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PRE–IMAGES OF EXTREME POINTS OF THE NUMERICAL RANGE, AND APPLICATIONS

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In memory of Leiba Rodman, a wonderful mathematician, an impeccable colleague, and a dear friend

(*Communicated by Y.-T. Poon*)

Abstract. We extend the pre-image representation of exposed points of the numerical range of a matrix to all extreme points. With that we characterize extreme points which are multiply generated, having at least two linearly independent pre-images, as the extreme points which are Hausdorff limits of flat boundary portions on numerical ranges of a sequence converging to the given matrix. These studies address the inverse numerical range map and the maximum-entropy inference map which are continuous functions on the numerical range except possibly at certain multiply generated extreme points. This work also allows us to describe closures of subsets of 3-by-3 matrices having the same shape of the numerical range.

1. Introduction

We denote the set of complex $d \times d$ matrices by M_d, $d \in \mathbb{N}$, with identity matrix $\mathbb{1}$. The *numerical range* of $A \in M_d$ is the subset

$$W(A) := \{f_A(x) \mid x \in S\mathbb{C}^d\}$$

of the complex plane $\mathbb{C}$ where $f_A : S\mathbb{C}^d \to \mathbb{C}, x \mapsto x^*Ax$ is the *numerical range map* defined on the unit sphere $S\mathbb{C}^d = \{x \in \mathbb{C}^d \mid x^*x = 1\}$ of $\mathbb{C}^d$. On $\mathbb{C} \cong \mathbb{R}^2$ we use the standard Euclidean scalar product $\langle \alpha, \beta \rangle = \Re(\alpha\overline{\beta})$ for $\alpha, \beta \in \mathbb{C}$. The numerical range is a compact and convex subset, the convexity statement being known as the Toeplitz-Hausdorff theorem [31, 15], more recent work includes [1, 17, 25].

We are interested in points $\alpha \in W(A)$ which are

- *extreme points*, that is α cannot be written as a proper convex combination of points in $W(A)$,
- *multiply generated* [23, 24, 29], that is $f_A^{-1}(\alpha)$ contains at least two linearly independent vectors.

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Part of our interest in these points comes from a continuity problem in operator theory. The (multi-valued) inverse $f_A^{-1} : W(A) \to S\mathbb{C}^d$ of f_A is *strongly continuous* at $\alpha \in W(A)$ if for all $x \in f_A^{-1}(\alpha)$ the function f_A is open at x. Strong continuity holds on $W(A)$ except at certain multiply generated extreme points. A *round boundary point* (see [7, 23, 24] and Lemma 6.1 of [29]) is an extreme point of $W(A)$ which lies on a unique *supporting line* $\{\alpha \in \mathbb{C} \mid \langle \alpha, e^{\mathrm{i}\theta} \rangle = h(e^{\mathrm{i}\theta})\}$ where $e^{\mathrm{i}\theta}$, $\theta \in \mathbb{R}$, is an outward pointing normal vector and $h(e^{\mathrm{i}\theta}) = \max_{\alpha \in W(A)} \langle \alpha, e^{\mathrm{i}\theta} \rangle$. The map f_A^{-1} is strongly continuous on $W(A)$ except possibly at multiply generated round boundary points [7] but may be strongly continuous also there. A characterization of strong continuity of f_A^{-1}, in terms of analytic eigenvalue curves [24] shows that f_A^{-1} has at most finitely many discontinuity points. The corresponding eigenvector curves will be discussed in Coro. 2.5 where we consider their intersection with pre-images of extreme points under f_A.

Further interest in multiply generated extreme points comes from an optimization problem in quantum mechanics. The *maximum-entropy inference*, going back to ideas by Boltzmann, is a method to select a quantum state from the expected values of a collection of physical quantities represented by hermitian matrices when all other information about the state is ignored [18]. The numerical range $W(A)$ is the set of expected values of two hermitian matrices given implicitly by the *real part* $\Re(A) = \frac{1}{2}(A + A^*)$ and *imaginary part* $\Im(A) = \frac{1}{2\mathrm{i}}(A - A^*)$ of A [2, 29]. The inference map is

$$\rho^* : W(A) \to \mathcal{M}_d, \quad \alpha \mapsto \operatorname{argmax}\{S(\rho) \mid \rho \in \mathcal{M}_d, \operatorname{tr}(A\rho) = \alpha\}$$

with *state space* $\mathcal{M}_d$ consisting of all positive semi-definite matrices of trace one and *von Neumann entropy* $S(\rho) = -\operatorname{tr}\rho\log(\rho)$. A discontinuity of ρ^* may occur [37]. Note, however, that for normal A its numerical range $W(A)$ is a polytope, $\Re(A)$ and $\Im(A)$ commute, and ρ^* is continuous [35]. So, a discontinuity belongs to the proper quantum domain where it is discussed in the context of quantum phases [5, 20].

As it happens, the described continuity problems are equivalent. The map ρ^* is continuous at $\alpha \in W(A)$ if and only if f_A^{-1} is strongly continuous at α [36], and ρ^* is indeed discontinuous at all *isolated* multiply generated round boundary points of $W(A)$, see Sec. 6 of [29]. Further, it is known that multiply generated round boundary points are isolated for $d = 3$. Calling $A \in M_d$ *unitarily reducible* if A is unitarily similar to a block diagonal matrix with two proper blocks and *unitarily irreducible* otherwise, multiply generated round boundary points are isolated for all irreducible matrices of size $d \leqslant 5$ [23]. But, the whole boundary of $W(A)$ may consist of multiply generated round boundary points for reducible 4-by-4 matrices and irreducible 6-by-6 matrices [23].

Here we study these continuity problems from a topological perspective of a variable matrix $A \in M_d$. To this end we define a *flat boundary portion* of $W(A)$ as a maximal proper segment in the boundary of $W(A)$ [21, 4, 28, 9]. If $\alpha \in W(A)$, then we say that *a flat boundary portion is born at* α if there exists a sequence $(A_i)_{i\in\mathbb{N}} \subset M_d$ converging to A such that a sequence $(s_i)_{i\in\mathbb{N}}$ of flat boundary portions converges to $\{\alpha\}$ in the Hausdorff distance, the segment s_i being a flat boundary portion of $W(A_i)$, $i \in \mathbb{N}$.

The birth of a flat boundary portion was conjectured in Sec. I.B of [37] to be a condition for a discontinuity of ρ^*. The above discussion and the following theorem prove that the birth of a flat boundary portion at a round boundary point is a necessary condition. It is not a sufficient condition for $d \geqslant 4$ ($d \geqslant 6$ if A is unitarily irreducible).

THEOREM 1.1. *Let α be an extreme point of $W(A)$. For a flat boundary portion to be born at α it is necessary and sufficient that α is multiply generated.*

The necessity in Thm. 1.1 follows from properties of the Hausdorff distance (Sec. 3). We prove the sufficiency in Sec. 2 using a newly developed representation of extreme points in terms of pre-images. Pre-images were well-understood [31] for exposed points. These extreme points can be represented as the intersection of $W(A)$ with a supporting line. If the outward pointing normal vector is $e^{\mathrm{i}\theta}$ then $h(e^{\mathrm{i}\theta})$ is the maximal eigenvalue of the hermitian matrix $\cos(\theta)\Re(A)+\sin(\theta)\Im(A)$, the corresponding eigenspace is the demanded pre-image.

Some extreme points may fail to be exposed. Consider, e.g., the convex hull of a circle and a point outside the circle, which is realized as the numerical range of $A=\left[\begin{smallmatrix}0&2&0\\0&0&0\\0&0&2\end{smallmatrix}\right]$. Then the intersections of the two tangents from the point to the circle with the circle are extreme but non-exposed points. We will obtain the pre-images of non-exposed points by viewing them as exposed points of some flat boundary portion whose supporting lines are given by directional derivatives of h [3]. Notice that viewing non-exposed points as exposed points of a convex subset is a familiar idea in convex geometry [13, 33] (cf. *poonem*), geometry of quantum states [32], and in the theory of exponential families [8, 34] (cf. *access sequence*).

Finally, in Sec. 4 we combine observations from [14, 22, 21, 28, 23, 26] with Thm. 1.1 to compute closures of subsets of 3-by-3 matrices having the same shape of the numerical range. In particular, we prove in Sec. 5 that a 3-by-3 matrix A with $W(A)$ having a non-empty interior lies in the closure of irreducible matrices with elliptical numerical range and in the closure of irreducible matrices with flat portion on the boundary of their numerical range if and only if $W(A)$ is an ellipse with an eigenvalue of A on the boundary. Remarkably, these are precisely the 3-by-3 matrices A where f_A^{-1} and ρ^* have a discontinuity [7, 23, 29].

2. Representation of extreme points

We provide a pre-image representation of extreme points of the numerical range. We rewrite it in terms of eigenvalue curves and we use it to prove one part of Thm. 1.1.

Recall that the *relative interior* of a subset C of a Euclidean space $(\mathbb{E},\langle\cdot,\cdot\rangle)$ is the interior of C in the affine hull of C. A *face* of a convex set C is a convex subset $F\subset C$ which contains every closed segment in C the relative interior of which intersects F. The element of a singleton face is called *extreme point*. The *support function* of a *convex body* C, that is a non-empty compact convex subset of $\mathbb{E}$, is

$$h_C(u) := \max_{x\in C}\langle x,u\rangle, \qquad u\in\mathbb{E}.$$

The *supporting hyperplane* of C with outward pointing normal vector $u \in \mathbb{E} \setminus \{0\}$ is the set of $x \in \mathbb{E}$ on the hyperplane $\langle x, u\rangle = h_C(u)$. The intersection of this hyperplane with C is the *exposed face*

$$\operatorname*{argmax}_{x \in C} \langle x, u\rangle$$

with outward pointing normal vector u. If $\{\alpha\}$ is an exposed face, then α is called an *exposed point*. All exposed points are extreme points. The remaining extreme points are called *non-exposed points*.

Notice that a supporting hyperplane of $W(A)$ is a supporting line. Since $W(A)$ has (real) dimension at most two, a subset is a one-dimensional face if and only if it is a flat boundary portion, and every flat boundary portion is an exposed face. Moreover, the boundary of $W(A)$ is the disjoint union of extreme points and relative interiors of flat boundary portions. So, every extreme point α is an exposed point, an endpoint of a flat boundary portion, or both. It is crucial in the sequel that every non-exposed point of $W(A)$ is an exposed point of some flat boundary portion of $W(A)$.

Let

$$A(\theta) := \Re(e^{-\mathrm{i}\theta} A) = \cos(\theta)\Re(A) + \sin(\theta)\Im(A), \qquad \theta \in \mathbb{R}.$$

We denote by $X_{\mathrm{m}}(\theta)$ the eigenspace of $A(\theta)$ corresponding to the maximal eigenvalue $\lambda_{\mathrm{m}}(\theta)$ of $A(\theta)$. An easy computation gives

$$\langle f_A(x), e^{\mathrm{i}\theta}\rangle = \Re(x^* A x e^{-\mathrm{i}\theta}) = f_{A(\theta)}(x), \qquad x \in S\mathbb{C}^d. \tag{2.1}$$

The maximal eigenvalue $\lambda_{\mathrm{m}}(\theta)$ of $A(\theta)$ has the geometric meaning of support function of the numerical range, see Sec. 4 of [31],

$$h_{W(A)}(e^{\mathrm{i}\theta}) = \lambda_{\mathrm{m}}(\theta). \tag{2.2}$$

Let $F_A(\theta)$ denote the exposed face of $W(A)$ with outward pointing normal vector $e^{\mathrm{i}\theta}$, $\theta \in \mathbb{R}$.

LEMMA 2.1. *The point $f_A(x)$, $x \in S\mathbb{C}^d$, lies in the exposed face $F_A(\theta)$, $\theta \in \mathbb{R}$, if and only if $x \in X_{\mathrm{m}}(\theta)$.*

Proof. We consider the orthogonal direct sum $\mathbb{C}^d = X_{\mathrm{m}}(\theta) \oplus X_{\mathrm{m}}(\theta)^{\perp}$. A unit vector $x \in S\mathbb{C}^d$ has a unique decomposition $x = y + z$ for $y \in X_{\mathrm{m}}(\theta)$ and $z \in X_{\mathrm{m}}(\theta)^{\perp}$. So

$$\langle f_A(x), e^{\mathrm{i}\theta}\rangle = \lambda_{\mathrm{m}}(\theta) + z^*(A(\theta) - \lambda_{\mathrm{m}}(\theta)\mathbb{1})z$$

has the maximal value $\lambda_{\mathrm{m}}(\theta)$ only for $z = 0$. This proves the claim. $\square$

Non-exposed points of the numerical range are not addressed in Lemma 2.1. To describe them in terms of pre-images, we view them as exposed points of a flat boundary portion. The two endpoints $p_{\pm}(\theta)$ (not necessarily distinct) of the exposed face $F_A(\theta)$ are characterized by their membership in $F_A(\theta)$ and by the equality

$$\langle p_{\pm}(\theta), \pm\mathrm{i}e^{\mathrm{i}\theta}\rangle = h_{F_A(\theta)}(\pm\mathrm{i}e^{\mathrm{i}\theta}).$$

To evaluate the support function of $F_A(\theta)$ we consider the directional derivative of a function $f:\mathbb{R}^m\to\mathbb{R}^n$ at a point $u\in\mathbb{R}^m$ in direction $v\in\mathbb{R}^m$, which is defined by

$$f'(u;v) := \lim_{t\to 0^+} \tfrac{1}{t}(f(u+tv)-f(u)),$$

if the limit exists. Given a convex body C in a Euclidean space $\mathbb{E}$ and $u\in\mathbb{E}\setminus\{0\}$, the support function h_G of the exposed face G of C with outward pointing normal vector u is $h_G(v)=h'_C(u;v)$, $v\in\mathbb{E}$, see Section 16 of [3].

Since the chain rule does not hold for directional derivatives, we need to go into some detail in the following proof.

LEMMA 2.2. *The support function of the exposed face $F_A(\theta)$, $\theta\in\mathbb{R}$, has the values $h_{F_A(\theta)}(\pm\mathrm{i}e^{\mathrm{i}\theta})=\lambda'_{\mathrm{m}}(\theta;\pm 1)$. The endpoints of $F_A(\theta)$ are $p_\pm(\theta)=e^{\mathrm{i}\theta}(\lambda_{\mathrm{m}}(\theta)\pm \mathrm{i}\lambda'_{\mathrm{m}}(\theta;\pm 1))$.*

Proof. The equation for support functions of exposed faces, recalled in the previous paragraph, proves

$$h_{F_A(\theta)}(\pm\mathrm{i}e^{\mathrm{i}\theta}) = h'_{W(A)}(e^{i\theta};\pm\mathrm{i}e^{\mathrm{i}\theta}).$$

For all $s,t>0$ with $s=\arctan(t)$, and writing $h=h_{W(A)}$, we have

$$\tfrac{1}{t}(h(e^{\mathrm{i}\theta}\pm t\,\mathrm{i}e^{\mathrm{i}\theta})-h(e^{\mathrm{i}\theta})) = \tfrac{1}{s}(h(e^{\mathrm{i}(\theta\pm s)})-h(e^{\mathrm{i}\theta}))+g(s)$$

where $g(s)\to 0$ for $s\to 0$, because h is positive homogeneous of the first degree [3]. Taking the limit $t\to 0^+$ gives

$$h'_{W(A)}(e^{\mathrm{i}\theta};\pm\mathrm{i}e^{\mathrm{i}\theta}) = (h_{W(A)}\circ e^{\mathrm{i}\theta})'(\theta;\pm 1).$$

The values of $h_{F_A(\theta)}$ follow from (2.2) which provides $h_{W(A)}\circ e^{\mathrm{i}\theta}=\lambda_{\mathrm{m}}(\theta)$. The formula for the endpoints is obvious. $\square$

We are ready to describe $p_\pm(\theta)$ in terms of its pre-image under f_A. Noticing $A'(\theta)=\Im(e^{-\mathrm{i}\theta}A)$, where “ $'$ ” denotes derivative with respect to θ, an easy calculation shows that for all $y\in S\mathbb{C}^d$

$$\langle f_A(y),\mathrm{i}e^{\mathrm{i}\theta}\rangle = f_{A'(\theta)}(y) \tag{2.3}$$

holds. Equations (2.1) and (2.3) show that for all $y\in S\mathbb{C}^d$ we have

$$f_A(y) = e^{\mathrm{i}\theta}(f_{A(\theta)}(y)+\mathrm{i}f_{A'(\theta)}(y)). \tag{2.4}$$

We denote by $B|_X$ the compression of $B\in M_d$ onto a subspace $X\subset\mathbb{C}^d$, that is $B|_X$ is the restriction of PBP to X where P is the orthogonal projection onto X. For $\theta\in\mathbb{R}$ define $g_\theta:\mathbb{C}\to\mathbb{R},\alpha\mapsto\langle\alpha,\mathrm{i}e^{\mathrm{i}\theta}\rangle$ and $h_\theta:\mathbb{R}\to\mathbb{C},\eta\mapsto e^{\mathrm{i}\theta}(\lambda_{\mathrm{m}}(\theta)+\mathrm{i}\eta)$.

THEOREM 2.3. *For all $\theta\in\mathbb{R}$ the maximal (respectively, minimal) eigenvalue of $A'(\theta)|_{X_{\mathrm{m}}(\theta)}$ is $\lambda'_{\mathrm{m}}(\theta;1)$ (respectively, $-\lambda'_{\mathrm{m}}(\theta;-1)$). For all unit vectors $x\in X_{\mathrm{m}}(\theta)$ we have $f_{A'(\theta)|_{X_{\mathrm{m}}(\theta)}}(x)=g_\theta\circ f_A(x)$. The map $g_\theta|_{F_A(\theta)}:F_A(\theta)\to s$ is a bijection to the segment $s\subset\mathbb{R}$ with endpoints $\pm\lambda'_{\mathrm{m}}(\theta;\pm 1)$, the inverse is $h_\theta|_s$.*

Proof. The pre-image of the exposed face $F_A(\theta)$ is by Lemma 2.1 equal to $S\mathbb{C}^d \cap X_{\mathrm{m}}(\theta)$. By definition of the support function of $F_A(\theta)$ we get for all $x \in S\mathbb{C}^d \cap X_{\mathrm{m}}(\theta)$ the inequalities

$$-h_{F_A(\theta)}(-\mathrm{i}e^{\mathrm{i}\theta}) \leqslant \langle f_A(x), \mathrm{i}e^{\mathrm{i}\theta}\rangle \leqslant h_{F_A(\theta)}(\mathrm{i}e^{\mathrm{i}\theta}).$$

Both equalities are attained because $F_A(\theta)$ is compact. Using Lemma 2.2 and (2.3), the above inequality is equivalent to

$$-\lambda_{\mathrm{m}}'(\theta;-1) \leqslant x^* A'(\theta) x \leqslant \lambda_{\mathrm{m}}'(\theta;+1)$$

which shows that the hermitian operator $A'(\theta)|_{X_{\mathrm{m}}}$ has minimal eigenvalue $-\lambda_{\mathrm{m}}'(\theta;-1)$ and maximal eigenvalue $\lambda_{\mathrm{m}}'(\theta;+1)$.

The numerical range of the hermitian operator $A'(\theta)|_{X_{\mathrm{m}}(\theta)}$ is the segment s between its extreme eigenvalues $\pm\lambda_{\mathrm{m}}'(\theta;\pm 1)$. By (2.2) and (2.4) we have for all $x \in S\mathbb{C}^d \cap X_{\mathrm{m}}(\theta)$

$$f_A(x) = e^{\mathrm{i}\theta}(\lambda_{\mathrm{m}}(\theta) + \mathrm{i} f_{A'(\theta)}(x)).$$

Since $f_{A'(\theta)}(x) = \langle f_A(x), \mathrm{i}e^{\mathrm{i}\theta}\rangle$ holds, again by (2.3), the functions g_θ and h_θ have the claimed properties. $\square$

We can now compute the pre-images of all extreme points. Thm. 2.3 proves that $\pm\lambda_{\mathrm{m}}'(\theta;\pm 1)$ is an eigenvalue of $A'(\theta)|_{X_{\mathrm{m}}}$. We denote the corresponding eigenspace by $X_\pm(\theta)$.

COROLLARY 2.4. *The point $f_A(x)$, $x \in S\mathbb{C}^d$, is the endpoint $p_\pm(\theta)$, $\theta \in \mathbb{R}$, of the exposed face $F_A(\theta)$ if and only if $x \in X_\pm(\theta)$.*

Proof. This follows from Thm. 2.3, and from Lemma 2.1 applied to the exposed points $\pm\lambda_{\mathrm{m}}'(\theta;\pm 1)$ of the numerical range of the compression $A'(\theta)|_{X_{\mathrm{m}}(\theta)}$. $\square$

We describe the eigenspace $X_\pm(\theta)$ of $A'(\theta)|_{X_{\mathrm{m}}}$ in terms of eigenvalue curves $\{\lambda_k(\theta)\}_{k=1}^d$ of $A(\theta)$ and mutually orthogonal eigenvectors $\{x_k(\theta)\}_{k=1}^d$ which depend real analytically on the parameter θ [27]. Thus $A(\theta) = \sum_{k=1}^d \lambda_k(\theta) x_k(\theta) x_k(\theta)^*$ and we have for $k = 1, \ldots, d$

$$\lambda_k(\theta) = f_{A(\theta)}(x_k(\theta)), \qquad \theta \in \mathbb{R}.$$

We recall that the derivative of λ_k with respect to θ (see Lemma 3.2 of [19] and Sec. 5 of [11]) is

$$\lambda_k'(\theta) = f_{A'(\theta)}(x_k(\theta)), \qquad \theta \in \mathbb{R}. \tag{2.5}$$

We define

$$I_{\mathrm{m}}(\theta) := \{i \in \{1, \ldots, d\} \mid \lambda_i(\theta) = \lambda_{\mathrm{m}}(\theta)\}$$

and

$$I_\pm(\theta) := \{i \in I_{\mathrm{m}}(\theta) \mid \lambda_i'(\theta) = \pm\lambda_{\mathrm{m}}'(\theta;\pm 1)\}.$$

COROLLARY 2.5. *The span of $\{x_k(\theta) \mid k \in I_\pm(\theta)\}$ is $X_\pm(\theta)$, $\theta \in \mathbb{R}$.*

Proof. By Thm. 2.3 the eigenvalue $\pm\lambda_{\mathrm{m}}'(\theta;\pm 1)$ of the hermitian operator $A'(\theta)|_{X_{\mathrm{m}}(\theta)}$ is extreme. Hence, (2.5) shows that $X_{\pm}(\theta)$ contains all eigenvectors $x_k(\theta)$ of $A(\theta)$ with $k \in I_{\pm}(\theta)$. For all $k \in I_{\mathrm{m}}(\theta) \setminus I_{\pm}(\theta)$ we have by (2.5)

$$x_k(\theta)^* A'(\theta) x_k(\theta) = \lambda_k'(\theta) \lessgtr \pm\lambda_{\mathrm{m}}'(\theta;\pm 1).$$

Therefore $X_{\pm}(\theta)$ is the span of $\{x_k(\theta) \mid k \in I_{\pm}(\theta)\}$. $\square$

We finish the section with an application. See the next section for the definition of the Hausdorff distance.

PROPOSITION 2.6. *Let α be a multiply generated extreme point of the numerical range $W(A)$. Then a flat boundary portion is born at α.*

Proof. Without loss of generality let $\alpha = p_\sigma(\theta)$ for some $\theta \in \mathbb{R}$ and some $\sigma \in \{+,-\}$. Since $X_\sigma(\theta)$ belongs to the eigenspace $X_{\mathrm{m}}(\theta)$ of $A(\theta)$ corresponding to the maximal eigenvalue $\lambda_{\mathrm{m}}(\theta)$ of $A(\theta)$, the equation (2.4) implies that for all $x \in S\mathbb{C}^d \cap X_\sigma(\theta)$

$$f_A(x) = e^{\mathrm{i}\theta}(\lambda_{\mathrm{m}}(\theta) + \mathrm{i} f_{A'(\theta)}(x)).$$

Let P denote the orthogonal projection onto $X_\sigma(\theta)$. Coro. 2.4 shows that $X_\sigma(\theta)$ is the pre-image of α under f_A. Hence, there exists $\lambda \in \mathbb{R}$ such that $A'(\theta)|_{X_\sigma(\theta)} = \lambda P|_{X_\sigma(\theta)}$, for otherwise $f_A(X_\sigma(\theta))$ could not be a singleton. As the numerical range of $\lambda P|_{X_\sigma(\theta)}$ is $\{\lambda\}$, Thm. 2.3 proves $\alpha = e^{\mathrm{i}\theta}(\lambda_{\mathrm{m}}(\theta) + \mathrm{i}\lambda)$.

By assumption, α is multiply generated, so $\dim_{\mathbb{C}}(X_\sigma(\theta)) \geqslant 2$ holds. Choose any hermitian matrix H with $H|_{X_\sigma(\theta)}$ not being a scalar multiple of the identity and denote its maximal eigenvalue by μ_+ and minimal eigenvalue by μ_-. Let $\varepsilon > 0$, and define

$$A_\varepsilon := e^{\mathrm{i}\theta}(A(\theta) + \varepsilon P + \mathrm{i}(A'(\theta) + \varepsilon H)).$$

The numerical range of $(A'(\theta) + \varepsilon H)|_{X_\sigma(\theta)}$ is the segment between the two distinct reals $\lambda + \varepsilon\mu_\pm$. Since $X_\sigma(\theta)$ is the eigenspace of $\Re(e^{-\mathrm{i}\theta}A_\varepsilon)$ corresponding to the maximal eigenvalue $\lambda_{\mathrm{m}}(\theta) + \varepsilon$ of $\Re(e^{-\mathrm{i}\theta}A_\varepsilon)$, the exposed face $F_{W(A_\varepsilon)}(\theta)$ of $W(A_\varepsilon)$ has by Thm. 2.3 the endpoints

$$e^{\mathrm{i}\theta}(\lambda_{\mathrm{m}}(\theta) + \varepsilon + \mathrm{i}(\lambda + \varepsilon\mu_\pm)).$$

Clearly, $F_{W(A_\varepsilon)}(\theta)$ is a flat boundary portion of $W(A_\varepsilon)$ which converges for $\varepsilon \to 0$ in the Hausdorff distance to $\{\alpha\}$ while A_ε converges to A. This completes the proof. $\square$

3. Hausdorff distance

We address the Hausdorff convergence of numerical ranges and show that a flat boundary portion can only be born at a point of the numerical range which is multiply generated.

We denote by $|\cdot|$ the Euclidean norm in a Euclidean space $(\mathbb{E}, \langle\cdot,\cdot\rangle)$. The set of non-empty compact subsets of $\mathbb{E}$ is a complete metric space with respect to the *Hausdorff distance*

$$d_H(K,L) := \max(\max_{x\in K}\min_{y\in L}|y-x|, \max_{y\in L}\min_{x\in K}|y-x|),$$

where $\emptyset \neq K, L \subset \mathbb{E}$ are compact. The set of all convex bodies in $\mathbb{E}$ is closed in this metric space by Thm. 1.8.6 of [30].

We first recall a condition for the Hausdorff convergence.

REMARK 3.1. (Thm. 1.8.8 in [30]) A sequence $(K_i)_{i\in\mathbb{N}}$ of convex bodies in $\mathbb{E}$ converges to a convex body K in $\mathbb{E}$ if and only if the following two conditions hold.

1. Each point in K is the limit of a sequence $(x_i)_{i\in\mathbb{N}}$ with $x_i \in K_i$ for $i \in \mathbb{N}$.
2. The limit of any convergent subsequence $(x_{i_j})_{j\in\mathbb{N}}$ with $x_{i_j} \in K_{i_j}$ for $j \in \mathbb{N}$ belongs to K.

The simplest example of a non-converging sequence illustrates that subsequences are needed in (2). Consider for example $K = \{0\}$ and the sequence defined by $K_{2i} = K$ and $K_{2i+1} = [0,1]$, $i \in \mathbb{N}$.

REMARK 3.2. (Hausdorff convergence of the numerical range)

1. Remark 3.1 and the continuity for every $x \in \mathbb{C}^d$ of the linear function $M_d \to \mathbb{C}$, $A \mapsto x^*Ax$ in A prove that

$$A_i \overset{i\to\infty}{\longrightarrow} A \quad \Longrightarrow \quad W(A_i) \overset{i\to\infty}{\longrightarrow} W(A), \qquad (A_i)_{i\in\mathbb{N}} \subset M_d.$$

   Here, the first limit is in any norm, the second limit is in the Hausdorff distance.

2. The support function (2.2) of $W(A)$ in the direction $e^{\mathrm{i}\theta}$ is jointly continuous in $A \in M_d$ and $\theta \in \mathbb{R}$ because the maximal eigenvalue of a hermitian matrix is a continuous function of the matrix. Therefore, if $\theta = \lim_{i\to\infty}\theta_i$, $(\theta_i)_{i\in\mathbb{N}} \subset \mathbb{R}$, $A = \lim_{i\to\infty} A_i$, $(A_i)_{i\in\mathbb{N}} \subset M_d$, and if the exposed faces $F_{A_i}(\theta_i)$ converge in the Hausdorff distance, then their limit is a subset of $F_A(\theta)$.

We now complete the proof of Thm. 1.1. Recall from Section 2, third paragraph, that a flat boundary portion of $W(A)$ is an exposed face of dimension one.

LEMMA 3.3. *If a flat boundary portion is born at $\alpha \in W(A)$, then α is multiply generated.*

Proof. Let $(s_i)_{i\in\mathbb{N}}$ be a sequence of segments, each s_i being a flat boundary portion of $W(A_i)$, $i \in \mathbb{N}$, and assume that $A = \lim_{i\to\infty} A_i$ and that $s_i \overset{i\to\infty}{\to} \{\alpha\}$ converges in the Hausdorff distance. The segments s_i are exposed faces of $W(A_i)$. So, for every

$i \in \mathbb{N}$ there exists $\theta_i \in \mathbb{R}$ such that $e^{\mathrm{i}\theta_i}$ is an outward pointing normal vector of s_i. Lemma 2.1 implies

$$f_{A_i}^{-1}(s_i) = S\mathbb{C}^d \cap H_i$$

for some subspace $H_i \subset \mathbb{C}^d$. The segments s_i are no singletons so $\dim_{\mathbb{C}}(H_i) \geqslant 2$. This proves that $f_{A_i}^{-1}(s_i)$ contains two orthogonal vectors p_i, q_i. The compactness of $S\mathbb{C}^d$ proves $p_{i_j} \overset{j\to\infty}{\to} p$ and $q_{i_j} \overset{j\to\infty}{\to} q$ for suitable $p,q \in S\mathbb{C}^d$ and a suitable subsequence. Note that p and q are orthogonal. Since $f_A(p)$ is jointly continuous in A and p, and since $f_{A_i}(p_i) \in s_i$ for all $i \in \mathbb{N}$, this shows

$$\begin{aligned} |f_A(p) - \alpha| &= \lim_{j\to\infty} |f_{A_{i_j}}(p_{i_j}) - \alpha| \leqslant \limsup_{j\to\infty} \max_{\beta \in s_{i_j}} |\beta - \alpha| \\ &\leqslant \limsup_{j\to\infty} d_H(s_{i_j}, \{\alpha\}) = 0. \end{aligned}$$

Similarly, $f_A(q) = \alpha$ holds and hence α is multiply generated. $\square$

Before turning to other subjects we add a statement about the Hausdorff convergence of ellipses in the plane. Thereby an *ellipse* is the zero set of a real quadratic form $q : \mathbb{R}^2 \to \mathbb{R}$, $x \mapsto x^*Sx + b^*x + c$ where $S \in M_2$ is a real symmetric and positive definite 2-by-2 matrix, $b \in \mathbb{R}^2$, and $c \in \mathbb{R}$. We call the level set $\{x \in \mathbb{R}^2 \mid f(x) \leqslant 0\}$, which is the convex hull of the ellipse, also an ellipse.

REMARK 3.4. If a sequence of ellipses in $\mathbb{R}^2$ converges in the Hausdorff distance then the limit is an ellipse, a segment, or a point. This can be proved by representing each ellipse as a linear image of the unit disk and by using Rem. 3.1 and compactness arguments.

4. 3-by-3 matrices

Kippenhahn's [22] representation of the numerical range of a $d \times d$ matrix A in terms of the convex hull of a planar real affine algebraic curve provides a classification of the possible shapes of the numerical range of a 3-by-3 matrix whose equivalence classes are well-understood [21, 28] in terms of matrix entries and of spectral data of A. We compute the closures of these equivalence classes.

Using the hermitian real and imaginary parts of $A = \Re(A) + \mathrm{i}\Im(A)$, the homogeneous polynomial

$$F(y_0, y_1, y_2) := \det(y_0\mathbb{1} + y_1\Re(A) + y_2\Im(A))$$

defines a curve $S_A := \{(y_0 : y_1 : y_2) \in \mathbb{PC}^3 \mid F(y_0, y_1, y_2) = 0\}$ in the complex projective plane $\mathbb{PC}^3$. The dual curve $S_A^\wedge$ is an algebraic curve in $\mathbb{PC}^3$ which consists roughly speaking of the tangent lines to S_A [12, 10]. Thereby, a line $\{(y_0 : y_1 : y_2) \in \mathbb{PC}^3 \mid x_0y_0 + x_1y_1 + x_2y_2 = 0\}$ is identified with the point $(x_0 : x_1 : x_2) \in \mathbb{PC}^3$. The *boundary generating curve* $S_A^\wedge(\mathbb{R})$ of A is the real part of the affine component $x_0 = 1$ of $S_A^\wedge$.

One can show that the convex hull of $S_A^\wedge(\mathbb{R})$ is the numerical range [6, 22]. See also [16] for further algebraic context of the numerical range.

We will discuss boundary generating curves separately for unitarily reducible and irreducible 3-by-3 matrices. We will say that $\alpha \in \mathbb{C}$ is a *normal eigenvalue* of A if there exists a non-zero vector x such that $Ax = \alpha x$, $A^*x = \overline{\alpha}x$. The 3-by-3 matrix A has a normal eigenvalue if and only if A is unitarily reducible. Every normal eigenvalue contributes as a point to the boundary generating curve, see §7 of [22].

REMARK 4.1. A complete list of boundary generating curves for a unitarily reducible matrix $A \in M_3$ is

1. three normal eigenvalues (not necessarily distinct),
2. one normal eigenvalue and one ellipse.

The corresponding shapes of $W(A)$ are (1) a point, a segment, or a triangle and (2) an ellipse or the convex hull of an ellipse and a point outside the ellipse.

REMARK 4.2. A complete list of boundary generating curves for a unitarily irreducible matrix $A \in M_3$ is

1. an ellipse and a point inside the ellipse,
2. a degree four curve with a double tangent,
3. a degree six curve consisting of two nested parts one inside another, the outer part having an 'ovular' shape.

Following [28] we denote the sets of irreducible matrices (1) by $\mathscr{E}_3$, their numerical ranges being ellipses, and the set of irreducible matrices (2) by $\mathscr{F}_3$, their numerical ranges having a flat boundary portion. Further, we denote the set of irreducible matrices (3) by $\mathscr{O}_3$ and the set of reducible 3-by-3 matrices by $\mathscr{R}_3$. This gives a disjoint union

$$M_3 = \mathscr{R}_3 \cup \mathscr{E}_3 \cup \mathscr{F}_3 \cup \mathscr{O}_3. \tag{4.1}$$

We begin with some observations following from already published results though not necessarily explicitly stated there.

REMARK 4.3.

1. The set of reducible matrices $\mathscr{R}_3$ is closed and nowhere dense in M_3. In fact, the analogous statement holds for all matrix sizes $d \in \mathbb{N}$, see Problem 8 in [14].
2. The set $\mathscr{E}_3$ is closed relative to the set of unitarily irreducible matrices, and nowhere dense. The closedness follows from Thms. 2.3, 2.4 of [21] which provide a constructive criterion for a 3-by-3 matrix to have an elliptical numerical range. Indeed, these conditions are in form of equalities which persist under taking limits, while staying within the set of non-normal matrices. This implies further that the set of 3-by-3 matrices who have an elliptical numerical range is closed relative to the set of non-normal matrices. The fact that $\mathscr{E}_3$ is nowhere dense was derived from the same criterion in [26], see Prop. 3.1 there.

3. The set $\mathscr{F}_3$ is closed relative to the set of unitarily irreducible matrices and nowhere dense. The closedness follows from Prop. 3.2 of [21], according to which a unitarily irreducible matrix A belongs to $\mathscr{F}_3$ if and only if $u\Re(A)+v\Im(A)+w\mathbb{1}$ has rank one for some real u,v,w. It remains to invoke the lower semi-continuity of the rank function. The fact that $\mathscr{F}_3$ is nowhere dense can be seen from an alternative description of this class, provided by [28, Theorem 1.2].

4. From (4.1) and already established (1)–(3) it directly follows that the set $\mathscr{O}_3$ is open and dense in M_3.

The statements (2) and (3) can be refined using Kippenhahn's classification in Remarks 4.1 and 4.2 and the property that the numerical ranges of a converging sequence of matrices converge to the numerical range of the limit matrix (Rem. 3.2).

LEMMA 4.4.

1. *If $A \in M_3 \setminus \mathscr{E}_3$ is the limit of a sequence in $\mathscr{E}_3$, then A is reducible and $W(A)$ is an ellipse, a segment, or a point.*

2. *If $A \in M_3 \setminus \mathscr{F}_3$ is the limit of a sequence in $\mathscr{F}_3$, then A is reducible. If $W(A)$ is an ellipse then the normal eigenvalue of A lies on the boundary of $W(A)$.*

Proof. (1). Rem. 4.3(2) shows that A is reducible and that $W(A)$ is an ellipse if A is not normal. The numerical range $W(A)$ cannot be a triangle because the limit of a sequence of ellipses is an ellipse, a segment, or a point (Rem. 3.4).

(2). Rem. 4.3(3) shows that A is reducible. The Blaschke selection theorem, Thm. 1.8.7 of [30], shows that there is a sequence $(A_i)_{i\in\mathbb{N}} \subset \mathscr{F}_3$ converging to A with flat boundary portions s_i of $W(A_i)$, $i \in \mathbb{N}$, such that $(s_i)_{i\in\mathbb{N}}$ converges in the Hausdorff distance. If $W(A)$ is an ellipse, then Rem. 3.2(2) shows that the limit is an extreme point α of $W(A)$ and Lemma 3.3 proves that α is multiply generated. Now Thm. 3.2 of [23] proves that α is an eigenvalue of A which is in fact a normal eigenvalue, see Theorem 1.6.6 of [17]. $\square$

5. Reducible 3-by-3 matrices

In this section we describe the intersection of the (norm) closures $\overline{\mathscr{E}_3}$ of $\mathscr{E}_3$ and $\overline{\mathscr{F}_3}$ of $\mathscr{F}_3$ with the set of reducible matrices.

To simplify possible limit points of sequences we consider the equivalence of matrices $A \in M_d$ modulo

- unitary similarity,
- substitution of A by a matrix in M_d while preserving the linear span of $\mathbb{1}, \Re(A), \Im(A)$. (5.1)

The numerical range $W(A)$ and the boundary generating curve $S_A^\wedge(\mathbb{R})$ are invariant under the action of the unitary group. The substitutions are realized by the group of invertible affine transformations of the real plane, whose action commutes with the operators $A \mapsto W(A)$ and $A \mapsto S_A^\wedge(\mathbb{R})$, see §2.4 and §4.18 of [22]. Hence, every equivalence class (5.1) is a subset of an equivalence class in Kippenhahn's classification provided in Remarks 4.1 and 4.2. In particular, the blocks $\mathscr{R}_3$, $\mathscr{E}_3$, $\mathscr{F}_3$, and $\mathscr{O}_3$ in the partition (4.1) of M_d are invariants of (5.1). Since the action of a fixed unitary (respectively, affine transformation) is a homeomorphism of M_d, the closures $\overline{\mathscr{E}_3}$ and $\overline{\mathscr{F}_3}$ are invariants, too.

LEMMA 5.1. *Any matrix $A \in M_3$ such that $\Re(A)$ and $\Im(A)$ commute is equivalent modulo* (5.1) *to the diagonal matrix*

$$0, \qquad \operatorname{diag}[0,\lambda,1] \textit{ for } 0 \leqslant \lambda \leqslant \tfrac{1}{2}, \quad \textit{or} \quad \operatorname{diag}[0,1,\mathrm{i}]. \tag{5.2}$$

Any reducible matrix $A \in M_3$ such that $\Re(A)$ and $\Im(A)$ do not commute is equivalent modulo (5.1) *to*

$$\begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a \end{bmatrix}, \qquad a \geqslant 0. \tag{5.3}$$

No two of the matrices in (5.2) *or* (5.3) *are equivalent modulo* (5.1).

Proof. We shall simplify A using transformations (5.1). Since A is unitarily reducible we can assume that its real part is $X \oplus a$ and its imaginary part $Y \oplus b$ where X,Y are self-adjoint 2-by-2 matrices and a,b reals. Using an affine transformation we can assume that $\operatorname{tr}(X)=0$, $\operatorname{tr}(Y)=0$, $a \geqslant 0$, and $b=0$. Thereby, the real and imaginary parts of the original matrix A commute if and only if X and Y commute.

If X and Y commute then, using unitary similarity, we assume that X and Y are scalar multiples of $\operatorname{diag}[1,-1]$. The three cases $A=0$, ($A \neq 0$, $Y=0$), and ($A \neq 0$, $Y \neq 0$) lead *via* affine transformations to the cases of (5.2) in the same order.

If X and Y do not commute then by adding scalar multiples of $Y \oplus 0$ to $X \oplus a$ we assume that X and Y are orthogonal with respect to the Hilbert-Schmidt inner product $M_2 \times M_2 \to \mathbb{C}$, $(x,y) \mapsto \operatorname{tr}(x^*y)$. Normalizing and applying an $SU(2)=SO(3)$ rotation we obtain a matrix of the form (5.3).

To see that no two of the matrices described in (5.2) and (5.3) are equivalent modulo (5.1) recall that the boundary generating curves of two equivalent matrices are affinely isomorphic. The boundary generating curves of the described matrices are one, two, or three points on a line, the vertices of a triangle, or the union of a circle and a point. This makes sure that no two of them are affinely isomorphic. $\square$

We shall compute the intersections of $\overline{\mathscr{E}_3}$ with the set of reducible matrices. It is shown in [28], Sec. 2, that the boundary generating curve of the matrix

$$A' := \begin{bmatrix} a & x & y \\ 0 & b & z \\ 0 & 0 & c \end{bmatrix}, \qquad a,b,c,x,y,z \in \mathbb{C}$$

consists of an ellipse and a point if and only if $d := |x|^2+|y|^2+|z|^2 > 0$ and the number

$$\lambda := (c|x|^2 + b|y|^2 + a|z|^2 - x\bar{y}z)/d \tag{5.4}$$

coincides with one of the eigenvalues a,b,c of A'. In this case the two other eigenvalues of A' are the foci of an ellipse with minor axis of length $\sqrt{d}$, and the boundary generating curve of A' is the union of this ellipse and of λ. Moreover, the eigenvalue λ is a normal eigenvalue if and only if A is reducible.

Observe that the boundary generating curve of the following matrix is the union of the unit circle and a point lying on or inside the unit circle. So the numerical range is the unit disk.

LEMMA 5.2. *If $a \in [0,1]$ then the matrix $\left[\begin{smallmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a \end{smallmatrix}\right]$ lies in $\overline{\mathscr{E}_3}$.*

Proof. If $a > 0$ then the matrix in the above statement equals $a \cdot A(\frac{2}{a})$ where

$$A(\gamma) := \left[\begin{smallmatrix} 0 & \gamma & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right], \qquad \gamma \geqslant 2.$$

Hence, it suffices to prove $A(\gamma) \in \overline{\mathscr{E}_3}$ for real $\gamma \geqslant 2$. The case $a = 0$ would follow by taking the limit $a \to 0$. We define

$$M(\alpha,\beta) := \begin{bmatrix} \alpha & (1-\alpha)(1+\beta^2)/\beta & \alpha \\ 0 & \alpha & -\alpha\beta \\ 0 & 0 & 1 \end{bmatrix}, \qquad \alpha \in \mathbb{R},\ \beta > 0.$$

Choose $\beta > 0$ such that $\gamma = (1+\beta^2)/\beta$. Then the matrix $A(\gamma)$ is the limit $\alpha \to 0$ of $M(\alpha,\beta)$. It suffices to prove $M(\alpha,\beta) \in \mathscr{E}_3$ for $\beta > 0$ and for α in a neighborhood of zero. The discussion in the paragraph of (5.4) shows that for $\alpha > 0$ the numerical range of $M(\alpha,\beta)$ is a disk centered at α.

Let us prove that $M(\alpha,\beta)$ is unitarily irreducible for all $\beta > 0$ and $\alpha \notin \{0,1\}$. For α different from one, the latter is a simple eigenvalue (for M, as well as for M^*), and $e_3 = [0,0,1]$, is the respective eigenvector for M^*. By contradiction, if M were unitarily reducible, then e_3 would lie in its reducing subspace, say L. Applying M to e_3 and assuming $\alpha \neq 0$ the vector $[1,-\beta,0]$ also lies in L, and by applying M again the vector $e_1 = [1,0,0]$ is in L. So, unless β is zero, L is 3-dimensional, proving that M is unitarily irreducible. $\square$

LEMMA 5.3. *If $0 \leqslant \lambda \leqslant \frac{1}{2}$ then the matrix* $\operatorname{diag}[0,\lambda,1]$ *lies in $\overline{\mathscr{E}_3}$.*

Proof. It suffices to show that for all $\lambda \in [0,\frac{1}{2}]$ the diagonal matrix $\operatorname{diag}[0,\lambda,1]$ is a limit of matrices obtainable from $A(a) := \left[\begin{smallmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a \end{smallmatrix}\right]$, $a \in [0,1]$, considered in Lemma 5.2, *via* transformations (5.1). To this end, introduce the matrix resulting from the affine transformation $(x,y) \mapsto ((1-x)/2, \varepsilon y)$, $\varepsilon > 0$, applied to $U^*A(1-2\lambda)U$, where $U := \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \end{bmatrix}$ is unitary. A straightforward computation shows that it equals $\left[\begin{smallmatrix} 0 & 0 & -\varepsilon \\ 0 & \lambda & 0 \\ \varepsilon & 0 & 1 \end{smallmatrix}\right]$ which converges for $\varepsilon \to 0$ to $\operatorname{diag}[0,\lambda,1]$. $\square$

Lemmas 4.4(1), 5.1, 5.2, and 5.3 show the following.

LEMMA 5.4. *A reducible 3-by-3 matrix lies in* $\overline{\mathscr{E}_3}$ *if and only if its numerical range is an ellipse, a segment, or a point.*

We shall compute the intersection of $\overline{\mathscr{F}_3}$ with the set of reducible matrices.

LEMMA 5.5. *A reducible 3-by-3 matrix* A *lies in* $\overline{\mathscr{F}_3}$ *if and only if* $W(A)$ *is not an ellipse having the normal eigenvalue of* A *in the interior.*

Proof. Lemma 4.4(2) excludes reducible matrices A from $\overline{\mathscr{F}_3}$ when $W(A)$ is an ellipse with the normal eigenvalue of A in the interior. Let us show that all other reducible matrices lie in $\overline{\mathscr{F}_3}$.

We can assume that A is a matrix listed in Lemma 5.1. Let A be of the form (5.2). Clearly $0 \in \overline{\mathscr{F}_3}$. Otherwise, if $A \neq 0$, then $A = \operatorname{diag}[0,\lambda,1]$ for $\lambda \in [0,\frac{1}{2}]$ or $A = \operatorname{diag}[0,1,\mathrm{i}]$. In both cases we define for $\varepsilon > 0$ a matrix $M(\varepsilon)_{j,k} := A_{j,k} + \mathrm{i}\varepsilon$, $j,k = 1,2,3$. In the first case $\Im(M(\varepsilon))$ and in the second case $\Re(M(\varepsilon))$ has a multiple eigenvalue. If $\varepsilon > 0$ then these matrices are irreducible because their real and imaginary parts have no common eigenvectors. This proves $A \in \overline{\mathscr{F}_3}$.

Let A be of the form (5.3) with $a \geqslant 1$. It suffices to consider $a > 1$, the case $a = 1$ would follow by taking the limit. We apply transformations (5.1), namely the affine transformation $\mathbb{R}^2 \to \mathbb{R}^2$, $[x,y] \mapsto [x + y\sqrt{a^2-1}, y]$ followed by a unitary transformation $A \mapsto u^*Au$ with unitary

$$u := \frac{1}{a\sqrt{2}} \begin{bmatrix} 0 & 1-\mathrm{i}\sqrt{a^2-1} & 1-\mathrm{i}\sqrt{a^2-1} \\ 0 & a & -a \\ a\sqrt{2} & 0 & 0 \end{bmatrix}.$$

By means of these transformations we may assume that

$$\Re(A) = \operatorname{diag}[a,a,-a] \quad \text{and} \quad \Im(A) = \frac{1}{a} \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sqrt{a^2-1} & \mathrm{i} \\ 0 & -\mathrm{i} & -\sqrt{a^2-1} \end{bmatrix}.$$

Now it is obvious that for all $\varepsilon > 0$ the matrix

$$\operatorname{diag}[a,a,-a] + \frac{\mathrm{i}}{a} \begin{bmatrix} 0 & 0 & \varepsilon \\ 0 & \sqrt{a^2-1} & \mathrm{i} \\ \varepsilon & -\mathrm{i} & -\sqrt{a^2-1} \end{bmatrix}$$

is unitarily irreducible and Rem. 4.3(3) shows that it lies in $\mathscr{F}_3$. Since the matrix converges to A for $\varepsilon \to 0$, the proof is complete. $\square$

Lemmas 4.4, 5.4 and 5.5 combined yield the following.

THEOREM 5.6. *The set* $\overline{\mathscr{E}_3} \cap \overline{\mathscr{F}_3}$ *is the subset of all reducible matrices* $A \in \mathscr{R}_3$ *where* $W(A)$ *is a point, a segment, or an ellipse with the normal eigenvalue of* A *on the boundary.*

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