\infty))^p \end{bmatrix}. \tag{5.3}$$

Moreover the operator A, which is given by (5.2), is invertible.

Let c_0 be the function c restricted to the interval $[-\tau,0]$, and put $c_0^*(t)=c_0(-t)^*$ for $0\leqslant t\leqslant\tau$. Then relative to the decomposition (5.1) the function $c^*\in L^1(\mathbb{R}_+)^{p\times q}$ is given $c^*=\begin{bmatrix}c_0^* & 0\end{bmatrix}^{\mathrm{T}}$. Let k be given by

$$k(t)=\begin{cases}(A^{-1}c_0^*)(t) & \text{for } 0\leqslant t\leqslant\tau,\\ 0 & \text{otherwise.}\end{cases} \tag{5.4}$$

Clearly $k\in L^1(\mathbb{R}_+)^{p\times q}$ and $\operatorname{supp}k\subset[0,\tau]$. Furthermore, we have

$$T_{\alpha^*}k=\begin{bmatrix}A & B\\ 0 & D\end{bmatrix}\begin{bmatrix}A^{-1}c_0^*\\ 0\end{bmatrix}=\begin{bmatrix}c_0^*\\ 0\end{bmatrix}=c^*. \tag{5.5}$$

Now put $g=-\mathscr{F}'k$. Then $g\in\mathscr{W}_{+,0}^{p\times q}$ and (5.5) tells us that $\alpha^*g+\gamma^*\in\mathscr{W}_{-,0}^{p\times q}$. In particular, g satisfies the second inclusion in (1.4). Note that g has finite support. It remains to show that g also satisfies the first inclusion in (1.4).

PART 2. Let g be as in the previous part, and put $\varphi=(\alpha+g\gamma-e)_+$. Let f be the inverse Fourier transform of φ, that is, $\varphi=\mathscr{F}'f$, where $f\in L^1(\mathbb{R}_+)^{p\times p}$. In this part we show that f has finite support. In order to do this, recall that the functions a, c and k all have finite support. In fact, without loss of generality we may assume that

$$\operatorname{supp}a\subset[0,\tau],\quad \operatorname{supp}c\subset[-\tau,0],\quad \operatorname{supp}k\subset[0,\tau].$$

Then on $\mathbb{R}_+$ the function f is given by

$$f(t)=a(t)+\int_{-\infty}^{\infty}k(t-s)\gamma(s)\,\mathrm{d}s=a(t)+\int_{-\tau}^{0}k(t-s)\gamma(s)\,\mathrm{d}s,\quad t\geqslant 0.$$

Now take $t>\tau$. Then $a(t)=0$ and $k(t-s)=0$ for all $-\tau\leqslant s\leqslant 0$. Hence $\operatorname{supp}f\subset[0,\tau]$.

PART 3. According to Lemma 3.4 we have $(\alpha^*\varphi)_+=0$. This fact implies that $T_{\alpha^*}f=0$. Thus $f\in\operatorname{Ker}T_{\alpha^*}$ and f has finite support. By Lemma 5.2 the function f is zero. But then $\varphi=0$. In other words, $(\alpha+g\gamma-e)_+=0$, that is, $\alpha+g\gamma-e\in\mathscr{W}_{-,0}^{p\times p}$. We conclude that g satisfies the first inclusion in (1.4). Thus $g=\mathscr{F}'k$ is a solution of the EG inverse problem with k having finite support.

PART 4. It remains to prove the uniqueness statement. Let $g_\circ=\mathscr{F}'k_\circ$, where $k_\circ\in L^1(\mathbb{R}_+)^{p\times q}$ and $k_\circ$ has finite support. Put $\psi=g-g_\circ\in\mathscr{W}_{+,0}^{p\times q}$. Then $\psi=\mathscr{F}'h$, where $h=k-k_\circ\in L^1(\mathbb{R}_+)^{p\times q}$ and h has finite support. Since g and $g_\circ$ both satisfy the second inclusion in (1.4), we know that both $\alpha^*g+\gamma^*$ and $\alpha^*g_\circ+\gamma^*$ belong to $\mathscr{W}_{-,0}^{p\times q}$. This yields

$$\alpha^*\psi=\alpha^*(g-g_\circ)=(\alpha^*g+\gamma^*)-(\alpha^*g_\circ+\gamma^*)\in\mathscr{W}_{-,0}^{p\times q}.$$

Thus $(\alpha^*\psi)_+=0$ and hence $T_{\alpha^*}\psi=0$. Since $\psi=\mathscr{F}'h$ and h has finite support, Lemma 5.2 shows that $h=0$. Thus $g=g_\circ$, which proves the uniqueness. $\square$

6. Rational data

In this section the results of Section 3 are specified further for the case when $\gamma \in \mathscr{W}_{-,0}^{q\times p}$ is rational. Our starting point is a minimal realization of γ, namely

$$\gamma(\lambda) = C_\gamma(\lambda I_{n_\gamma} - A_\gamma)^{-1}B_\gamma. \tag{6.1}$$

Here A_γ, B_γ, C_γ are matrices of size $n_\gamma \times n_\gamma$, $n_\gamma \times p$, and $q \times n_\gamma$, respectively. Since all the poles of γ are in the upper half plane, the minimality of the realization implies that $\sigma(A_\gamma) \subset \mathbb{C}_+$. But then, without loss of generality, we may assume that

$$A_\gamma - A_\gamma^* = iC_\gamma^* C_\gamma. \tag{6.2}$$

Indeed, if $\gamma(\lambda) = C(\lambda I_n - A)^{-1}B$ is an arbitrary minimal realization, then $\gamma \in \mathscr{W}_{-,0}^{q\times p}$ implies that $\sigma(A) \subset \mathbb{C}_+$. Furthermore, because of minimality, the pair (C,A) is observable, and hence the Lyaponov equation $PA - A^*P = iC^*C$ has a (unique) positive definite solution P; see, for instance, [17, Theorem 13.4] or [8, Theorem I.5.5]. Put $S = P^{\frac{1}{2}}$, and let $A_\gamma = SAS^{-1}$, $C_\gamma = CS^{-1}$ and $B_\gamma = SB$. Then both (6.1) and (6.2) are satisfied.

A realization for the right spectral factor. Put $w(\lambda) = I_p + \gamma^*(\lambda)\gamma(\lambda)$. We first compute a realization for the right spectral factor $w_{\rm sp}$ of w. Using the realization (6.1) together with (6.2), it follows that

$$w(\lambda) = (-iB_\gamma)^*(\lambda I_{n_\gamma} - A_\gamma^*)^{-1}B_\gamma + I_p + B_\gamma^*(\lambda I_{n_\gamma} - A_\gamma)^{-1}(-iB_\gamma). \tag{6.3}$$

Since w is positive definite on the real line and at infinity, this allows us to use theory of algebraic Ricatti equations to compute a realization for $w_{\rm sp}$ and its inverse.

PROPOSITION 6.1. *Let γ be given by (6.1), (6.2). Then the algebraic Ricatti equation*

$$XiB_\gamma B_\gamma^* X + X(A_\gamma^* - iB_\gamma B_\gamma^*) - (A_\gamma + iB_\gamma B_\gamma^*)X + iB_\gamma B_\gamma^* = 0 \tag{6.4}$$

has a (unique) Hermitian solution $X = Q_\gamma$ such that

$$\sigma(A_\gamma + (I_{n_\gamma} - Q_\gamma)iB_\gamma B_\gamma^*) \subset \mathbb{C}_+. \tag{6.5}$$

Furthermore, the right spectral factor $w_{\rm sp}$ of $w = e + \gamma^*\gamma$ and the inverse of $w_{\rm sp}$ are the rational matrix functions given by

$$w_{\rm sp}(\lambda) = I_p + C_{\rm sp}(\lambda I_{n_\gamma} - A_\gamma^*)^{-1}B_\gamma, \tag{6.6}$$

$$w_{\rm sp}(\lambda)^{-1} = I_p - C_{\rm sp}\left(\lambda I_{n_\gamma} - (A_\gamma^\times)^*\right)^{-1}B_\gamma, \tag{6.7}$$

where $C_{\rm sp} = iB_\gamma^*(I_{n_\gamma} - Q_\gamma)$, and

$$A_\gamma^\times = A_\gamma + (I_{n_\gamma} - Q_\gamma)iB_\gamma B_\gamma^*. \tag{6.8}$$

Proof. The fact that equation (6.4) has a unique solution $X = Q_\gamma$ satisfying (6.8) follows from [1, Theorem 13.2]. To make the connection with this theorem put

$$A = iA_\gamma, \quad B = B_\gamma, \quad C = B_\gamma^*, \quad D = I_p.$$

Then A is stable, that is, all the eigenvalues of A are in the open left half plane, and the function W in [1, Eq. (13.6)] is given by $w(\lambda) = W(i\lambda)$, which implies that the function W has no zeros on the imaginary axis. Applying [1, Theorem 13.2] for this Hermitian case, we conclude that the Riccati equation

$$QC^*CQ - Q(A-BC)^* - (A-BC)Q + BB^* = 0$$

has a unique Hermitian solution with $A^* - BC - C^*CQ$ stable. Put $Q_\gamma = -Q$. Then one checks that Q_γ is indeed a solution of (6.4) with $\sigma(A_\gamma^\times) \subset \mathbb{C}_+$.

It remains to check that $w = w_{\rm sp}^* w_{\rm sp}$. This can be done by a direct computation in which one uses that $X = Q_\gamma$ is a solution of (6.4), which implies that

$$A_\gamma Q_\gamma - Q_\gamma A_\gamma^* = iC_{\rm sp}^* C_{\rm sp}.$$

The formula for $w_{\rm sp}^{-1}$ follows from the formula for $w_{\rm sp}$. Notice that the fact that $\sigma(A_\gamma^\times) \subset \mathbb{C}_+$ indeed gives that also $w_{\rm sp}^{-1} \in \mathscr{W}_+^{p\times p}$. $\square$

The Hermitian solution $X = Q_\gamma$ in the above proposition is actually positive definite. To see this we can use Proposition 2.2 in [7] which implies that

$$Q_\gamma = \rho^* T_w^{-1} \rho. \tag{6.9}$$

Here T_w is the Wiener-Hopf integral operator on $L^2(\mathbb{R}_+)^p$ defined by w and ρ is the operator mapping $\mathbb{C}^{n_\gamma}$ into $L^2(\mathbb{R}_+)^p$ given by

$$(\rho x)(t) = B_\gamma^* e^{-itA_\gamma^*} x, \quad 0 \leqslant t < \infty \quad (x \in \mathbb{C}^{n_\gamma}).$$

Since w is positive definite on the real line and at infinity, the operator T_w is strictly positive, and hence the same holds true for T_w^{-1}. Furthermore, the minimality of the realization (6.1) implies that ρ is one-to-one. But then the identity (6.9) shows that Q_γ is positive definite.

EXAMPLE 6.2. In this example, as in Example 3.7, the function γ is given by

$$\gamma(\lambda) = \frac{1}{\lambda - \lambda_\circ} M. \tag{6.10}$$

As before, $\lambda_\circ \in \mathbb{C}_+$ and M is a $q \times p$ matrix. We first derive a minimal realization of γ of the form (6.1), (6.2), and next we use Proposition 6.1 to compute the spectral factor $w_{\rm sp}$.

Let $r = \operatorname{rank} M$, and write $\lambda_\circ = x + iy$, with $y > 0$. Using the singular value decomposition of M, there exist an isometry $U : \mathbb{C}^r \to \mathbb{C}^q$, a co-isometry $V : \mathbb{C}^p \to \mathbb{C}^r$

and a diagonal operator

$$\Delta = \begin{bmatrix} s_1 & & & \\ & s_2 & & \\ & & \ddots & \\ & & & s_r \end{bmatrix} : \mathbb{C}^r \to \mathbb{C}^r,$$

where $s_1 \geqslant s_2 \geqslant \cdots \geqslant s_r > 0$ are the non-zero singular values of M, such that $M = U\Delta V$. Next put

$$A_\gamma = \lambda_\circ I_r, \quad B_\gamma = \frac{1}{\sqrt{2y}}\Delta V, \quad C_\gamma = \sqrt{2y}U. \tag{6.11}$$

Then, using $M = U\Delta V$, we see that

$$\gamma(\lambda) = C_\gamma(\lambda I_r - A_\gamma)^{-1}B_\gamma \quad \text{and} \quad A_\gamma - A_\gamma^* = iC_\gamma^* C_\gamma. \tag{6.12}$$

Furthermore, the realization given above is minimal.

Next, we derive the right spectral factor $w_{\rm sp}$ of the matrix function $w(\lambda) = I_p + \gamma^*(\lambda)\gamma(\lambda)$. It is easy to do this by a direct computation, but we wish to illustrate the method given by Proposition 6.1. Using the minimal realization in (6.11) we see that

$$w(\lambda) = -(iB_\gamma)^*(\lambda I_r - A_\gamma^*)^{-1}B_\gamma^* + I_p - B_\gamma^*(\lambda I_r - A_\gamma)^{-1}(iB_\gamma).$$

Since $VV^* = I_r$, it follows that in this case $B_\gamma B_\gamma^* = (1/2y)\Delta^2$. Hence equation (6.4) reduces to

$$X\frac{i}{2y}\Delta^2 X + X\left(\overline{\lambda_\circ} I_r - \frac{i}{2y}\Delta^2\right) - \left(\lambda_\circ I_r + \frac{i}{2y}\Delta^2\right) + \frac{i}{2y}\Delta^2 = 0. \tag{6.13}$$

By Proposition 6.1, specified for this case, equation (6.13) has a unique Hermitian solution X with

$$\sigma\left(\lambda_\circ I_r + \frac{i}{2y}(I_r - X)\Delta^2\right) \subset \mathbb{C}_+. \tag{6.14}$$

To find this solution, we solve (6.13) with X being diagonal using the fact that Δ is diagonal. Indeed, with $X = \operatorname{diag}(x_1, \cdots, x_r)$ equation (6.13) reduces to

$$x_j^2 - 2x_j\left(\frac{2y^2}{s_j^2} + 1\right) + 1 = 0, \quad j = 1, \cdots r.$$

Furthermore, using $\lambda_\circ = x + iy$, condition (6.14) requires

$$x_j < \left(\frac{2y^2}{s_j^2} + 1\right), \quad j = 1, \ldots, r.$$

This leads to

$$X = \operatorname{diag}(x_1, \cdots, x_r) \quad \text{where}$$
$$x_j = \left(\frac{2y^2}{s_j^2} + 1\right) - \frac{2y}{s_j^2}\sqrt{y^2 + s_j^2}, \quad j = 1, \cdots, r.$$

Thus, in the present setting, $X = Q_\gamma$ is the solution of (6.4) with $\sigma(A_\gamma^\times) \subset \mathbb{C}_+$, where $A_\gamma^\times$ is given by (6.8).

We proceed with computing the realization of the spectral factor $w_{\rm sp}$ of w. To do this note that

$$C_{\rm sp} = iB_\gamma^*(I - Q_\gamma) = V^*\mathrm{diag}\left(\frac{is_1}{\sqrt{2y}}, \dots, \frac{is_r}{\sqrt{2y}}\right) \mathrm{diag}\,(1 - x_1, \dots, 1 - x_r).$$

Put $\lambda_j = x + i\sqrt{y^2 + s_j^2}$. Then one gets

$$C_{\rm sp} = V^*\mathrm{diag}\,(\overline{\lambda_\circ - \lambda_1}, \dots, \overline{\lambda_\circ - \lambda_r})\ \mathrm{diag}\left(\frac{\sqrt{2y}}{s_1}, \dots, \frac{\sqrt{2y}}{s_r}\right). \tag{6.15}$$

This yields

$$w_{\rm sp}(\lambda) = I_p + C_{\rm sp}(\lambda I_r - A_\gamma^*)^{-1}B_\gamma = I_p - V^*V + V^*\mathrm{diag}\left(\frac{\lambda - \overline{\lambda_1}}{\lambda - \overline{\lambda_\circ}}, \dots, \frac{\lambda - \overline{\lambda_r}}{\lambda - \overline{\lambda_\circ}}\right) V. \tag{6.16}$$

The bi-inner function Θ and the function $\alpha = \Theta w_{\rm sp}$. Next, let $\alpha = \Theta w_{\rm sp}$, where Θ is a (rational) bi-inner function in $\mathscr{W}_+^{p\times p}$ and $w_{\rm sp}$ is the right spectral factor of $w = e + \gamma^*\gamma$ introduced above. Then α is rational and condition (C1) is fulfilled for α and c.

Since $\Theta^*(\lambda)\Theta(\lambda) = I_p$ for all $\lambda \in \mathbb{R}$ and at infinity, we may assume that Θ admits a minimal realization of the form:

$$\Theta(\lambda) = I_p + C_\theta(\lambda I_{n_\theta} - A_\theta)^{-1}B_\theta \quad \text{with} \quad B_\theta = -iC_\theta^*. \tag{6.17}$$

Here A_θ and C_θ are matrices of size $n_\theta \times n_\theta$ and $p \times n_\theta$, respectively. Since all the poles of Θ are in $\mathbb{C}_-$, the minimality of the realization implies that $\sigma(A_\theta) \subset \mathbb{C}_-$. Furthermore, using arguments similar to those used in the paragraph after (6.2) we may assume without loss of generality that

$$A_\theta^* - A_\theta = iC_\theta^*C_\theta. \tag{6.18}$$

PROPOSITION 6.3. *Let γ be given by* (6.1), (6.2), *and let $\alpha = \Theta w_{\rm sp}$, where $w_{\rm sp}$ is the spectral factor given by* (6.6) *and Θ is the bi-inner function given by* (6.17). *Then α and γ satisfy condition* (C1) *and*

$$\left(\alpha^{-1}\right)_{-,0}(\lambda) = C_+(\lambda I_{n_+} - A_+)^{-1}B_+, \tag{6.19}$$

where

$$A_+ = A_\theta^*, \quad B_+ = iC_\theta^*, \quad C_+ = C_\theta + B_\gamma^*(I_{n_\gamma} - Q_\gamma)P_1^*, \quad n_+ = n_\theta. \tag{6.20}$$

Here Q_γ is the solution of (6.4), (6.5), *and P_1 is the unique $n_\theta \times n_\gamma$ matrix solution of the Sylvester equation*

$$A_\theta P_1 - P_1 A_\gamma^\times = -C_\theta^*(iB_\gamma)^*, \tag{6.21}$$

where $A_\gamma^\times$ is given by (6.8). *Moreover, the realization* (6.19) *is minimal.*

Proof. Since $\sigma(A_\theta) \subset \mathbb{C}_-$ and $\sigma(A_\gamma^\times) \subset \mathbb{C}_+$, the Sylvester equation (6.21) has a unique solution P_1 (cf., [16, Theorem 5.2.2]). Both $w_{\rm sp}$ and Θ are in $\mathscr{W}_+^{p\times p}$, and therefore $\alpha \in e + \mathscr{W}_{+,0}^{p\times p}$. Recall that $e+\gamma^*\gamma = w_{\rm sp}^* w_{\rm sp}$ on the real axis and that Θ is unitary on the real axis. It follows that condition (C1) is satisfied and $\alpha(\lambda)^{-1} = w_{\rm sp}(\lambda)^{-1}\Theta(\lambda)^*$ for $\lambda \in \mathbb{R}$. Using the realizations (6.6) and (6.17) we obtain

$$\begin{aligned}\alpha^{-1}(\lambda) &= w_{\rm sp}^{-1}(\lambda)\Theta^*(\lambda)\\ &= w_{\rm sp}^{-1}(\lambda) + iC_\theta(\lambda I_{n_\theta} - A_\theta^*)^{-1}C_\theta^* - C_{\rm sp}\left(\lambda I_{n_\gamma} - (A_\gamma^\times)^*\right)^{-1} B_\gamma(iC_\theta)(\lambda I_{n_\theta} - A_\theta^*)^{-1}C_\theta^*.\end{aligned} \tag{6.22}$$

The Sylvester equation (6.21) yields

$$B_\gamma(iC_\theta) = \left(C_\theta^*(iB_\gamma)^*\right)^* = P_1^*(\lambda I_{n_\theta} - A_\theta^*) - \left(\lambda I_{n_\gamma} - (A_\gamma^\times)^*\right)P_1^*.$$

Replacing $B_\gamma(iC_\theta)$ in (6.22) by the right hand side of the previous identity we get

$$\begin{aligned}&-C_{\rm sp}\left(\lambda I_{n_\gamma} - (A_\gamma^\times)^*\right)^{-1} B_\gamma(iC_\theta)(\lambda I_{n_\theta} - A_\theta^*)^{-1}C_\theta^*\\ &\quad= -C_{\rm sp}\left(\lambda I_{n_\gamma} - (A_\gamma^\times)^*\right)^{-1} P_1^*C_\theta^* + C_{\rm sp}P_1^*(\lambda I_{n_\theta} - A_\theta^*)^{-1}C_\theta^*.\end{aligned} \tag{6.23}$$

Next using $w_{\rm sp}^{-1} \in \mathscr{W}_+^{p\times p}$ and $\sigma(A_\gamma^\times) \subset \mathbb{C}_+$ we arrive at

$$(\alpha^{-1})_{-,0}(\lambda) = (iC_\theta + C_{\rm sp}P_1^*)(\lambda I_{n_\theta} - A_\theta^*)^{-1}C_\theta^*.$$

Finally, since $C_{\rm sp} = iB_\gamma^*(I_{n_\gamma} - Q_\gamma)$, the definitions of A_+, B_+, and C_+ in (6.20) yield (6.19).

It remains to show that the realization in (6.19) is minimal. Using (6.20) and taking adjoints we see that it suffices to show that the realization

$$\left(\alpha^{-*}\right)_{+,0}(\lambda) = C_\theta(\lambda I_{n_\theta} - A_\theta)^{-1}(-iC_+^*) \tag{6.24}$$

is minimal. But the latter realization has the same state space dimension as the minimal realization of $(\alpha^{-*})_{+,0}$ in (2.11). Thus the realization in (6.24) is minimal, and hence the same is true for the realization in (6.19). $\square$

Main results specified for the rational case. We proceed by specifying Theorem 3.2 in this rational matrix function setting using the realizations of the functions involved.

THEOREM 6.4. *Let γ be given by* (6.1), (6.2), *and let Θ be given by* (6.17), (6.18). *Put $\alpha = \Theta w_{\rm sp}$ with $w_{\rm sp}$ given by* (6.6). *Furthermore, let Q_γ be the unique Hermitian solution of* (6.4) *satisfying the spectral condition* (6.5), *and let P_1 be the solution of* (6.21). *Then the EG inverse problem associated with α and γ is solvable if and only if there exists $R \in \mathbb{C}^{n_\theta\times n_\gamma}$ such that*

$$RB_\gamma = C_\theta^* + P_1(I_{n_\gamma} - Q_\gamma)B_\gamma \quad \text{and} \quad \operatorname{Ker} C_\gamma \subset \operatorname{Ker}(RA_\gamma - A_\theta R). \tag{6.25}$$

Moreover, in that case all solutions g are given by

$$g(\lambda) = -(\alpha^{-*}\gamma^*)_+(\lambda) - iC_\theta(\lambda I_{n_\theta} - A_\theta)^{-1}Y^*, \tag{6.26}$$

where Y is any matrix such that $Y^*C_\gamma = RA_\gamma - A_\theta R$ with R satisfying (6.25).

Proof. We use the minimal realization (6.19) with A_+, B_+, and C_+ being given by (6.20). Then, according to Theorem 3.2, the EG inverse problem associated with α and γ has a solution if and only if there exists a $q \times n_+$ matrix Y such that

$$\frac{1}{2\pi i} \int_\Gamma \gamma^*(\lambda) Y (\lambda I_{n_+} - A_+)^{-1} \, \mathrm{d}\lambda = -C_+. \tag{6.27}$$

Here Γ is a Cauchy contour in $\mathbb{C}_+$ such that all the eigenvalues of A_+ are in the inner domain of Γ.

Let the EG inverse problem be solvable. Then there exists $Y \in \mathbb{C}^{q \times n_+}$ such that (6.27) holds. Given this Y put

$$R^* = -\frac{1}{2\pi i} \int_\Gamma (\lambda I_{n_\gamma} - A_\gamma^*)^{-1} C_\gamma^* Y (\lambda I_{n_+} - A_\theta^*)^{-1} \, \mathrm{d}\lambda \,. \tag{6.28}$$

Recall (see (6.20)) that

$$A_+ = A_\theta^*, \quad B_+ = iC_\theta^*, \quad C_+ = C_\theta + B_\gamma^*(I_{n_\gamma} - Q_\gamma) P_1^*.$$

Since γ is given by (6.1), (6.2), it follows that

$$B_\gamma^* R^* = -\frac{1}{2\pi i} \int_\Gamma \gamma^*(\lambda) Y (\lambda I_{n_+} - A_+)^{-1} \, \mathrm{d}\lambda = C_+ = C_\theta + B_\gamma^*(I_{n_\gamma} - Q_\gamma) P_1^*.$$

Hence $RB_\gamma = C_\theta^* + P_1(I_{n_\gamma} - Q_\gamma)B_\gamma$, and the first identity in (6.25) is proved. The fact that the spectrum of A_γ is in $\mathbb{C}_-$ and that of A_θ^* in $\mathbb{C}_+$ implies that these spectra are disjoint, and we can use the operator functional calculus (see, e.g., [8, Theorem I.4.1]) to show that

$$\begin{aligned} \text{the identity (6.28)} &\iff A_\gamma^* R^* - R^* A_\theta^* = C_\gamma^* Y \\ &\iff Y^* C_\gamma = RA_\gamma - A_\theta R. \\ &\implies \operatorname{Ker} C_\gamma \subset \operatorname{Ker}(RA_\gamma - A_\theta R), \end{aligned}$$

which proves the second part of (6.25).

To prove the reverse implication, assume that there exists $R \in \mathbb{C}^{n_\theta \times n_\gamma}$ such that both parts of (6.25) are satisfied. Given such a matrix R, the inclusion in the second part of (6.25) implies that there exist $Y \in \mathbb{C}^{q \times n_+}$ such that $Y^* C_\gamma = RA_\gamma - A_\theta R$, and hence $C_\gamma^* Y = A_\gamma^* R^* - R^* A_\theta^*$. Using the functional calculus again, we conclude that (6.28) holds. Since $A_+ = A_\theta^*$, this implies

$$\begin{aligned} \frac{1}{2\pi i} \int_\Gamma \gamma^*(\lambda) Y (\lambda I_{n_+} - A_+)^{-1} \, \mathrm{d}\lambda &= B_\gamma^* \left(\frac{1}{2\pi i} \int_\Gamma (\lambda I_{n_\gamma} - A_\gamma^*)^{-1} C_\gamma^* Y (\lambda I_{n_+} - A_\theta^*)^{-1} \, \mathrm{d}\lambda \right) \\ &= B_\gamma^* R^* = -\big(C_\theta + B_\gamma^*(I_{n_\gamma} - Q_\gamma) P_1^* \big) = -C_+. \end{aligned}$$

Hence the identity (6.27) is satisfied, and thus the EG inverse problem associated with α and γ is solvable.

To complete the proof we use (3.3) in Theorem 3.2 and $B_+ = iC_\theta^*$. Since (6.27) is satisfied, it follows that (6.26) gives all solutions of the EG inverse problem. $\square$

Note that there exists a matrix R satisfying the first identity in (6.25) if and only if $\operatorname{Ker} B_\gamma \subset \operatorname{Ker} C_\theta^*$. Furthermore, the second condition in (6.25) is automatically fulfilled if C_γ is one-to-one. This yields the following corollary.

COROLLARY 6.5. *Let γ, Θ, α be as in the previous theorem. If C_γ is one-to-one, then the EG inverse problem is solvable if and only if* $\operatorname{Ker} B_\gamma \subset \operatorname{Ker} C_\theta^*$.

In general, the condition $\operatorname{Ker} B_\gamma \subset \operatorname{Ker} C_\theta^*$ appearing in the above corollary is not sufficient for the EG inverse problem to be solvable. This is already clear in the scalar case. Indeed, let us assume that $p = q = 1$ and γ does not vanish identically. Then necessarily the matrix B_γ in the realization (6.1) must be one-to-one, that is, $\operatorname{Ker} B_\gamma = \{0\}$, and hence condition $\operatorname{Ker} B_\gamma \subset \operatorname{Ker} C_\theta^*$ is satisfied for any $\alpha = \Theta w_{\rm sp}$. On the other hand, we know from Theorem 4.1 that in the scalar case the EG inverse problem is solvable if and only if α (or, equivalently, Θ) has no zero in common with γ^* in $\mathbb{C}_+$. Thus the condition $\operatorname{Ker} B_\gamma \subset \operatorname{Ker} C_\theta^*$ is not sufficient. To provide a concrete counter example, take

$$\gamma(\lambda) = \frac{\lambda + i\sqrt{2}}{(\lambda - i\sqrt{2})(\lambda - i)}, \quad \Theta(\lambda) = \frac{\lambda - i\sqrt{2}}{(\lambda + i\sqrt{2})}, \quad \alpha(\lambda) = \frac{\lambda - i\sqrt{2}}{(\lambda + i)}. \tag{6.29}$$

In this case, γ does not vanish identically, Θ is bi-inner, and $\alpha = \Theta\gamma$. In particular, condition (C1) is satisfied. Furthermore, $i\sqrt{2}$ is a common zero of α and γ^* in $\mathbb{C}_+$. Thus the EG inverse problem associate with this α and γ is not solvable by Theorem 4.1, but the condition $\operatorname{Ker} B_\gamma \subset \operatorname{Ker} C_\theta^*$ is satisfied trivially.

In the right hand side of the formula for all solutions of the EG inverse problem in (6.26) appears the function $-(\alpha^{-*}\gamma^*)_+$. In the special case when when $p = q$ and $\det\alpha$ and $\det\gamma^*$ have no common zero, this function is given by

$$\begin{aligned} -\left(\alpha^{-*}\gamma^*\right)_+(\lambda) = &-(B_\gamma^* + C_\theta P_1)P_2(\lambda I_{n_\gamma} - A_\gamma^*)^{-1}C_\gamma^* \\ &+ C_\theta(\lambda I_{n_\theta} - A_\theta)^{-1}(C_\theta^* + P_1(I_{n_\gamma} - Q_\gamma)B_\gamma + P_3 C_\gamma^*) \\ &- iC_\theta P_3(\lambda I_{n_\gamma} - A_\gamma^*)^{-1}C_\gamma^*. \end{aligned} \tag{6.30}$$

Here Q_γ is the solution of the equation (6.4) with (6.5), P_1 is the solution of (6.21) and P_2 and P_3 are the solutions of

$$\begin{aligned} A_\gamma^\times P_2 + P_2 A_\gamma^* &= -i(I_{n_\gamma} - Q_\gamma)B_\gamma B_\gamma^*, \\ A_\theta P_3 - P_3 A_\gamma^* &= (C_\theta^* + P_1(I_{n_\gamma} - Q_\gamma)B_\gamma)B_\gamma^*, \end{aligned}$$

respectively. To prove the identity (6.30) one first shows that

$$-\left(\alpha^{-*}\gamma^*\right)_+ = -\left((\alpha^{-*})_{-,0}\gamma^*\right)_+ - \left((\alpha^{-*})_+\gamma^*\right)_+. \tag{6.31}$$

Next one computes separately realizations for each term in the right hand side of the preceding identity. We omit the details.

Additional comments on Example 6.2. Let us return to Example 6.2. Thus γ is given by (6.10) and $w_{\rm sp}$ is given by (6.16). First we take $\alpha = w_{\rm sp}$. Then condition

(C1) is satisfied and $\det\alpha$ has no zero in $\mathbb{C}_+$, and by Theorem 3.1 the solution of the EG inverse problem for $\alpha = w_{\rm sp}$ and γ exists. Moreover, this solution is given by $g = -\left(w_{\rm sp}^{-*}\gamma^*\right)_+$. In the setting of Example 6.2 this yields

$$g(\lambda) = \frac{1}{\lambda - \overline{\lambda_\circ}} V^* \mathrm{diag}\left(\frac{ys_1}{y+\sqrt{y^2+s_1^2}}, \ldots, \frac{ys_r}{y+\sqrt{y^2+s_r^2}}\right) U^*.$$

Next we assume that $\alpha = \Theta w_{\rm sp}$ where Θ is an arbitrary rational bi-inner function in $\mathscr{W}_+^{p\times p}$, and we use that γ is given by the realization (6.12). From (6.11) we know that the matrix C_γ in the realization (6.12) is one-to-one. Thus we can apply Corollary 6.5 to show that the EG inverse problem is solvable if and only if $\operatorname{Ker} B_\gamma \subset \operatorname{Ker} C_\theta^*$. From (6.11) we see that $M = C_\gamma B_\gamma$, and hence $\operatorname{Ker} B_\gamma = \operatorname{Ker} M$ because C_γ is one-to-one. Thus, in this example, the EG inverse problem is solvable if and only if $\operatorname{Ker} M \subset \operatorname{Ker} C_\theta^*$.

The final result of the previous paragraph also follows from Example 3.7. Indeed, in Example 3.7 we concluded that the EG inverse problem is solvable if and only $\operatorname{Ker} M \subset \operatorname{Ker} C_+^*$, where $C_+^* = C_\theta^* + P_1(I_{n_\gamma} - Q_\gamma)B_\gamma$. Thus

$$\operatorname{Ker} M \subset \operatorname{Ker} C_\theta^* \iff \operatorname{Ker} M \subset \operatorname{Ker} C_+^*,$$

However, since $\operatorname{Ker} M = \operatorname{Ker} B_\gamma$, the above equivalence directly follows from the fact that $C_+^* = C_\theta^* + P_1(I_{n_\gamma} - Q_\gamma)B_\gamma$. Therefore the result of the previous paragraph also follows from Example 3.7. Finally, note that the condition $\operatorname{Ker} M \subset \operatorname{Ker} C_\theta^*$ is trivially satisfied if M is one-to-one. Such a case we met in Example 4.2, where $M = \begin{bmatrix} 1 \end{bmatrix}$.

A. Appendix

In this appendix we present three auxiliary results. The first is used in the proofs of Propositon 2.4 and Proposition 2.6, the second second is used a few times in the proof of Theorem 3.2, and the third plays a role in Section 5, proof of Lemma 5.2.

LEMMA A.1. *Let A be an $n\times n$ matrix with all eigenvalues in $\mathbb{C}_-$, let B be an $n\times k$ matrix, and let $\rho \in \mathscr{W}_-^{k\times m}$. Put*

$$\varphi(\lambda) = (\lambda I_n - A)^{-1} B\rho(\lambda) \quad (\Im\lambda \geqslant 0).$$

Then $\varphi \in \mathscr{W}_{+,0}^{n\times m} \dot{+} \mathscr{W}_{-,0}^{n\times m}$ and $\varphi_{+,0}$ is given by

$$\varphi_{+,0}(\lambda) = (\lambda I_n - A)^{-1} Y, \quad \text{where } Y \in \mathbb{C}^{n\times k} \text{ is defined by} \tag{A.1}$$

$$Y = \frac{1}{2\pi i}\int_\Gamma (\mu I_n - A)^{-1} B\rho(\mu)\, \mathrm{d}\mu\,. \tag{A.2}$$

Here Γ is a contour in $\mathbb{C}_-$ around the eigenvalues of A.

Proof. Let $\rho = \mathscr{F}' r$ with $r \in L^1(\mathbb{R}_-)^{k\times m}$. First we will prove that formula (A.1) holds with Y given by

$$Y = \int_{-\infty}^0 e^{isA} Br(s)\, \mathrm{d}s \in \mathbb{C}^{n\times m}. \tag{A.3}$$

To prove this we begin with a general remark. Let $\eta \in \mathscr{W}_{+,0}^{m\times k}$, and let us assume that $\eta = \mathscr{F}' f$, where f is any function in $L^1(\mathbb{R}_+)^{m\times k}$. Then, according to (1.1), the function $\eta\rho$ is equal to the function $\mathscr{F}'(f\star r)$. In this case, since the support of the function r is contained in $\mathbb{R}_-$, we have

$$(f\star r)(t) = \int_{-\infty}^{\infty} f(t-s)r(s)\,\mathrm{d}s = \int_{-\infty}^{0} f(t-s)r(s)\,\mathrm{d}s, \quad t\in\mathbb{R}.$$

Now put

$$g_+(t) = \begin{cases} \displaystyle\int_{-\infty}^{0} f(t-s)r(s)\,\mathrm{d}s & \text{when } t \geqslant 0,\\ 0 & \text{when } t<0. \end{cases} \tag{A.4}$$

Then $(\eta\rho)_+ = \mathscr{F}' g_+$.

In the remaining part we apply the result of the previous paragraph with $\eta(\lambda) = (\lambda I_n - A)^{-1}B$. Using $\sigma(A)\subset\mathbb{C}_-$ and formula (3) in [8, Section XIII.4] we see that

$$(\lambda I_n - A)^{-1} = -i\int_0^{\infty} e^{i\lambda t}e^{-itA}\,\mathrm{d}t, \quad \lambda\in\mathbb{R}. \tag{A.5}$$

Hence

$$(\lambda I_n - A)^{-1}B = (\mathscr{F}' f)(\lambda), \text{where } f(t) = -ie^{-itA}B. \tag{A.6}$$

Next, we compute g_+ with f being given by (A.6). In this case, using (A.4) and the definition of Y in (A.3), we obtain

$$g_+(t) = -i\int_{-\infty}^{0} e^{-i(t-s)A}Br(s)\,\mathrm{d}s = -ie^{-itA}Y, \quad t\geqslant 0.$$

It follows that

$$\begin{aligned}\varphi_+(\lambda) = (\mathscr{F}' g_+)(\lambda) &= -i\int_0^{\infty} e^{i\lambda t}e^{-itA}Y\,\mathrm{d}t\\ &= \left(-i\int_0^{\infty} e^{i\lambda t}e^{-itA}\,\mathrm{d}t\right)Y, \quad \lambda\in\mathbb{R}.\end{aligned}$$

But then, again using (A.5), we see that the identity in (A.1) is proved with Y given by (A.3).

It remains to prove that this implies that Y is given by (A.2). Note that

$$e^{isA} = \frac{1}{2\pi i}\int_{\Gamma} e^{is\lambda}(\lambda I_n - A)^{-1}\,\mathrm{d}\lambda.$$

Thus

$$\begin{aligned}\int_{-\infty}^{0} e^{isA}Br(s)\,\mathrm{d}s &= \frac{1}{2\pi i}\int_{-\infty}^{0}\Big(\int_{\Gamma} e^{is\lambda}(\lambda I_n - A)^{-1}\,\mathrm{d}\lambda\Big)Br(s)\,\mathrm{d}s\\ &= \frac{1}{2\pi i}\int_{\Gamma}(\lambda I_n - A)^{-1}B\Big(\int_{-\infty}^{0} e^{is\lambda}r(s)\,\mathrm{d}s\Big)\,\mathrm{d}\lambda\\ &= \frac{1}{2\pi i}\int_{\Gamma}(\lambda I_n - A)^{-1}B\rho(\lambda)\,\mathrm{d}\lambda. \quad \square\end{aligned}$$

We shall also need the following dual version of Lemma A.1.

LEMMA A.2. *Let A be an $n\times n$ matrix with all eigenvalues in $\mathbb{C}_+$, let C be a $k\times n$ matrix, and let ρ belong to $\mathscr{W}_+^{m\times k}$. Put*

$$\varphi(\lambda)=\rho(\lambda)C(\lambda I_n-A)^{-1}\quad(\lambda\in\mathbb{R}).$$

Then $\varphi\in\mathscr{W}_{+,0}^{m\times n}\dot{+}\mathscr{W}_{-,0}^{m\times n}$ and $\varphi_{-,0}$ is given by

$$\varphi_{-,0}(\lambda)=Y(\lambda I_n-A)^{-1},\quad \textit{where } Y\in\mathbb{C}^{k\times n} \textit{ is defined by} \tag{A.7}$$

$$Y=\frac{1}{2\pi i}\int_\Gamma \rho(\mu)C(\mu I_n-A)^{-1}\,\mathrm{d}\mu\,. \tag{A.8}$$

Here Γ is a Cauchy contour in $\mathbb{C}_+$ around the eigenvalues of A.

The third lemma is about lower or upper triangular integral operators on a finite interval.

LEMMA A.3. *Let $h\in L^1(\mathbb{R})^{p\times p}$, and assume that the support of h belongs to $[0,\tau]$ or to $[-\tau,0]$. Then the integral operator K on $L^1([0,\tau])^p$ given by*

$$K=I+H,\ \textit{where}\ (Hf)(t)=\int_0^\tau h(t-s)f(s)\,\mathrm{d}s,\quad 0\leqslant t\leqslant\tau, \tag{A.9}$$

is invertible.

Proof. We split the proof into two parts.

PART 1. In this first part we only assume that the support of h belongs to $[-\tau,\tau]$. Let n be a positive integer, and put $\tau_0=\tau/n$. For each $f\in L^1([0,\tau])^p$ and $j\in\{1,2,\ldots,n\}$ let f_j be the function in $L^1([0,\tau_0])^p$ given by

$$f_j(t)=f\left((j-1)\tau_0+t\right),\quad 0\leqslant t\leqslant\tau_0.$$

Put $Jf=(f_1,f_2,\cdots,f_n)$. Then J is an isometry mapping $L^1([0,\tau])^p$ in a one-to-way onto the direct sum $L^1([0,\tau_0])^p\dot{+}L^1([0,\tau_0])^p\dot{+}\cdots\dot{+}L^1([0,\tau_0])^p$. Next, for $\nu=-(n-1),\ldots,(n-1)$, we let H_ν be the operator on $L^1([0,\tau_0])^p$ defined by

$$(H_\nu f)(t)=\int_0^{\tau_0} h(\nu\tau_0+t-s)f(s)\,\mathrm{d}s,\quad f\in L^1([0,\tau_0])^p. \tag{A.10}$$

We claim that

$$JK=\begin{bmatrix} I+H_0 & H_{-1} & \cdots & H_{-(n-1)}\\ H_1 & I+H_0 & & H_{-(n-2)}\\ \vdots & & \ddots & \\ H_{n-1} & H_{n-2} & \cdots & I+H_0\end{bmatrix}J. \tag{A.11}$$

To see this, note that for $0 \leqslant t \leqslant \tau$ we have

$$
\begin{aligned}
(Hf)(t) &= \int_0^\tau h(t-s)f(s)\,\mathrm{d}s = \sum_{\nu=1}^{n} \int_{(\nu-1)\tau_0}^{\nu\tau_0} h(t-s)f(s)\,\mathrm{d}s \\
&= \sum_{\nu=1}^{n} \int_0^{\tau_0} h\left(t-s'-(\nu-1)\tau_0\right) f\left((\nu-1)\tau_0+s'\right)\,\mathrm{d}s' \\
&= \sum_{\nu=1}^{n} \int_0^{\tau_0} h\left(t-s-(\nu-1)\tau_0\right) f_\nu(s)\,\mathrm{d}s.
\end{aligned}
$$

Put $g = Hf$. Then, by definition, for $j \in \{1,2,\dots,n\}$ we have

$$
g_j(t) = g\left((j-1)\tau_0+t\right), \quad 0 \leqslant t \leqslant \tau_0.
$$

Hence for $0 \leqslant t \leqslant \tau_0$ we see that

$$
\begin{aligned}
g_j(t) &= (Hf)\left((j-1)\tau_0+t\right) \\
&= \sum_{\nu=1}^{n} \int_0^{\tau_0} h\left((j-1)\tau_0+t-s-(\nu-1)\tau_0\right) f_\nu(s)\,\mathrm{d}s \\
&= \sum_{\nu=1}^{n} \int_0^{\tau_0} h\left((j-\nu)\tau_0+t-s\right) f_\nu(s)\,\mathrm{d}s = \sum_{\nu=0}^{n-1} \left(H_{j-\nu} f_\nu\right)(t).
\end{aligned}
$$

This proves (A.11).

PART 2. In this part we assume that the support of h belongs to $[-\tau,0]$. The case $\operatorname{supp} h \subset [0,\tau]$ is treated in a similar way. The assumption $\operatorname{supp} h \subset [-\tau,0]$ implies that that the operator H_ν defined (A.10) is zero for $\nu = 1,\dots,(n-1)$, and hence the equality (A.11) reduces to

$$
JKJ^{-1} = \begin{bmatrix} I+H_0 & H_{-1} & \cdots & H_{-(n-1)} \\ & I+H_0 & & H_{-(n-2)} \\ & & \ddots & \vdots \\ & & & I+H_0 \end{bmatrix}. \tag{A.12}
$$

Since the right hand side of (A.12) is a block upper triangular finite operator matrix, it suffices to show that $I+H_0$ is invertible. In general this will not be true. However, if we choose the positive integer n we started with large enough, then $I+H_0$ will be invertible. To see this, let $\chi_{[0,\tau_0]}$ be the function equal to one on the interval $[0,\tau_0]$ and zero otherwise. Recall $\tau_0 = \tau/n$. Thus

$$
\|H_0\| = \int_0^{\tau/n} \|h(t)\|\,\mathrm{d}t = \int_0^\tau \|h(t)\|\chi_{[0,\tau/n]}(t)\,\mathrm{d}t \downarrow 0 \quad (n \to \infty),
$$

by Lebesgue's theorem on integration of monotone sequences. Hence $\|H_0\|$ will be strictly less than one for n sufficiently large. But then $I+H_0$ will be invertible as desired. Since we are free in the choice of n, this proves that K is invertible. $\square$

REFERENCES

[1] H. BART, I. GOHBERG, M. A. KAASHOEK AND A. C. M. RAN, *A state space approach to canonical factorization*, Birkhäuser Verlag, Basel, 2010.

[2] M. J. CORLESS AND A. E. FRAZHO, *Linear systems and control, An operator perspective*, Marcel Dekker, Inc, New York, 2003.

[3] R. L. ELLIS, *An Identity Satisfied by Certain Orthogonal Vector-Valued Functions*, in A Panorama of Modern Operator Theory and Related Topics. The Israel Gohberg Memorial Volume, OT **218**, Birkhäuser Verlag, Basel, 2012, pp. 329–344.

[4] R. L. ELLIS AND I. GOHBERG, *Orthogonal systems related to infinite Hankel matrices*, J. Functional Analysis **109** (1992), 155–198.

[5] R. L. ELLIS AND I. GOHBERG, *Orthogonal systems and convolution operators*, OT **140**, Birkhäuser Verlag, Basel, 2003.

[6] R. L. ELLIS, I. GOHBERG, AND D. C. LAY, *Infinite analogues of block Toeplitz matrices and related orthogonal functions*, Integral Equations and Operator Theory **22** (1995), 375–419.

[7] A. E. FRAZHO, M. A. KAASHOEK, AND A. C. M. RAN, *Rational Matrix Solutions of a Bezout Type Equation on the Half-plane*, in: Advances in Structured Operator Theory and Related Areas. The Leonid Lerer Anniversary Volume, Oper. Theory Adv. Appl. **237**, Springer Basel AG 2012, pp. 145–160.

[8] I. GOHBERG, S. GOLDBERG, AND M. A. KAASHOEK, *Classes of Linear Operators*, Volume I, Birkhäuser Verlag, Basel, 1990.

[9] I. GOHBERG, S. GOLDBERG, AND M. A. KAASHOEK, *Classes of Linear Operators*, Volume II, Birkhäuser Verlag, Basel, 1993.

[10] I. GOHBERG, M. A. KAASHOEK AND F. VAN SCHAGEN, *Partially Specified Matrices and Operators: Classification, Completion, Applications*, Birkhäuser Verlag, Basel, 1990.

[11] C. HEIJ, A. C. M. RAN AND F. VAN SCHAGEN, *Introduction to Mathematical Systems Theory*, Birkhäuser Verlag, Basel, 2007.

[12] M. A. KAASHOEK AND F. VAN SCHAGEN, *Ellis-Gohberg identities for certain orthogonal functions I: Block matrix generalizations and ℓ^2-setting*, Indag. Math. **23** (2012) 777–795.

[13] M. A. KAASHOEK AND F. VAN SCHAGEN, *Ellis-Gohberg identities for certain orthogonal functions II: Algebraic setting and asymmetric versions*, West Memorial Issue, Proc. Math. Royal Irish Acad., **113A** (2) (2013), 107–130.

[14] M. A. KAASHOEK AND F. VAN SCHAGEN, *The inverse problem for Ellis-Gohberg orthogonal matrix functions*, Integral Equ. Oper. Theory **80** (2014), 527–555.

[15] M. G. KREIN, *On the distribution of the roots of polynomials which are orthogonal on the unit circle with respect to an alternating weight*, Teor. Functsiĭ Funkstional Anal. i Prlozhen. **2** (1966) 131–137, (in Russian).

[16] P. LANCASTER AND L. RODMAN, *Algebraic Riccati Equations*, Oxford University Press, Oxford, 1995.

[17] P. LANCASTER AND M. TISMENETSKY, *The Theory of Matrices*, Academic Press, Orlando, 1985.

[18] S. LANG, *Algebra*, Addison-Wesley Publ. Co, Reading MA, 1965.

[19] V. V. PELLER, *Hankel operators and their applications*, Springer-Verlag, New York, 2003.

[20] E. D. SONTAG, *Mathematical Control Theory*, Springer-Verlag, New York, 1998.

(Received September 13, 2015)

M. A. Kaashoek
Department of Mathematics, VU University
Amsterdam, The Netherlands
e-mail: m.a.kaashoek@vu.nl

F. van Schagen
Department of Mathematics, VU University
Amsterdam, The Netherlands
e-mail: f.van.schagen@vu.nl