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GENERALIZED γ–GENERATING MATRICES AND NEHARI–TAKAGI PROBLEM

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To the Memory of Leiba Rodman

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Abstract. Let $\Gamma(f)$ be the block Hankel matrix of negative Fourier coefficients of a matrix valued function (mvf) $f \in L_\infty^{p\times q}(\mathbb{T})$ defined on the unit circle $\mathbb{T}$. In the present paper a matrix Nehari-Takagi problem is considered: Given a Hankel matrix Γ and $\kappa \in \mathbb{N}\cup\{0\}$ find a mvf $f \in L_\infty^{p\times q}(\mathbb{T})$, such that $\|f\|_\infty \leqslant 1$ and $\operatorname{rank}(\Gamma(f)-\Gamma) \leqslant \kappa$. Under certain mild assumption, we establish a one-to-one correspondence between solutions of the Nehari-Takagi problem and solutions of some Takagi-Sarason interpolation problem. The resolvent matrix of the Nehari-Takagi problem is shown to belong to the class of so-called generalized γ-generating matrices, which is introduced and studied in the paper.

1. Introduction

For a summable function f defined on $\mathbb{T} = \{z : |z| = 1\}$ let us set

$$\gamma_k(f) = \frac{1}{2\pi}\int_{\mathbb{T}} e^{ik\theta} f(e^{i\theta})d\theta \quad (k = 1,2,\dots). \tag{1.1}$$

The Nehari problem consists of the following: given a sequence of complex numbers γ_k $(k \in \mathbb{N})$ find a function $f \in L_\infty(\mathbb{T})$ such that $\|f\|_\infty \leqslant 1$ and

$$\gamma_k(f) = \gamma_k, \quad (k = 1,2,\dots). \tag{1.2}$$

By Nehari theorem [22] this problem is solvable if and only if the Hankel matrix $\Gamma = (\gamma_{i+j-1})_{i,j=1}^\infty$ determines a bounded operator in $l_2(\mathbb{N})$ with $\|\Gamma\| \leqslant 1$. The problem (1.2) is called indeterminate if it has infinitely many solutions. A criterion for the Nehari problem to be indeterminate and a full description of the set of its solutions was given in [1], [2].

In [2] Adamyan, Arov and Kreĭn considered the following indefinite version of the Nehari problem, so called Nehari-Takagi problem $\mathbf{NTP}_\kappa(\Gamma)$: Given $\kappa \in \mathbb{N}$ and a

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sequence $\{\gamma_k\}_{k=1}^{\infty}$ of complex numbers, find a function $f \in L_\infty(\mathbb{T})$, such that $\|f\|_\infty \leqslant 1$ and

$$\mathrm{rank}\,(\Gamma(f) - \Gamma) \leqslant \kappa.$$

Here $\Gamma(f)$ is the Hankel matrix $\Gamma(f) := (\gamma_{i+j-1}(f))_{i,j=1}^{\infty}$. As was shown in [2], the problem $\mathbf{NTP}_\kappa(\Gamma)$ is solvable if and only if the total multiplicity $\nu_-(I - \Gamma^*\Gamma)$ of the negative spectrum of the operator $I - \Gamma^*\Gamma$ does not exceed κ. In the case when the operator $I - \Gamma^*\Gamma$ is invertible and $\nu_-(I - \Gamma^*\Gamma) = \kappa$, the set of solutions of this problem was parameterized by the formula

$$f(\mu) = (a_{11}(\mu)\varepsilon(\mu) + a_{12}(\mu))(a_{21}(\mu)\varepsilon(\mu) + a_{22}(\mu))^{-1}, \tag{1.3}$$

where $\mathfrak{A}(\mu) = (a_{ij}(\mu))_{i,j=1}^{2}$ is the so-called γ-generating matrix and the parameter ε ranges over the Schur class of functions holomorphic on $\mathbb{D} = \{z : \|z\| < 1\}$ and bounded by one. In [2] applications of the Nehari-Takagi problem to various approximation and interpolation problems were presented. Matrix and operator versions of Nehari problem were considered in [25] and [3]. In the rational case matrix Nehari and Nehari-Takagi problems were studied in [10]. A complete exposition of these results can be found also in [24] and [8].

In the present paper we consider the general matrix Nehari-Takagi problem and show that under some assumptions this problem can be reduced to Takagi-Sarason interpolation problem studied earlier in [14]. Using the results from [14], [15] we obtain in Theorem 5.3 a description of the set of solutions of the matrix Nehari-Takagi problem in the form (1.3).

The resolvent matrix $\mathfrak{A}(\mu) = (a_{i,j}(\mu))_{i,j=1}^{2}$ in (1.3) is shown to belong to the class of generalized γ-generating matrices, introduced in Definition 4.1. Connections between the class of generalized γ-generating matrices and the class of generalized j-inner matrix valued functions (mvf's) introduced in [13] is established in Theorem 4.3. Using this connection we present another proof of the formula for the resolvent matrix $\mathfrak{A}(\mu)$ from [10] in the case when the Hankel matrix Γ corresponds to a rational mvf. All the results, except the last section, are presented in unified notations both for the unit circle $\mathbb{T}$ and the real line $\mathbb{R}$.

2. Preliminaries

2.1. Notations

Let Ω_+ be either $\mathbb{D} = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$ or $\mathbb{C}_+ = \{\lambda \in \mathbb{C} : \mathrm{Im}\,\lambda| > 0\}$. Let us set for arbitrary $\lambda, \omega \in \mathbb{C}$

$$\rho_\omega(\lambda) = \begin{cases} 1 - \lambda\bar{\omega}, & \Omega_+ = \mathbb{D}, \\ -i(\lambda - \bar{\omega}), & \Omega_+ = \mathbb{C}_+, \end{cases} \qquad \lambda^\circ = \begin{cases} 1/\bar{\lambda}, & \Omega_+ = \mathbb{D}, \\ \bar{\lambda}, & \Omega_+ = \mathbb{C}_+. \end{cases}$$

Thus, $\Omega_+ = \{\omega \in \mathbb{C} : \rho_\omega(\omega) > 0\}$ and let

$$\Omega_0 = \{\omega \in \mathbb{C} : \rho_\omega(\omega) = 0\}, \quad \Omega_- = \{\omega \in \mathbb{C} : \rho_\omega(\omega) < 0\}.$$

The following basic classes of mvf's will be used in this paper: $H_2^{p\times q}$ (resp., $H_\infty^{p\times q}$) is the class of $p\times q$ mvf's with entries in the Hardy space H_2 (resp., H_∞); $H_2^p := H_2^{p\times 1}$, and $(H_2^p)^\perp = L_2^p \ominus H_2^p$, $\mathcal{S}^{p\times q}$ is the Schur class of $p\times q$ mvf's holomorphic and contractive on Ω_+, $\mathcal{S}_{in}^{p\times q}$ (resp., $\mathcal{S}_{out}^{p\times q}$) is the class of inner (resp., outer) mvf's in $\mathcal{S}^{p\times q}$:

$$\mathcal{S}_{in}^{p\times q} = \{s\in\mathcal{S}^{p\times q} : s(\mu)^*s(\mu) = I_p \text{ a.e. on } \Omega_0\};$$
$$\mathcal{S}_{out}^{p\times q} = \{s\in\mathcal{S}^{p\times q} : \overline{sH_2^q} = H_2^p\},$$

The Nevanlinna class $\mathcal{N}^{p\times q}$ and the Smirnov class $\mathcal{N}_+^{p\times q}$ are defined by

$$\begin{aligned}\mathcal{N}^{p\times q} &= \{f = h^{-1}g : g\in H_\infty^{p\times q}, h\in\mathcal{S} := \mathcal{S}^{1\times 1}\},\\ \mathcal{N}_+^{p\times q} &= \{f = h^{-1}g : g\in H_\infty^{p\times q}, h\in\mathcal{S}_{out} := \mathcal{S}_{out}^{1\times 1}\}.\end{aligned} \tag{2.1}$$

For a mvf $f(\lambda)$ let us set $f^\#(\lambda) = f(\lambda^\circ)^*$. Denote by $\mathfrak{h}_f$ the domain of holomorphy of the mvf f and let $\mathfrak{h}_f^\pm = \mathfrak{h}_f\cap\Omega_\pm$.

A $p\times q$ mvf f_- in Ω_- is said to be a pseudocontinuation of a mvf $f\in\mathcal{N}^{p\times q}$, if

(1) $f_-^\# \in \mathcal{N}^{p\times q}$;

(2) $\lim_{\nu\downarrow 0} f_-(\mu - i\nu) = \lim_{\nu\downarrow 0} f_+(\mu+i\nu) (= f(\mu))$ a.e. on Ω_0.

The subclass of all mvf's $f\in\mathcal{N}^{p\times q}$ that admit pseudocontinuations f_- into Ω_- will be denoted $\Pi^{p\times q}$.

Let $\varphi(\lambda)$ be a $p\times q$ mvf that is meromorphic on Ω_+ with a Laurent expansion

$$\varphi(\lambda) = (\lambda-\lambda_0)^{-k}\varphi_{-k} + \cdots + (\lambda-\lambda_0)^{-1}\varphi_{-1} + \varphi_0 + \cdots$$

in a neighborhood of its pole $\lambda_0\in\Omega_+$. The pole multiplicity $\mathcal{M}_\pi(\varphi,\lambda_0)$ is defined by (see [20])

$$\mathcal{M}_\pi(\varphi,\lambda_0) = \operatorname{rank} L(\varphi,\lambda_0), \quad T(\varphi,\lambda_0) = \begin{bmatrix}\varphi_{-k} & & \mathbf{0}\\ \vdots & \ddots & \\ \varphi_{-1} & \dots & \varphi_{-k}\end{bmatrix}.$$

The pole multiplicity of φ over Ω_+ is given by

$$\mathcal{M}_\pi(\varphi,\Omega_+) = \sum_{\lambda\in\Omega_+}\mathcal{M}_\pi(\varphi,\lambda).$$

This definition of pole multiplicity coincides with that based on the Smith-McMillan representation of φ (see [10]).

Let $b_\omega(\lambda)$ be a Blaschke factor ($b_\omega(\lambda) = \frac{\lambda-\omega}{1-\lambda\bar\omega}$, if $\Omega_+ = \mathbb{D}$, and $b_\omega(\lambda) = \frac{\lambda-\omega}{\lambda-\bar\omega}$, if $\Omega_+ = \mathbb{C}_+$), and let P be an orthogonal projection in $\mathbb{C}^p$. Then the mvf

$$B_\alpha(\lambda) = I_p - P + b_\alpha(\lambda)P, \quad \omega\in\Omega_+,$$

belongs to the Schur class $\mathscr{S}^{p\times p}$ and is called *the elementary Blaschke–Potapov (BP) factor* and $B(\lambda)$ is called *primary* if *rank* $P = 1$. The product

$$B(\lambda) = \prod_{j=1}^{\overset{\kappa}{\curvearrowright}} B_{\alpha_j}(\lambda),$$

where $B_{\alpha_j}(\lambda)$ are primary Blaschke–Potapov factors, is called *a Blaschke–Potapov product* of degree κ.

REMARK 2.1. For a Blaschke-Potapov product b the following statements are equivalent:

(1) the degree of b is equal κ;

(2) $\mathscr{M}_\pi(b^{-1}, \Omega_+) = \kappa$.

2.2. The generalized Schur class

Let $\kappa \in \mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. Recall, that a Hermitian kernel $\mathsf{K}_\omega(\lambda) : \Omega \times \Omega \to \mathbb{C}^{m\times m}$ is said to have κ negative squares, if for every positive integer n and every choice of $\omega_j \in \Omega$ and $u_j \in \mathbb{C}^m$ $(j = 1, \dots, n)$ the matrix

$$(\langle \mathsf{K}_{\omega_j}(\omega_k) u_j, u_k \rangle)_{j,k=1}^n$$

has at most κ negative eigenvalues, and for some choice of $\omega_1, \dots, \omega_n \in \Omega$ and $u_1, \dots, u_n \in \mathbb{C}^m$ exactly κ negative eigenvalues (see [20]).

Let $\mathscr{S}_\kappa^{q\times p}$ denote *the generalized Schur class* of $q \times p$ mvf's s that are meromorphic in Ω_+ and for which the kernel

$$\Lambda_\omega^s(\lambda) = \frac{I_p - s(\lambda)s(\omega)^*}{\rho_\omega(\lambda)} \tag{2.2}$$

has κ negative squares on $\mathfrak{h}_s^+ \times \mathfrak{h}_s^+$. In the case where $\kappa = 0$, the class $\mathscr{S}_0^{q\times p}$ coincides with the Schur class $\mathscr{S}^{q\times p}$ of contractive mvf's holomorphic in Ω_+. As was shown in [20] every mvf $s \in \mathscr{S}_\kappa^{q\times p}$ admits factorizations of the form

$$s(\lambda) = b_\ell(\lambda)^{-1} s_\ell(\lambda) = s_r(\lambda) b_r(\lambda)^{-1}, \quad \lambda \in \mathfrak{h}_s^+, \tag{2.3}$$

where $b_\ell \in \mathscr{S}^{q\times q}$, $b_r \in \mathscr{S}^{p\times p}$ are Blaschke–Potapov products of degree κ, $s_\ell, s_r \in \mathscr{S}^{q\times p}$ and the factorizations (2.3) are left coprime and right coprime, respectively, i.e.

$$\operatorname{rank}\left[b_\ell(\lambda) \;\, s_\ell(\lambda) \right] = q \quad (\lambda \in \Omega_+) \tag{2.4}$$

and

$$\operatorname{rank}\left[b_r(\lambda)^* \;\, s_r(\lambda)^* \right] = p \quad (\lambda \in \Omega_+). \tag{2.5}$$

The following matrix identity was established in the rational case in [16], in general case see [13].

THEOREM 2.2. *Let $s \in \mathscr{S}_\kappa^{q\times p}$ have Kreĭn-Langer factorizations*

$$s = b_\ell^{-1} s_\ell = s_r b_r^{-1}. \tag{2.6}$$

Then there exists a set of mvf's $c_\ell \in H_\infty^{q\times q}$, $d_\ell \in H_\infty^{p\times q}$, $c_r \in H_\infty^{p\times p}$ and $d_r \in H_\infty^{p\times q}$, such that

$$\begin{bmatrix} c_r & d_r \\ -s_\ell & b_\ell \end{bmatrix} \begin{bmatrix} b_r & -d_\ell \\ s_r & c_\ell \end{bmatrix} = \begin{bmatrix} I_p & 0 \\ 0 & I_q \end{bmatrix}. \tag{2.7}$$

2.3. The generalized Smirnov class

Let $\mathscr{R}^{p\times q}$ denote the class of rational $p\times q$ mvf's and let $\kappa \in \mathbb{N}$. A $p\times q$ mvf $\varphi(z)$ is said to belong to the generalized Smirnov class $\mathscr{N}_{+,\kappa}^{p\times q}$, if it admits the representation

$$\varphi(z) = \varphi_0(z) + r(z), \quad \text{where} \quad \varphi_0 \in \mathscr{N}_+^{p\times q}, r \in \mathscr{R}^{p\times q} \quad \text{and} \quad M_\pi(r, \Omega_+) \leqslant \kappa.$$

If $\kappa = 0$, then the class $\mathscr{N}_{+,0}^{p\times q}$ coincides with the Smirnov class $\mathscr{N}_+^{p\times q}$, defined in (2.1). The generalized Smirnov class $\mathscr{N}_{+,\kappa}^{p\times q}$ was introduced in [23]. In [15], mvf's φ from $\mathscr{N}_{+,\kappa}^{p\times q}$ were characterized by the following left coprime factorization

$$\varphi(\lambda) = b_\ell(\lambda)^{-1}\varphi_\ell(\lambda),$$

where $b_\ell \in S_{in}^{p\times p}$ is a Blaschke–Potapov product of degree κ, $\varphi_\ell \in \mathscr{N}_+^{p\times q}$ and

$$\operatorname{rank}\begin{bmatrix} b_\ell(\lambda) & \varphi_\ell(\lambda) \end{bmatrix} = p \quad \text{for } \lambda \in \Omega_+.$$

Clearly, for $\varphi \in \mathscr{N}_{+,\kappa}^{p\times q}$ there exists a right coprime factorization with similar properties. This implies, in particular, that the class $\mathscr{S}_\kappa^{p\times q}$ is contained in $\mathscr{N}_{+,\kappa}^{p\times q}$.

2.4. Generalized j_{pq}-inner mvf's

Let j_{pq} be an $m\times m$ signature matrix

$$j_{pq} = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}, \quad \text{where } p+q = m,$$

DEFINITION 2.3. [4] An $m\times m$ mvf $W(\lambda) = [w_{ij}(\lambda)]_{i,j=1}^2$ that is meromorphic in Ω_+ is said to belong to the class $\mathscr{U}_\kappa(j_{pq})$ of *generalized j_{pq}-inner* mvf's, if:

(i) the kernel

$$\mathsf{K}_\omega^W(\lambda) = \frac{j_{pq} - W(\lambda) j_{pq} W(\omega)^*}{\rho_\omega(\lambda)}$$

has κ negative squares in $\mathfrak{h}_W^+ \times \mathfrak{h}_W^+$;

(ii) $j_{pq} - W(\mu) j_{pq} W(\mu)^* = 0$ a.e. on Ω_0.

As is known [4, Theorem 6.8] for every $W \in \mathcal{U}_\kappa(j_{pq})$ the block $w_{22}(\lambda)$ is invertible for all $\lambda \in \mathfrak{h}_W^+$ except for at most κ points in Ω_+. Thus the Potapov-Ginzburg transform of W

$$S(\lambda) = PG(W) := \begin{bmatrix} w_{11}(\lambda) & w_{12}(\lambda) \\ 0 & I_q \end{bmatrix} \begin{bmatrix} I_p & 0 \\ w_{21}(\lambda) & w_{22}(\lambda) \end{bmatrix}^{-1} \tag{2.8}$$

is well defined for those $\lambda \in \mathfrak{h}_W^+$, for which $w_{22}(\lambda)$ is invertible. It is well known that $S(\lambda)$ belongs to the class $\mathcal{S}_\kappa^{m\times m}$ and $S(\mu)$ is unitary for a.e. $\mu \in \Omega_0$ (see [4], [13]).

DEFINITION 2.4. [13] A mvf $W \in \mathcal{U}_\kappa(j_{pq})$ is said to be in the class $\mathcal{U}_\kappa^r(j_{pq})$, if

$$s_{21} := -w_{22}^{-1}w_{21} \in \mathcal{S}_\kappa^{q\times p}. \tag{2.9}$$

Let $W \in \mathcal{U}_\kappa^r(j_{pq})$ and let the Kreĭn-Langer factorization of s_{21} be written as

$$s_{21}(\lambda) = b_\ell(\lambda)^{-1}s_\ell(\lambda) = s_r(\lambda)b_r(\lambda)^{-1} \quad (\lambda \in \mathfrak{h}_{s_{21}}^+),$$

where $b_\ell \in \mathcal{S}_{in}^{q\times q}$, $b_r \in \mathcal{S}_{in}^{p\times p}$, $s_\ell, s_r \in \mathcal{S}^{q\times p}$. Then, as was shown in [13], the mvf's $b_\ell s_{22}$ and $s_{11}b_r$ are holomorphic in Ω_+, and

$$b_\ell s_{22} \in \mathcal{S}^{q\times q} \quad \text{and} \quad s_{11}b_r \in \mathcal{S}^{p\times p}.$$

DEFINITION 2.5. [13] Consider inner-outer factorization of $s_{11}b_r$ and outer-inner factorization of $b_\ell s_{22}$

$$s_{11}b_r = b_1a_1, \qquad b_\ell s_{22} = a_2b_2, \tag{2.10}$$

where $b_1 \in \mathcal{S}_{in}^{p\times p}$, $b_2 \in \mathcal{S}_{in}^{q\times q}$, $a_1 \in \mathcal{S}_{out}^{p\times p}$, $a_2 \in \mathcal{S}_{out}^{q\times q}$. The pair $\{b_1, b_2\}$ of inner factors in the factorizations (2.10) is called *the associated pair* of the mvf $W \in \mathcal{U}_\kappa^r(j_{pq})$.

From now onwards this pair $\{b_1, b_2\}$ will be called also a right associated pair since it is related to the right linear fractional transformation

$$T_W[\varepsilon] := (w_{11}\varepsilon + w_{12})(w_{21}\varepsilon + w_{22})^{-1}, \tag{2.11}$$

see [5], [7], [8]. Such transformations play important role in description of solutions of different interpolation problems, see [2], [5], [10], [9], [12], [14]. In the case $\kappa = 0$ the definition of the associated pair was given in [5].

For every $W \in \mathcal{U}_\kappa^r(j_{pq})$ and $\varepsilon \in \mathcal{S}^{p\times q}$ the mvf $T_W[\varepsilon]$ admits the dual representation

$$T_W[\varepsilon] = (w_{11}^\# + \varepsilon w_{12}^\#)^{-1}(w_{21}^\# + \varepsilon w_{22}^\#).$$

As was shown in [13], for $W \in \mathcal{U}_\kappa^r(j_{pq})$ and c_r, d_r, c_ℓ and d_ℓ as in (2.7) the mvf

$$K^\circ := (-w_{11}d_\ell + w_{12}c_\ell)(-w_{21}d_\ell + w_{22}c_\ell)^{-1}, \tag{2.12}$$

belongs to $H_\infty^{p\times q}$. It is clear that $(K^\circ)^\# \in H_\infty^{q\times p}(\Omega_-)$.

In the future we will need the following factorization formula for the mvf $W \in \mathcal{U}_\kappa^r(j_{pq})$, obtained in [13, Theorem 4.12]:

$$W = \Theta^\circ \Phi^\circ \quad in\ \Omega_+, \tag{2.13}$$

where

$$\Theta^\circ = \begin{bmatrix} b_1 & K^\circ b_2^{-1} \\ 0 & b_2^{-1} \end{bmatrix}, \quad \Phi^\circ, (\Phi^\circ)^{-1} \in \mathcal{N}_+.$$

3. The Takagi-Sarason interpolation problem

Problem TSP$_\kappa(b_1,b_2,K)$ Let $b_1 \in \mathcal{S}_{in}^{p\times p}$, $b_2 \in \mathcal{S}_{in}^{q\times q}$ be inner mvf's, let $K \in H_\infty^{p\times q}$ and let $\kappa \in \mathbb{Z}_+$. A $p\times q$ mvf s is called a solution of the Takagi-Sarason problem **TSP**$_\kappa(b_1,b_2,K)$, if s belongs to $\mathcal{S}_{\kappa'}^{p\times q}$ for some $\kappa' \leqslant \kappa$ and satisfies

$$b_1^{-1}(s-K)b_2^{-1} \in \mathcal{N}_{+,\kappa}^{p\times q}. \tag{3.1}$$

The set of solutions of the Takagi-Sarason problem will be denoted by

$$\mathcal{TS}_\kappa(b_1,b_2,K) = \bigcup_{\kappa'\leqslant\kappa} \{s \in \mathcal{S}_{\kappa'}^{p\times q} : b_1^{-1}(s-K)b_2^{-1} \in \mathcal{N}_{+,\kappa}^{p\times q}\}.$$

The problem **TSP**$_\kappa(b_1,b_2,K)$ has been studied in [11], in the rational case ($K \in \mathcal{R}^{p\times q}$) the set $\mathcal{TS}_\kappa(b_1,b_2,K)\cap\mathcal{R}^{p\times q}$ was described in [10]. In the completely indeterminate case explicit formulas for the resolvent matrix can be found in [14], [15]. In the general positive semidefinite case the problem was solved in [17], [18].

We now recall the construction of the resolvent matrix from [15]. Let

$$\mathcal{H}(b_1) = H_2^p \ominus b_1 H_2^p, \quad \mathcal{H}_*(b_2) := (H_2^q)^\perp \ominus b_2^*(H_2^q)^\perp,$$

$$\mathcal{H}(b_1,b_2) := \mathcal{H}(b_1) \oplus \mathcal{H}_*(b_2).$$

and let the operators $K_{11} : H_2^q \to \mathcal{H}(b_1)$, $K_{12} : \mathcal{H}_*(b_2) \to \mathcal{H}(b_1)$, $K_{22} : \mathcal{H}_*(b_2) \to (H_2^p)^\perp$ and $P : \mathcal{H}(b_1,b_2) \to \mathcal{H}(b_1,b_2)$ be defined by the formulas

$$\begin{aligned} K_{11}h_+ &= \Pi_{\mathcal{H}(b_1)} K h_+, \quad && h_+ \in H_2^q, \\ K_{12}h_2 &= \Pi_{\mathcal{H}(b_1)} K h_2, \quad && h_2 \in \mathcal{H}_*(b_2), \\ K_{22}h_2 &= \Pi_- K h_2, \quad && h_2 \in \mathcal{H}_*(b_2), \end{aligned} \tag{3.2}$$

$$P = \begin{bmatrix} I - K_{11}K_{11}^* & -K_{12} \\ -K_{12}^* & I - K_{22}^*K_{22} \end{bmatrix}. \tag{3.3}$$

The data set b_1, b_2, K considered in [15] is subject to the following constraints:

(H1) $b_1 \in \mathcal{S}_{in}^{p\times p}$, $b_2 \in \mathcal{S}_{in}^{q\times q}$, $K \in H_\infty^{p\times q}$.

(H2) $\kappa_1 = \nu_-(P) < \infty$.

(H3) $0 \in \rho(P)$.

(H4) $\mathfrak{h}_{b_1} \cap \mathfrak{h}_{b_2^\#} \cap \Omega_0 \neq \emptyset$.

Define also the operator

$$F = \begin{bmatrix} I & K_{22} \\ K_{11}^* & I \end{bmatrix} : \begin{array}{c} \mathscr{H}(b_1) \\ \oplus \\ \mathscr{H}_*(b_2) \end{array} \to \begin{array}{c} b_1(H_2^p)^\perp \\ \oplus \\ b_2^*(H_2^q) \end{array} \stackrel{def}{=} \mathscr{K}. \tag{3.4}$$

As was shown in [15] for every $h_1 \in \mathscr{H}(b_1)$ and $h_2 \in \mathscr{H}_*(b_2)$ the vvf's $(K_{11}^* h_1)(\lambda)$ and $(K_{22}h_2)(\lambda)$ admit pseudocontinuations of bounded type which are holomorphic on $\mathfrak{h}_{b_1}$ and $\mathfrak{h}_{b_2^\#}$, respectively. This allows to define the operator

$$F(\lambda) = E(\lambda)F : \mathscr{H}(b_1,b_2) \to \mathscr{K} \quad \text{for } \lambda \in \mathfrak{h}_{b_1} \cap \mathfrak{h}_{b_2^\#}$$

as the composition of the operator $F : \mathscr{H}(b_1,b_2) \to \mathbb{C}^m$ and the evaluation operator

$$E(\lambda) : f \in \mathscr{K} \to f(\lambda) \in \mathbb{C}^m.$$

Let $\mu \in \mathfrak{h}_{b_1} \cap \mathfrak{h}_{b_2^\#} \cap \Omega_0$. Then the mvf $W(\lambda)$ defined by

$$W(\lambda) = I - \rho_\mu(\lambda)F(\lambda)P^{-1}F(\mu)^* j_{pq} \quad \text{for } \lambda \in \mathfrak{h}_{b_1} \cap \mathfrak{h}_{b_2^\#} \tag{3.5}$$

belongs to the class $\mathscr{U}_{\kappa_1}^r(j_{pq})$ of *generalized* j_{pq}*-inner* mvf's and takes values in $L_2^{m\times m}$. The following theorem presents a description of the set $\mathscr{T}\mathscr{S}_\kappa(b_1,b_2,K)$.

THEOREM 3.1. *Let (H1)–(H4) be in force and let $W(\lambda)$ be the mvf, defined by (3.5). Then $W \in \mathscr{U}_{\kappa_1}^r(j_{pq}) \cap L_2^{m\times m}$ and*

(1) $\mathscr{T}\mathscr{S}_\kappa(b_1,b_2;K) \neq \emptyset \Longleftrightarrow \nu_-(P) \leqslant \kappa$.

(2) *If $\kappa_1 = \nu_-(P) \leqslant \kappa$, then*

$$\mathscr{T}\mathscr{S}_\kappa(b_1,b_2;K) = T_W[\mathscr{S}_{\kappa-\kappa_1}^{p\times q}] := \{T_W[\varepsilon] : \varepsilon \in \mathscr{S}_{\kappa-\kappa_1}^{p\times q}\}, \tag{3.6}$$

where $T_W[\varepsilon]$ is the linear fractional transformation given by (2.11).

Proof. The proof of this statement can be derived from the proof of Theorem 5.7 in [15]. However, we would like to present here a shorter proof based on the description of the set $\mathscr{T}\mathscr{S}_\kappa(b_1,b_2;K)$, given in [14, Theorem 5.17].

As was shown in [15, see Theorem 4.2 and Corollary 4.4] the mvf $W(z)$ belongs to the class $\mathscr{U}_{\kappa_1}^r(j_{pq})$ of generalized j_{pq}-inner mvf's with the property (2.9) and $\{b_1,b_2\}$ is the associated pair of W. Moreover, by construction $W(z)$ takes values in $L_2^{m\times m}$.

Let c_ℓ and d_ℓ be mvf's defined in Theorem 2.2 and let K° be given by (2.12). Then W admits the factorization (2.13) (see [13, Theorem 4.12]). This proves that all the assumptions of Theorem 5.17 from [14] with K replaced by K° are satisfied and by that theorem

$$\mathcal{T}\mathcal{S}_\kappa(b_1,b_2;K^\circ) = T_W[\mathcal{S}^{p\times q}_{\kappa-\kappa_1}]. \tag{3.7}$$

On the other hand it follows from [15, Theorem 4.2] that the mvf W admits the factorization

$$W = \Theta\Phi = \begin{bmatrix} b_1 & Kb_2^{-1} \\ 0 & b_2^{-1}\end{bmatrix}\begin{bmatrix}\varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22}\end{bmatrix} \tag{3.8}$$

with $\Phi,\Phi^{-1} \in \mathcal{N}_+^{m\times m}$. Comparing (3.8) with (2.13) one obtains

$$\begin{bmatrix} I & b_1^{-1}(K-K^\circ)b_2^{-1} \\ 0 & I\end{bmatrix} = \Phi^\circ\Phi^{-1} \in \mathcal{N}_+^{m\times m}$$

and hence

$$b_1^{-1}(K-K^\circ)b_2^{-1} \in \mathcal{N}_+^{p\times q}.$$

This implies the equality $\mathcal{T}\mathcal{S}_\kappa(b_1,b_2;K) = \mathcal{T}\mathcal{S}_\kappa(b_1,b_2;K^\circ)$, that in combination with (3.6) completes the proof. □

4. Generalized γ-generating mvf's

DEFINITION 4.1. Let $\mathfrak{M}^r_\kappa(j_{pq})$ denote the class of $m\times m$ mvf's $\mathfrak{A}(\mu)$ on Ω_0 of the form

$$\mathfrak{A}(\mu) = \begin{bmatrix} a_{11}(\mu) & a_{12}(\mu) \\ a_{21}(\mu) & a_{22}(\mu)\end{bmatrix},$$

with blocks a_{11} and a_{22} of size $p\times p$ and $q\times q$, respectively, such that:

(1) $\mathfrak{A}(\mu)$ is a measurable mvf on Ω_0 and j_{pq}-unitary a.e. on Ω_0;

(2) the mvf's $a_{22}(\mu)$ and $a_{11}(\mu)^*$ are invertible for a.e. $\mu\in\Omega_0$ and the mvf

$$s_{21}(\mu) = -a_{22}(\mu)^{-1}a_{21}(\mu) = -a_{12}(\mu)^*(a_{11}(\mu)^*)^{-1} \tag{4.1}$$

is the boundary value of a mvf $s_{21}(\lambda)$ that belongs to the class $\mathcal{S}^{q\times p}_\kappa$;

(3) $a_{11}(\mu)^*$ and $a_{22}(\mu)$, are the boundary values of mvf's $a^\#_{11}(\lambda)$ and $a_{22}(\lambda)$ that are meromorphic in $\mathbb{C}_+$ and, in addition,

$$a_1 := (a^\#_{11})^{-1}b_r \in \mathcal{S}^{p\times p}_{out}, \quad a_2 := b_\ell a_{22}^{-1} \in \mathcal{S}^{q\times q}_{out}, \tag{4.2}$$

where b_ℓ, b_r are Blaschke-Potapov products of degree κ, determined by Kreĭn-Langer factorizations of s_{21}.

Mvf's in the class $\mathfrak{M}^r_\kappa(j_{pq})$ are called *generalized right γ-generating* mvf's. The class $\mathfrak{M}^r(j_{pq}) := \mathfrak{M}^r_0(j_{pq})$ was introduced in [6], in this case conditions (2) and (3) in Definition 4.1 are simplified to:

(2′) $s_{21} \in \mathscr{S}^{q\times p}$;

(3′) $a_1 := (a_{11}^{\#})^{-1} \in \mathscr{S}_{out}^{p\times p}$, $a_2 := a_{22}^{-1} \in \mathscr{S}_{out}^{q\times q}$.

Mvf's from the class $\mathfrak{M}^r(j_{pq})$ play an important role in the description of solutions of the Nehari problem and are called right γ-generating mvf's, [6, 8].

DEFINITION 4.2. [8] An ordered pair $\{b_1, b_2\}$ of inner mvf's $b_1 \in \mathscr{S}^{p\times p}$, $b_2 \in \mathscr{S}^{q\times q}$ is called a denominator of the mvf $f \in \mathscr{N}^{p\times q}$, if

$$b_1 f b_2 \in \mathscr{N}_+^{p\times q}.$$

The set of denominators of the mvf $f \in \mathscr{N}^{p\times q}$ is denoted by den (f).

THEOREM 4.3. *Let* $\mathfrak{A} \in \Pi^{m\times m} \cap \mathfrak{M}_\kappa^r(j_{pq})$, *and let* c_r, d_r, c_ℓ *and* d_ℓ *be as in Theorem 2.2,*

$$f_0 = (-a_{11}d_\ell + a_{12}c_\ell)a_2. \tag{4.3}$$

Then the mvf f_0 *admits the dual representation*

$$f_0 = a_1(c_r a_{21}^{\#} - d_r a_{22}^{\#}). \tag{4.4}$$

If, in addition, $\{b_1, b_2\} \in den(f_0)$ *and*

$$W(z) = \begin{bmatrix} b_1 & 0 \\ 0 & b_2^{-1} \end{bmatrix} \mathfrak{A}(z), \tag{4.5}$$

then $W \in \mathscr{U}_\kappa^r(j_{pq})$ *and* $\{b_1, b_2\}$ *is the associated pair of* W.

Conversely, if $W \in \mathscr{U}_\kappa^r(j_{pq})$ *and* $\{b_1, b_2\}$ *is the associated pair of* W*, then*

$$\mathfrak{A}(z) = \begin{bmatrix} b_1^{-1} & 0 \\ 0 & b_2 \end{bmatrix} W(z) \in \Pi^{m\times m} \cap \mathfrak{M}_\kappa^r(j_{pq}) \quad and \quad \{b_1, b_2\} \in den(f_0).$$

Proof. Let $\mathfrak{A} \in \Pi^{m\times m} \cap \mathfrak{M}_\kappa^r(j_{pq})$. It follows from (4.1), (4.2) and (2.3) that

$$\begin{aligned} -a_{21}d_\ell + a_{22}c_\ell &= \begin{bmatrix} a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} -d_\ell \\ c_\ell \end{bmatrix} = \begin{bmatrix} -a_{22}s_{21} & a_{22} \end{bmatrix} \begin{bmatrix} -d_\ell \\ c_\ell \end{bmatrix} \\ &= a_{22}b_\ell^{-1} \begin{bmatrix} -s_\ell & b_\ell \end{bmatrix} \begin{bmatrix} -d_\ell \\ c_\ell \end{bmatrix} = a_2^{-1}(s_\ell d_\ell + b_\ell c_\ell) = a_2^{-1}. \end{aligned}$$

Let f_0 be defined by the equation (4.3). Then

$$f_0 = (-a_{11}d_\ell + a_{12}c_\ell)(-a_{21}d_\ell + a_{22}c_\ell)^{-1}.$$

The identity

$$\begin{bmatrix} c_r & -d_r \end{bmatrix} \mathfrak{A}^{\#} j_{pq} \mathfrak{A} \begin{bmatrix} -d_\ell \\ c_\ell \end{bmatrix} = \begin{bmatrix} c_r & -d_r \end{bmatrix} j_{pq} \begin{bmatrix} -d_\ell \\ c_\ell \end{bmatrix} = 0$$

implies that

$$(c_r a_{11}^{\#} - d_r a_{12}^{\#})(-a_{11}d_\ell + a_{12}c_\ell) = (c_r a_{21}^{\#} - d_r a_{22}^{\#})(-a_{21}d_\ell + a_{22}c_\ell),$$

and hence that f_0 admits the dual representation

$$f_0 = (c_r a_{11}^{\#} - d_r a_{12}^{\#})^{-1}(c_r a_{21}^{\#} - d_r a_{22}^{\#}).$$

Using the identity

$$\begin{bmatrix} c_r & -d_r \end{bmatrix} \begin{bmatrix} a_{11}^{\#} \\ a_{12}^{\#} \end{bmatrix} = \begin{bmatrix} c_r & -d_r \end{bmatrix} \begin{bmatrix} a_{11}^{\#} \\ -s_{21}a_{11}^{\#} \end{bmatrix} = \begin{bmatrix} c_r & -d_r \end{bmatrix} \begin{bmatrix} I_p \\ -s_r b_r^{-1} \end{bmatrix} b_r a_1^{-1} = a_1^{-1}$$

one obtains the equality (4.4).

Let $\{b_1, b_2\} \in den(f_0)$, i.e. $b_1 f_0 b_2 \in \mathcal{N}_+^{p\times q}$. Since $b_1 f_0 b_2 \in L_\infty^{p\times q}$ then by Smirnov theorem

$$b_1 f_0 b_2 \in H_\infty^{p\times q}.$$

Let us find the Potapov-Ginzburg transform $S = PG(W)$ of W, see (2.8). The formula (4.5) implies that

$$s_{21} = -w_{22}^{-1}w_{21} = -a_{22}^{-1}a_{21} = -b_\ell^{-1}s_\ell, \tag{4.6}$$

$$s_{22} = w_{22}^{-1} = a_{22}^{-1}b_2 = b_\ell^{-1}a_2 b_2, \tag{4.7}$$

$$\begin{aligned} s_{11} &= w_{11}^{-*} = b_1 a_1 a_1^{-1} b_1^{-1} w_{11}^{-*} \\ &= b_1 a_1 (c_r a_{11}^{*} - d_r a_{12}^{*}) b_1^{-1} w_{11}^{-*} \\ &= b_1 a_1 (c_r w_{11}^{*} - d_r w_{12}^{*}) w_{11}^{-*} \\ &= b_1 a_1 (c_r + d_r s_{21}), \end{aligned} \tag{4.8}$$

$$\begin{aligned} s_{12} &= -w_{11}^{-*}w_{21}^{*} = b_1 a_1 (c_r w_{11}^{*} - d_r w_{12}^{*}) w_{11}^{-*} w_{21}^{*} \\ &= b_1 a_1 (c_r w_{11}^{*} - d_r w_{22}^{*} + d_r s_{22}) \\ &= b_1 f_0 b_2 + b_1 a_1 d_r s_{22}. \end{aligned} \tag{4.9}$$

The equalities (4.6)-(4.9) lead to the formula

$$\begin{aligned} S(z) &= \begin{bmatrix} b_1 a_1 c_r + b_1 a_1 d_r s_{21} & b_1 f_0 b_2 + b_1 a_1 d_r s_{22} \\ s_{21} & s_{22} \end{bmatrix} \\ &= \begin{bmatrix} b_1 a_1 c_r & b_1 f_0 b_2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} b_1 a_1 d_r \\ I \end{bmatrix} \begin{bmatrix} s_{21} & s_{22} \end{bmatrix} \\ &= T(z) + \begin{bmatrix} b_1 a_1 d_r \\ I \end{bmatrix} b_\ell^{-1} \begin{bmatrix} -s_\ell & a_2 b_2 \end{bmatrix}, \end{aligned} \tag{4.10}$$

where $T(z) \in H_\infty^{m\times m}$. It follows from (4.10) that $M_\pi(S, \Omega_+) \leqslant \kappa$. On the other hand

$$M_\pi(s_{21}, \Omega_+) = M_\pi(-b_\ell^{-1}s_\ell, \Omega_+) = \kappa,$$

and, consequently,

$$M_\pi(S, \Omega_+) = \kappa.$$

Thus, $S \in \mathcal{S}_\kappa^{m\times m}$ and, hence, $W \in \mathcal{U}_\kappa^r(j_{pq})$. $\square$

5. A Nehari-Takagi problem

Let $f \in L_\infty^{p\times q}$ and let $\Gamma(f)$ be the Hankel operator associated with f_0:

$$\Gamma(f) := \Pi_- M_f|_{H_2^q}, \tag{5.1}$$

where M_f denotes the operator of multiplication by f, acting from L_2^q into L_2^p and let Π_- denote the orthogonal projection of L_2^p onto $(H_2^p)^\perp$. The operator $\Gamma(f)$ is bounded as an operator from H_2^q to $(H_2^p)^\perp$, moreover,

$$\|\Gamma(f)\| \leqslant \|f\|_\infty.$$

Consider the following Nehari-Takagi problem

Problem NTP$_\kappa(f_0)$: Given a mvf $f_0 \in L_\infty^{p\times q}$. Find $f \in L_\infty^{p\times q}$, such that

$$\operatorname{rank}(\Gamma(f) - \Gamma(f_0)) \leqslant \kappa \quad \text{and} \quad \|f\|_\infty \leqslant 1. \tag{5.2}$$

In the scalar case, the problem **NTP**$_\kappa(f_0)$ has been solved by V.M. Adamyan, D.Z. Arov and M.G. Kreĭn in [1] for the case $\kappa = 0$ and in [2] for arbitrary $\kappa \in \mathbb{N}$. In the matrix case a description of solutions of the problem **NTP**$_0(f_0)$ was obtained in the completely indeterminate case by V.M. Adamyan, [3], and in the general positive-semidefinite case by A. Kheifets, [19]. The indefinite case $(\kappa \in \mathbb{N})$ was treated in [11] (see also [10], where an explicit formula for the resolvent matrix was obtained in the rational case).

In what follows we confine ourselves to the case when $\operatorname{den}(f_0) \neq \emptyset$ and give a description of all solutions of the problem **NTP**$_\kappa(f_0)$. Let us set for $f_0 \in L_\infty^{p\times q}$

$$\mathcal{N}_\kappa(f_0) = \{f \in L_\infty^{p\times q} : f - f_0 \in \mathcal{N}_{+,\kappa}^{p\times q}, \|f\|_\infty \leqslant 1\}$$

and let us denote the set of solutions of the problem **NTP**$_\kappa(f_0)$ by

$$\mathcal{N}\mathcal{T}_\kappa(f_0) = \{f \in L_\infty^{p\times q} : \operatorname{rank}(\Gamma(f) - \Gamma(f_0)) \leqslant \kappa \text{ and } \|f\|_\infty \leqslant 1\}.$$

By Kronecker Theorem ([21]), the condition $f - f_0 \in \mathcal{N}_{+,\kappa}^{p\times q}$ is equivalent to

$$\operatorname{rank}(\Gamma(f) - \Gamma(f_0)) = \kappa,$$

Therefore, the set $\mathcal{N}\mathcal{T}_\kappa(f_0)$ is represented as

$$\mathcal{N}\mathcal{T}_\kappa(f_0) = \bigcup_{\kappa' \leqslant \kappa} \mathcal{N}_{\kappa'}(f_0). \tag{5.3}$$

In the following theorem relations between the set of solutions of the Nehari-Takagi problem and the set of solutions of a Takagi-Sarason problem is established in the case when $\operatorname{den}(f_0) \neq \emptyset$.

THEOREM 5.1. *Let $f_0 \in L_\infty^{p\times q}$, $\Gamma = \Gamma(f_0)$, $\kappa \in \mathbb{Z}_+$, $\{b_1, b_2\} \in den(f_0)$ and $K = b_1 f_0 b_2$. Then*

$$f \in \mathcal{N}_\kappa(f_0) \Leftrightarrow s = b_1 f b_2 \in \mathcal{T}\mathcal{S}_\kappa(b_1, b_2, K).$$

Proof. Let $f \in \mathcal{N}_\kappa(f_0)$. Then the mvf's $\varphi(\mu) := f(\mu) - f_0(\mu)$, $f_0(\mu)$ and $f(\mu)$ admit meromorphic continuations $\varphi(z)$, $f_0(z)$ and $f(z)$ on Ω_+, such that

$$M_\pi(f - f_0, \Omega_+) = \kappa. \tag{5.4}$$

Let $s = b_1 f b_2$ and $K = b_1 f_0 b_2$. Then the equality (5.4) yields $M_\pi(s - K, \Omega_+) \leqslant \kappa$. Since $K \in H_\infty^{p\times q}$, then

$$\kappa' := M_\pi(s, \Omega_+) = M_\pi(s - K, \Omega_+) \leqslant \kappa.$$

Taking into account that $\|s\|_\infty = \|f\|_\infty \leqslant 1$, one obtains $s \in \mathcal{S}_{\kappa'}$. Moreover, the condition (5.4) is equivalent to the condition (3.1), i.e. $s \in \mathcal{T}\mathcal{S}_\kappa(b_1, b_2, K)$.

Conversely, if $s \in \mathcal{S}_{\kappa'}^{p\times q}$ with $\kappa' \leqslant \kappa$ and the condition (3.1) is in force, then for $f = b_1^{-1} s b_2^{-1}$, $f_0 = b_1^{-1} K b_2^{-1}$ one obtains that (5.4) holds and $\|f\|_\infty \leqslant 1$. Therefore, $f \in \mathcal{N}_\kappa(f_0)$. $\square$

LEMMA 5.2. *Let $f_0 \in L_\infty^{p\times q}$, $\Gamma = \Gamma(f_0)$, $\{b_1, b_2\} \in den(f_0)$, $K = b_1 f_0 b_2$ and let* $\mathbf{P}$ *be the operator in $\mathcal{H}(b_1) \oplus \mathcal{H}_*(b_2)$, defined by formulas (3.2) and (3.3). Then*

$$\nu_-(\mathbf{P}) = \nu_-(I - \Gamma^*\Gamma).$$

Moreover, if $\nu_-(I - \Gamma^*\Gamma) < \infty$, then

$$0 \in \rho(\mathbf{P}) \Longleftrightarrow 0 \in \rho(I - \Gamma^*\Gamma).$$

Proof. Let us decompose the spaces H_2^q and $(H_2^p)^\perp$:

$$H_2^q = b_2(H_2^q) \oplus \mathcal{H}(b_2), \quad (H_2^p)^\perp = \mathcal{H}_*(b_1) \oplus b_1(H_2^p)^\perp$$

and let us decompose the operator $\Gamma : H_2^q \to (H_2^p)^\perp$, accordingly

$$\Gamma \overset{def}{=} \begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ 0 & \Gamma_{22} \end{pmatrix} : \begin{matrix} b_2(H_2^q) \\ \oplus \\ \mathcal{H}(b_2) \end{matrix} \to \begin{matrix} \mathcal{H}_*(b_1) \\ \oplus \\ b_1^*(H_2^p)^\perp \end{matrix}, \tag{5.5}$$

where the operators

$$\Gamma_{11} : b_2(H_2^q) \to \mathcal{H}_*(b_1), \quad \Gamma_{12} : \mathcal{H}(b_2) \to \mathcal{H}_*(b_1), \quad \Gamma_{22} : \mathcal{H}(b_2) \to b_1^*(H_2^p)^\perp$$

are defined by the formulas

$$\begin{aligned} \Gamma_{11} h_+ &= \Pi_{\mathcal{H}_*(b_1)} K h_+, & h_+ &\in b_2(H_2^q), \\ \Gamma_{12} h_2 &= \Pi_{\mathcal{H}_*(b_1)} K h_2, & h_2 &\in \mathcal{H}(b_2), \\ \Gamma_{22} h_2 &= (b_1^* \Pi_- b_1) K h_2, & h_2 &\in \mathcal{H}(b_2). \end{aligned} \tag{5.6}$$

It follows from (5.5), (5.6) and (3.2) that the operator $\Gamma : H_2^q \to (H_2^p)^\perp$ and the operator

$$\mathbf{K} \overset{def}{=} \begin{pmatrix} K_{11} & K_{12} \\ 0 & K_{22} \end{pmatrix} : \begin{matrix} H_2^q \\ \oplus \\ \mathcal{H}_*(b_2) \end{matrix} \to \begin{matrix} \mathcal{H}(b_1) \\ \oplus \\ (H_2^p)^\perp \end{matrix}$$

are connected by

$$\Gamma = (\mathscr{M}_{b_1^*}|_{b_1(H_2^p)^\perp})\mathbf{K}(\mathscr{M}_{b_2}|_{H_2^q})$$

and, hence, the operators Γ and $\mathbf{K}$ are unitary equivalent. Now the statements are implied by [15, Lemma 5.10]. □

THEOREM 5.3. *Let* $f_0 \in L_\infty^{p\times q}$, $\Gamma = \Gamma(f_0)$, $\kappa \in \mathbb{Z}_+$, $\kappa_1 := \nu_-(I - \Gamma^*\Gamma)$, $\{b_1, b_2\} \in$ *den*(f_0), $K = b_1 f_0 b_2$, *let* $\mathbf{P}$ *be defined by formulas* (3.3)*, let (H1)–(H4) be in force, let the mvf* $W(z)$ *be defined by* (3.5) *and let*

$$\mathfrak{A}(\mu) = \begin{bmatrix} b_1(\mu)^{-1} & 0 \\ 0 & b_2(\mu) \end{bmatrix} W(\mu), \quad \mu \in \mathfrak{h}_{b_1} \cap \mathfrak{h}_{b_2^\#} \cap \Omega_0. \tag{5.7}$$

Then:

(1) $\mathfrak{A} \in \mathfrak{M}_{\kappa_1}^r(j_{pq})$;

(2) $\mathscr{N}_\kappa(f_0) \neq \emptyset$ *if and only if* $\kappa \geqslant \kappa_1$;

(3) $\mathscr{N}_\kappa(f_0) = T_{\mathfrak{A}}[\mathscr{S}_{\kappa-\kappa_1}^{p\times q}]$,

(4) $\mathscr{N}\mathscr{T}_\kappa(f_0) = \cup_{k=\kappa_1}^{\kappa} T_{\mathfrak{A}}[\mathscr{S}_{k-\kappa_1}^{p\times q}]$.

Proof. (1) By [15, Theorem 4.2] the rows of $W(z)$ admit factorizations

$$\begin{bmatrix} w_{11} & w_{12} \end{bmatrix} = b_1 \begin{bmatrix} a_{11} & a_{12} \end{bmatrix},$$

$$\begin{bmatrix} w_{21} & w_{22} \end{bmatrix} = b_2^{-1} \begin{bmatrix} a_{21} & a_{22} \end{bmatrix},$$

where $a_{11} \in (H_2^{p\times p})^\perp, a_{12} \in (H_2^{p\times q})^\perp$, $a_{21} \in H_2^{q\times p}$, $a_{22} \in H_2^{q\times q}$ and

$$s_{21} = -w_{22}^{-1} w_{21} = -a_{22}^{-1} a_{21} \in \mathscr{S}_{\kappa_1}^{p\times q}.$$

If the mvf's b_ℓ^{-1}, s_ℓ, b_r, s_r are determined by Kreĭn-Langer factorizations of s_{21}

$$s_{21} = b_\ell^{-1} s_\ell = s_r b_r^{-1},$$

then in accordance with [15, Theorem 4.3] (see (4.26), (4.27))

$$a_2 := b_\ell a_{22}^{-1} \in \mathscr{S}_{out}^{q\times q}, \quad a_1 := (a_{11}^\#)^{-1} b_r \in \mathscr{S}_{out}^{p\times p}.$$

Thus

$$\mathfrak{A}(z) = \begin{bmatrix} b_1^{-1} & 0 \\ 0 & b_2 \end{bmatrix} \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix}$$

belongs to the class $\mathfrak{M}_{\kappa_1}^r(j_{pq})$.

(2) By Theorem 5.1 $\mathscr{N}_\kappa(f_0)$ is nonempty if and only if $\mathscr{T}\mathscr{S}_\kappa(b_1, b_2, K)$ is nonempty. Therefore (2) is implied by Theorem 3.1 and Lemma 5.2.

(3) The statement (3) follows from the formula (3.6) proved in Theorem 3.1 and from the equivalence

$$f \in \mathcal{N}_\kappa(f_0) \Longleftrightarrow b_1 f b_2 \in \mathcal{T}\mathcal{S}_\kappa(b_1, b_2, K) = T_W[\mathcal{S}_{\kappa-\kappa_1}]$$

(Theorem 5.1). This means that for every $f \in \mathcal{N}_\kappa(f_0)$ the mvf $s = b_1 f b_2$ belongs to $\mathcal{T}\mathcal{S}_\kappa(b_1, b_2, K)$ and hence it admits the representation

$$s = (w_{11}\varepsilon + w_{12})(w_{21}\varepsilon + w_{22})^{-1} = T_W[\varepsilon]$$

for some $\varepsilon \in \mathcal{S}_{\kappa-\kappa_1}$. Therefore, the mvf $f = b_1^{-1} s b_2^{-1}$ can be represented as

$$f = b_1^{-1}(w_{11}\varepsilon + w_{12})(b_2 w_{21}\varepsilon + b_2 w_{22})^{-1} = T_{\mathfrak{A}}[\varepsilon].$$

(4) As follows from (2) $\mathcal{N}_{\kappa'}(f_0) = \emptyset$ for $\kappa' < \kappa_1$. Therefore, (4) is implied by (5.3) and by the statement (3). □

6. Resolvent matrix in the case of a rational mvf f_0

Assume now that $\Omega_+ = \mathbb{D}$ and f_0 is a rational mvf with a minimal realization

$$f_0(z) = C(zI_n - A)^{-1}B, \tag{6.1}$$

where $n \in \mathbb{N}$, $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times q}$, $C \in \mathbb{C}^{p\times n}$,

$$\sigma(A) \subset \mathbb{D}. \tag{6.2}$$

Then the corresponding Hankel operator $\Gamma = \Gamma(f_0) : H_2^q \to (H_2^p)^\perp$ in (5.1) admits in the standard basis the following block matrix representation

$$(\gamma_{j+k-1})_{j,k=1}^\infty = (CA^{j+k-2}B)_{j,k=1}^\infty = \Omega\Xi,$$

where γ_j are given by (1.1) and

$$\Xi = \begin{bmatrix} B \ AB \ \ldots \ A^{n-1}B \end{bmatrix} \quad \text{and} \quad \Omega = \begin{bmatrix} CA^0 \\ \vdots \\ CA^{n-1} \end{bmatrix}.$$

Representation (6.1) is called minimal, if the dimension of the matrix A in (6.1) is minimal. As is known see [10, Thm 4.1.4] the representation (6.1) is minimal if and only if the pair (A, B) is controllable and the pair (C, A) is observable, i.e.

$$\operatorname{ran}\Xi = \mathbb{C}^n \quad \text{and} \quad \ker\Omega = \{0\}, \tag{6.3}$$

The controllability gramian P and the observability gramian Q, defined by

$$P = \sum_{k=0}^\infty A^k BB^*(A^*)^k = \Xi\Xi^*, \quad Q = \sum_{k=0}^\infty (A^*)^k CC^*(A)^k = \Omega^*\Omega,$$

are solutions to the following Lyapunov-Stein equations

$$P - APA^* = BB^*, \quad Q - A^*QA = C^*C. \tag{6.4}$$

As was shown in [14, Remark 4.2], a denominator of the mvf $f_0(z)$ may be selected as (I_p, b_2), where

$$b_2(z) = I_q - (1-z)B^*(I_n - zA^*)^{-1}P^{-1}(I_n - A)^{-1}B \tag{6.5}$$

Straightforward calculations show that

$$(zI_n - A)^{-1}Bb_2(z) = P(I_n - A^*)(I_n - zA^*)^{-1}P^{-1}(I_n - A)^{-1}B. \tag{6.6}$$

Since the mvf $b_2(z)$ is inner, then $b_2(z)^{-1} = b_2(\frac{1}{\bar{z}})^*$, and hence

$$b_2(z)^{-1} = I_q + (1-z)B^*(I_n - A^*)^{-1}P^{-1}(zI_n - A)^{-1}B. \tag{6.7}$$

PROPOSITION 6.1. *Let $f_0(z)$ be a mvf of the form (6.1), where $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times q}$, $C \in \mathbb{C}^{p\times n}$ satisfy (6.2) and (6.3), and let*

$$M = \begin{bmatrix} -A & 0 \\ 0 & I_n \end{bmatrix}, \quad N = \begin{bmatrix} -I_n & 0 \\ 0 & A^* \end{bmatrix}, \quad \Lambda = \begin{bmatrix} -Q & I_n \\ I_n & -P \end{bmatrix}, \tag{6.8}$$

$$G(z) = \begin{bmatrix} C & 0 \\ 0 & B^* \end{bmatrix} (M - zN)^{-1}. \tag{6.9}$$

Assume that $1 \notin \sigma(PQ)$. Then:

(1) $\mathcal{N}_\kappa(f_0) \neq \emptyset$ if and only if $\kappa_1 := \nu_-(I - PQ) \leqslant \kappa$;

(2) If (1) holds then the matrix Λ is invertible and $\mathcal{N}_\kappa(f_0) = T_{\mathfrak{A}}[\mathcal{S}_{\kappa-\kappa_1}]$, where

$$\mathfrak{A}(\mu) = I_m - (1-\mu)G(\mu)\Lambda^{-1}G(1)^* j_{pq}; \tag{6.10}$$

(3) The mvf $\mathfrak{A}(\mu)$ is a generalized right γ-generating mvf of the class $\mathfrak{M}^r_{\kappa_1}(j_{pq})$.

The statements (1), (2) of Proposition 6.1 and the formula (6.10) for the resolvent matrix $\mathfrak{A}(\mu)$ are well known from [10, Theorem 20.5.1]. We will show here that (6.10) can be derived from the general formula (3.5) for the resolvent matrix of the problem $\mathbf{TSP}_\kappa(I_p, b_2, K)$ with

$$K(z) = f_0(z)b_2(z) = C(zI_n - A)^{-1}Bb_2(z). \tag{6.11}$$

Proof. (1) By Theorem 5.1 $f \in \mathcal{N}_\kappa(f_0)$ if and only if $s = fb_2 \in \mathcal{T}\mathcal{S}_\kappa(I_p, b_2, K)$. Alongside with $\mathbf{TSP}_\kappa(I_p, b_2, K)$ consider also the problem $\mathbf{GSTP}_\kappa(I_p, b_2, K)$: find a $p \times q$ mvf s, such that:

$$s \in \mathcal{S}_\kappa^{p\times q} \quad \text{and} \quad b_1^{-1}(s - K)b_2^{-1} \in \mathcal{N}_{+,\kappa}^{p\times q}. \tag{6.12}$$

As is known [14, Theorem 5.17], these problems have the same resolvent matrix. Assume that s satisfies (6.12). Then

$$\mathscr{M}_\pi((s-K)b_2^{-1},\Omega_+) = \mathscr{M}_\pi(s,\Omega_+) = \kappa.$$

By the noncancellation lemma [15, Lemma 2.3]

$$\mathscr{M}_\pi(b_\ell(s-K)b_2^{-1},\Omega_+) = \mathscr{M}_\pi(b_\ell s,\Omega_+) = \mathscr{M}_\pi(s_\ell,\Omega_+) = 0. \tag{6.13}$$

By (6.7) and (6.11) the expression $b_\ell(s-K)b_2^{-1} = (s_\ell - b_\ell K)b_2^{-1}$ takes the form

$$s_\ell(I_q + (1-z)B^*(I_n-A^*)^{-1}P^{-1}(zI_n-A)^{-1}B) - b_\ell C(zI_n-A)^{-1}B.$$

and hence, the condition (6.13) can be rewritten as

$$\{s_\ell B^*(I_n-A^*)^{-1}P^{-1}(I_n-A) - b_\ell C\}(zI_n-A)^{-1}B \in \mathscr{N}_+. \tag{6.14}$$

Since the pair (A,B) is controllable, then (6.14) can be rewritten as

$$\begin{bmatrix} b_\ell & -s_\ell \end{bmatrix} F \in \mathscr{N}_+, \tag{6.15}$$

where

$$F(z) = \widetilde{C}(A-zI_n)^{-1}, \quad \widetilde{C} = \begin{bmatrix} C \\ B^*(I_n-A^*)^{-1}P^{-1}(I_n-A) \end{bmatrix}. \tag{6.16}$$

Thus, the problem $\mathbf{GSTP}_\kappa(I_p,b_2,K)$ is equivalent to the interpolation problem (6.15) considered in [14]. As was shown in [14, (1.14)], the Pick matrix $\widetilde{P}$, corresponding to the problem (6.15), is the unique solution of the Lyapunov-Stein equation

$$A^*\widetilde{P}A - \widetilde{P} = \widetilde{C}^* j_{pq}\widetilde{C} \tag{6.17}$$

and the problem (6.15) is solvable if and only if $\kappa_1 := \nu_-(\widetilde{P}) \leqslant \kappa$. Since by (6.4)

$$\widetilde{C}^* j_{pq}\widetilde{C} = (Q-P^{-1}) - A^*(Q-P^{-1})A,$$

one gets

$$\widetilde{P} = P^{-1} - Q = P^{-1/2}(I - P^{1/2}QP^{1/2})P^{-1/2}. \tag{6.18}$$

It follows from (6.18) and Theorem 3.1 that $\mathscr{T}\mathscr{S}_\kappa(I_p,b_2,K) \neq \emptyset$ if and only if

$$\kappa_1 := \nu_-(I - P^{1/2}QP^{1/2}) \leqslant \kappa.$$

Now it remains to note that $\sigma(I-P^{1/2}QP^{1/2}) = \sigma(I-PQ)$. In view of Theorem 5.1 this proves (1).

(2) By [14, Theorem 3.1 and Theorem 5.17] the resolvent matrix $\widetilde{W}(z)$, which describes the set $\mathscr{T}\mathscr{S}_\kappa(I_p,b_2,K)$ via the formula (3.6), takes the form

$$\widetilde{W}(z) = I_m - (1-z)F(z)\widetilde{P}^{-1}F(1)^* j_{pq},$$

where $\widetilde{P}$ is given by (6.17). By [14, Lemma 4.8] $\widetilde{W} \in \mathscr{U}_\kappa^r(j_{pq})$. Let us set

$$\widetilde{\mathfrak{A}}(\mu) := \begin{bmatrix} I_p & 0 \\ 0 & b_2(\mu) \end{bmatrix} W(\mu) \tag{6.19}$$

and show that the mvf

$$\widetilde{\mathfrak{A}}(\mu) := \begin{bmatrix} I_p & 0 \\ 0 & b_2(\mu) \end{bmatrix} + (\mu - 1) \begin{bmatrix} I_p & 0 \\ 0 & b_2(\mu) \end{bmatrix} F(\mu)\widetilde{P}^{-1}F(1)^* j_{pq} \tag{6.20}$$

coincides with the mvf $\mathfrak{A}(\mu)$ from (6.10). It follows from (6.6) that

$$b_2(\mu)B^*(I_n - A^*)^{-1}P^{-1}(\mu I_n - A)^{-1}(I_n - A) = B^*(I_n - \mu A^*)^{-1}P^{-1}.$$

In view of (6.5), (6.16), (6.8) and (6.9) this implies

$$\begin{bmatrix} I_p & 0 \\ 0 & b_2(\mu) \end{bmatrix} F(\mu) = \begin{bmatrix} C(\mu I_n - A)^{-1} \\ B^*(I_n - \mu A^*)^{-1}P^{-1} \end{bmatrix} = G(\mu) \begin{bmatrix} I_p \\ P^{-1} \end{bmatrix}. \tag{6.21}$$

Next, in view of (6.16) and (6.5)

$$F(1)^* = \left[(I_n - A^*)^{-1}C^* \;\; P^{-1}(I_n - A)^{-1}B\right] = \left[I_n \;\; P^{-1}\right] G(1)^*, \tag{6.22}$$

$$\begin{bmatrix} I_p & 0 \\ 0 & b_2(\mu) \end{bmatrix} = I_m - (1-\mu)G(\mu) \begin{bmatrix} 0 & 0 \\ 0 & -P^{-1} \end{bmatrix} G(1)^* j_{pq}. \tag{6.23}$$

Substituting (6.21), (6.22) and (6.23) into (6.20) one obtains (6.10).

By [14, Theorem 3.1 and Theorem 5.17] and Theorem 3.1 the set $\mathscr{T}\mathscr{S}_\kappa(I_p, b_2, K)$ is described by the formula

$$\mathscr{T}\mathscr{S}_\kappa(b_1, b_2; K) = T_W[\mathscr{S}_{\kappa-\kappa_1}^{p\times q}] = \{T_W[\varepsilon] : \varepsilon \in \mathscr{S}_{\kappa-\kappa_1}^{p\times q}\}.$$

Therefore, the statement (2) is implied by Theorem 5.3 (3).

(3) Since $\widetilde{W} \in \mathscr{U}_\kappa^r(j_{pq})$ it follows from (6.19) and Theorem 5.3 that $\widetilde{\mathfrak{A}} \in \mathfrak{M}_{\kappa_1}^r(j_{pq})$. □

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