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WEAKLY CLOSED LIE MODULES OF NEST ALGEBRAS

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Abstract. Let $\mathcal{T}(\mathcal{N})$ be a nest algebra of operators on Hilbert space and let $\mathcal{L}$ be a weakly closed Lie $\mathcal{T}(\mathcal{N})$-module. We construct explicitly the largest possible weakly closed $\mathcal{T}(\mathcal{N})$-bimodule $\mathcal{J}(\mathcal{L})$ and a weakly closed $\mathcal{T}(\mathcal{N})$-bimodule $\mathcal{K}(\mathcal{L})$ such that

$$\mathcal{J}(\mathcal{L}) \subseteq \mathcal{L} \subseteq \mathcal{K}(\mathcal{L}) + \mathcal{D}_{\mathcal{K}(\mathcal{L})},$$

$[\mathcal{K}(\mathcal{L}), \mathcal{T}(\mathcal{N})] \subseteq \mathcal{L}$ and $\mathcal{D}_{\mathcal{K}(\mathcal{L})}$ is a von Neumann subalgebra of the diagonal $\mathcal{T}(\mathcal{N}) \cap \mathcal{T}(\mathcal{N})^*$.

1. Introduction

It has been established in [5] that any weakly closed Lie ideal $\mathcal{L}$ of a nest algebra $\mathcal{T}(\mathcal{N})$ of operators on Hilbert space contains a weakly closed associative ideal of $\mathcal{T}(\mathcal{N})$ and is contained in a sum of this ideal with a von Neumann subalgebra of the diagonal $\mathcal{D}(\mathcal{N})$ of the nest algebra. That is to say that there exist a weakly closed associative ideal $\mathcal{K}(\mathcal{L})$ and a von Neumann subalgebra $\mathcal{D}_{\mathcal{K}(\mathcal{L})}$ of $\mathcal{D}(\mathcal{N})$ such that

$$\mathcal{K}(\mathcal{L}) \subseteq \mathcal{L} \subseteq \mathcal{K}(\mathcal{L}) + \mathcal{D}_{\mathcal{K}(\mathcal{L})}. \tag{1}$$

The purpose of the present work is to show that a similar result holds when we pass from ideals to modules. More precisely, the main result Theorem 1 asserts that, if $\mathcal{L}$ is a weakly closed Lie $\mathcal{T}(\mathcal{N})$-module, then

$$\mathcal{J}(\mathcal{L}) \subseteq \mathcal{L} \subseteq \mathcal{K}(\mathcal{L}) + \mathcal{D}_{\mathcal{K}(\mathcal{L})}, \tag{2}$$

where $\mathcal{D}_{\mathcal{K}(\mathcal{L})}$ is a von Neumman subalgebra of the diagonal $\mathcal{D}(\mathcal{N})$, $\mathcal{J}(\mathcal{L})$ is explicitly constructed as the largest weakly closed $\mathcal{T}(\mathcal{N})$-bimodule contained in $\mathcal{L}$ and $\mathcal{K}(\mathcal{L})$ is a weakly closed $\mathcal{T}(\mathcal{N})$-bimodule such that $[\mathcal{K}(\mathcal{L}), \mathcal{T}(\mathcal{N})] \subseteq \mathcal{L}$, a result reminiscent of [4], Theorem 2.

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Neither is it necessarily the case that $\mathscr{J}(\mathscr{L})$ be a subset of $\mathscr{K}(\mathscr{L})$ nor that $\mathscr{L}$ be contained in $\mathscr{K}(\mathscr{L})$, as Example 1 shows. However, when $\mathscr{L}$ is in fact a weakly closed Lie ideal, a refinement of both (1) and (2) can be obtained, as is outlined in Remark 2. In this situation, (1) and (2) coalesce yielding

$$\mathscr{K}(\mathscr{L}) \subseteq \mathscr{J}(\mathscr{L}) \subseteq \mathscr{L} \subseteq \mathscr{K}(\mathscr{L}) + \mathscr{D}_{\mathscr{K}(\mathscr{L})}, \tag{3}$$

and $\mathscr{K}(\mathscr{L})$ might even be a proper subset of $\mathscr{J}(\mathscr{L})$.

The notation is set in this final part of Section 1 and some facts needed in the sequel are also recalled. Theorem 1 is proved in Section 2.

Let $\mathscr{H}$ be a complex Hilbert space, let $\mathscr{B}(\mathscr{H})$ be the complex Banach space of bounded linear operators on $\mathscr{H}$ and let $\mathscr{F}_1(\mathscr{H})$ be the set of rank one operators in $\mathscr{B}(\mathscr{H})$. A totally ordered family $\mathscr{N}$ of projections in $\mathscr{B}(\mathscr{H})$ containing 0 and the identity I is said to be a *nest*. If, furthermore, $\mathscr{N}$ is a complete sublattice of the lattice of projections in $\mathscr{B}(\mathscr{H})$, then $\mathscr{N}$ is called a *complete nest*. The *nest algebra* $\mathscr{T}(\mathscr{N})$ associated with a nest $\mathscr{N}$ is the subalgebra of all operators T in $B(\mathscr{H})$ such that, for all projections P in $\mathscr{N}$, $T(P(\mathscr{H})) \subseteq P(\mathscr{H})$, or, equivalently, an operator T in $B(\mathscr{H})$ lies in $\mathscr{T}(\mathscr{N})$ if and only if, for all projections P in the nest $\mathscr{N}$, $P^{\perp}TP = 0$, where $P^{\perp} = I - P$. Each nest is contained in a complete nest which generates the same nest algebra (cf. [2, 7]). Henceforth only complete nests will be considered.

The algebra $\mathscr{T}(\mathscr{N})$ is a weakly closed subalgebra of $B(\mathscr{H})$, the *diagonal* $\mathscr{D}(\mathscr{N})$ of which is the von Neumann algebra defined by $\mathscr{D}(\mathscr{N}) = \mathscr{T}(\mathscr{N}) \cap \mathscr{T}(\mathscr{N})^*$.

A nest algebra $\mathscr{T}(\mathscr{N})$ together with the product defined, for all operators T and S in $\mathscr{T}(\mathscr{N})$, by $[T,S] = TS - ST$ is a Lie algebra. A complex subspace $\mathscr{M}$ of $\mathscr{B}(\mathscr{H})$ is said to be a $\mathscr{T}(\mathscr{N})$*-bimodule* if $\mathscr{M}\mathscr{T}(\mathscr{N}), \mathscr{T}(\mathscr{N})\mathscr{M} \subseteq \mathscr{M}$ and is called a *Lie* $\mathscr{T}(\mathscr{N})$*-module* if $[\mathscr{M}, \mathscr{T}(\mathscr{N})] \subseteq \mathscr{M}$. Lie $\mathscr{T}(\mathscr{N})$-modules and $\mathscr{T}(\mathscr{N})$-bimodules contained in the nest algebra $\mathscr{T}(\mathscr{N})$ are called, respectively, Lie ideals and ideals of $\mathscr{T}(\mathscr{N})$. In the sequel, Lie $\mathscr{T}(\mathscr{N})$-modules may be referred to as Lie modules for simplicity. For the same reason, $\mathscr{T}(\mathscr{N})$-bimodules may be called simply bimodules.

Let x and y be elements of the Hilbert space $\mathscr{H}$ and let $x \otimes y$ be the rank one operator defined, for all z in $\mathscr{H}$, by $z \mapsto \langle z,x \rangle y$, where $\langle \cdot,\cdot \rangle$ denotes the inner product of $\mathscr{H}$. Let P be a projection in the nest $\mathscr{N}$ and let P_- (respectively, P_+) be the projection in $\mathscr{N}$ defined by $P_- = \vee\{Q \in \mathscr{N} : Q < P\}$ (respectively, $P_+ = \wedge\{Q \in \mathscr{N} : P < Q\}$). A rank one operator $x \otimes y$ lies in $\mathscr{T}(\mathscr{N})$ if, and only if, there exists a projection P such that $P_- x = 0$ and $Py = y$; moreover, P can be chosen to be equal to $\bigwedge\{Q \in \mathscr{N} : Qy = y\}$ (cf. [7]). For the general theory of nest algebras, the reader is referred to [2, 7].

In what follows, the closure in the weak operator topology of a subset $\mathscr{X}$ of $\mathscr{B}(\mathscr{H})$ will be denoted by $\overline{\mathscr{X}}^{w}$, and the closure in the same topology of the span of $\mathscr{X}$ will be denoted by $\overline{\text{span}}^{w}(\mathscr{X})$. All subspaces either of $\mathscr{H}$ or of $\mathscr{B}(\mathscr{H})$ are assumed to be complex subspaces.

2. Lie $\mathscr{T}(\mathscr{N})$-modules

This section is devoted to the proof of the main result Theorem 1. To this purpose, some lemmas are firstly obtained concerning the $\mathscr{T}(\mathscr{N})$-bimodules $\mathscr{K}(\mathscr{L})$ and $J(\mathscr{L})$ in (2).

LEMMA 1. *Let $\mathscr{L}$ be a Lie $\mathscr{T}(\mathscr{N})$-module and let $P,Q \in \mathscr{T}(\mathscr{N})$ be mutually orthogonal projections. Then, for all $T \in \mathscr{L}$, the operators PTQ, QTP lie in $\mathscr{L}$.*

Proof. Since $PQ = 0$, it is easily seen that

$$QTP = \frac{1}{2}([[[T,P],Q],Q] - [[T,P],Q]),$$

from which follows that $QTP \in \mathscr{L}$. The remaining assertion can be similarly proved. □

LEMMA 2. *Let $\mathscr{L}$ be a weakly closed Lie $\mathscr{T}(\mathscr{N})$-module and let P be a projection in $\mathscr{N}$. If $P^{\perp}\mathscr{L}P \neq \{0\}$, then $P\mathscr{L}P^{\perp} = P\mathscr{B}(\mathscr{H})P^{\perp}$.*

Proof. Let $P \in \mathscr{N}$ and $T \in \mathscr{L}$ be such that $P^{\perp}TP \neq 0$. Notice that Lemma 1 guarantees that $P^{\perp}TP \in \mathscr{L}$. To prove the assertion, it suffices to show that, for all $x,y \in \mathscr{H}$, the operator $P(x\otimes y)P^{\perp}$ lies in $\mathscr{L}$. This trivially holds when $P(x\otimes y)P^{\perp} = 0$. Assume now that $P(x\otimes y)P^{\perp}$ is a rank one operator. Then

$$[[P(x\otimes y)P^{\perp}, P^{\perp}TP], P(x\otimes y)P^{\perp}] = 2P(x\otimes y)P^{\perp}TP(x\otimes y)P^{\perp} \tag{4}$$

and, therefore,

$$[[P(x\otimes y)P^{\perp}, P^{\perp}TP], P(x\otimes y)P^{\perp}] = 2\langle P^{\perp}TPy, x\rangle P(x\otimes y)P^{\perp} \tag{5}$$

lies in $\mathscr{L}$. It follows that $P(x\otimes y)P^{\perp} \in \mathscr{L}$, whenever $\langle P^{\perp}TPy, x\rangle \neq 0$.

On the other hand, if $x \perp P^{\perp}TPy$, then suppose firstly that $P^{\perp}TPy \neq 0$. In this case, replacing $x\otimes y$ by $P^{\perp}TPy\otimes Py$ in the above computations yields that the operator $P^{\perp}TPy\otimes Py$ lies in $\mathscr{L}$. Notice that the condition under which it can be deduced from (5) that $P^{\perp}TPy\otimes Py \in \mathscr{L}$ is, in this case, that

$$\langle P^{\perp}TPy, P^{\perp}TPy\rangle \neq 0,$$

which clearly holds. Moreover, since $\langle P^{\perp}TPy - x, P^{\perp}TPy\rangle \neq 0$, it also follows from (5) that $(P^{\perp}TPy - P^{\perp}x)\otimes Py$ lies in $\mathscr{L}$. Hence,

$$P(x\otimes y)P^{\perp} = P^{\perp}TPy\otimes Py - (P^{\perp}TPy - P^{\perp}x)\otimes Py$$

lies in $\mathscr{L}$.

Assume now that $P^{\perp}TPy = 0$. Since $P^{\perp}TP \neq 0$, there exists $z \in \mathscr{H}$ such that $P^{\perp}TPz \neq 0$, from which follows that $P^{\perp}TP(z-y) \neq 0$. Applying a reasoning similar to that of the preceding paragraph, it follows that both $P(x\otimes z)P^{\perp}$ and $P(x\otimes(z-y))P^{\perp}$ lie in $\mathscr{L}$. Hence,

$$P(x\otimes y)P^{\perp}=P(x\otimes z)P^{\perp}-P(x\otimes(z-y))P^{\perp}$$

lies in $\mathscr{L}$, which concludes the proof. □

Let $\mathscr{L}$ be a Lie $\mathscr{T}(\mathscr{N})$-module and let $\mathscr{K}(\mathscr{L})$ be the subspace of $\mathscr{B}(\mathscr{H})$ defined by

$$\mathscr{K}(\mathscr{L})=\mathscr{K}_V(\mathscr{L})+\mathscr{K}_L(\mathscr{L})+\mathscr{K}_D(\mathscr{L})+\mathscr{K}_\Delta(\mathscr{L}), \tag{6}$$

where

$$\mathscr{K}_V(\mathscr{L})=\overline{\operatorname{span}}^w\{PTP^{\perp}\colon P\in\mathscr{N},T\in\mathscr{L}\}, \tag{7}$$

$$\mathscr{K}_L(\mathscr{L})=\overline{\operatorname{span}}^w\{P^{\perp}TP\colon P\in\mathscr{N},T\in\mathscr{L}\}, \tag{8}$$

$$\mathscr{K}_D(\mathscr{L})=\overline{\operatorname{span}}^w\{PSP^{\perp}TP\colon P\in\mathscr{N},T\in\mathscr{L},S\in\mathscr{T}(\mathscr{N})\}, \tag{9}$$

$$\mathscr{K}_\Delta(\mathscr{L})=\overline{\operatorname{span}}^w\{P^{\perp}TPSP^{\perp}\colon P\in\mathscr{N},T\in\mathscr{L},S\in\mathscr{T}(\mathscr{N})\}. \tag{10}$$

LEMMA 3. *Let $\mathscr{L}$ be a weakly closed Lie $\mathscr{T}(\mathscr{N})$-module and let $\mathscr{K}(\mathscr{L})$ and $\mathscr{K}_V(\mathscr{L})$ be as in (6) and (7), respectively. Then, $\mathscr{K}(\mathscr{L})$ is a weakly closed $\mathscr{T}(\mathscr{N})$-bimodule and $\mathscr{K}_V(\mathscr{L})$ is a weakly closed ideal of $\mathscr{T}(\mathscr{N})$.*

REMARK 1. Notice that $\mathscr{K}_V(\mathscr{L})$ is a subspace of $\mathscr{T}(\mathscr{N})$ and that, by Lemma 1, the spaces $\mathscr{K}_V(\mathscr{L})$ and $\mathscr{K}_L(\mathscr{L})$ are contained in $\mathscr{L}$.

Proof. It is clear that $\mathscr{K}(\mathscr{L})$ and $\mathscr{K}_V(\mathscr{L})$ are weakly closed subspaces of $\mathscr{B}(\mathscr{H})$ and, as observed in Remark 1, $\mathscr{K}_V(\mathscr{L})\subseteq\mathscr{T}(\mathscr{N})$.

To see that $\mathscr{K}_V(\mathscr{L})$ is an ideal of $\mathscr{T}(\mathscr{N})$, it suffices to show that, for all $T\in\mathscr{L},P\in\mathscr{N},S\in\mathscr{T}(\mathscr{N})$ one has that both $PTP^{\perp}S$ and $SPTP^{\perp}$ lie in $\mathscr{K}_V(\mathscr{L})$. Since $P^{\perp}SP^{\perp}\in\mathscr{T}(\mathscr{N})$ and since, by Lemma 1, $PTP^{\perp}$ lies in $\mathscr{L}$, it follows that

$$PTP^{\perp}S=PTP^{\perp}P^{\perp}SP^{\perp}=[PTP^{\perp},P^{\perp}SP^{\perp}]$$

lies in $\mathscr{L}$. But $PTP^{\perp}S=P(PTP^{\perp}S)P^{\perp}$, which shows that $PTP^{\perp}S$ lies in $\mathscr{K}_V(\mathscr{L})$.

Similarly,

$$SPTP^{\perp}=PSPPTP^{\perp}=[PSP,PTP^{\perp}]$$

lies in $\mathscr{L}$ and, therefore,

$$SPTP^{\perp}=P(SPTP^{\perp})P^{\perp}$$

lies in $\mathscr{K}_V(\mathscr{L})$.

It will be shown next that $\mathscr{K}_L(\mathscr{L})\mathscr{T}(\mathscr{N}),\mathscr{T}(\mathscr{N})\mathscr{K}_L(\mathscr{L})\subseteq\mathscr{K}(\mathscr{L})$. It suffices to show that, for all $T\in\mathscr{L},P\in\mathscr{N}$ and $S\in\mathscr{T}(\mathscr{N})$, the operators $P^{\perp}TPS$, $SP^{\perp}TP$ lie in $\mathscr{K}(\mathscr{L})$. Observe also that, if T is an operator in the Lie module $\mathscr{L}$, then, by Lemma 1, the operator $P^{\perp}TP$ lies in $\mathscr{L}$. Hence, it suffices to assume that $T\in\mathscr{L}$ is such that $T=P^{\perp}TP$, for some $P\in\mathscr{N}$, and then prove that $TS,ST\in\mathscr{K}(\mathscr{L})$, for all $S\in\mathscr{T}(\mathscr{N})$.

Let T be an operator in $\mathscr{L}$ such that $T = P^{\perp}TP$, and let S be an operator in the nest algebra. It follows that

$$TS = P^{\perp}TPSP + P^{\perp}TPSP^{\perp}.$$

It is clear that $P^{\perp}TPSP^{\perp} \in \mathscr{K}_{\Delta}(\mathscr{L})$. On the other hand,

$$\begin{aligned} P^{\perp}TPSP &= [P^{\perp}TP, PSP] \\ &= P^{\perp}[T, PSP]P. \end{aligned}$$

Since $[T, PSP] \in \mathscr{L}$, it follows that $P^{\perp}TPSP \in \mathscr{K}_L(\mathscr{L})$. Hence, TS lies in $\mathscr{K}(\mathscr{L})$, as required.

Similarly,

$$ST = P^{\perp}ST + PST = [P^{\perp}SP^{\perp}, T] + PSP^{\perp}TP$$

lies in $\mathscr{K}(\mathscr{L})$, since $PSP^{\perp}TP \in \mathscr{K}_D(\mathscr{L})$ and

$$[P^{\perp}SP^{\perp}, T] = P^{\perp}[P^{\perp}SP^{\perp}, T]P$$

lies in $\mathscr{K}_L(\mathscr{L})$.

To show that $\mathscr{K}_D(\mathscr{L})\mathscr{T}(\mathscr{N}), \mathscr{T}(\mathscr{N})\mathscr{K}_D(\mathscr{L}) \subseteq \mathscr{K}(\mathscr{L})$, it suffices to prove that, for all $T \in \mathscr{L}$, $S, R \in \mathscr{T}(\mathscr{N})$ and $P \in \mathscr{N}$, the operators $PSP^{\perp}TPR$ and $RPSP^{\perp}TP$ lie in $\mathscr{K}(\mathscr{L})$.

As to the operator $RPSP^{\perp}TP$, observe that

$$RPSP^{\perp}TP = P(RPS)P^{\perp}TP$$

and, since $RPS \in \mathscr{T}(\mathscr{N})$, it immediately follows that $RPSP^{\perp}TP \in \mathscr{K}_D(\mathscr{L})$. Hence, $\mathscr{T}(\mathscr{N})\mathscr{K}_D(\mathscr{L}) \subseteq \mathscr{K}(\mathscr{L})$.

It only remains to show that $PSP^{\perp}TPR \in \mathscr{K}(\mathscr{L})$. Observe that, by Lemma 2, either $P\mathscr{L}P^{\perp} = P\mathscr{B}(\mathscr{H})P^{\perp}$ or $P^{\perp}\mathscr{L}P = \{0\}$. In the latter case, it is obvious that the assertion to be proved trivially holds. In the former case, notice that, by Lemma 1, $P\mathscr{B}(\mathscr{H})P^{\perp} \subseteq \mathscr{L}$.

Let T, S, R be as above and let $P \in \mathscr{N}$ be such that $P\mathscr{B}(\mathscr{H})P^{\perp} \subseteq \mathscr{L}$. Then,

$$\begin{aligned} PSP^{\perp}TPR &= PSP^{\perp}TPRP + PSP^{\perp}TPRP^{\perp} \\ &= PSP^{\perp}[P^{\perp}TP, PRP]P + PSP^{\perp}TPRP^{\perp}. \end{aligned}$$

As seen above, $P\mathscr{B}(\mathscr{H})P^{\perp} \subseteq \mathscr{L}$ yielding that the operator $PSP^{\perp}TPRP^{\perp}$ lies in $\mathscr{L}$. Consequently,

$$PSP^{\perp}TPRP^{\perp} = P(PSP^{\perp}TPRP^{\perp})P^{\perp}$$

lies in $\mathscr{K}_V(\mathscr{L})$. Moreover, by Lemma 1, $P^{\perp}TP \in \mathscr{L}$, from which follows that $[P^{\perp}TP, PRP] \in \mathscr{L}$. Hence, $PSP^{\perp}[P^{\perp}TP, PRP]P \in \mathscr{K}_D(\mathscr{L})$. It follows that $\mathscr{K}_D(\mathscr{L})\mathscr{T}(\mathscr{N}) \subseteq \mathscr{K}(\mathscr{L})$.

Finally, it will be shown that $\mathscr{K}_{\Delta}(\mathscr{L})\mathscr{T}(\mathscr{N}), \mathscr{T}(\mathscr{N})\mathscr{K}_{\Delta}(\mathscr{L}) \subseteq \mathscr{K}(\mathscr{L})$. That is to say that, it must be proved that, for all $T \in \mathscr{L}, S, R \in \mathscr{T}(\mathscr{N})$ and $P \in \mathscr{N}$, the operators $P^{\perp}TPSP^{\perp}R$ and $RP^{\perp}TPSP^{\perp}$ lie in $\mathscr{K}(\mathscr{L})$.

Suppose again that $P\mathscr{L}P^{\perp} = P\mathscr{B}(\mathscr{H})P^{\perp}$. Recall that, by Lemma 2, the only other possibility is $P^{\perp}\mathscr{L}P = \{0\}$, in which case the assertions to be proved trivially hold.

Since $SP^{\perp}R \in \mathscr{T}(\mathscr{N})$, it follows that

$$P^{\perp}TPSP^{\perp}R = P^{\perp}TP(SP^{\perp}R)P^{\perp}$$

lies in $\mathscr{K}_{\Delta}(\mathscr{L})$. Furthermore,

$$\begin{aligned} RP^{\perp}TPSP^{\perp} &= PRP^{\perp}TPSP^{\perp} + P^{\perp}RP^{\perp}TPSP^{\perp} \\ &= PRP^{\perp}TPSP^{\perp} + P^{\perp}[P^{\perp}RP^{\perp}, P^{\perp}TP]PSP^{\perp}. \end{aligned}$$

Observe that $PRP^{\perp}TPSP^{\perp} \in P\mathscr{B}(\mathscr{H})P^{\perp} \subseteq \mathscr{K}_V(\mathscr{L})$, since it is assumed that $P\mathscr{L}P^{\perp} = P\mathscr{B}(\mathscr{H})P^{\perp}$. Moreover, $P^{\perp}[P^{\perp}RP^{\perp}, P^{\perp}TP]PSP^{\perp}$ lies in $\mathscr{K}_{\Delta}(\mathscr{L})$, since $[P^{\perp}RP^{\perp}, P^{\perp}TP] \in \mathscr{L}$. □

LEMMA 4. *Let $\mathscr{L}$ be a weakly closed Lie $\mathscr{T}(\mathscr{N})$-module and let $\mathscr{K}(\mathscr{L})$ be the weakly closed $\mathscr{T}(\mathscr{N})$-bimodule associated with $\mathscr{L}$ in (6).*

Then $[\mathscr{K}(\mathscr{L}), \mathscr{T}(\mathscr{N})] \subseteq \mathscr{L}$.

Proof. Since $\mathscr{K}_V(\mathscr{L}), \mathscr{K}_L(\mathscr{L}) \subseteq \mathscr{L}$, it is enough to prove that

$$[\mathscr{K}_D(\mathscr{L}), \mathscr{T}(\mathscr{N})], [\mathscr{K}_{\Delta}(\mathscr{L}), \mathscr{T}(\mathscr{N})] \subseteq \mathscr{L}.$$

That is to say that it suffices to show that for all $T \in \mathscr{L}, P \in \mathscr{N}$ and $R, S \in \mathscr{T}(\mathscr{N})$, the operators $[PSP^{\perp}TP, R]$ and $[P^{\perp}TPSP^{\perp}, R]$ lie in $\mathscr{L}$.

Recall once again that, given $P \in \mathscr{N}$, by Lemma 2, either $P\mathscr{L}P^{\perp} = P\mathscr{B}(\mathscr{H})P^{\perp}$ or $P^{\perp}\mathscr{L}P = \{0\}$. In the latter case, for all $T \in \mathscr{L}$, $P^{\perp}TP = 0$, from which follows that the assertions to be proved are trivially true.

Suppose now that $P\mathscr{L}P^{\perp} = P\mathscr{B}(\mathscr{H})P^{\perp}$ and that $T \in \mathscr{L}$ is such that $P^{\perp}TP \neq 0$, in which case, by Lemma 1, $P^{\perp}TP \in \mathscr{L}$. Then, for all $R, S \in \mathscr{N}$,

$$\begin{aligned} [PSP^{\perp}TP, R] &= [PSP^{\perp}TP, RP] + [PSP^{\perp}TP, PRP^{\perp}] + [PSP^{\perp}TP, P^{\perp}RP^{\perp}] \\ &= [PSP^{\perp}TP - P^{\perp}TPSP^{\perp}, RP] + PSP^{\perp}TPRP^{\perp} \\ &= [[PSP^{\perp}, P^{\perp}TP], RP] + PSP^{\perp}TPRP^{\perp} \end{aligned}$$

lies in $\mathscr{L}$. Similarly,

$$\begin{aligned} [P^{\perp}TPSP^{\perp}, R] &= [P^{\perp}TPSP^{\perp}, RP] + [P^{\perp}TPSP^{\perp}, PRP^{\perp}] + [P^{\perp}TPSP^{\perp}, P^{\perp}RP^{\perp}] \\ &= -PRP^{\perp}TPSP^{\perp} + [P^{\perp}TPSP^{\perp} - PSP^{\perp}TP, P^{\perp}RP^{\perp}] \\ &= -PRP^{\perp}TPSP^{\perp} + [[P^{\perp}TP, PSP^{\perp}], P^{\perp}RP^{\perp}] \end{aligned}$$

is an operator in $\mathscr{L}$, which concludes the proof. □

Recall that it is possible to associate with each weakly closed $\mathscr{T}(\mathscr{N})$-bimodule $\mathscr{K}$ a (not necessarily unique) left order continuous homomorphism $\phi : \mathscr{N} \to \mathscr{N}$ such that

$$\mathscr{K} = \{T \in \mathscr{B}(\mathscr{H}) \colon \phi(P)^{\perp} T P = 0\}$$

(see [3]).

LEMMA 5. *Let $\mathscr{L}$ be a weakly closed Lie $\mathscr{T}(\mathscr{N})$-module, let $\mathscr{K}(\mathscr{L})$ be the weakly closed $\mathscr{T}(\mathscr{N})$-bimodule defined in (6)–(10), and let $\phi : \mathscr{N} \to \mathscr{N}$ be a left order continuous homomorphism associated with $\mathscr{K}(\mathscr{L})$. If $P \in \mathscr{N}$ is such that $\phi(P) < P$, then, for all $T \in \mathscr{L}$ and all $Q \in \mathscr{N}$ with $\phi(P) < Q < P$,*

$$(Q - \phi(P))T(P - Q) = 0.$$

Proof. Let T be an operator in $\mathscr{L}$ and let that $P, Q \in \mathscr{N}$. Since, by the definition (6)–(10) of $\mathscr{K}(\mathscr{L})$, $QTQ^{\perp} \in \mathscr{K}(\mathscr{L})$, it follows that

$$\phi(P)^{\perp}(QTQ^{\perp})P = 0.$$

Hence, if $\phi(P) < Q < P$, then

$$(Q - \phi(P))T(P - Q) = 0,$$

as required. □

DEFINITION 1. Given a weakly closed $\mathscr{T}(\mathscr{N})$-bimodule $\mathscr{K}$, define $\mathscr{D}_{\mathscr{K}}$ as the algebra consisting of all operators $T \in \mathscr{D}(\mathscr{N})$ such that, for every $P \in \mathscr{N}$ for which $\phi(P) < P_{-}$, there exists λ_P in $\mathbb{C}$ satisfying the equality

$$T\big(P - \phi(P)\big) = \lambda_P\big(P - \phi(P)\big).$$

The algebra $\mathscr{D}_{\mathscr{K}}$ is a von Neumann subalgebra of $\mathscr{D}(\mathscr{N})$ and, when $\mathscr{K}$ is a weakly closed Lie ideal of $\mathscr{T}(\mathscr{N})$, the algebra $\mathscr{D}_{\mathscr{K}}$ is that defined in [5].

The next lemma is inspired by results of [5].

LEMMA 6. *Let $\mathscr{L}$ be a weakly closed Lie $\mathscr{T}(\mathscr{N})$-module. Then $\mathscr{L} \subseteq \mathscr{K}(\mathscr{L}) + \mathscr{D}_{\mathscr{K}(\mathscr{L})}$.*

Proof. Let π be an expectation of $\mathscr{T}(\mathscr{N})$ on $\mathscr{D}(\mathscr{N})$ (see [2], Corollary 8.5). Given $T \in \mathscr{L}$, let

$$T = T_{\pi} + \pi(T),$$

where

$$T_{\pi} = T - \pi(T).$$

Firstly, it will be shown that $T_{\pi} \in \mathscr{K}(\mathscr{L})$; that is to say that, for all $P \in \mathscr{N}$,

$$\phi(P)^{\perp} T_{\pi} P = 0,$$

where $\phi : \mathscr{N} \to \mathscr{N}$ is a left order continuous homomorphism on $\mathscr{N}$ associated with the bimodule $\mathscr{K}(\mathscr{L})$.

Let Q be a projection in $\mathscr{N}$. Notice that

$$\phi(P)^{\perp} Q^{\perp} T Q P = 0,$$

since $Q^{\perp} T Q \in \mathscr{K}(\mathscr{L})$ (see (6)–(10)). Then,

$$\begin{aligned} Q^{\perp}(\phi(P)^{\perp} T_{\pi} P) Q &= \phi(P)^{\perp}(Q^{\perp} T_{\pi} Q) P \\ &= \phi(P)^{\perp} Q^{\perp} T Q P - \phi(P)^{\perp} Q^{\perp} \pi(T) Q P \\ &= -\phi(P)^{\perp} Q^{\perp} \pi(T) Q P. \end{aligned}$$

But, since by [2], Theorem 8.1, $\phi(P)^{\perp} Q^{\perp} \pi(T) Q P = \pi\big(\phi(P)^{\perp} Q^{\perp} T Q P\big)$, it follows that, for all $Q \in \mathscr{N}$, $Q^{\perp}(\phi(P)^{\perp} T P) Q = 0$. Similarly,

$$\begin{aligned} Q\phi(P)^{\perp} T_{\pi} P Q^{\perp} &= \phi(P)^{\perp}(Q T Q^{\perp}) P - \phi(P)^{\perp} Q \pi(T) Q^{\perp} P \\ &= -\pi\big(\phi(P)^{\perp} Q T Q^{\perp} P\big) = 0. \end{aligned}$$

Hence, for all $P, Q \in \mathscr{N}$,

$$\phi(P)^{\perp} T_{\pi} P = Q\phi(P)^{\perp} T_{\pi} P Q + Q^{\perp}\phi(P)^{\perp} T_{\pi} P Q^{\perp}, \tag{11}$$

from which follows that $\phi(P)^{\perp} T_{\pi} P \in \mathscr{D}(\mathscr{N})$. Hence, by [2], Theorem 8.1,

$$\begin{aligned} \phi(P)^{\perp} T_{\pi} P &= \pi(\phi(P)^{\perp} T_{\pi} P) \\ &= \phi(P)^{\perp} \pi(T_{\pi}) P \\ &= \phi(P)^{\perp} \pi(T - \pi(T)) P. \end{aligned}$$

Since $\pi(T - \pi(T)) = 0$, it follows that, for all $P \in \mathscr{N}$, $\phi(P)^{\perp} T_{\pi} P = 0$ or, in other words, T_{π} lies in $\mathscr{K}(\mathscr{L})$.

It remains to show that $\pi(T)$ lies in $\mathscr{D}_{\mathscr{K}(\mathscr{L})}$. Let $P \in \mathscr{N}$ be such that $\phi(P) < P_{-}$. Then, there exists a projection $Q \in \mathscr{N}$ such that $\phi(P) < Q < P$.

Since $Q T Q^{\perp} \in \mathscr{K}(\mathscr{L})$ (see (6)–(10)), it follows that, for all $P \in \mathscr{N}$,

$$\phi(P)^{\perp}(Q T Q^{\perp}) P = 0.$$

Observe also that, since $T_{\pi} \in \mathscr{K}(\mathscr{L})$, by Lemma 4, $[\pi(T), \mathscr{T}(\mathscr{N})] \subseteq \mathscr{L}$. Hence, for all $x, y \in \mathscr{H}$, the operator

$$[\pi(T), (Q - \phi(P))(x \otimes y)(P - Q)]$$

lies in $\mathscr{L}$. It follows, by Lemma 5, that

$$(Q - \phi(P))[\pi(T), (Q - \phi(P))(x \otimes y)(P - Q)](P - Q) = 0$$

and, consequently,

$$((P - Q)x \otimes (Q - \phi(P))\pi(T)(Q - \phi(P))y) = ((P - Q)\pi(T)^{*}(P - Q)x \otimes (Q - \phi(P))y)$$

Choosing $x, y \in \mathcal{H}$ such that $x = (P-Q)x$ and $y = (Q - \phi(P))y$, it is easy to see that there must exist $\lambda_P \in \mathbb{C}$ such that

$$\pi(T)(P-Q) = \lambda_P(P-Q)$$

and

$$\pi(T)\big(Q-\phi(P)\big) = \lambda_P\big(Q-\phi(P)\big).$$

It follows that

$$\pi(T)\big(P-\phi(P)\big) = \lambda_P\big(P-\phi(P)\big),$$

yielding that $\pi(T)$ lies in $\mathcal{D}_{\mathcal{K}(\mathcal{L})}$, as required. □

The characterisation of the largest weakly closed $\mathcal{T}(\mathcal{N})$-bimodule contained in a weakly closed Lie $\mathcal{T}(\mathcal{N})$-module $\mathcal{L}$ will be obtained in Lemma 8 below.

Let z be an element of the Hilbert space $\mathcal{H}$ and let P_z and $\hat{P}_z$ be the projections defined by

$$P_z = \wedge\{Q \in \mathcal{N} : Qz = z\}, \qquad \hat{P}_z = \vee\{Q \in \mathcal{N} : Qz = 0\}.$$

The projections P_z and $\hat{P}_z$ lie in the nest $\mathcal{N}$ and $P_z z = z$, $\hat{P}_z z = 0$. Following [6], each rank one operator $x \otimes y$ will be associated with the projections $\hat{P}_x$ and P_y.

LEMMA 7. *Let $\mathcal{U}$ be a norm closed $T(\mathcal{N})$-bimodule and let $x \otimes y$ be a rank one operator in $\mathcal{B}(\mathcal{H})$. Then $x \otimes y$ lies in $\mathcal{U}$ if and only if $P_y \mathcal{B}(\mathcal{H})\hat{P}_x^{\perp}$ is contained in $\overline{\mathcal{U}}^w$.*

Proof. Since $\overline{\mathcal{U}}^w$ is a weakly closed bimodule, by [3], Theorem 1.5, there exists a left order continuous homomorphism $P \mapsto \tilde{P}$ on $\mathcal{N}$ such that an operator $T \in \mathcal{B}(\mathcal{H})$ lies in $\overline{\mathcal{U}}^w$ if and only if, for all $P \in \mathcal{N}$, $\tilde{P}^{\perp} T P = 0$.

Let $x \otimes y$ be a rank one operator in $\mathcal{U}$, let T lie in $P_y B(\mathcal{H})\hat{P}_x^{\perp}$ and suppose that $P \in \mathcal{N}$ is a projection such that $P \leqslant \hat{P}_x$. Then,

$$\tilde{P}^{\perp} T P = \tilde{P}^{\perp} P_y T \hat{P}_x^{\perp} P = 0. \tag{12}$$

Suppose now that $P \in \mathcal{N}$ is a projection such that $\hat{P}_x < P$. Since $x \otimes y \in \mathcal{U}$, by the definition of $\tilde{P}$, $P_y \leqslant \tilde{P}$ (see [3], p. 221). Hence

$$\tilde{P}^{\perp} T P = \tilde{P}^{\perp} P_y T \hat{P}_x^{\perp} P = 0. \tag{13}$$

Combining (12)-(13) yields that $P_y B(\mathcal{H})\hat{P}_x^{\perp} \subseteq \overline{\mathcal{U}}^w$.

Conversely, if $x \otimes y$ is a rank one operator such that $P_y B(\mathcal{H})\hat{P}_x^{\perp} \subseteq \overline{\mathcal{U}}^w$, then it is clear that $x \otimes y \in \overline{\mathcal{U}}^w$. By [3], Lemma 1.3, $x \otimes y \in \mathcal{U}$. □

Let $\mathcal{L}$ be a weakly closed Lie $\mathcal{T}(\mathcal{N})$-module and let $\mathcal{C}(\mathcal{L})$ be the subset of $\mathcal{B}(\mathcal{H})$ defined by

$$\mathcal{C}(\mathcal{L}) = \{x \otimes y \in \mathcal{B}(\mathcal{H}) : P_y \mathcal{B}(\mathcal{H})\hat{P}_x^{\perp} \subseteq \mathcal{L}\}. \tag{14}$$

Observe that the set $\mathscr{C}(\mathscr{L})$ can be properly contained in $\mathscr{L}\cap\mathscr{F}_1(\mathscr{H})$. For example, let $\mathscr{T}(\mathscr{N})$ be the nest algebra of the 8×8 upper triangular complex matrices and let $\mathscr{L}$ be the subspace of the 8×8 complex matrices consisting of those having null trace. It is clear that $\mathscr{L}$ is a Lie $\mathscr{T}(\mathscr{N})$-module and that the matrix unit E_{65} lies in $\mathscr{L}$ but not in $\mathscr{C}(\mathscr{L})$.

LEMMA 8. *Let $\mathscr{L}$ be a weakly closed Lie $\mathscr{T}(\mathscr{N})$-module and let $\mathscr{C}(\mathscr{L})$ be as in* (14)*. Then the largest weakly closed $\mathscr{T}(\mathscr{N})$-bimodule contained in $\mathscr{L}$ is*

$$\mathscr{J}(\mathscr{L})=\{T\in\mathscr{B}(\mathscr{H})\colon\ \phi(P)^{\perp}TP=0\},\tag{15}$$

where $\phi\colon\mathscr{N}\to\mathscr{N}$ is the left order continuous homomorphism defined by

$$\phi(P)=\vee\{P_y\colon\ \exists x\in\mathscr{H}\ x\otimes y\in\mathscr{C}(\mathscr{L})\wedge\hat{P}_x<P\}.\tag{16}$$

Proof. It is easy to see that a set defined as in (15) by any, not even necessarily order-preserving, map $\phi\colon\mathscr{N}\to\mathscr{N}$ is a weakly closed $\mathscr{T}(\mathscr{N})$-bimodule. It will be shown next that $\mathscr{C}(\mathscr{L})$ coincides with the subset of rank one operators contained in $\mathscr{J}(\mathscr{L})$.

Let $x\otimes y$ be a rank one operator in $\mathscr{C}(\mathscr{L})$, let P be a projection in $\mathscr{N}$ and suppose initially that $\hat{P}_x<P$. It follows from the definition of ϕ that $P_y\leqslant\phi(P)$ and, consequently,

$$\phi(P)^{\perp}(x\otimes y)P=\phi(P)^{\perp}P_y(x\otimes y)\hat{P}_x^{\perp}P=0.$$

It can be similarly shown that $\phi(P)^{\perp}(x\otimes y)P=0$, when $P\leqslant\hat{P}_x$. Hence $\mathscr{C}(\mathscr{L})\subseteq\mathscr{J}(\mathscr{L})$.

Conversely, let $x\otimes y$ be an operator lying in the weakly closed bimodule $\mathscr{J}(\mathscr{L})$. Hence, by Lemma 7, for all $T\in P_y\mathscr{B}(\mathscr{H})\hat{P}_x^{\perp}$ and all $P\in\mathscr{N}$, $\phi(P)^{\perp}TP=0$. It follows that $P_y\leqslant\phi(P)$, whenever $P\in\mathscr{N}$ is such that $\hat{P}_x<P$.

Case 1. $P\in\mathscr{N}$ is such that $\hat{P}_x<P$ and $P_y<\phi(P)$.

In this case, by (16), there exists a rank one operator $z_P\otimes w_P\in\mathscr{C}(\mathscr{L})$ such that $\hat{P}_{z_P}<P$ and $P_y<P_{w_P}$. Hence $P_{w_P}\mathscr{B}(\mathscr{H})\hat{P}_{z_P}^{\perp}\subseteq\mathscr{L}$ and, consequently, $P_{w_P}(x\otimes y)\hat{P}_{z_P}^{\perp}=\hat{P}_{z_P}^{\perp}x\otimes y$ lies in $\mathscr{C}(\mathscr{L})$.

Since $P_{w_P}\mathscr{B}(\mathscr{H})P^{\perp}\subseteq P_{w_P}\mathscr{B}(\mathscr{H})\hat{P}_{z_P}^{\perp}$, it follows that $P^{\perp}x\otimes y$ lies also in $\mathscr{C}(\mathscr{L})$.

Case 2. $P\in\mathscr{N}$ is such that $\hat{P}_x<P$ and $P_y=\phi(P)$.

By (16), there exists a set $\{z_j\otimes w_j\colon\ j\in\Lambda\}$ contained in $\mathscr{C}(\mathscr{L})$ such that (P_{w_j}) is an increasing net converging to P_y in the strong operator topology and, for all j, $\hat{P}_{z_j}<P$. Consequently, for all j, the operator $\hat{P}_{z_j}^{\perp}x\otimes P_{w_j}y$ lies in $\mathscr{C}(\mathscr{L})$. Observing that $P_{w_j}\mathscr{B}(\mathscr{H})P^{\perp}\subseteq P_{w_j}\mathscr{B}(\mathscr{H})\hat{P}_{z_j}^{\perp}\subseteq\mathscr{L}$, it follows that $P^{\perp}x\otimes P_{w_j}y$ also lies in $\mathscr{C}(\mathscr{L})$.

Since $\mathscr{L}$ is weakly closed, it is also the case that $P^{\perp}x\otimes y\in\mathscr{C}(\mathscr{L})$.

If $\hat{P}_x<\hat{P}_x^{+}$, then set $P=\hat{P}_x^{+}$. If Case 1 applies, then there exists a rank one operator $z_P\otimes w_P\in\mathscr{C}(\mathscr{L})$ such that $\hat{P}_{z_P}<\hat{P}_x^{+}$ and $P_y<P_{w_P}$. Consequently,

$$\hat{P}_{z_P}^{\perp}x\otimes y=\hat{P}_x^{\perp}x\otimes y=x\otimes y$$

lies in $\mathscr{C}(\mathscr{L})$. On the other hand, if Case 2 holds, then, for all j, $\hat{P}_{z_j} \leqslant \hat{P}_x$. Hence, for all j,

$$\begin{aligned}\hat{P}_{z_j}^{\perp} x \otimes P_{w_j} y &= \hat{P}_x^{\perp} x \otimes P_{w_j} y \\ &= x \otimes P_{w_j} y,\end{aligned}$$

from which follows that $x \otimes P_{w_j} y$ lies in $\mathscr{C}(\mathscr{L})$. A limit argument similar to that above finally yields that $x \otimes y \in \mathscr{C}(\mathscr{L})$.

If $\hat{P}_x = \hat{P}_x^+$, then there exists a decreasing net (P_j) in $\mathscr{N}$ converging to $\hat{P}_x$ in the strong operator topology and such that, for all j, $\hat{P}_x < P_j$. Since, either by Case 1 or Case 2, for all j, the operator $P_j^{\perp} x \otimes y$ lies in $\mathscr{C}(\mathscr{L})$, taking limits it follows that $P_y \mathscr{B}(\mathscr{H}) \hat{P}_x^{\perp} \subseteq \mathscr{L}$. Hence, $\hat{P}_x^{\perp} x \otimes y = x \otimes y \in \mathscr{C}(\mathscr{L})$, as required.

It has been shown that a rank one operator lies in the weakly closed bimodule $\mathscr{J}(\mathscr{L})$ if and only if it lies in $\mathscr{C}(\mathscr{L})$. Hence, by [3], Lemma 1.2 and Theorem 1.5, $\mathscr{J}(\mathscr{L}) = \overline{\text{span}}^w(\mathscr{C}(\mathscr{L}))$ and, consequently, $\mathscr{J}(\mathscr{L}) \subseteq \mathscr{L}$. Notice that it is implicit in the proof of [3], Theorem 1.5 that a weakly closed $\mathscr{T}(\mathscr{N})$-bimodule coincides with the closure in the weak operator topology of its subset of finite rank operators.

Suppose that there exists a weakly closed bimodule $\mathscr{U}$ contained in $\mathscr{L}$ which properly contains $\mathscr{J}(\mathscr{L})$. By Lemma 7 and [3], Lemma 1.2 and Theorem 1.5, there exists a rank one operator $x \otimes y \in \mathscr{U} \setminus \mathscr{J}(\mathscr{L})$ such that $P_y \mathscr{B}(\mathscr{H}) \hat{P}_x^{\perp} \subseteq \mathscr{U}$ and, therefore,

$$P_y \mathscr{B}(\mathscr{H}) \hat{P}_x^{\perp} \subseteq \mathscr{L}. \tag{17}$$

But this is impossible since, as shown above, $\mathscr{J}(\mathscr{L})$ contains all operators $x \otimes y$ satisfying (17). Hence $\mathscr{J}(\mathscr{L})$ is the largest weak operator closed bimodule contained in $\mathscr{L}$.

To end the proof, it will be shown next that the map ϕ is a left order continuous homomorphism on $\mathscr{N}$.

If $P_1 \leqslant P_2$, then $\{P_y \colon \exists x \in \mathscr{H}\ x \otimes y \in \mathscr{C}(\mathscr{L}) \wedge \hat{P}_x < P_1\}$ is a subset of $\{P_y \colon \exists x \in \mathscr{H}\ x \otimes y \in \mathscr{C}(\mathscr{L}) \wedge \hat{P}_x < P_2\}$, from which immediately follows that $\phi(P_1) \leqslant \phi(P_2)$. Hence ϕ is an order homomorphism on $\mathscr{N}$.

It only remains to show that the map ϕ is left order continuous; that is to say that, for every subset $\mathscr{X}$ of $\mathscr{N}$, $\phi(\vee \mathscr{X}) = \vee \phi(\mathscr{X})$. This trivially holds for the empty set. Suppose then that in what follows $\mathscr{X} \neq \emptyset$.

If $\vee \mathscr{X} \in \mathscr{X}$, then the equality $\phi(\vee \mathscr{X}) = \vee \phi(\mathscr{X})$ is obvious, since ϕ is an order-preserving map. If, on the other hand, $\vee \mathscr{X} \notin \mathscr{X}$ then $(\vee \mathscr{X})_- = \vee \mathscr{X}$.

Hence, suppose now that $P \in \mathscr{N}$ is such that $P_- = P$. In this case,

$$P = \vee \{R \in \mathscr{N} : R < P\}$$

and, since ϕ is an order homomorphism, it is clear that

$$\vee \{\phi(R) \in \mathscr{N} : R < P\} \leqslant \phi(P).$$

If $\vee \{\phi(R) \in \mathscr{N} : R < P\} < \phi(P)$, then by (16) there would exist a rank one operator $x \otimes y \in \mathscr{L}$ such that $\hat{P}_x < P$, $P_y \mathscr{B}(\mathscr{H}) \hat{P}_x^{\perp} \subseteq \mathscr{L}$ and

$$\vee \{\phi(R) \in \mathscr{N} : R < P\} < P_y.$$

But this cannot happen since, by the definition of supremum,

$$P_y \leqslant \vee\{\phi(R) \in \mathscr{N} : R < P\}.$$

Hence

$$\vee\{\phi(R) \in \mathscr{N} : R < P\} = \phi(P),$$

as required. Letting $P = \vee\mathscr{X}$, we finally have $\phi(\vee\mathscr{X}) = \vee\phi(\mathscr{X})$, which concludes the proof. □

Given a weakly closed Lie $\mathscr{T}(\mathscr{N})$-module $\mathscr{L}$, let $\mathscr{K}(\mathscr{L})$ and $\mathscr{D}_{\mathscr{K}(\mathscr{L})}$ be defined, respectively, by (6)–(10) and Definition 1. The next theorem summarises the results of Section 2.

THEOREM 1. *Let $\mathscr{L}$ be a weakly closed Lie $\mathscr{T}(\mathscr{N})$-module and let $\mathscr{C}(\mathscr{L})$ be as in* (14)*. Then, there exist weakly closed $\mathscr{T}(\mathscr{N})$-bimodules $\mathscr{J}(\mathscr{L})$ and $\mathscr{K}(\mathscr{L})$ and a von Neumman subalgebra $\mathscr{D}_{\mathscr{K}(\mathscr{L})}$ of the diagonal $\mathscr{D}(\mathscr{N})$ such that*

$$\mathscr{J}(\mathscr{L}) \subseteq \mathscr{L} \subseteq \mathscr{K}(\mathscr{L}) + \mathscr{D}_{\mathscr{K}(\mathscr{L})},$$

where $\mathscr{J}(\mathscr{L}) = \{T \in \mathscr{B}(\mathscr{H}) \colon \phi(P)^{\perp}TP = 0\}$ and $\phi \colon \mathscr{N} \to \mathscr{N}$ is the left order continuous homomorphism defined by

$$\phi(P) = \vee\{P_y \colon \exists x \in \mathscr{H}\ x \otimes y \in \mathscr{C}(\mathscr{L}) \wedge \hat{P}_x < P\}.$$

Moreover, $\mathscr{J}(\mathscr{L})$ is the largest weakly closed $\mathscr{T}(\mathscr{N})$-bimodule contained in $\mathscr{L}$ and $\mathscr{K}(\mathscr{L})$ is such that $[\mathscr{K}(\mathscr{L}), \mathscr{T}(\mathscr{N})] \subseteq \mathscr{L}$.

EXAMPLE 1. Notice that neither is it necessarily the case that $\mathscr{J}(\mathscr{L}) \subseteq \mathscr{K}(\mathscr{L})$ nor that $\mathscr{L} \subseteq \mathscr{K}(\mathscr{L})$. A simple counter-example can be given in the nest algebra of the 5×5 upper triangular complex matrices. Consider the Lie module $\mathscr{L} = \operatorname{span}\{I\} + \mathscr{J}(\mathscr{L})$, where $\mathscr{J}(\mathscr{L})$ is the bimodule consisting of the 5×5 complex matrices such that $a_{i1} = 0$, if $1 \leqslant i \leqslant 5$, and $a_{i2} = 0$, if $3 \leqslant i \leqslant 5$. In this case, $\mathscr{K}(\mathscr{L})$ consists of the matrices in $\mathscr{J}(\mathscr{L})$ such that $a_{22} = 0$.

REMARK 2. When $\mathscr{L}$ is a weakly closed Lie ideal, $\mathscr{K}_L(\mathscr{L})$, $\mathscr{K}_D(\mathscr{L})$, $\mathscr{K}_\Delta(\mathscr{L}) = \{0\}$. In this situation, it has been shown in [1] that

$$\mathscr{K}(\mathscr{L}) \subseteq \mathscr{J}(\mathscr{L}) \subseteq \mathscr{L} \subseteq \mathscr{K}(\mathscr{L}) + \mathscr{D}_{\mathscr{K}(\mathscr{L})} = \mathscr{J}(\mathscr{L}) \oplus \breve{\mathscr{D}}(\mathscr{L}),$$

where $\breve{\mathscr{D}}(\mathscr{L})$ is an appropriate unital weakly closed $*$-subalgebra of $\mathscr{D}_{\mathscr{K}(\mathscr{L})}$.

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