

WEAVING FRAMES

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Abstract. We study an intriguing question in frame theory we call *Weaving Frames* that is partially motivated by preprocessing of Gabor frames. Two frames $\{\varphi_i\}_{i \in I}$ and $\{\psi_i\}_{i \in I}$ for a Hilbert space \mathbb{H} are *woven* if there are constants $0 < A \leq B$ so that for every subset $\sigma \subset I$, the family $\{\varphi_i\}_{i \in \sigma} \cup \{\psi_i\}_{i \in \sigma^c}$ is a frame for \mathbb{H} with frame bounds A, B . Fundamental properties of woven frames are developed and key differences between weaving Riesz bases and weaving frames are considered. In particular, it is shown that a Riesz basis cannot be woven with a redundant frame. We also introduce an apparently weaker form of *weaving* but show that it is equivalent to weaving. Weaving frames has potential applications in wireless sensor networks that require distributed processing under different frames, as well as preprocessing of signals using Gabor frames.

1. Introduction

This paper introduces a new problem in frame theory called *weaving frames*. Two frames $\{\varphi_i\}_{i \in I}$ and $\{\psi_i\}_{i \in I}$ for a Hilbert space \mathbb{H} are (weakly) *woven* if for every subset $\sigma \subset I$, the family $\{\varphi_i\}_{i \in \sigma} \cup \{\psi_i\}_{i \in \sigma^c}$ is a frame for \mathbb{H} .

The concept is motivated by the following question in distributed signal processing: given are two sets $\{\varphi_i\}_{i \in I}$ and $\{\psi_i\}_{i \in I}$ of linear measurements with stable recovery guarantees, in mathematical terminology each set is a frame labeled by a node or sensor $i \in I$. At each sensor we measure a signal x either with φ_i or with ψ_i , so that the collected information is the set of numbers $\{\langle x, \varphi_i \rangle\}_{i \in \sigma} \cup \{\langle x, \psi_i \rangle\}_{i \in \sigma^c}$ for some subset $\sigma \subseteq I$. Can x still be recovered robustly from these measurements, no matter which kind of measurement has been made at each node? In other words, is the set $\{\varphi_i\}_{i \in \sigma} \cup \{\psi_i\}_{i \in \sigma^c}$ a frame for all subsets $\sigma \subseteq I$? This question led us to the definition of woven frames.

In this paper, we develop the fundamental properties of woven frames for their own sake. Naturally we hope that the notion of woven frames will be applicable to wireless sensor networks which may be subjected to distributed processing under different frames and possibly in the preprocessing of signals using Gabor frames.

Let us briefly describe the main results about woven frames. Let $\Phi = \{\varphi_i\}_{i \in I}$ and $\Psi = \{\psi_i\}_{i \in I}$ be two frames for a separable Hilbert space \mathbb{H} .

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(A) *Uniform frame bounds.* If the weaving $\{\varphi_i\}_{i \in \sigma} \cup \{\psi_i\}_{i \in \sigma^c}$ is a frame for every subset $\sigma \subset I$, then there exist uniform frame bounds $A, B > 0$ that work simultaneously for *all* weavings. In other words, the stability of the reconstruction does not depend on how the measurements are chosen from the two frames Φ and Ψ . This surprising fact shows that the formal distinction of weakly woven frames and woven frames, which we make in Sections 3 and 4, is not necessary.

(B) *Woven Riesz bases.* If Φ and Ψ are two Riesz bases such that every weaving $\{\varphi_i\}_{i \in \sigma} \cup \{\psi_i\}_{i \in \sigma^c}$ is a frame, then, in fact, $\{\varphi_i\}_{i \in \sigma} \cup \{\psi_i\}_{i \in \sigma^c}$ must already be a Riesz basis for every $\sigma \subset I$. In Section 5 we also give a characterization of woven Riesz bases with a geometric flavor.

(C) *Existence of woven frames and perturbations.* The property that every weaving $\{\varphi_i\}_{i \in \sigma} \cup \{\psi_i\}_{i \in \sigma^c}$ is a frame is rather strong, and it seems that Φ and Ψ must resemble each other in some sense. We will show that if Ψ is a perturbation of Φ , then Φ and Ψ are indeed woven. For the technical statements we will use several notions of perturbation, such as the distance of the corresponding synthesis operators, or the almost diagonalization of cross-Gramian matrix of the two frames, or the perturbation by an invertible operator. As a consequence, every frame with a reasonable condition number is woven with its canonical dual frame.

In the literature one finds other concepts which use multiple frames called *quilted frames* in [4]. These concepts require an underlying phase space and a notion of localization. Quilted Gabor frames are then systems constructed from several globally defined frames by restricting these to certain sufficiently large regions in the time-frequency or time-scale plane. Such frame constructions were investigated in detail by Dörfler and Romero in [4, 5, 8]. Except for the use of multiple frames, quilted frames are seemingly unrelated to weaving frames.

The paper is organized as follows: Section 2 contains the basic definitions about frames, and Section 3 introduces the new notion of weaving frames. In Section 4 we give a characterization of weaving frames that does not require universal frame bounds. In Section 5 we consider the case of weaving Riesz bases and provide an abstract characterization of when two Riesz bases are woven. In Sections 6 and 7 we provide sufficient conditions for weaving frames by means of perturbation theory and diagonal dominance. Finally we speculate about possible applications.

2. Frame theory preliminaries

A brief introduction to frame theory is given in this section, which contains the necessary background for this paper. For a thorough approach to the basics of frame theory, see [1, 2]. Throughout the paper, \mathbb{H} will denote either a finite or infinite dimensional Hilbert space while \mathbb{H}^M will denote an M -dimensional Hilbert space. Also, I can represent a finite or countably infinite index set unless otherwise noted.

DEFINITION 1. A family of vectors $\Phi = \{\varphi_i\}_{i \in I}$ in a Hilbert space \mathbb{H} is said to be a *Riesz sequence* if there are constants $0 < A \leq B < \infty$ so that for all $\{c_i\}_{i \in I} \in \ell^2(I)$,

$$A \sum_{i \in I} |c_i|^2 \leq \left\| \sum_{i \in I} c_i \varphi_i \right\|^2 \leq B \sum_{i \in I} |c_i|^2$$

where A and B are the *lower Riesz bound* and *upper Riesz bound*, respectively. If, in addition, Φ is complete in \mathbb{H} , then it is a *Riesz basis* for \mathbb{H} .

An important, equivalent formulation for a Riesz basis is that the vectors are the image of an orthonormal basis under some invertible operator. That is, $\{\varphi_i\}_{i \in I}$ is a Riesz basis for \mathbb{H} if and only if there is an orthonormal basis $\{e_i\}_{i \in I}$ for \mathbb{H} and an invertible operator $T : \mathbb{H} \rightarrow \mathbb{H}$ satisfying $Te_i = \varphi_i$ for all $i \in I$.

Riesz bases have proved to be useful in those applications in which the assumption of orthonormality is too extreme, but the uniqueness and stability of the associated series expansion is still required.

There are times when assuming the sequence is a Riesz basis is even too strong. In these cases, we work with *frames* which are redundant family of vectors having proper subsets that span the space. Redundancy is the fundamental property of frames which makes them so useful in practice since it can be used to mitigate losses during transmission of a signal, noise in the signal, and quantization errors, as well as being able to be adapted to particular signal characteristics.

DEFINITION 2. A family of vectors $\Phi = \{\varphi_i\}_{i \in I}$ in a Hilbert space \mathbb{H} is said to be a *frame* if there are constants $0 < A \leq B < \infty$ so that for all $x \in \mathbb{H}$,

$$A\|x\|^2 \leq \sum_{i \in I} |\langle x, \varphi_i \rangle|^2 \leq B\|x\|^2,$$

where A and B are a choice *lower frame bound* and *upper frame bound*, respectively. If only B is assumed to exist, then Φ is called a *Bessel sequence*. If $A = B = 1$, then Φ is a *Parseval frame*. The values $\{\langle x, \varphi_i \rangle\}_{i \in I}$ are called the *frame coefficients* of the vector $x \in \mathbb{H}$ with respect to the frame Φ .

If $\Phi = \{\varphi_i\}_{i \in I}$ is a Bessel sequence for \mathbb{H} , then the *analysis operator* of Φ is the operator $T : \mathbb{H} \rightarrow \ell^2(I)$ given by

$$Tx = \{\langle x, \varphi_i \rangle\}_{i \in I}$$

and the associated *synthesis operator* is given by the adjoint operator $T^* : \ell^2(I) \rightarrow \mathbb{H}$ and satisfies

$$T^* \{c_i\}_{i \in I} = \sum_{i \in I} c_i \varphi_i.$$

The *frame operator* $S : \mathbb{H} \rightarrow \mathbb{H}$ is the positive, self-adjoint operator defined by $S = T^*T$ and satisfies

$$Sx = T^*Tx = \sum_{i \in I} \langle x, \varphi_i \rangle \varphi_i$$

for every $x \in \mathbb{H}$. On the other hand, the *Gramian operator* $G : \ell^2(I) \rightarrow \ell^2(I)$ is the operator defined by $G = TT^*$ and has the matrix representation

$$G = (\langle \varphi_i, \varphi_j \rangle)_{i, j \in I}.$$

These four operators are well-defined when the sequence Φ is assumed to be at least a B -Bessel sequence.

When Φ is a frame with bounds A and B , the frame operator satisfies for every $x \in \mathbb{H}$,

$$\langle Ax, x \rangle \leq \langle Sx, x \rangle = \|Tx\|^2 = \sum_{i \in I} |\langle x, \varphi_i \rangle|^2 \leq \langle Bx, x \rangle,$$

and hence operator inequality $A \cdot Id \leq S \leq B \cdot Id$ holds and S is invertible¹. Also, note that $\{S^{-1/2}\varphi_i\}_{i \in I}$ is a Parseval frame, called the *canonical Parseval frame of Φ* . Finally, the norm of S is $\|S\| = \|T^*T\| = \|T\|^2$.

3. Getting started

In this section, the definition of woven frames is introduced. Then some results and examples are presented in regards to weaving families of vectors. Throughout the rest of the paper for ease of notation, let

$$[m] = \{1, \dots, m\} \quad \text{and} \quad [m]^c = \mathbb{N} \setminus [m] = \{m + 1, m + 2, \dots\}$$

for a given natural number m . Also, denote by $[m, k] = [m + k] \setminus [m] = \{m + 1, \dots, m + k\}$ for every $m, k \in \mathbb{N}$. Here $\mathbb{N} = \{1, 2, \dots\}$.

DEFINITION 3. A family of frames $\{\varphi_{ij}\}_{i \in I}$ for $j \in [M]$ for a Hilbert space \mathbb{H} is said to be *woven* if there are universal constants A and B so that for every partition $\{\sigma_j\}_{j \in [M]}$ of I , the family $\{\varphi_{ij}\}_{i \in \sigma_j, j \in [M]}$ is a frame for \mathbb{H} with lower and upper frame bounds A and B , respectively. Each family $\{\varphi_{ij}\}_{i \in \sigma_j, j \in [M]}$ is called a *weaving*.

The first proposition of this section gives that every weaving automatically has a universal upper frame bound.

PROPOSITION 3.1. *If each $\Phi_j = \{\varphi_{ij}\}_{i \in I}$ is a Bessel sequence for \mathbb{H} with bounds B_j for all $j \in [M]$, then every weaving is a Bessel sequence with $\sum_{j=1}^M B_j$ as a Bessel bound.*

Proof. For every partition $\{\sigma_j\}_{j \in [M]}$ of I and every $x \in \mathbb{H}$,

$$\sum_{j=1}^M \sum_{i \in \sigma_j} |\langle x, \varphi_{ij} \rangle|^2 \leq \sum_{j=1}^M \sum_{i \in I} |\langle x, \varphi_{ij} \rangle|^2 \leq \sum_{j=1}^M B_j \cdot \|x\|^2,$$

yielding the desired bound. \square

To verify that a finite number of frames Φ_j with $j \in [M]$, is woven, we therefore only need to check that there is a universal lower bound for all weavings.

If all Φ_j are frames for \mathbb{H} , then the bound in Proposition 3.1 cannot be obtained, that is, the sum $\sum_{j \in [M]} B_j$ cannot be the smallest upper weaving bound if B_j is optimal for its respective frame. Since the concept of weakly woven is used to show this, the proof is deferred until Section 4, Remark 2. However, the sum of the upper bounds can

be approached arbitrarily, as the next example shows. Example 1 also shows that one cannot hope to classify woven frames by placing restrictions on the weaving bounds, even if the frames considered are Parseval.

EXAMPLE 1. There exist two Parseval frames that give weavings with arbitrary weaving bounds. Let $\varepsilon > 0$, set $\delta = (1 + \varepsilon^2)^{-1/2}$, and let $\{e_1, e_2\}$ be the standard orthonormal basis of \mathbb{R}^2 . Then the two sets

$$\Phi = \{\varphi_i\}_{i=1}^4 = \{\delta e_1, \delta \varepsilon e_1, \delta e_2, \delta \varepsilon e_2\}$$

and

$$\Psi = \{\psi_i\}_{i=1}^4 = \{\delta \varepsilon e_1, \delta e_1, \delta \varepsilon e_2, \delta e_2\}$$

are Parseval frames, which are woven since any choice of σ gives a spanning set (see Section 4, Theorem 4.1). Since they are Parseval, the universal upper frame bound for every weaving can be chosen to be 2 by Proposition 3.1. If $\sigma = \{2, 4\}$ and $x \in \mathbb{R}^2$, then

$$\sum_{i \in \sigma} |\langle x, \varphi_i \rangle|^2 + \sum_{i \in \sigma^c} |\langle x, \psi_i \rangle|^2 = 2\delta^2 \varepsilon^2 \|x\|^2 = \frac{2\varepsilon^2}{1 + \varepsilon^2} \|x\|^2$$

which can be anywhere between 0 and 2 for arbitrary choice of $\varepsilon \in (0, \infty)$.

Clearly the property of woven frames is preserved under a bounded invertible operator. This observation sometimes helps to simplify proofs.

COROLLARY 3.2. *When considering whether two frames $\Phi = \{\varphi_i\}_{i \in I}$ and $\Psi = \{\psi_i\}_{i \in I}$ are woven, it may always be assumed that one of them is Parseval by instead considering $\{S^{-1/2}\varphi_i\}_{i \in I}$ and $\{S^{-1/2}\psi_i\}_{i \in I}$ where S is the frame operator of Φ .*

The next example shows that in general, frames may be woven without their canonical Parseval frames being woven. This also shows that applying two different operators to woven frames can give frames that are not woven.

EXAMPLE 2. There exist Riesz bases that are woven in which their canonical Parseval frames are not woven. Let $\Phi = \{e_1, e_2\}$ be the standard orthonormal basis for \mathbb{R}^2 and let $\Psi = \{\psi_1, \psi_2\}$ be defined by

$$\psi_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \psi_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Note that Φ and Ψ are woven. The frame operator of Ψ is

$$S = \psi_1 \psi_1^* + \psi_2 \psi_2^* = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}.$$

The canonical Parseval frame of Φ is itself. Note that because Φ is Parseval, any orthonormal basis forms a set of eigenvectors for its frame operator, the identity operator, and hence has the same eigenbasis as S . Finally, computing $S^{-1/2}$ gives

$$S^{-1/2} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

giving that the canonical Parseval frame of Ψ is $\{S^{-1/2}\psi_1, S^{-1/2}\psi_2\} = \{e_2, e_1\}$ which is clearly not woven with Φ .

4. Weakly woven frames

In this section, it will be shown that if each weaving is a frame, then necessarily there exists a universal lower frame bound for all weavings. Consequently it only needs to be checked that each weaving has a lower frame bound to show that a family of frames are woven.

DEFINITION 4. A family of frames $\{\varphi_{ij}\}_{i \in \mathbb{N}, j \in [M]}$ for a Hilbert space \mathbb{H} is said to be *weakly woven* if for every partition $\{\sigma_j\}_{j \in [M]}$ of \mathbb{N} , the family $\{\varphi_{ij}\}_{i \in \sigma_j, j \in [M]}$ is a frame for \mathbb{H} .

The main question is whether or not weakly woven is equivalent to woven. The answer to this question in the finite dimensional case is obvious since there are only finitely many ways to partition the index set.

THEOREM 4.1. *Two frames $\{\varphi_i\}_{i=1}^N$ and $\{\psi_i\}_{i=1}^N$ for a finite-dimensional Hilbert space \mathbb{H}^M are woven if and only if for every $\sigma \subset [N]$, $\{\varphi_i\}_{i \in \sigma} \cup \{\psi_i\}_{i \in \sigma^c}$ spans the space.*

Recall that a finite frame for \mathbb{H}^M is said to be *full spark*, if every M element subset of the frame is linearly independent. We obtain the following immediate condition for weaving finite frames.

COROLLARY 4.2. *If $\{\varphi_i\}_{i=1}^N$ is full spark in \mathbb{H}^M and every subset of $\{\psi_i\}_{i=1}^N$ with $N - M$ elements spans, then these two frames are woven. In particular, if two full spark frames each have $N \geq 2M - 1$ elements in an M dimensional space, then they are necessarily woven.*

Proof. Let $\sigma \subset [N]$. If $|\sigma| \geq M$, then $\{\varphi_i\}_{i \in \sigma}$ spans and so $\{\varphi_i\}_{i \in \sigma} \cup \{\psi_i\}_{i \in \sigma^c}$ is a frame. If $|\sigma| \leq M - 1$, then $|\sigma^c| \geq N - M + 1 \geq M$ and so $\{\psi_i\}_{i \in \sigma^c}$ spans. \square

REMARK 1. The corresponding statement fails in infinite dimensions: two frames with the property that every finite subset is independent may not be woven. To see this, take an orthonormal basis and a non-trivial permutation of it.

The equivalence of woven and weakly woven frames is significantly more difficult to show for frames in an infinite dimensional space. As a preparation we need the following lemma.

LEMMA 4.3. *Let $\{\varphi_i\}_{i=1}^\infty$ and $\{\psi_i\}_{i=1}^\infty$ be frames for a Hilbert space \mathbb{H} . Assume that for every two disjoint finite sets $I, J \subset \mathbb{N}$ and every $\varepsilon > 0$ there are subsets $\sigma, \delta \subset \mathbb{N} \setminus (I \cup J)$ with $\delta = \mathbb{N} \setminus (I \cup J \cup \sigma)$ so that the lower frame bound of $\{\varphi_i\}_{i \in I \cup \sigma} \cup \{\psi_i\}_{i \in J \cup \delta}$ is less than ε . Then there is a subset $\mathcal{A} \subset \mathbb{N}$ so that $\{\varphi_i\}_{i \in \mathcal{A}} \cup \{\psi_i\}_{i \in \mathcal{A}^c}$ is not a frame, i.e., these frames are not woven.*

Proof. Fix $\varepsilon > 0$. By assumption, by letting $I_0 = J_0 = \emptyset$, we can choose $\sigma_1 \subset \mathbb{N}$ so that if $\delta_1 = \sigma_1^c$, a lower frame bound of $\{\varphi_i\}_{i \in \sigma_1} \cup \{\psi_i\}_{i \in \delta_1}$ is less than ε . Hence, there is a vector $h_1 \in \mathbb{H}$ with $\|h_1\| = 1$ so that

$$\sum_{i \in \sigma_1} |\langle h_1, \varphi_i \rangle|^2 + \sum_{i \in \delta_1} |\langle h_1, \psi_i \rangle|^2 < \varepsilon.$$

Since

$$\sum_{i=1}^{\infty} |\langle h_1, \varphi_i \rangle|^2 + \sum_{i=1}^{\infty} |\langle h_1, \psi_i \rangle|^2 < \infty,$$

there is a $k_1 \in \mathbb{N}$ so that

$$\sum_{i=k_1+1}^{\infty} |\langle h_1, \varphi_i \rangle|^2 + \sum_{i=k_1+1}^{\infty} |\langle h_1, \psi_i \rangle|^2 < \varepsilon.$$

Let $I_1 = \sigma_1 \cap [k_1]$ and $J_1 = \delta_1 \cap [k_1]$. Then $I_1 \cap J_1 = \emptyset$ and $I_1 \cup J_1 = [k_1]$. By assumption, there are subsets $\sigma_2, \delta_2 \subset [k_1]^c$ with $\delta_2 = [k_1]^c \setminus \sigma_2$ such that a lower frame bound of $\{\varphi_i\}_{i \in I_1 \cup \sigma_2} \cup \{\psi_i\}_{i \in J_1 \cup \delta_2}$ is less than $\varepsilon/2$. That is, there is a vector $h_2 \in \mathbb{H}$ with $\|h_2\| = 1$ so that

$$\sum_{i \in I_1 \cup \sigma_2} |\langle h_2, \varphi_i \rangle|^2 + \sum_{i \in J_1 \cup \delta_2} |\langle h_2, \psi_i \rangle|^2 < \frac{\varepsilon}{2}.$$

Similar to above, there is a $k_2 > k_1$ so that

$$\sum_{i=k_2+1}^{\infty} |\langle h_2, \varphi_i \rangle|^2 + \sum_{i=k_2+1}^{\infty} |\langle h_2, \psi_i \rangle|^2 < \frac{\varepsilon}{2}.$$

Set $I_2 = I_1 \cup (\sigma_2 \cap [k_2])$ and $J_2 = J_1 \cup (\delta_2 \cap [k_2])$. Note that $I_2 \cap J_2 = \emptyset$ and their union is $I_2 \cup J_2 = [k_2]$. Continue inductively to obtain

- natural numbers $k_1 < k_2 < \dots < k_n < \dots$,
- vectors $h_n \in \mathbb{H}$ with $\|h_n\| = 1$,
- $\sigma_n \subset [k_{n-1}]^c$, $\delta_n = [k_{n-1}]^c \setminus \sigma_n$, and
- $I_n = I_{n-1} \cup (\sigma_n \cap [k_n])$, $J_n = J_{n-1} \cup (\delta_n \cap [k_n])$

which satisfies both

$$\sum_{i \in I_{n-1} \cup \sigma_n} |\langle h_n, \varphi_i \rangle|^2 + \sum_{i \in J_{n-1} \cup \delta_n} |\langle h_n, \psi_i \rangle|^2 < \frac{\varepsilon}{n} \quad (1)$$

and

$$\sum_{i=k_n+1}^{\infty} |\langle h_n, \varphi_i \rangle|^2 + \sum_{i=k_n+1}^{\infty} |\langle h_n, \psi_i \rangle|^2 < \frac{\varepsilon}{n}. \quad (2)$$

By construction $I_n \cap J_n = \emptyset$ and $I_n \cup J_n = [k_n]$ for all n so that

$$\bigcup_{i=1}^{\infty} I_i \sqcup \bigcup_{j=1}^{\infty} J_j = \mathbb{N},$$

where “ \sqcup ” represents a disjoint union. Now set

$$\mathcal{A} = \bigcup_{i=1}^{\infty} I_i, \text{ and note } \mathcal{A}^c = \bigcup_{j=1}^{\infty} J_j.$$

It follows by construction along with (1) and (2) that

$$\begin{aligned} & \sum_{i \in \mathcal{A}} |\langle h_n, \varphi_i \rangle|^2 + \sum_{i \in \mathcal{A}^c} |\langle h_n, \psi_i \rangle|^2 \\ &= \left(\sum_{i \in I_n} |\langle h_n, \varphi_i \rangle|^2 + \sum_{j \in J_n} |\langle h_n, \psi_j \rangle|^2 \right) \\ &+ \left(\sum_{i \in \mathcal{A} \cap [k_n]^c} |\langle h_n, \varphi_i \rangle|^2 + \sum_{i \in \mathcal{A}^c \cap [k_n]^c} |\langle h_n, \psi_i \rangle|^2 \right) \\ &\leq \left(\sum_{i \in I_{n-1} \cup \sigma_n} |\langle h_n, \varphi_i \rangle|^2 + \sum_{i \in J_{n-1} \cup \delta_n} |\langle h_n, \psi_i \rangle|^2 \right) \\ &+ \left(\sum_{i=k_n+1}^{\infty} |\langle h_n, \varphi_i \rangle|^2 + \sum_{i=k_n+1}^{\infty} |\langle h_n, \psi_i \rangle|^2 \right) \\ &< \frac{\varepsilon}{n} + \frac{\varepsilon}{n} = \frac{2\varepsilon}{n} \end{aligned}$$

so that a lower frame bound of $\{\varphi_i\}_{i \in A} \cup \{\psi_i\}_{i \in A^c}$ is zero. Therefore, it is not a frame and the two original frames are not woven. \square

COROLLARY 4.4. *If the frames $\{\varphi_i\}_{i=1}^{\infty}$ and $\{\psi_i\}_{i=1}^{\infty}$ in \mathbb{H} are weakly woven, then there are disjoint finite sets $I, J \subset \mathbb{N}$ and a constant $A > 0$ so that for all $\sigma, \delta \subset \mathbb{N} \setminus (I \cup J)$ and $\delta = \mathbb{N} \setminus (I \cup J \cup \sigma)$, the family $\{\varphi_i\}_{i \in I \cup \sigma} \cup \{\psi_i\}_{i \in J \cup \delta}$ has lower frame bound A .*

Using Corollary 4.4, we can now prove the main result of this section.

THEOREM 4.5. *Given two frames $\{\varphi_i\}_{i=1}^{\infty}$ and $\{\psi_i\}_{i=1}^{\infty}$ for \mathbb{H} , the following are equivalent:*

- (i) *The two frames are woven.*
- (ii) *The two frames are weakly woven.*

Proof. Only (ii) \Rightarrow (i) needs to be shown. By Corollary 4.4, there are subsets $I, J \subset \mathbb{N}$ with $|I|, |J| < \infty$ and $I \cap J = \emptyset$, and $A > 0$ satisfying:

(†) For every subset $\sigma \subset \mathbb{N} \setminus (I \cup J)$ with $\delta = \mathbb{N} \setminus (I \cup J \cup \sigma)$ the family $\{\varphi_i\}_{i \in I \cup \sigma} \cup \{\psi_i\}_{i \in J \cup \delta}$ has lower frame bound A .

To simplify the notation, we permute both frames so that $I = [q]$ and $J = [m] \setminus [q]$, or one of I or J is empty and the other is all of $[m]$. Note that permuting does not affect weaving as long as it is done to both frames simultaneously.

Step 1. If for each partition I_α, J_α of $[m]$ there is a constant $D_\alpha > 0$ so that for every $\sigma \subset [m]^c$ and $\delta = [m]^c \setminus \sigma$ the family $\{\varphi_i\}_{i \in I_\alpha \cup \sigma} \cup \{\psi_i\}_{i \in J_\alpha \cup \delta}$ has lower frame bound D_α , then the frames are woven with universal lower frame bound $A_0 = \min\{D_\alpha : \alpha\}$, which is positive since it is a minimum of a finite number of positive numbers.

Assume that the above does not hold. That is, there is a partition I_1, J_1 of $[m]$ such that for every $\varepsilon > 0$ there are subsets $\sigma \subset [m]^c$ and $\delta = [m]^c \setminus \sigma$ so that a lower frame bound of $\{\varphi_i\}_{i \in I_1 \cup \sigma} \cup \{\psi_i\}_{i \in J_1 \cup \delta}$ has lower frame bound less than ε . We will show that this yields a contradiction.

Step 2. It is shown that for all $n \in \mathbb{N}$, there are subsets $\sigma_n \subset [m]^c$ and $\delta_n = [m]^c \setminus \sigma_n$ and unit vectors $h_n \in \ell^2$ so that

$$\sum_{i \in I_1 \cup \sigma_n} |\langle h_n, \varphi_i \rangle|^2 + \sum_{i \in J_1 \cup \delta_n} |\langle h_n, \psi_i \rangle|^2 < \frac{1}{n}. \quad (3)$$

Furthermore, the sets σ_n and δ_n satisfy the following properties.

- (a) For every $k = 1, 2, \dots$, either $m+k \in \sigma_n$ for all $n \geq k$ or $m+k \in \delta_n$ for all $n \geq k$.
- (b) There is a $\sigma \subset [m]^c$ with $\delta = [m]^c \setminus \sigma$ so that $m+k \in \sigma$ implies that $m+k \in \sigma_n$ for all $n \geq k$ or if $m+k \in \delta$ then $m+k \in \delta_n$ for all $n \geq k$.

Proof of Step 2. By assumption, for each n there exist subsets $\sigma_n, \delta_n \subset [m]^c$ with $\delta_n = [m]^c \setminus \sigma_n$ such that the lower frame bound of $\{\varphi_i\}_{i \in I_1 \cup \sigma_n} \cup \{\psi_i\}_{i \in J_1 \cup \delta_n}$ is less than $1/n$. Hence, there are vectors h_n of norm one so that

$$\sum_{i \in I_1 \cup \sigma_n} |\langle h_n, \varphi_i \rangle|^2 + \sum_{i \in J_1 \cup \delta_n} |\langle h_n, \psi_i \rangle|^2 < \frac{1}{n}, \text{ for all } n \in \mathbb{N}.$$

Note that for each n , either $m+1 \in \sigma_n$ or $m+1 \in \delta_n$, so we can choose a subsequence $L_1 = \{l_{1j}\}_{j=1}^\infty$ of \mathbb{N} such that one of the following must hold:

- For every $k \in L_1$, $m+1 \in \sigma_k$.
- For every $k \in L_1$, $m+1 \in \delta_k$.

Similarly, there are subsequences $\{L_i\}_{i=1}^\infty$ of \mathbb{N} with $L_i = \{l_{ij}\}_{j=1}^\infty$ satisfying:

- L_{i+1} is a subsequence of L_i for all $i = 1, 2, \dots$
- For every $i = 1, 2, \dots$, either $m+i \in \sigma_k$ for all $k \in L_i$ with $k \geq l_{i,i}$ or $m+i \in \delta_k$ for all $k \in L_i$ with $k \geq l_{i,i}$.

Now set $L = \{l_{ii}\}_{i=1}^\infty$, so that $i \leq l_{ii}$ for all i and note that this is a subsequence of \mathbb{N} satisfying: $\{l_{ii}\}_{i=k}^\infty$ is a subsequence of L_k for every $k \in \mathbb{N}$.

Now reindex the sigmas, $\sigma_{l_{nn}} \mapsto \sigma_n$ to obtain for each $n \in \mathbb{N}$, a subset $\sigma_n \subset [m]^c$ satisfying

$$\sum_{i \in I_1 \cup \sigma_n} |\langle h_n, \varphi_i \rangle|^2 + \sum_{i \in J_1 \cup \delta_n} |\langle h_n, \psi_i \rangle|^2 < \frac{1}{l_{nn}} \leq \frac{1}{n}. \tag{4}$$

Note that (4) is satisfied by construction of σ_n . To obtain (4), define σ by putting $m+k \in \sigma$ if $m+k \in \sigma_n$ for all $n \geq k$ and $m+k \in \delta$ if $m+k \in \delta_n$ for all $n \geq k$.

Step 3. It will be shown that by switching to a subsequence and reindexing, it can be assumed that

$$h_n \rightarrow_w h \text{ and } h \neq 0.$$

Proof of Step 3. Since the sequence $\{h_n\}_{n=1}^\infty$ is bounded, it has a weakly convergent subsequence $\{h_{n_i}\}_{i=1}^\infty$ that converges to some h . Reindex, $h_{n_i} \mapsto h_i$ and $\sigma_{n_i} \mapsto \sigma_i$ and notice that the properties proved in Step 2 still hold.

Fix $k \in \mathbb{N}$ so that $k > 2/A$, where A is the constant in (\dagger) . Note that since $\{h_n\}_{n=1}^\infty$ converges weakly to h , it converges in norm to h on finite dimensional subspaces. In particular, there is an $N_k \in \mathbb{N}$ so that for all $n \geq N_k > k$,

$$\sum_{i \in [m+k]} |\langle h - h_n, \varphi_i \rangle|^2 + \sum_{i \in [m+k]} |\langle h - h_n, \psi_i \rangle|^2 < \frac{1}{2k}. \tag{5}$$

Now (\dagger) implies that, for every n ,

$$\sum_{i \in I \cup \sigma_n} |\langle h_n, \varphi_i \rangle|^2 + \sum_{i \in J \cup \delta_n} |\langle h_n, \psi_i \rangle|^2 \geq A. \tag{6}$$

Therefore, if $n \geq N_k > k$, then by definition of σ and δ , $\sigma_n \cap [m, k] = \sigma \cap [m, k]$ and $\delta_n \cap [m, k] = \delta \cap [m, k]$, and thus inequalities (4), (5), and (6), imply

$$\begin{aligned} & \sum_{i \in I \cup \sigma} |\langle h, \varphi_i \rangle|^2 + \sum_{i \in J \cup \delta} |\langle h, \psi_i \rangle|^2 \\ & \geq \sum_{i \in I \cup (\sigma \cap [m, k])} |\langle h, \varphi_i \rangle|^2 + \sum_{i \in J \cup (\delta \cap [m, k])} |\langle h, \psi_i \rangle|^2 \\ & = \sum_{i \in I \cup (\sigma_n \cap [m, k])} |\langle h, \varphi_i \rangle|^2 + \sum_{i \in J \cup (\delta_n \cap [m, k])} |\langle h, \psi_i \rangle|^2 \\ & \geq \frac{1}{2} \left(\sum_{i \in I \cup (\sigma_n \cap [m, k])} |\langle h_n, \varphi_i \rangle|^2 + \sum_{i \in J \cup (\delta_n \cap [m, k])} |\langle h_n, \psi_i \rangle|^2 \right) \\ & \quad - \left(\sum_{i \in I \cup (\sigma_n \cap [m, k])} |\langle h - h_n, \varphi_i \rangle|^2 + \sum_{i \in J \cup (\delta_n \cap [m, k])} |\langle h - h_n, \psi_i \rangle|^2 \right) \end{aligned}$$

$$\begin{aligned}
&\geq \frac{1}{2} \left(\sum_{i \in I \cup \sigma_n} |\langle h_n, \varphi_i \rangle|^2 + \sum_{i \in J \cup \delta_n} |\langle h_n, \psi_i \rangle|^2 \right) \\
&\quad - \frac{1}{2} \left(\sum_{i \in \sigma_n \cap [m+k]^c} |\langle h_n, \varphi_i \rangle|^2 + \sum_{i \in \delta_n \cap [m+k]^c} |\langle h_n, \psi_i \rangle|^2 \right) - \frac{1}{2k} \\
&\geq \frac{1}{2} \cdot A - \frac{1}{2} \cdot \frac{1}{n} - \frac{1}{2k} \\
&\geq \frac{A}{2} - \frac{1}{k} > 0
\end{aligned}$$

showing that $h \neq 0$.

Step 4. It is now shown that

$$\sum_{i \in I_1 \cup \sigma} |\langle h, \varphi_i \rangle|^2 + \sum_{i \in J_1 \cup \delta} |\langle h, \psi_i \rangle|^2 = 0.$$

Proof of Step 4. By definition of σ and δ , $\sigma_{N_k} \cap [m, k] = \sigma \cap [m, k]$ and $\delta_{N_k} \cap [m, k] = \delta \cap [m, k]$, and therefore inequalities (4) and (5) imply that

$$\begin{aligned}
&\sum_{i \in I_1 \cup \sigma} |\langle h, \varphi_i \rangle|^2 + \sum_{i \in J_1 \cup \delta} |\langle h, \psi_i \rangle|^2 \\
&= \lim_{k \rightarrow \infty} \left(\sum_{i \in I_1 \cup (\sigma \cap [m, k])} |\langle h, \varphi_i \rangle|^2 + \sum_{i \in J_1 \cup (\delta \cap [m, k])} |\langle h, \psi_i \rangle|^2 \right) \\
&= \lim_{k \rightarrow \infty} \left(\sum_{i \in I_1 \cup (\sigma_{N_k} \cap [m, k])} |\langle h, \varphi_i \rangle|^2 + \sum_{i \in J_1 \cup (\delta_{N_k} \cap [m, k])} |\langle h, \psi_i \rangle|^2 \right) \\
&\leq 2 \lim_{k \rightarrow \infty} \left(\sum_{i \in I_1 \cup \sigma_{N_k}} |\langle h_{N_k}, \varphi_i \rangle|^2 + \sum_{i \in J_1 \cup \delta_{N_k}} |\langle h_{N_k}, \psi_i \rangle|^2 \right) \\
&\quad + 2 \lim_{k \rightarrow \infty} \left(\sum_{i \in [m+k]} |\langle h - h_{N_k}, \varphi_i \rangle|^2 + \sum_{i \in [m+k]} |\langle h - h_{N_k}, \psi_i \rangle|^2 \right) \\
&\leq 2 \lim_{k \rightarrow \infty} \frac{1}{N_k} + 2 \lim_{k \rightarrow \infty} \frac{1}{2k} = 0.
\end{aligned}$$

Therefore, h is not in the span of $\{\varphi_i\}_{i \in I_1 \cup \sigma} \cup \{\psi_i\}_{i \in J_1 \cup \delta}$ and hence this weaving is not frame. It follows that the original frames are not weakly woven, so a contradiction is met, concluding the proof. \square

REMARK 2. This section is concluded by showing that the upper bound in Proposition 3.1 cannot be obtained for woven frames. The case of two frames is given, but the argument is easily extended to finitely many. Suppose that $\Phi = \{\varphi_i\}_{i \in I}$ and $\Psi = \{\psi_i\}_{i \in I}$ are frames for a Hilbert space \mathbb{H} with optimal upper frame bounds B_1 and B_2 . Assume by way of contradiction that $B_1 + B_2$ is the optimal upper weaving bound. That is, the smallest upper weaving bound for all possible weavings. Then for

a fixed $\varepsilon > 0$, one can choose a $\sigma \subset I$ and $\|x\| = 1$ so that

$$\sum_{i \in \sigma} |\langle x, \varphi_i \rangle|^2 + \sum_{i \in \sigma^c} |\langle x, \psi_i \rangle|^2 \geq B_1 + B_2 - \varepsilon.$$

Since

$$\sum_{i \in I} |\langle x, \varphi_i \rangle|^2 + \sum_{i \in I} |\langle x, \psi_i \rangle|^2 \leq B_1 + B_2,$$

it follows that

$$\sum_{i \in \sigma^c} |\langle x, \varphi_i \rangle|^2 + \sum_{i \in \sigma} |\langle x, \psi_i \rangle|^2 \leq \varepsilon. \tag{7}$$

Now (7) implies that there are weavings for which the lower frame bounds approach zero. Therefore, Theorem 4.5 gives that Φ and Ψ are not woven, which is a contradiction.

5. Weaving Riesz bases

In this section we classify when Riesz bases and Riesz basic sequences can be woven and provide a characterization in terms of distances between subspaces.

We need a lemma in the case that σ is finite in order to prove Theorem 5.2, which is in terms of general partitions.

LEMMA 5.1. *Suppose $\{\varphi_i\}_{i=1}^\infty$ and $\{\psi_i\}_{i=1}^\infty$ are Riesz bases for \mathbb{H} for which there are uniform constants $0 < A \leq B < \infty$ so that for every $\sigma \subset \mathbb{N}$, the family $\{\varphi_i\}_{i \in \sigma} \cup \{\psi_i\}_{i \in \sigma^c}$ is a Riesz sequence with Riesz bounds A and B . Then for every $\sigma \subset \mathbb{N}$ with $|\sigma| < \infty$, the family $\{\varphi_i\}_{i \in \sigma} \cup \{\psi_i\}_{i \in \sigma^c}$ is actually a Riesz basis, that is, it spans \mathbb{H} .*

Proof. We proceed by induction on the cardinality of σ . The case $|\sigma| = 0$ being obvious, we assume the result holds for every σ with $|\sigma| = n$.

Now let $\sigma \subset \mathbb{N}$ with $|\sigma| = n + 1$ and choose $i_0 \in \sigma$. Let $\sigma_1 = \sigma \setminus \{i_0\}$, then $\{\varphi_i\}_{i \in \sigma_1} \cup \{\psi_i\}_{i \in \sigma_1^c}$ is a Riesz basis by the induction hypothesis.

We proceed by way of contradiction and assume that $\{\varphi_i\}_{i \in \sigma} \cup \{\psi_i\}_{i \in \sigma^c}$ is not a Riesz basis. However, it is at least a Riesz sequence by assumption. If

$$\psi_{i_0} \in \text{span}(\{\varphi_i\}_{i \in \sigma} \cup \{\psi_i\}_{i \in \sigma^c}),$$

then

$$\overline{\text{span}}(\{\varphi_i\}_{i \in \sigma} \cup \{\psi_i\}_{i \in \sigma^c}) \supset \overline{\text{span}}(\{\varphi_i\}_{i \in \sigma_1} \cup \{\psi_i\}_{i \in \sigma_1^c}) = \mathbb{H},$$

i.e., $\{\varphi_i\}_{i \in \sigma} \cup \{\psi_i\}_{i \in \sigma^c}$ would be a basis, which is assumed to not be the case. So it must be that

$$\psi_{i_0} \notin \text{span}(\{\varphi_i\}_{i \in \sigma} \cup \{\psi_i\}_{i \in \sigma^c})$$

from which it follows that

$$\{\varphi_i\}_{i \in \sigma} \cup \{\psi_i\}_{i \in \sigma^c} \cup \{\psi_{i_0}\}$$

is a Riesz sequence in \mathbb{H} . Hence, because $\sigma_1^c = \sigma^c \cup \{i_0\}$,

$$\{\varphi_i\}_{i \in \sigma_1} \cup \{\psi_i\}_{i \in \sigma_1^c}$$

cannot be a Riesz basis, since we obtained it by deleting the element φ_{i_0} from a Riesz sequence, yielding a contradiction. \square

Next, the result from the previous lemma is extended to σ of arbitrary cardinality.

THEOREM 5.2. *Suppose $\{\varphi_i\}_{i=1}^\infty$ and $\{\psi_i\}_{i=1}^\infty$ are Riesz bases so that there are uniform constants $0 < A \leq B < \infty$ so that for every $\sigma \subset \mathbb{N}$, the family $\{\varphi_i\}_{i \in \sigma} \cup \{\psi_i\}_{i \in \sigma^c}$ is a Riesz sequence with Riesz bounds A and B . Then for every $\sigma \subset \mathbb{N}$ the family $\{\varphi_i\}_{i \in \sigma} \cup \{\psi_i\}_{i \in \sigma^c}$ is actually a Riesz basis.*

Proof. Assume, by way of contradiction, there is a $\sigma \subset \mathbb{N}$ with both σ and σ^c infinite, so that

$$K = \overline{\text{span}}(\{\varphi_i\}_{i \in \sigma} \cup \{\psi_i\}_{i \in \sigma^c}) \neq \mathbb{H}.$$

Choose a nonzero $x \in K^\perp$. Since $\{\psi_i\}_{i \in I}$ is Bessel, by taking the tail of the series, there exists a $\sigma_1 \subset \sigma$ with $|\sigma_1| < \infty$ and

$$\sum_{i \in \sigma \setminus \sigma_1} |\langle x, \psi_i \rangle|^2 < \frac{A}{2} \|x\|^2.$$

By Lemma 5.1, the family $\{\varphi_i\}_{i \in \sigma_1} \cup \{\psi_i\}_{i \in \sigma \setminus \sigma_1} \cup \{\psi_i\}_{i \in \sigma^c}$ is a Riesz basis with Riesz basis bounds A, B and therefore

$$\begin{aligned} A\|x\|^2 &\leq \sum_{i \in \sigma_1} |\langle x, \varphi_i \rangle|^2 + \sum_{i \in \sigma \setminus \sigma_1} |\langle x, \psi_i \rangle|^2 + \sum_{i \in \sigma^c} |\langle x, \psi_i \rangle|^2 \\ &= \sum_{i \in \sigma \setminus \sigma_1} |\langle x, \psi_i \rangle|^2 \\ &\leq \frac{A}{2} \|x\|^2 \end{aligned}$$

giving a contradiction. \square

REMARK 3. If every weaving of two Riesz bases is a frame sequence, it does not follow that every weaving is a Riesz basis, nor even a Riesz sequence. To see this, let $\{\varphi_i\}_{i=1}^\infty$ be a Riesz basis and let π be a non-trivial permutation so that $\pi(i_0) \neq i_0$. Each weaving is a frame sequence but the weaving given by $\sigma = \mathbb{N} \setminus \{i_0\}$ does not span.

The next theorem says that if two Riesz bases are woven, then every weaving is in fact a Riesz basis, and not just a frame.

THEOREM 5.3. *Suppose $\Phi = \{\varphi_i\}_{i=1}^\infty$ and $\Psi = \{\psi_i\}_{i=1}^\infty$ are Riesz bases and that there is a uniform constant $A > 0$ so that for every $\sigma \subset \mathbb{N}$, the family $\{\varphi_i\}_{i \in \sigma} \cup \{\psi_i\}_{i \in \sigma^c}$ is a frame with lower frame bound A . Then for every $\sigma \subset \mathbb{N}$, the family $\{\varphi_i\}_{i \in \sigma} \cup \{\psi_i\}_{i \in \sigma^c}$ is actually a Riesz basis.*

Proof. The proof is carried out in steps.

Step 1. For every $|\sigma| < \infty$, the family $\{\varphi_i\}_{i \in \sigma} \cup \{\psi_i\}_{i \in \sigma^c}$ is a Riesz basis.

Proof of Step 1. We do the proof by induction on $|\sigma|$ with $|\sigma| = 0$ clear. Now assume the result is true for all $|\sigma| = n$. Let $\sigma \subset I$ be so that $|\sigma| = n + 1$ and let $i_0 \in \sigma$. Then $\{\varphi_i\}_{i \in \sigma \setminus \{i_0\}} \cup \{\psi_i\}_{i \in \sigma^c \cup \{i_0\}}$ is a Riesz basis and therefore

$$\{\varphi_i\}_{i \in \sigma \setminus \{i_0\}} \cup \{\psi_i\}_{i \in \sigma^c}$$

is a Riesz sequence spanning a subspace of codimension at least one.

Now by assumption, $\{\varphi_i\}_{i \in \sigma} \cup \{\psi_i\}_{i \in \sigma^c}$ is at least a frame. Since the removal of the single vector φ_{i_0} yields a set that does not longer span \mathbb{H} , $\{\varphi_i\}_{i \in \sigma} \cup \{\psi_i\}_{i \in \sigma^c}$ must actually be a Riesz basis [6]. Furthermore, its lower Riesz bound is A . This concludes the proof of Step 1.

Step 2. Now consider the case that $|\sigma| = \infty$.

For this step choose $\sigma_1 \subset \sigma_2 \subset \dots \subset \sigma$ so that

$$\sigma = \bigcup_{j=1}^{\infty} \sigma_j,$$

and $|\sigma_j| < \infty$. Now, for every $j = 1, 2, \dots$ the family

$$\{\varphi_i\}_{i \in \sigma_j} \cup \{\psi_i\}_{i \in \sigma \setminus \sigma_j} \cup \{\psi_i\}_{i \in \sigma^c} = \{\varphi_i\}_{i \in \sigma_j} \cup \{\psi_i\}_{i \in \sigma_j^c},$$

is a Riesz basis with lower Riesz basis constant A . If $\{a_i\}_{i=1}^{\infty} \in \ell^2$ and

$$\sum_{i \in \sigma} a_i \varphi_i + \sum_{i \in \sigma^c} a_i \psi_i = 0,$$

then

$$\begin{aligned} 0 &= \left\| \sum_{i \in \sigma} a_i \varphi_i + \sum_{i \in \sigma^c} a_i \psi_i \right\|^2 \\ &= \lim_{j \rightarrow \infty} \left\| \sum_{i \in \sigma_j} a_i \varphi_i + \sum_{i \in \sigma_j^c} a_i \psi_i \right\|^2 \\ &\geq \lim_{j \rightarrow \infty} A \left(\sum_{i \in \sigma_j} |a_i|^2 + \sum_{i \in \sigma_j^c} |a_i|^2 \right) \end{aligned}$$

where the last inequality follows from the Riesz basis property of $\{\varphi_i\}_{i \in \sigma_j} \cup \{\psi_i\}_{i \in \sigma_j^c}$. So $a_i = 0$ for every $i = 1, 2, \dots$, implying that the synthesis operator for the family $\{\varphi_i\}_{i \in \sigma} \cup \{\psi_i\}_{i \in \sigma^c}$ is bounded, linear, onto, and by the above it is also one-to-one. Therefore, it is invertible and so the family $\{\varphi_i\}_{i \in \sigma} \cup \{\psi_i\}_{i \in \sigma^c}$ is a Riesz basis. \square

The next results says that a frame (which is not a Riesz basis) cannot be woven with a Riesz basis.

THEOREM 5.4. *Let $\Phi = \{\varphi_i\}_{i=1}^\infty$ be a Riesz basis and let $\Psi = \{\psi_i\}_{i=1}^\infty$ be a frame for \mathbb{H} . If Φ and Ψ are woven, then Ψ must actually be a Riesz basis.*

Proof. Note that by Corollary 3.2, it may be assumed that $\{\varphi_i\}_{i=1}^\infty$ is an orthonormal basis. By way of contradiction, assume that $\{\psi_i\}_{i=1}^\infty$ is not a Riesz basis. Without loss of generality it may be assumed that $\psi_1 \in \overline{\text{span}}\{\psi_i\}_{i=2}^\infty$. Now, choose $n \in \mathbb{N}$ so that

$$0 \leq d(\psi_1, \text{span}\{\psi_i\}_{i=2}^n) \leq \varepsilon$$

and let

$$K_n = [\text{span}\{\psi_i\}_{i=2}^n]^\perp.$$

Then K_n has codimension at most $n - 1$ in \mathbb{H} and since $\{\varphi_i\}_{i=1}^\infty$ is an orthonormal basis,

$$\dim \text{span}\{\varphi_i\}_{i=1}^n = n.$$

So there exists $x \in \text{span}\{\varphi_i\}_{i=1}^n \cap K_n$ with $\|x\| = 1$. Now, if $\sigma^c = [n]$ then

$$\sum_{i \in \sigma^c} |\langle x, \varphi_i \rangle|^2 = 0,$$

while

$$\sum_{i \in \sigma^c} |\langle x, \psi_i \rangle|^2 = |\langle x, \psi_1 \rangle|^2 \leq \varepsilon.$$

So these two families are not woven. \square

The next lemma and theorem provide some geometric intuition of what it takes for Riesz bases to weave. First a notion of distance between subspaces is introduced.

DEFINITION 5. If W_1 and W_2 are nontrivial subspaces of \mathbb{H} , let

$$d_{W_1}(W_2) = \inf\{\|x - y\| : x \in W_1, y \in S_{W_2}\}$$

and

$$d_{W_2}(W_1) = \inf\{\|x - y\| : x \in S_{W_1}, y \in W_2\}$$

where $S_{W_i} = S_{\mathbb{H}} \cap W_i$ and $S_{\mathbb{H}}$ is the unit sphere in \mathbb{H} . Then the *distance between W_1 and W_2* is defined as

$$d(W_1, W_2) = \min\{d_{W_1}(W_2), d_{W_2}(W_1)\}.$$

LEMMA 5.5. *Suppose $\{\varphi_i\}_{i \in I}$ and $\{\psi_j\}_{j \in J}$ are Riesz sequences in \mathbb{H} . The following are equivalent:*

(i) $\{\varphi_i\}_{i \in I} \cup \{\psi_j\}_{j \in J}$ is a Riesz sequence.

(ii) $D = d(\overline{\text{span}}\{\varphi_i\}_{i \in I}, \overline{\text{span}}\{\psi_j\}_{j \in J}) > 0$.

Proof. To prove (i) \Rightarrow (ii), let A and B be lower and upper Riesz bounds of $\{\varphi_i\}_{i \in I} \cup \{\psi_j\}_{j \in J}$, respectively. Note that each original sequence $\{\varphi_i\}_{i \in I}$ and $\{\psi_j\}_{j \in J}$ also has these bounds. If

$$x = \sum_{i \in I} a_i \varphi_i \quad \text{and} \quad y = \sum_{j \in J} b_j \psi_j$$

with $\|x\| = 1$ or $\|y\| = 1$, then

$$\begin{aligned} \|x - y\|^2 &= \left\| \sum_{i \in I} a_i \varphi_i - \sum_{j \in J} b_j \psi_j \right\|^2 \\ &\geq A \left(\sum_{i \in I} |a_i|^2 + \sum_{j \in J} |b_j|^2 \right) \\ &\geq A \left(\frac{1}{B} \left\| \sum_{i \in I} a_i \varphi_i \right\|^2 + \frac{1}{B} \left\| \sum_{j \in J} b_j \psi_j \right\|^2 \right) \\ &\geq \frac{A}{B} \end{aligned}$$

so that $D^2 \geq A/B$, proving (ii).

Now to prove (ii) \Rightarrow (i), let A and B be universal lower and upper Riesz bounds, respectively, of the two original sequences. Let $\{a_i\}_{i \in I}$ and $\{b_j\}_{j \in J}$ be sequences of scalars so that

$$\sum_{i \in I} |a_i|^2 + \sum_{j \in J} |b_j|^2 = 1.$$

First assume that $\sum_{i \in I} |a_i|^2 \geq 1/2$. Then

$$\begin{aligned} \left\| \sum_{i \in I} a_i \varphi_i + \sum_{j \in J} b_j \psi_j \right\|^2 &= \left\| \sum_{i \in I} a_i \varphi_i \right\|^2 \left\| \frac{\sum_{i \in I} a_i \varphi_i}{\left\| \sum_{i \in I} a_i \varphi_i \right\|} + \frac{\sum_{j \in J} b_j \psi_j}{\left\| \sum_{i \in I} a_i \varphi_i \right\|} \right\|^2 \\ &\geq A \sum_{i \in I} |a_i|^2 D^2 \\ &\geq \frac{AD^2}{2}. \end{aligned}$$

If $\sum_{j \in J} |b_j|^2 \geq 1/2$, then a similar argument works, and thus $\{\varphi_i\}_{i \in I} \cup \{\psi_j\}_{j \in J}$ is a Riesz sequence with lower and upper bounds $AD^2/2$ and B , respectively. \square

One more lemma is needed for the proof of Theorem 5.7.

LEMMA 5.6. *Let W_1 and W_2 be subspaces of \mathbb{H} and let $\{\varphi_i\}_{i \in I}$ and $\{\psi_j\}_{j \in J}$ be a Riesz basis for W_1 and W_2 , respectively. Then for every $\varepsilon > 0$, there is an $x \in S_{W_1}$ and $y \in W_2$ that are finitely supported on $\{\varphi_i\}_{i \in I}$ and $\{\psi_j\}_{j \in J}$, respectively, so that*

$$\|x - y\| \leq d_{W_2}(W_1) + \varepsilon.$$

Proof. The proof is a straightforward approximation argument. \square

The following theorem gives a geometric characterization of woven Riesz bases.

THEOREM 5.7. *If $\Phi = \{\varphi_i\}_{i=1}^\infty$ and $\Psi = \{\psi_i\}_{i=1}^\infty$ are Riesz bases in \mathbb{H} , then the following are equivalent:*

- (i) Φ and Ψ are woven.
- (ii) For every $\emptyset \neq \sigma \subsetneq \mathbb{N}$, $D_\sigma = d(\overline{\text{span}}\{\varphi_i\}_{i \in \sigma}, \overline{\text{span}}\{\psi_i\}_{i \in \sigma^c}) > 0$.
- (iii) There is a constant $C > 0$ so that for every $\emptyset \neq \sigma \subsetneq \mathbb{N}$,

$$D_\sigma = d(\overline{\text{span}}\{\varphi_i\}_{i \in \sigma}, \overline{\text{span}}\{\psi_i\}_{i \in \sigma^c}) \geq C.$$

Proof. The implication (i) \Rightarrow (iii) follows from the proof of Lemma 5.5 since if A and B are universal weaving bounds and $C = A/B$, then $D_\sigma \geq C$ for all $\sigma \subset \mathbb{N}$.

Conversely, for the implication (iii) \Rightarrow (i), the proof of Lemma 5.5 can be applied again, since if A and B are universal Riesz bounds of Φ and Ψ , then $AD_\sigma/2$ and B , are Riesz bounds for $\{\varphi_i\}_{i \in \sigma} \cup \{\psi_i\}_{i \in \sigma^c}$. Because $C \leq D_\sigma$, each weaving has Riesz bounds $AC/2$ and B , which are independent of σ . Furthermore, Theorem 5.2 gives that each weaving is actually a Riesz basis, and thus Φ and Ψ are woven.

Since (iii) \Rightarrow (ii) is obvious, all that remains is (ii) \Rightarrow (iii). Assume (ii) and by way of contradiction assume that for every $n \in \mathbb{N}$, there are subsets $\emptyset \neq \sigma_n \subsetneq \mathbb{N}$ and elements $x_n, y_n \in \mathbb{H}$ of the form

$$x_n = \sum_{i \in \sigma_n} a_{in} \varphi_i \quad \text{and} \quad y_n = \sum_{i \in \sigma_n^c} b_{in} \psi_i$$

with either $\|x_n\| = 1$ or $\|y_n\| = 1$ that satisfy $\|x_n - y_n\| \leq 1/n$. Without loss of generality, it may be assumed that $\|x_n\| = 1$ for all n , by passing to a subsequence and possibly switching the roles of Φ and Ψ . Furthermore, by Lemma 5.6, x_n and y_n may be assumed to be finitely supported, and by switching to another subsequence and reindexing it may be assumed that

$$x_n \rightarrow_w x = \sum_{i=1}^\infty a_i \varphi_i \quad \text{and} \quad y_n \rightarrow_w y = \sum_{i=1}^\infty b_i \psi_i.$$

The rest of the proof will be done in two steps.

Step 1. $x = y = 0$.

Proof of Step 1. For every $n \in \mathbb{N}$, either $1 \in \sigma_n$ or $1 \in \sigma_n^c$, so by switching to a subsequence, it may be assumed that $1 \in \sigma_n$ for all n or that $1 \in \sigma_n^c$ for all n . Furthermore, switching to yet another subsequence, it may be assumed that $2 \in \sigma_n$ for all $n \geq 2$ or $2 \in \sigma_n^c$ for all $n \geq 2$. Continuing, a subsequence can be found (call it σ_n again) satisfying:

- for every $i \leq k$, either $i \in \sigma_n$ for all $n \geq k$ or $i \in \sigma_n^c$ for all $n \geq k$.

Now define $\sigma \subset \mathbb{N}$ by

$$\sigma = \{i \in \mathbb{N} : i \in \sigma_n \text{ for infinitely many } n\}.$$

Since $x_n \rightarrow_w x$ and Φ is a Riesz bases, $a_{in} \rightarrow a_i$ for any fixed i . Therefore, if $i \in \sigma$ is fixed, then $i \notin \sigma_n^c$ and $b_{in} = 0$ for all sufficiently large n . Likewise, if $i \in \sigma^c$, then $a_{in} = 0$ for all large n . Hence, if $i \in \sigma$, then $b_i = 0$, and if $i \in \sigma^c$, then $a_i = 0$. Consequently,

$$x = \sum_{i \in \sigma} a_i \varphi_i \quad \text{and} \quad y = \sum_{i \in \sigma^c} b_i \psi_i. \tag{8}$$

Next, since $x_n - y_n$ converges in norm to zero, it also does so weakly and thus

$$\|x - y\| \leq \liminf_{n \rightarrow \infty} \|x_n - y_n\| = 0 \tag{9}$$

implying that $x = y$. Finally, the assumption

$$d(\overline{\text{span}}\{\varphi_i\}_{i \in \sigma}, \overline{\text{span}}\{\psi_i\}_{i \in \sigma^c}) > 0$$

gives that $x = y = 0$ by (8) and (9). Thus, $x_n \rightarrow_w 0$ and $y_n \rightarrow_w 0$.

Step 2. $d(\overline{\text{span}}\{\varphi_i\}_{i \in \sigma}, \overline{\text{span}}\{\psi_i\}_{i \in \sigma^c}) = 0$, giving a contradiction.

Proof of Step 2. First some notation is introduced. If η and μ are finite subsets of \mathbb{N} , write $\eta \prec \mu$ if

$$\max\{i : i \in \eta\} \leq \min\{i : i \in \mu\}.$$

Since x_n and y_n are finitely supported, there are finite subsets of \mathbb{N} , $\{\eta_k\}_{k=1}^\infty$ subsets of σ , and $\{\mu_k\}_{k=1}^\infty$ subsets of σ^c satisfying for every $k \in \mathbb{N}$:

- (a) $\eta_k \cap \mu_k = \emptyset$.
- (b) $\eta_k, \mu_k \prec \eta_{k+1}, \mu_{k+1}$.
- (c) There is an n_k so that

$$\left\| \sum_{i \in \cup_{j=1}^{k-1} \eta_j} a_{in_k} \varphi_i \right\| < \frac{1}{k} \quad \text{and} \quad \left\| \sum_{i \in \cup_{j=1}^{k-1} \mu_j} b_{in_k} \psi_i \right\| < \frac{1}{k}.$$

With these properties in hand along with the fact that x_{n_k} and y_{n_k} are finitely supported, a standard epsilon thirds argument gives that

$$d(\overline{\text{span}}\{\varphi_i\}_{i \in \eta_k}, \overline{\text{span}}\{\psi_i\}_{i \in \mu_k}) \leq C_k$$

where $C_k \rightarrow 0$ as $k \rightarrow \infty$, but

$$d(\overline{\text{span}}\{\varphi_i\}_{i \in \sigma}, \overline{\text{span}}\{\psi_i\}_{i \in \sigma^c}) \leq d(\overline{\text{span}}\{\varphi_i\}_{i \in \eta_k}, \overline{\text{span}}\{\psi_i\}_{i \in \mu_k})$$

for all k , giving the desired contradiction. \square

REMARK 4. In the literature one finds several notions for the distance or the angle between two subspaces of a Hilbert space. For instance, the angle between two subspaces $W_1, W_2 \subset \mathbb{H}$ is defined as

$$\alpha(W_1, W_2) = \inf \left\{ \arccos \left(\frac{|\langle v, w \rangle|}{\|v\| \|w\|} \right) : v \in W_1, w \in W_2 \right\},$$

see, e.g., [3]. It is easy to see that $\alpha(W_1, W_2) > 0$ if and only if $d(W_1, W_2) > 0$. Consequently, Theorem 5.7 could also be formulated by means of the angle between subspaces: two Riesz bases Φ and Ψ are woven, if and only if for every $\sigma \subset \mathbb{N}$, $\alpha(\overline{\text{span}}\{\varphi_i\}_{i \in \sigma}, \overline{\text{span}}\{\psi_i\}_{i \in \sigma^c}) > 0$.

6. Weavings and perturbations

In this section it is shown frames that are small perturbations of each other are woven. We state the results for two frames, but then give how they can be generalized to any finite number of them.

THEOREM 6.1. Let $\Phi = \{\varphi_i\}_{i \in I}$ (respectively, $\Psi = \{\psi_i\}_{i \in I}$) be frames for a Hilbert space \mathbb{H} with frame bounds A_1, B_1 (respectively, A_2, B_2). Assume there is a $0 < \lambda < 1$ so that

$$\lambda(\sqrt{B_1} + \sqrt{B_2}) \leq \frac{A_1}{2}$$

and for all sequences of scalars $\{a_i\}_{i \in I}$ we have

$$\left\| \sum_{i \in I} a_i(\varphi_i - \psi_i) \right\| \leq \lambda \|\{a_i\}_{i \in I}\|. \quad (10)$$

Then for every $\sigma \subset I$, the family $\{\varphi_i\}_{i \in \sigma^c} \cup \{\psi_i\}_{i \in \sigma}$ is a frame for \mathbb{H} with frame bounds $\frac{A_1}{2}, B_1 + B_2$. That is, Φ and Ψ are woven.

Proof. The proof will be carried out in steps, but first some notation is introduced. Let T (respectively R) be the synthesis operator for the frame $\{\varphi_i\}_{i \in I}$ (respectively, $\{\psi_i\}_{i \in I}$), and let P_σ denote the orthogonal projection onto $\overline{\text{span}}\{e_i\}_{i \in \sigma}$, where $\{e_i\}_{i \in I}$ is the standard orthonormal basis for $\ell^2(I)$ and $\sigma \subset I$. For each $\sigma \subset I$ let

$$T_\sigma(\{a_i\}_{i \in I}) = TP_\sigma(\{a_i\}_{i \in I}) = \sum_{i \in \sigma} a_i \varphi_i,$$

and

$$R_\sigma(\{a_i\}_{i \in I}) = RP_\sigma(\{a_i\}_{i \in I}) = \sum_{i \in \sigma} a_i \psi_i.$$

In this notation, notice that the inequality in (10) becomes

$$\|T - R\| \leq \lambda.$$

Observe that $\|T_\sigma - R_\sigma\| \leq \|T - R\|$, and $\|T_\sigma\| \leq \|T\|$ and $\|R_\sigma\| \leq \|R\|$ since $T_\sigma = TP_\sigma$ and $R_\sigma = RP_\sigma$.

Step 1. For every $x \in \mathbb{H}$,

$$\left\| \sum_{i \in \sigma} \langle x, \psi_i \rangle \psi_i - \sum_{i \in \sigma} \langle x, \phi_i \rangle \phi_i \right\| = \|T_\sigma T_\sigma^* x - R_\sigma R_\sigma^* x\| \leq \frac{A_1}{2} \|x\|.$$

Proof of Step 1. Computing gives for every $x \in \mathbb{H}$

$$\begin{aligned} \|T_\sigma T_\sigma^* x - R_\sigma R_\sigma^* x\| &\leq \|(T_\sigma T_\sigma^* - T_\sigma R_\sigma^*)x\| + \|(T_\sigma R_\sigma^* - R_\sigma R_\sigma^*)x\| \\ &\leq \|T_\sigma\| \|T_\sigma^* - R_\sigma^*\| \|x\| + \|T_\sigma - R_\sigma\| \|R_\sigma^*\| \|x\| \\ &\leq \|T\| \|T - R\| \|x\| + \|T - R\| \|R\| \|x\| \\ &\leq (\|T\| \lambda + \|R\| \lambda) \|x\| \\ &\leq \lambda (\|T\| + \|R\|) \|x\| \\ &\leq \lambda (\sqrt{B_1} + \sqrt{B_2}) \|x\| \\ &\leq \frac{A_1}{2} \|x\|. \end{aligned}$$

Step 2. The lower frame bound is $A_1/2$ for every weaving.

Proof of Step 2. For every $x \in \mathbb{H}$, it follows by applying Step 1 that

$$\begin{aligned} &\left\| \sum_{i \in \sigma} \langle x, \psi_i \rangle \psi_i + \sum_{i \in \sigma^c} \langle x, \phi_i \rangle \phi_i \right\| \\ &= \left\| \sum_{i \in I} \langle x, \phi_i \rangle \phi_i + \left(\sum_{i \in \sigma} \langle x, \psi_i \rangle \psi_i - \sum_{i \in \sigma} \langle x, \phi_i \rangle \phi_i \right) \right\| \\ &\geq \left\| \sum_{i \in I} \langle x, \phi_i \rangle \phi_i \right\| - \left\| \sum_{i \in \sigma} \langle x, \psi_i \rangle \psi_i - \sum_{i \in \sigma} \langle x, \phi_i \rangle \phi_i \right\| \\ &\geq A_1 \|x\| - \left\| \sum_{i \in \sigma} \langle x, \psi_i \rangle \psi_i - \sum_{i \in \sigma} \langle x, \phi_i \rangle \phi_i \right\| \\ &\geq A_1 \|x\| - \frac{A_1}{2} \|x\| \\ &= \frac{A_1}{2} \|x\|. \end{aligned}$$

The upper frame bound of $\{\psi_i\}_{i \in \sigma} \cup \{\phi_i\}_{i \in \sigma^c}$ is at most $B_1 + B_2$ by Proposition 3.1. Thus Φ and Ψ are woven. \square

REMARK 5. Theorem 6.1 can be generalized to a finite number of frames by either requiring each of the frames to be very close to one of them or by creating a “chain” where the first is close to the second, the second to the third, and so on.

In the next proposition the perturbed frames is obtained as the image of a bounded, invertible operator of a given frame.

PROPOSITION 6.2. Let $\{\varphi_i\}_{i \in I}$ be a frame with bounds A, B and let T be a bounded operator. If $\|Id - T\|^2 < \frac{A}{B}$ then $\{\varphi_i\}_{i \in I}$ and $\{T\varphi_i\}_{i \in I}$ are woven.

Proof. Note that T is invertible and thus $\{T\varphi_i\}_{i \in I}$ is automatically a frame. For every $\sigma \subset I$ and for every $x \in \mathbb{H}$ we have by Minkowski's inequality and subadditivity of the square root function

$$\begin{aligned} & \left(\sum_{i \in \sigma} |\langle x, \varphi_i \rangle|^2 + \sum_{i \in \sigma^c} |\langle x, T\varphi_i \rangle|^2 \right)^{1/2} \\ &= \left(\sum_{i \in \sigma} |\langle x, \varphi_i \rangle|^2 + \sum_{i \in \sigma^c} |\langle x, \varphi_i \rangle - \langle (I - T^*)x, \varphi_i \rangle|^2 \right)^{1/2} \\ &\geq \left(\sum_{i \in I} |\langle x, \varphi_i \rangle|^2 \right)^{1/2} - \left(\sum_{i \in \sigma^c} |\langle (I - T^*)x, \varphi_i \rangle|^2 \right)^{1/2} \\ &\geq \sqrt{A}\|x\| - \sqrt{B}\|(I - T^*)x\| \\ &\geq (\sqrt{A} - \sqrt{B}\|I - T^*\|)\|x\|. \end{aligned}$$

Thus, $\{\varphi_i\}_{i \in \sigma} \cup \{T\varphi_i\}_{i \in \sigma^c}$ forms a frame having

$$(\sqrt{A} - \sqrt{B}\|I - T^*\|)^2 > 0$$

as its lower frame bound. \square

REMARK 6. Proposition 6.2 can be extended to a finite number of invertible operators by making the assumption that the sum over all j of $\|Id - T_j\|$ is less than $\sqrt{A/B}$. A nearly identical proof gives $\{\varphi_i\}_{i \in I}$, $\{T_j\varphi_i\}_{i \in I, j \in [n]}$ are woven.

By applying Proposition 6.2 to a power of the frame operator, we obtain many examples where a frame and its canonical dual frame are woven.

COROLLARY 6.3. Let $\Phi = \{\varphi_i\}_{i \in I}$ be a frame with frame constants $A, B > 0$ and frame operator S . If the condition number B/A is sufficiently close to one, e.g., $B/A < 2$, then Φ and the scaled canonical dual frame $\Psi = \{\frac{2AB}{A+B}S^{-1}\varphi_i\}_{i \in I}$ are woven. Likewise Φ and the scaled canonical Parseval frame $\tilde{\Psi} = \{\frac{2\sqrt{AB}}{\sqrt{A+B}}S^{-1/2}\varphi_i\}_{i \in I}$ are woven.

Proof. We apply Proposition 6.2 to the operators $T = \frac{2AB}{A+B}S^{-1}$ for the scaled dual frame and to $\tilde{T} = \frac{2\sqrt{AB}}{\sqrt{A+B}}S^{-1/2}$ for the scaled Parseval frame. Since the spectrum of S is contained in the interval $[A, B]$, the spectrum of $Id - T$ is contained in the interval $[\frac{A-B}{A+B}, \frac{B-A}{A+B}]$ and thus $\|Id - T\| \leq \frac{B-A}{B+A}$. This norm is majorized by $(A/B)^{1/2}$, whenever $B/A \leq 2$. The proof for \tilde{T} is similar and, in fact, yields the condition $B/A \leq (\sqrt{2} + 1)^2$. \square

REMARK 7. Theorem 6.1 and Proposition 6.2 should be compared with the corresponding statements for the perturbation of frames (e.g., in [2]). For instance, if $\Phi = \{\varphi_i\}_{i \in I}$ is a frame with frame bounds $A, B > 0$ and T is a bounded operator satisfying $\|Id - T\| < 1$, then the set $\Psi = \{T\varphi_i\}_{i \in I}$ is also a frame (because T is invertible). Under the stronger condition $\|Id - T\| < A/B$, the frame Φ and Ψ are even woven. A similar remark applies to Theorem 6.1.

7. Gramians

In this section, the Gramian and its relation to weaving frames is considered. The first result says that two Riesz bases are woven as long as the cross Gramian is almost diagonal.

PROPOSITION 7.1. *Let $\{e_k\}_{k=1}^\infty$ be an orthonormal basis for \mathbb{H} and let $\{\varphi_\ell\}_{\ell=1}^\infty$ be a Riesz basis for \mathbb{H} . Let*

$$A = (\langle \varphi_\ell, e_k \rangle)_{k,\ell=1}^\infty = D + R,$$

be the cross Gramian where D is the diagonal of A . If the diagonal entries satisfy $\inf_{1 \leq i \leq \infty} |D_{ii}| \geq \lambda$ and $\|R\| \leq \frac{\lambda}{2}$, then $\{e_k\}_{k=1}^\infty, \{\varphi_\ell\}_{\ell=1}^\infty$ are woven.

Proof. Given $a \in \ell^2$ and $\sigma \subset \mathbb{N}$ we have

$$\begin{aligned} \sum_{\ell \in \sigma^c} a_\ell \varphi_\ell &= \sum_{\ell \in \sigma^c} \sum_{k=1}^\infty a_\ell \langle \varphi_\ell, e_k \rangle e_k \\ &= \sum_{k=1}^\infty \left(\sum_{\ell \in \sigma^c} a_\ell \langle \varphi_\ell, e_k \rangle \right) e_k \\ &= \sum_{k=1}^\infty (A(I - P_\sigma)a)_k e_k, \end{aligned}$$

where P_σ is the diagonal projection onto span $\{e_k\}_{k \in \sigma}$. Now we compute:

$$\begin{aligned} \left\| \sum_{k \in \sigma} a_k e_k + \sum_{\ell \in \sigma^c} a_\ell \varphi_\ell \right\| &= \|P_\sigma a + A(I - P_\sigma)a\| \\ &= \|P_\sigma a + D(I - P_\sigma)a + R(I - P_\sigma)a\| \\ &\geq \|P_\sigma a + D(I - P_\sigma)a\| - \|R(I - P_\sigma)a\| \\ &\geq (\|P_\sigma a\|^2 + \|D(I - P_\sigma)a\|^2)^{1/2} - \|R\| \|(I - P_\sigma)a\| \\ &\geq (\|P_\sigma a\|^2 + \lambda^2 \|(I - P_\sigma)a\|^2)^{1/2} - \frac{\lambda}{2} \|(I - P_\sigma)a\| \\ &\geq \frac{1}{2} (\|P_\sigma a\|^2 + \lambda^2 \|(I - P_\sigma)a\|^2)^{1/2} \\ &\geq \frac{1}{2} \min(1, \lambda) \|a\|. \end{aligned}$$

This proves that $\{e_k\}_{k \in \sigma} \cup \{\phi_\ell\}_{\ell \in \sigma^c}$ is a Riesz sequence with lower Riesz bound $\min(1, \lambda^2)/4$ independent of $\sigma \subseteq \mathbb{N}$. Theorem 5.2 now implies that the weavings are actually Riesz bases with uniform bounds, and so the two sets are woven. \square

REMARK 8. Replacing $\{e_k\}_{k=1}^\infty$ with a Riesz basis is possible with an appropriate change in required bounds. If it is replaced by a Riesz basis $\{\psi_k\}_{k=1}^\infty$, then there is a bounded invertible operator so that $e_k = T^{-1}\psi_k$. Therefore, the two sets $\{e_k\}_{k=1}^\infty$ and $\{T^{-1}\phi_\ell\}_{\ell=1}^\infty$ will be woven if the correct bounds hold according to Proposition 7.1 and thus applying T gives $\{\psi_k\}_{k=1}^\infty$ and $\{\phi_\ell\}_{\ell=1}^\infty$ are woven.

8. Weaving Gabor frames

A potential application is the preprocessing of signals using Gabor frames. First, recall that a Gabor system for $L^2(\mathbb{R})$ is of the form

$$\{M_{bn}T_{am}g : m, n \in \mathbb{Z}\}$$

where $a, b > 0$ are fixed parameters, $g \in L^2(\mathbb{R})$ is a fixed window function, and the time-frequency shifts $M_{bn}T_{am}$ of g are given by

$$M_{bn}T_{am}g(t) = e^{2\pi ibnt}g(t - am).$$

for $a, b \in \mathbb{R}$, $m, n \in \mathbb{Z}$. If such a system forms a frame, then it is called a Gabor frame. See [7] for a thorough approach to time-frequency analysis and Gabor systems. The problem to consider is as follows.

PROBLEM 1. Given a fixed lattice generated by $a, b > 0$ with $ab < 1$ and rotated Gaussians $U_jg_{\alpha_j}$, where $g_{\alpha_i}(x) = e^{-\alpha_i x^2}$, are the Gabor frames

$$\{T_{am}M_{bn}U_jg_{\alpha_j}\}_{m, n \in \mathbb{Z}, j \in [M]}$$

woven?

We believe that Problem 1 has a positive answer and that any such family of Gabor frames is always woven.

REMARK 9. It seems that for two frames to be woven, in some sense they need to be close to one another. However, the problem is if \mathcal{B}_1 and \mathcal{B}_2 are arbitrary Bessel sequences, then $\Phi \cup \mathcal{B}_1$ and $\Psi \cup \mathcal{B}_2$ are still woven. That is, one would need to somehow reduce the frames to a “minimal” woven state for them to truly resemble each other. In this paper we presented two sufficient conditions for weaving: *perturbation theory* and *diagonal dominance*. It is possible that some kind of *localization* of the cross Gramian (without diagonal dominance) would be sufficient. This would also answer Problem 1. This also suggests that perhaps there is a converse to these results. That is, if two frames are *intrinsically* localized and woven, then is their cross Gramian also localized (in the same matrix algebra?).

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