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ON NON COMMUTATIVE TAYLOR INVERTIBILITY

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In memory also of Joe Taylor

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Abstract. The act of proof for the left, and similarly the right, spectral mapping theorem in several variables is carried out on a stage known as a "residual quotient". With some modification, this also works for the Taylor spectrum. Here we set this out, considering also Taylor spectrum for general non commuting systems of Banach algebra elements, for "quasi-commuting" systems, and also the generalization from Banach to "Waelbroeck algebras".

0. Introduction

In a general semigroup A, with an associative binary operation

$$(x,y) \mapsto xy = x \cdot y : A \times A \to A \ ,$$

we can multiply not only single elements, but also subsets, setting

$$KH = K \cdot H = \{x \cdot y : (x,y) \in K \times H\} \ . \tag{0.1}$$

This generates [14], [17] a kind of "fraction", known as a *residual quotient*; there are both "left" and "right" quotients:

$$K^{-1}H = \{x \in A : Kx \subseteq H\} \tag{0.2}$$

and

$$HK^{-1} = \{x \in H : xK \subseteq H\} \ . \tag{0.3}$$

Evidently general statements about left imply corresponding statements about right residuals, and vice versa; we shall choose to focus here on left quotients. Trivial observations are that (0.2), and dually (0.3), are monotonically increasing in H, and monotonically decreasing in K:

$$(H' \subseteq H \ \& \ K \subseteq K') \Longrightarrow K'^{-1}H' \subseteq K^{-1}H \ . \tag{0.4}$$

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Paper OaM-10-62

When there is an identity $1 \in A$, there ([14] Theorem 3) is implication

$$H \subseteq K \Longrightarrow (K^{-1}H)(K^{-1}H) \subseteq K^{-1}H\ ; \tag{0.5}$$

$$K \subseteq H \Longrightarrow 1 \in K^{-1}H\ ; \tag{0.6}$$

$$1 \in K \Longrightarrow K^{-1}H \subseteq H\ . \tag{0.7}$$

The extension of spectral theory from single elements to finite or infinite systems is mostly confined to *commuting* systems, although usually the definitions survive without this restriction. In a linear algebra, or more generally a "linear category" A, a *spectrum* $\omega(a) \subseteq \mathbf{C}^X$ is derived from some collection $H \subseteq A$ of "invertible" or more generally non-singular, systems of elements $a \in A^X$:

$$\omega(a) = \{\lambda \in \mathbf{C}^X : a - \lambda \notin H\}\ . \tag{0.8}$$

For such a "joint spectrum" we look for the *spectral mapping theorem*

$$p\omega(a) = \omega p(a) \subseteq \mathbf{C}^X, \tag{0.9}$$

for element $a \in A^X$ and systems $p \in \mathrm{Poly}_X^Y$ of "non commutative polynomials". Equality (0.9) divides into a *forward* spectral mapping theorem,

$$p\omega(a) \subseteq \omega p(a)\ , \tag{0.10}$$

and a *backward* spectral mapping theorem,

$$\omega p(a) \subseteq p\omega(a)\ . \tag{0.11}$$

Typically the forward theorem (0.10) is easier, and survives for other than commutative systems of elements, combining the remainder theorem for non commutative polynomials with some kind of reverse semi-group property of the non singulars H; the harder backward theorem (0.11) needs the "fundamental theorem of algebra", or more generally Liouville's theorem from complex analysis. In this note however we start by observing that, for general non commuting systems, the forward spectral mapping theorem (0.10) is liable to fail for the Taylor spectrum.

1. Taylor invertibility

Suppose $a \in A$ and $b \in A$, for a complex linear algebra A with identity 1; then we shall say that the pair $(a,b) \in A^2$ is *Taylor invertible* if it is at once *left*, *right* and *middle invertible*: here

$$(a,b) \in A^{-2}_{\mathrm{left}} \Longleftrightarrow 1 \in \begin{pmatrix} A & A \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}\ ; \tag{1.1}$$

$$(a,b) \in A^{-2}_{\mathrm{right}} \Longleftrightarrow 1 \in \begin{pmatrix} b & -a \end{pmatrix} \begin{pmatrix} A \\ A \end{pmatrix}\ ; \tag{1.2}$$

$$(a,b) \in A^{-2}_{\text{middle}} \Longleftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} A & A \end{pmatrix} + \begin{pmatrix} A \\ A \end{pmatrix} \begin{pmatrix} b & -a \end{pmatrix} . \tag{1.3}$$

Necessary for left invertibility is the implication, for arbitrary $x \in A$,

$$ax = bx = 0 \Longrightarrow x = 0 , \tag{1.4}$$

and for right invertibility the implication, for arbitrary $y \in A$,

$$ya = yb = 0 \Longrightarrow y = 0 . \tag{1.5}$$

If the elements a,b commute, in the sense $ab = ba$, and A is a Banach algebra, then (1.1)-(1.3) add up to the condition that $(0,0)$ is not in the "Taylor split spectrum" of the pair (a,b): the point here is that we are withholding commutivity. Without commutivity there is a "one way spectral mapping theorem" for left and for right invertibility, and it would be nice to be able to say the same for "Taylor invertibility". The sequence of matrices

$$\left(0, \begin{pmatrix} b & -a \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix}, 0\right) \tag{1.6}$$

may [21], [16], [7], [12] be referred to as the *Koszul complex* of the pair $(a,b) \in A^2$; of course it will not truly be a "complex" unless

$$\begin{pmatrix} b & -a \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \equiv ba - ab = 0 , \tag{1.7}$$

which says that a and b commute.

2. Special Lie algebra

If $A = \mathbf{C}^{2\times 2}$ and ([7] (11.2..4), [12] (5.1.9))

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \; f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} , \tag{2.1}$$

then for arbitrary $(\lambda,\mu) \in \mathbf{C}^2$ the conditions (1.1) and (1.2) are satisfied by $(a,b) = (e-\lambda, f-\mu)$: in words both the left and the right spectrum of the pair (e,f) are empty. It is also true that the condition (1.3) is satisfied unless $(\lambda,\mu) = (0,0)$: however

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \notin \begin{pmatrix} e \\ f \end{pmatrix} \begin{pmatrix} A & A \end{pmatrix} + \begin{pmatrix} A \\ A \end{pmatrix} \begin{pmatrix} f & -e \end{pmatrix} = \begin{pmatrix} eA + Af & eA + Ae \\ fA + Af & fA + Ae \end{pmatrix} , \tag{2.2}$$

and hence the right hand side of (1.3) implies inclusion

$$1 \in (eA + Af) \cap (Ae + fA) , \tag{2.3}$$

which says that each of the pairs (f,e) and (e,f) are "splitting exact". However each of the pairs (f,e) and (e,f) are [13] "skew exact", so that if they were also exact then e and f would have to be left or right invertible. Alternatively notice

$$\begin{pmatrix} f & -e \end{pmatrix} \begin{pmatrix} f \\ e \end{pmatrix} = 0 ; \; \begin{pmatrix} f \\ e \end{pmatrix} \notin \begin{pmatrix} e \\ f \end{pmatrix} A . \tag{2.4}$$

Indeed

$$\begin{pmatrix} f & -e \end{pmatrix} \begin{pmatrix} f \\ e \end{pmatrix} = f^2 - e^2 = 0 - 0 , \tag{2.5}$$

while

$$\begin{pmatrix} f \\ e \end{pmatrix} = \begin{pmatrix} e \\ f \end{pmatrix} g \Longrightarrow g = (fe + ef)g = f^2 + e^2 = 0 \,; \Longrightarrow f = e = 0 \,. \tag{2.6}$$

Thus the middle spectrum, and hence the Taylor spectrum, of this pair does contain a point, and is given by the singleton $\{(0,0)\}$. From one point of view this might seem to be a good thing: the Taylor spectrum of this unruly pair of matrices is nonempty. There are however consequences: without commutivity, the "one way" spectral mapping theorem (0.10) fails for the Taylor spectrum.

3. Spectral mapping theorems

Suppose $p \equiv p(z_1, z_2) \in \mathrm{Poly}_2$ is a "polynomial" in two free variables, with in particular $p(0,0) = 0$: then in general, with no assumption of commutivity, there is implication

$$1 \in Ap(a,b) \Longrightarrow 1 \in \begin{pmatrix} A & A \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \tag{3.1}$$

and

$$1 \in p(a,b)A \Longrightarrow 1 \in \begin{pmatrix} b & -a \end{pmatrix} \begin{pmatrix} A \\ A \end{pmatrix} \;; \tag{3.2}$$

thus if in particular $p(a,b) \in A^{-1}$ is invertible then $(a,b) \in A^2$ is both left and right invertible. In general however this may not be enough to ensure middle invertibility. Indeed, with (e,f) as in (2.1),

$$p = z_2 z_1 + z_1 z_2 \;,\; (a,b) = (e,f) \;\Longrightarrow\; p(a,b) = 1 \notin (aA + Ab) \cup (Aa + bA) \,, \tag{3.3}$$

which implies that the right hand side of (1.3) cannot hold. With

$$\sigma^{\mathrm{left}}(a,b) = \{(\lambda,\mu) \in \mathbf{C}^2 : (a-\lambda, b-\mu) \notin A^{-2}_{\mathrm{left}}\} \,, \tag{3.4}$$

$$\sigma^{\mathrm{right}}(a,b) = \{(\lambda,\mu) \in \mathbf{C}^2 : (a-\lambda, b-\mu) \notin A^{-2}_{\mathrm{right}}\} \tag{3.5}$$

and

$$\sigma^{\mathrm{middle}}(a,b) = \{(\lambda,\mu) \in \mathbf{C}^2 : (a-\lambda, b-\mu) \notin A^{-2}_{\mathrm{middle}}\} \,, \tag{3.6}$$

(3.1) and (3.2) give inclusions

$$p\sigma^{\mathrm{left}}(a,b) \subseteq \sigma^{\mathrm{left}} p(a,b) \tag{3.7}$$

and

$$p\sigma^{\mathrm{right}}(a,b) \subseteq \sigma^{\mathrm{right}} p(a,b) \,. \tag{3.8}$$

However, with $(a,b) = (e,f)$ and $p = z_2 z_1 + z_1 z_2$ we have

$$p\sigma^{\mathrm{middle}}(a,b) = \{p(0,0)\} = \{0\} \not\subseteq \{1\} = \sigma p(a,b) \,. \tag{3.9}$$

4. Shifts

In search of further examples of such misbehaviour we recall the backward and forward *shifts*. If for example $ba \in A^{-1}$, $(a,b) \in A^2$ is both left and right invertible: with

$$cba = 1 = bac \ , \tag{4.1}$$

it is clear that

$$\begin{pmatrix} cb & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 1 = \begin{pmatrix} b & -a \end{pmatrix} \begin{pmatrix} ac \\ 0 \end{pmatrix} \ . \tag{4.2}$$

If also $ba = ab$ invertible then $(a,b) \in A^2$ will also be middle invertible; generally however there is equality

$$\begin{pmatrix} b'' \\ a'' \end{pmatrix} \begin{pmatrix} b & -a \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} a' & b' \end{pmatrix} \equiv \begin{pmatrix} aa' + b''b & ab' - b''a \\ ba' + a''b & bb' - a''a \end{pmatrix} \ , \tag{4.3}$$

and then equivalence

$$\begin{pmatrix} aa' + b''b & ab' - b''a \\ ba' + a''b & bb' - a''a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Longleftrightarrow \begin{matrix} aa' + b''b = 1 \ ; \ ab' - b''a = 0 \ ; \\ ba' + a''b = 0 \ ; \ bb' - a''a = 1 \ . \end{matrix} \tag{4.4}$$

If for example

$$ba = 1 \neq ab \tag{4.5}$$

then the top left hand condition on the right hand side of (4.4) can easily fail; the condition (4.1) holds with $c = 1$, while there are $(x,y) \in A^2$ violating an obvious necessary condition for (1.3):

$$x = y = 1 - ab \Longrightarrow bx = ya = 0 \neq yx \ , \tag{4.6}$$

and hence $1 \notin aA + Ab$.

(4.5) holds when

$$(a,b) = (u,v) \in A^2 = B(X)^2$$

are the forward and backward *shifts* ([7] (2.8.2.2), (2.8.2.3); (7.2.6.12), (7.6.3.13)) on $X = \ell_2$ or more generally; we recall

$$\sigma^{\text{left}}(u) = \sigma^{\text{right}}(v) = \mathbf{S} = \partial \mathbf{D} \subseteq \mathbf{D} = \sigma^{\text{right}}(u) = \sigma^{\text{left}}(v) \ , \tag{4.7}$$

where $\mathbf{D}$ is the closed unit disc and $\mathbf{S}$ its boundary the circle:

$$|\lambda| < 1 \Longrightarrow (v - \lambda, u - \lambda) = (v(1 - \lambda u), (1 - \lambda v)u) \in (vA^{-1}) \times (A^{-1}u) \ . \tag{4.8}$$

For the shifts the algebra $A = B(X) = B(\ell_2)$ has a hermitian *involution* $a \mapsto a^*$:

$$a = a^* \Longrightarrow \omega(a) \in \mathbf{R} \ ; \tag{4.9}$$

we have (*cf* [7] (5.1.12); [12])

$$\omega(a, a^*) = \{(\alpha + i\beta, \alpha - i\beta) : (\alpha, \beta) \in \omega((a + a^*)/2, (a - a^*)/2i)\} \ ,$$

giving

$$a \in A^n \Longrightarrow \omega(a,a^*) \subseteq \{(\lambda,\lambda^*) : \lambda \in \omega(a)\} \ . \tag{4.10}$$

Now with

$$\mathbf{D}^{(*)} = \{(\lambda,\lambda^*) : \lambda \in \mathbf{D}\} \ , \ \mathbf{S}^{(*)} = \mathbf{D}^{(*)} \cap (\mathbf{S} \times \mathbf{S}) \ , \tag{4.11}$$

we also have

$$\sigma^{\text{left}}(u,v) \cup \sigma^{\text{right}}(u,v) \subseteq \mathbf{S}^{(*)} \ . \tag{4.12}$$

We go on to claim that

$$(0,0) \in \sigma^{\text{middle}}(u,v) \subseteq \sigma^{\text{Taylor}}(u,v) \ : \tag{4.13}$$

observe

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1+u^2 \end{pmatrix} (1-uv) \Longrightarrow \begin{pmatrix} v & -u \end{pmatrix} \begin{pmatrix} y \\ x \end{pmatrix} = 0 \ ,$$

while

$$y \notin uA \ ; \Longrightarrow \begin{pmatrix} y \\ x \end{pmatrix} \notin \begin{pmatrix} u \\ v \end{pmatrix} A \ .$$

Now

$$p = z_2 z_1 + z_1 z_2 \Longrightarrow p(u,v) = 1 + vu \Longrightarrow \sigma p(u,v) = \{1,2\}$$

and hence

$$0 \in p\sigma^{\text{Taylor}}(u,v) \setminus \sigma p(u,v) \ . \tag{4.14}$$

5. Exactness

More general than either left or right invertibility is *self exactness.* We shall say that the pair $(b,a) \in A^2$ is *splitting exact*, and write

$$(b,a) \in A^{-(1,1)}_{\text{left,right}} \ , \tag{5.1}$$

provided

$$1 \in Ab + aA \ . \tag{5.2}$$

More generally (*cf* [10]) we might write

$$A^{-(m,n)}_{\text{left,right}} = \{(b,a) \in A^m \times A^n : 1 \in A^m \cdot b + a \cdot A^n \equiv \sum_{k=1}^{m} Ab_k + \sum_{j=1}^{n} a_j A\} \ . \tag{5.3}$$

Now $a \in A$ is to be *self exact* provided (a,a) is exact:

$$A^{-1}_{\text{left,right}} = \{a \in A : (a,a) \in A^{-(1,1)}_{\text{left,right}}\} \ , \tag{5.4}$$

and more generally

$$A^{-n}_{\text{left,right}} = \{a \in A^n : 1 \in A^n \cdot a + a \cdot A^n\} \ . \tag{5.5}$$

Exactness (5.1) makes sense in a ring; in a more general additive category it is necessary that

$$\exists\, ba \in A\ , \tag{5.6}$$

the product is defined. We do not however include the requirement that the *chain condition*

$$ba = 0 \in A \tag{5.7}$$

is satisfied; for some readers therefore (5.2) might be referred to as "non commutative exactness". Self exactness in a linear algebra would seem to generate another kind of spectrum, writing, for $a \in A^n$,

$$\sigma^{\text{left,right}}(a) = \{\lambda \in \mathbf{C}^n : 1 \notin A^n \cdot (a-\lambda) + (a-\lambda)\cdot A^n\}\ . \tag{5.8}$$

We can now enquire whether (0.10) or (0.11) hold with $\omega = \sigma^{\text{left,right}}$. For the forward version (0.10) observe that if $p \in \text{Poly}_n^m$ with $p(0) = 0$, there is inclusion

$$A^m \cdot p(a) + p(a)\cdot A^m \subseteq A^n \cdot a + a \cdot A^n\ . \tag{5.9}$$

Notice however that, with e and f as in (2.1),

$$\sigma^{\text{left,right}}(e) = \sigma^{\text{left,right}}(f) = \emptyset\ ; \tag{5.10}$$

in the notation of (0.2), $Ae + eA = (Ae)^{-1}Ae$ is the set of upper triangles, and $Af + fA = (Af)^{-1}Af$ the lower triangles.

Of course, in (5.2), $N = Ab + aA$ is neither a left nor a right ideal, and we have to replace the residual quotient $N^{-1}N$ by something more complicated. We find what we are looking for in some approximation theory [15], [18], [19]. Generally, if $N \subseteq A$ is a subring, closed under multiplication, we define

$$N : N = \{c \in A : Nc + cN \subseteq N\}\ ; \tag{5.11}$$

now $N \subseteq N : N$ is again a two-sided ideal and we can form the quotient $(N : N)/N$. Provided $1 \notin N$ then the ideal $N \subseteq N : N$ will be proper; if further A is a Banach algebra and $N = \text{cl}(N)$ is closed then $B = (N : N)/N$ is a non trivial Banach algebra in its own right. Now if $c \in \text{comm}(N) \subseteq N : N$ then

$$\lambda \in \partial\sigma_B[c]_N \Longrightarrow 1 \notin N + A(c-\lambda) + (c-\lambda)A\ ; \tag{5.12}$$

if $\lambda_n \to \lambda$ with $[c-\lambda_n]_N \in B^{-1}$ and the right hand side of (5.12) fails then $\lambda \notin \sigma_B[c]_N$: if $1 \in c'(c-\lambda) + (c-\lambda)c'' + N$ then there is inequality

$$\|[c-\lambda_n]_N^{-1}\| \leqslant (\|c'\| + \|c''\|)\|[c-\lambda]_N[c-\lambda_n]_N^{-1}\| \leqslant (\|c'\| + \|c''\|)(1 + |\lambda_n - \lambda|)\ . \tag{5.13}$$

6. Koszul matrices

The problem for the "left,right invertibility" of (5.8) is that it is not clear, for $a \in A^n$, that $N = A^n \cdot a + a \cdot A^n \subseteq A$ is a subring, closed under multiplication. In a Banach algebra A it is also not clear that it is norm closed; we would like, for $N = A^n \cdot a + a \cdot A^n$, implication

$$1 \in \mathrm{cl}(N) \Longrightarrow 1 \in N \; . \tag{6.1}$$

For "Taylor invertibility" the self exactness is applied not directly to the primary element $a \in A$ or system $a \in A^n$, but rather to its *Koszul matrix*. It is possible [16], [20] to pile up the Koszul complex of an n tuple $a \in A^n$ of linear algebra elements into a single matrix Λ_a in a larger algebra D, which is now potentially self exact; the definition is inductive. For a single element $a \in A$, whose Koszul complex is just the triple $(0, a, 0)$ we set

$$\Lambda_a = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} \; ; \tag{6.2}$$

we could alternatively make an "upper triangular" version. For example if $a = 1 \in \mathbf{C}$ then $\Lambda_a = f$ as in (2.1); more generally if $a \in A$ then

$$\sigma^{\text{left,right}}(\Lambda_a) \subseteq \{0\} \; ; \; a \in A^{-1} \Longleftrightarrow \sigma^{\text{left,right}}(\Lambda_a) = \emptyset \; . \tag{6.3}$$

Inductively define, with $b \in A^k$ and $c \in A$,

$$\Lambda_{(b,c)} = \begin{pmatrix} \Lambda_b & O \\ \triangle_c & -\Lambda_b \end{pmatrix} \in D^{2\times 2} \; , \tag{6.4}$$

where

$$\Lambda_b \in D \; , \; \triangle_c \in D \tag{6.5}$$

are respectively what has already been defined, and the block diagonal generated by the single element $c \in A$. Generally if $a \in A^n$ and $p \in \mathrm{Poly}_n^m$ we define

$$\Lambda_p(\Lambda_a) = \Lambda_{p(a)} \; . \tag{6.6}$$

Inductively we claim, for $a \in A^n$, that

$$a \textit{ commutative} \Longleftrightarrow \Lambda_a^2 = O \; : \tag{6.7}$$

note that if $c \in \mathrm{comm}(b) \subseteq A$ then $\triangle_c \in \mathrm{comm}(\Lambda_b) \subseteq D$ and

$$\Lambda_b^2 = O \in D \Longrightarrow \Lambda_{(b,c)}^2 = O \in D^{2\times 2} \; .$$

We now claim that

$$(a, a') \in A^{2n} \textit{ commutative} \; , \; a' \cdot a = 1 \in A \Longrightarrow I \in D\Lambda_a + \Lambda_a D \subseteq D \; , \tag{6.8}$$

so that Λ_a is splitting self exact. We again argue by induction: if (6.8) holds with $a = b \in A^k$ then it continues to hold with $a = (b, c) \in A^k \times A$. In turn if $a \in A^n$ is commutative and $p \in \mathrm{Poly}_n$ with $p(0) = 0$ then there is $q \in \mathrm{Poly}_n^n$ with $p(a) = q(a) \cdot a$ and hence

$$p(a) \in A^{-1} \Longrightarrow I \in D\Lambda_a + \Lambda_a D\ , \tag{6.9}$$

and the extension to $p \in \mathrm{Poly}_n^m$ is induction on m. Next ([9] Theorem 1; [14] (3.12))

$$I \in D\Lambda_b + \Lambda_b D \subseteq D \Longleftrightarrow \begin{pmatrix} I & O \\ O & I \end{pmatrix} \in \begin{pmatrix} \Lambda_b & O \\ \triangle_c & -\Lambda_b \end{pmatrix} D^{2\times 2} + D^{2\times 2} \begin{pmatrix} \Lambda_b & O \\ \triangle_c & -\Lambda_b \end{pmatrix}\ , \tag{6.10}$$

and we look for $\lambda \in \mathbf{C}$ for which

$$I \notin D\Lambda_b + \Lambda_b D \subseteq D \Longrightarrow \begin{pmatrix} I & O \\ O & I \end{pmatrix} \notin \begin{pmatrix} \Lambda_b & O \\ \triangle_{c-\lambda} & -\Lambda_b \end{pmatrix} D^{2\times 2} + D^{2\times 2} \begin{pmatrix} \Lambda_b & O \\ \triangle_{c-\lambda} & -\Lambda_b \end{pmatrix}\ . \tag{6.11}$$

Implication (6.1), and the multiplicative property $N \cdot N \subseteq N$ are clear when $N = Aa + aA \subseteq A$ is replaced by $N = D\Lambda_a + \Lambda_a D \subseteq D$ and extend from $\Lambda_b \in D$ to $\Lambda_{(b,c)} \in D^{2\times 2}$ whenever $c \in \mathrm{comm}(b) \subseteq A$.

7. Quasicommutivity

One of the reasons for at least trying to state problems for non commutative systems is the feeling that the commutative theory ought to extend to *quasicommutative* systems [6], [7], [12]: associated with $(a,b,c) \in A^3$ for which

$$[b,a] \equiv ba - ab = c\ ;\ [a,c] = 0 = [b,c]\ , \tag{7.1}$$

Enrico Boasso has noticed that we have a true complex

$$0, \begin{pmatrix} b & -a & c \end{pmatrix}, \begin{pmatrix} a & -c & 0 \\ b & 0 & -c \\ -1 & b & -a \end{pmatrix}, \begin{pmatrix} c \\ a \\ b \end{pmatrix}, 0\ , \tag{7.2}$$

which can also be written

$$0, \begin{pmatrix} T^{\sim} & c \end{pmatrix}, \begin{pmatrix} T & -\triangle_c \\ -1 & T^{\sim} \end{pmatrix}, \begin{pmatrix} c \\ T \end{pmatrix}, 0 \tag{7.3}$$

where

$$T = \begin{pmatrix} a \\ b \end{pmatrix}, T^{\sim} = \begin{pmatrix} b & -a \end{pmatrix}, c = T^{\sim}T, \triangle_c = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} \tag{7.4}$$

and the quasicommutivity says

$$Tc = \triangle_c T,\ T^{\sim}\triangle_c = cT^{\sim}\ . \tag{7.5}$$

Provided $c \neq 0$ then the Lie algebra generated by $(a,b) \in A^2$ is a *Heisenberg algebra*. If we were able to argue that, since we have a true complex here, the spectral mapping theorem holds for the Taylor (split) spectrum (in particular for one polynomial in three variables), then two things would follow: the Taylor split spectral mapping theorem for the quasicommuting pair (a,b), and hence of course also again quasinilpotency for the commutator $ab-ba$. A new challenge would be to relax the quasicommutativity of (a,b) to commutivity $ac=ca$ and still have a "spectral" proof that $\sigma(ab-ba)=\{0\}$. If we extend the definition of "quasicommutative" from n tuples $a\in A^n$ to arbitrary systems $a\in A^X$, in particular [7], [12] to A itself, then A is "quasicommutative" iff

$$[A,[A,A]]=\{0\}\ ; \tag{7.6}$$

explicitly, for arbitrary $(a,b,c)\in A^3$,

$$(ab-ba)c=c(ab-ba)\ . \tag{7.7}$$

Evidently [6], [7], [12] Gelfand's theorem holds for quasicommutative Banach algebras. More generally, according to Feinstein, the spectral mapping theorem holds for Banach algebras which are *nilpotent* Lie [4], [3]. The idea of Boasso seems to be generally to consider

$$\sigma(a,b,ab-ba)\ . \tag{7.8}$$

If instead $(a,b)=(e,f)\in A^2$ as in (2.1), with $A=\mathbf{C}^{2\times 2}$, then the Lie algebra generated by $\{a,b\}$ is the *special Lie algebra* $\mathbf{sl}_2$.

For example if $a=(b,c)\in A^2$ is arbitrary then

$$\Lambda_a^2=\Lambda_{(b,c)}^2=\begin{pmatrix}\Lambda_b^2 & O\\ \Lambda_{cb-bc} & \Lambda_b^2\end{pmatrix}=\begin{pmatrix}O & O\\ \Lambda_{cb-bc} & O\end{pmatrix}\ ; \tag{7.9}$$

still with no restriction on (b,c) we get

$$\Lambda_a^3=\begin{pmatrix}O & O\\ O & O\end{pmatrix}\ , \tag{7.10}$$

and we can look [11] for a *Jordan decomposition*

$$\Lambda_a\sim T_1\oplus T_0 \textit{ with } T_1'T_1+T_1^2T_1''=I \textit{ and } T_0^2=O\ . \tag{7.11}$$

8. Waelbroeck algebras

Instead of relaxing the commutivity of $a\in A^n$, Wawrzyńczyk [22], [23] has extended the projection property of the left spectrum to locally convex *Waelbroeck algebras* A, for which the invertible group $A^{-1}\subseteq A$ is topologically open and the inversion map z^{-1} continuous; it is tempting to try and do the same thing for the Taylor split spectrum. Now [2] the argument of (5.12) is no longer available. Thus if $1\notin N\cdot N\subseteq N=\mathrm{cl}(N)\subseteq A$ and $c\in\mathrm{comm}(N)\subseteq N:N$ and if, for a contradiction, we have

$$1\in\bigcap_{\lambda\in\mathbf{C}}N+A(c-\lambda)+(c-\lambda)A \tag{8.1}$$

put $M=N:N$ and $B=\mathrm{comm}^2[c]_N\subseteq M/N$ and argue there will be (c_λ') and (c_λ'') in A for which, for arbitrary $\lambda\in\mathbf{C}$,

$$c_\lambda'(c-\lambda)+(c-\lambda)c_\lambda''\in 1+N\ , \tag{8.2}$$

and then, mimicking the argument of [22], observe, with $\gamma(z) \equiv 1-(z-\lambda)(c'_\lambda + c''_\lambda)$,

$$c'_\lambda(c-z)+(c-z)c''_\lambda \equiv c'_\lambda(c-\lambda)+(c-\lambda)c''_\lambda-(z-\lambda)(c'_\lambda+c''_\lambda) \in \gamma(z)+N\ , \quad (8.3)$$

so that on sufficiently small $U_\lambda \in \mathrm{Nbd}(\lambda)$,

$$\alpha = \gamma^{-1}c'_\lambda, \beta = c''_\lambda\gamma^{-1} \Longrightarrow \alpha(z)(c-z)+(c-z)\beta(z) \in 1+N\ . \quad (8.4)$$

Now, on the whole of $\mathbf{C}$, look for

$$\alpha(z)(c-z)+(c-z)\beta(z) \in 1+N\ . \quad (8.5)$$

If we take $D(c)$ to be the set of $\lambda \in \mathbf{C}$ for which (8.5) works holomorphically near $z=\lambda$ we look for $\partial D(c) = \emptyset$ and hence $D(c) = \mathbf{C}$.

Remember of course the example (5.10); the spectral mapping theorem is liable to fail for the "left,right spectrum", and we should therefore be applying this argument to Koszul matrices. Notice also that the link between the projection property and the backward inclusion (0.11) goes through the forward inclusion (0.10), which in general fails. Taking $c''_\lambda = 0$ in (8.2) reproduces the argument of Theorem 7 of [14], and indeed corrects a small but significant misprint there.

An affirmative solution to (8.5), in either Banach or Waelbroeck algebras A, would be the tip of the iceberg of an extension of Allan's theorem [1]: if $a = a(z) : G \to A$ and $b = b(z) : G \to A$ are holomorphic on an open connected set $G \subseteq \mathbf{C}$, and if

$$\lambda \in G \Longrightarrow 1 \in Ab(\lambda)+a(\lambda)A \subseteq A\ , \quad (8.6)$$

does the holomorphic pair of functions (a,b) have a holomorphic left,right inverse, $(b^\wedge, a^\wedge) : G \to A^2$ for which

$$b^\wedge(z)b(z)+a(z)a^\wedge(z) \equiv 1 : G \to A\ ? \quad (8.7)$$

Taking $b = a$ and then substituting $\Lambda_a \in D$ for $a \in A^n$, one might hope [7] for some kind of Cauchy integral formula for a "functional calculus" $f \mapsto f(a)$.

9. Weak and strong exactness

At the opposite extreme from the "splitting exactness" (5.1) for $(b,a) \in A^2$ is *weak exactness*, the implication (*cf* (4.6)), for arbitrary $(u,v) \in A^2$,

$$bu = 0 = va \Longrightarrow vu = 0\ . \quad (9.1)$$

For a general additive category A, implication (9.1) is subject to the compatibility conditions on (u,v) that the products bu and va, and hence also vu, exist. More generally it is possible to extend (9.1) to tuples $(b,a) \in A^m \times A^n$, and of course also specialise to the case $b = a$, giving "weak self-exactness" of $a \in A$. If in particular A is a *normed ring*, or the analagous specialization of an additive category, then *strong exactness* for $(b,a) \in A^2$ says that there are $k > 0$ and $h > 0$ participating in inequality, for arbitrary $(u,v) \in A^2$,

$$\|vu\| \leqslant k\|v\|\ \|bu\| + h\|va\|\ \|u\|\ . \quad (9.2)$$

Generally

$$\textit{splitting} \implies \textit{strong} \implies \textit{weak exactness}\,;$$

if however $a \in aAa$ and $b \in bAb$ are *relatively regular* then weak exactness for (b,a) implies splitting exactness for (b,a).

If in particular A is the category of all bounded linear operators between Banach spaces then ([7] Theorem 10.4.1) for $a : X \to Y$ and $b : Y \to Z$ it is necessary and sufficient (at least when $ba = 0$), for (b,a) to be strongly exact, that

$$b^{-1}(0) \subseteq a(X) = \mathrm{cl}\, a(X) \ \text{ and } \ b(Y) = \mathrm{cl}\, b(Y)\,. \tag{9.3}$$

Necessary and sufficient for the weak exactness of (b,a) is ([7] Theorem 10.4.2) the inclusion

$$b^{-1}(0) \subseteq \mathrm{cl}\, a(X)\,. \tag{9.4}$$

These exactness conditions convert to Taylor conditions when applied to Koszul matrices: we declare $a \in A^n$ to be *weakly Taylor non singular* provided Λ_a is weakly self exact in the sense derived from (9.1), and *strongly Taylor non singular* provided Λ_a is strongly self exact in the sense (9.2).

For example if $a \in A$ is a single element then necessary and sufficient for Λ_a to be weakly self exact is, analagous to (6.3), that $a \in A$ is both *monomorphic* and *epimorphic*, neither a left nor a right zero divisor. Similarly necessary and sufficient for Λ_a to be strongly self exact is that $a \in A$ is neither a topological left nor a topological right zero divisor. In a Banach algebra A these "strong monomorphisms" and "strong epimorphisms" form open sets; and the set of $a \in A$ generating strongly self-exact matrices $\Lambda_a \in D$ also form an open subset of A.

Strong exactness, with $\Lambda_a \in D$ in place of $a \in A^n$, potentially gives the true extension, to $A \subseteq B(E)$, from Taylor non singularity for systems of bounded operators on Banach spaces E, to systems of Banach algebra elements $a \in A^n$. Notice that as $A \subseteq B(E)$ decreases, "Taylor invertibility" for $a \in A^n$ gets more difficult, so that the Taylor split spectrum $\sigma_A^{\mathrm{Taylor}}(a)$ gets bigger, and a functional calculus works for potentially less holomorphic functions. At the same time "Taylor non singularity" for $a \in A^n$ gets easier, so that the Taylor spectrum $\tau_A^{\mathrm{Taylor}}(a)$ gets smaller, and if there was a functional calculus it would work for potentially more holomorphic functions.

We have however not been able to settle either the weak or the strong analogue of (6.10), and the failure of the index-of-product theorem for "weakly Fredholm" operators [5] suggests that they fail. Also Vladimir Müller keeps constructing Banach algebras A and commuting tuples $a \in A^n$ for this Taylor non singularity appears to misbehave.

10. Determinant and adjugate

Comparing the Koszul complex of a commuting system $a \in A^k$ with that of its polynomial image offers [8] a curious way to reach determinants and adjugates of operator matrices. Careful discussion of the case $k = 2$ is what is involved in the inductive step from k to $k+1$.

For example if $(a,b) \in A^2$ is a pair of single elements then their Koszul complex (1.6) is given by $(0, T^\sim, T, 0)$ where

$$T = \begin{pmatrix} a \\ b \end{pmatrix} \ , \ T^\sim = \begin{pmatrix} b & -a \end{pmatrix} \ ; \tag{10.1}$$

if we now attack (a,b) with a pair (p,q) in Poly_2 of two-variable polynomials without constant term, then we will replace T and $T^\sim$ with S and $S^\sim$, where

$$S = UT \ ; \ S^\sim U = |U|T \ ; \ S^\sim = T^\sim U^\sim \ ; \ U^\sim S = T|U| \ . \tag{10.2}$$

With

$$R^\sim S = 1 = S^\sim R \ , \tag{10.3}$$

and

$$RS^\sim + SR^\sim = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ , \tag{10.4}$$

we get left and right invertibility for (a,b);

$$(R^\sim U)T = 1 = T^\sim (U^\sim R) \ ; \tag{10.5}$$

and then also

$$RT^\sim U^\sim + UTR^\sim = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ . \tag{10.6}$$

We claim that (10.6) can be replaced by

$$U^\sim RT^\sim + TR^\sim U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ , \tag{10.7}$$

making $(T^\sim, T)$ exact and hence (a,b) also middle invertible. Specifically, if

$$\begin{pmatrix} w_1 & w_2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 1 = \begin{pmatrix} b & -a \end{pmatrix} \begin{pmatrix} w_2 \\ -w_1 \end{pmatrix}$$

with $\{a,b\} \subseteq \mathrm{comm}\{w_1, w_2\}$, then also

$$\begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} w_1 & w_2 \end{pmatrix} + \begin{pmatrix} w_2 \\ -w_1 \end{pmatrix} \begin{pmatrix} b & -a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ .$$

If in (10.2) we have

$$U = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \tag{10.8}$$

then [8], fixing U as in (10.8), with mutually commuting (u_{ij}) and varying a, b in $\mathrm{comm}\{u_{ij}\}$, (10.2) is uniquely satisfied by

$$|U| = u_{11}u_{22} - u_{21}u_{12} \ , \ U^\sim = \begin{pmatrix} u_{22} & -u_{12} \\ -u_{21} & u_{11} \end{pmatrix} \ . \tag{10.9}$$

For the Koszul matrix $\Lambda_a = \Lambda_{(b,c)}$ of (6.4) we get

$$\Lambda_a = \begin{pmatrix} 0 & 0 & 0 \\ T & 0 & 0 \\ 0 & T^{\sim} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ b & 0 & 0 & 0 \\ c & 0 & 0 & 0 \\ 0 & c & -b & 0 \end{pmatrix} . \tag{10.10}$$

The factorization (10.2) and the derivation of (10.9) depend heavily on the commutivity of $(a,b) \in A^2$, and does not even seem to extend to the quasi-commuting pair of (7.1). For example if

$$U = \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix} , \; S = \begin{pmatrix} ba \\ ab \end{pmatrix} \tag{10.11}$$

then

$$S^{\sim} U = |U| T^{\sim}$$

with $|U| \in A$ would require

$$\begin{pmatrix} ab^2 & -ba^2 \end{pmatrix} = \begin{pmatrix} |U|b & -|U|a \end{pmatrix} ,$$

and hence

$$(ab - |U|)b = 0 = (ba - |U|)a ;$$

now if also $\{a,b\} \subseteq A^{-1}_{\text{right}}$ are each right invertible then

$$ba = |U| = ab . \tag{10.12}$$

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