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FOURIER MULTIPLIERS ASSOCIATED WITH SINGULAR PARTIAL DIFFERENTIAL OPERATORS

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Abstract. We prove the Hörmander-Mikhlin multiplier theorem for the Fourier transform associated with the Riemann-Liouville operator.

1. Introduction

Given a measurable bounded function m on $\mathbb{R}^n$, the multiplier operator T_m is defined by

$$\widehat{T_m f} = m\widehat{f},$$

where $\widehat{f}$ denotes the classical Fourier transform. One of the most common problem connected with these operators is to characterize the function m for which the multiplier operator T_m is bounded from $L^p(\mathbb{R}^n)$ into itself for every $1 < p < +\infty$. Such multiplier operator is referred to as an L^p-Fourier multiplier. This problem is considered to be difficult and one of the most response is due to Hörmander [18] who took forward results obtained firstly by Mikhlin [21], it states that if m is a bounded function on $\mathbb{R}^n$, satisfying

$$\sup_{\substack{R>0 \\ |\alpha|\leqslant[\frac{n}{2}]+1}} R^{2|\alpha|-n} \int_{R<|x|<2R} \left|\frac{\partial^\alpha m}{\partial x^\alpha}(x)\right|^2 dx < A,$$

for some constant A, then T_m is an L^p-Fourier multiplier.

This result is known as Hörmander-Mikhlin multiplier theorem. Recently, similar results have been investigated for different Fourier type transforms. Indeed, Gosselin, Stempak [11] and Kapelko [19] investigated the Hörmander-Mikhlin multiplier theorem for the Hankel transform; whereas Bloom, Xu [7] showed an analogue version for Chébli-Trimèche hypergroup. In the same context, Fischer, Ruzhansky and Wirth studied the Fourier multipliers on compact and nilpotent Lie groups (see [10, 23, 24]).

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In this paper, we deal with the Fourier transform associated with the Riemann-Liouville operator investigated in [4] and we prove for it the Hörmander-Mikhlin multiplier theorem.

The Riemann-Liouville operator is defined for a continuous function f on $\mathbb{R}^2$, even with respect to the first variable as follow

$$\mathscr{R}_\alpha(f)(r,x)$$
$$= \begin{cases} \dfrac{\alpha}{\pi}\displaystyle\int_{-1}^{1}\int_{-1}^{1} f\left(rs\sqrt{1-t^2}, x+rt\right)\left(1-t^2\right)^{\alpha-\frac{1}{2}}\left(1-s^2\right)^{\alpha-1} dtds; & \text{if } \alpha>0, \\ \dfrac{1}{\pi}\displaystyle\int_{-1}^{1} f\left(r\sqrt{1-t^2}, x+rt\right)\dfrac{dt}{\sqrt{1-t^2}}; & \text{if } \alpha=0. \end{cases}$$

The particularity of this operator is that it generalizes the well known mean operator defined by

$$\mathscr{R}_0(f)(r,x) = \frac{1}{2\pi}\int_0^{2\pi} f(r\sin\theta, x+r\cos\theta)\,d\theta,$$

which means that $\mathscr{R}_0(f)(r,x)$ is the mean value of f on the circle centered at $(0,x)$ and radius r. This operator plays an important role and have many applications, for example, in image processing of so-called synthetic aperture radar (SAR) data [1, 16, 17], or in the linearized inverse scattering problem in acoustics [9].

The Fourier transform associated with $\mathscr{R}_\alpha$, is defined on Υ by

$$\mathscr{F}_\alpha(f)(\mu,\lambda) = \int_0^{+\infty}\int_{\mathbb{R}} f(r,x)\,j_\alpha\left(r\sqrt{\mu^2+\lambda^2}\right)\exp(-i\lambda x)\,d\nu_\alpha(r,x),$$

where j_α is the modified Bessel function of first kind and index α, ν_α is the measure defined on $[0,+\infty[\times\mathbb{R}$ by

$$d\nu_\alpha(r,x) = \frac{r^{2\alpha+1}}{\sqrt{2\pi}2^\alpha\Gamma(\alpha+1)}drdx, \tag{1.1}$$

and Υ is the set defined by

$$\Upsilon = \mathbb{R}^2 \cup \left\{(i\mu,\lambda);\ (\mu,\lambda)\in\mathbb{R}^2,\ |\mu|\leqslant|\lambda|\right\}. \tag{1.2}$$

Many harmonic analysis results related to the Riemann-Liouville operator have been established see for example [3, 5, 6, 13, 14, 15, 22] and the references therein.

The main result of this paper is the Hörmander-Mikhlin multiplier theorem for $\mathscr{F}_\alpha$, that is if m is a function on the set Υ such that $m\circ\theta^{-1}$ is a measurable, bounded function on $[0,+\infty[\times\mathbb{R}$ satisfying the following condition

$$\sup_{R>0}\sum_{(k,l)\in\mathscr{H}_{2p,2q}} R^{k+l+\frac{1}{2}}\left(\int_{B_R^c\cap B_{2R}}\left|\frac{\partial^{k+l}}{\partial r^k\partial x^l}\left(m\circ\theta^{-1}\right)(r,x)\right|^2 r^{-3}drdx\right)^{\frac{1}{2}} < +\infty,$$

where θ is the bijective function defined on Υ^+ by

$$\theta(\mu,\lambda) = (\sqrt{\mu^2+\lambda^2},\lambda), \tag{1.3}$$

and Υ^+ is the subspace of Υ given by

$$\Upsilon^+ = [0,+\infty[\times\mathbb{R}\cup\{(i\mu,\lambda);(\mu,\lambda)\in\mathbb{R}^2;0\leqslant\mu\leqslant|\lambda|\}.$$

Then the multiplier operator T_m defined by

$$\mathscr{F}_\alpha(T_m f) = m\mathscr{F}_\alpha(f)$$

is of a weak type $(1,1)$ (see [12]), which will allows us to deduce that T_m is of strong type (p,p) for every $1<p<+\infty$. Here B_R^c denotes the complementary of

$$B_R = \left\{(r,x)\in\mathbb{R}^2 \mid r^2+x^2\leqslant R^2\right\},$$

and

$$\mathscr{H}_{2p,2q} = \{(k,l)\in\mathbb{N}\times\mathbb{N} \mid k\leqslant 2p,\ l\leqslant 2q\},$$

with $p=\left[\frac{\alpha+1}{2}\right]+1$ and $q=\frac{1}{2}\left[\alpha+\frac{1}{2}\right]+\frac{1}{2}$.

This paper is organized as follows. In the second section, we recall some harmonic analysis results related to the Riemann-Liouville operator $\mathscr{R}_\alpha$ and its associated Fourier transform $\mathscr{F}_\alpha$. In the third section, we establish a Bernstein type inequality for the generalized translation associated with the Riemann-Liouville operator. The last section is devoted to the main result of this paper that is the Hörmander-Mikhlin multiplier theorem for the Riemann-Liouville operator.

2. Harmonic analysis results related to the Riemann-Liouville operator

In this section, we recall some harmonic analysis results related to the Fourier transform associated with the Riemann-Liouville operator. For this, we denote by

• $\mathscr{C}^k_{e,b}\left(\mathbb{R}^2\right)$ the space of bounded functions of classe C^k on $\mathbb{R}^2$, even with respect to the first variable.

• $S_e\left(\mathbb{R}^2\right)$ the space of smooth functions on $\mathbb{R}^2$, even with respect to the first variable, rapidly decreasing together with all their derivatives.

• $\mathscr{D}_e\left(\mathbb{R}^2\right)$ the space of smooth functions on $\mathbb{R}^2$ with compact support, even with respect to the first variable.

Let $\Delta_1=\dfrac{\partial}{\partial x}$ and Δ_2 be the singular partial differential operator defined by

$$\Delta_2 = \frac{\partial^2}{\partial r^2}+\frac{2\alpha+1}{r}\frac{\partial}{\partial r}-\frac{\partial^2}{\partial x^2} \qquad (r,x)\in]0,+\infty[\times\mathbb{R}, \qquad \alpha\geqslant 0.$$

In [4], the authors showed that for all $(\mu,\lambda)\in\mathbb{C}^2$, the following system

$$\begin{cases}\Delta_1 u(r,x) = -i\lambda u(r,x)\\ \Delta_2 u(r,x) = -\mu^2 u(r,x)\\ u(0,0)=1, \quad \dfrac{\partial u}{\partial r}(0,x)=0; \qquad \forall x\in\mathbb{R}\end{cases}$$

admits a unique solution given by

$$\varphi_{\mu,\lambda}(r,x) = j_\alpha\left(r\sqrt{\mu^2+\lambda^2}\right)\exp(-i\lambda x),$$

where j_α is the modified Bessel function of the first kind and index α, (see [2, 20]).

It is known that the function $\varphi_{\mu,\lambda}$ is bounded on $[0,+\infty[\times\mathbb{R}$ if, and only if (μ,λ) belongs to the set Υ defined by relation (1.2). In this case

$$\sup_{(r,x)\in\mathbb{R}^2}\left|\varphi_{\mu,\lambda}(r,x)\right| = 1.$$

For all $(r,x)\in[0,+\infty[\times\mathbb{R}$, the translation operator $\tau_{(r,x)}$ associated with the Riemann-Liouville transform is defined on $L^p(d\nu_\alpha)$, $p\in[1,+\infty]$ (the Lebesgue space on $[0,+\infty[\times\mathbb{R}$ with respect to the measure ν_α given by the formula (1.1) equipped with the L^p-norm denoted by $\|.\|_{p,\nu_\alpha}$) and for all $(s,y)\in[0,+\infty[\times\mathbb{R}$, by

$$\mathscr{T}_{(r,x)}(f)(s,y) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma\left(\alpha+\frac{1}{2}\right)}\int_0^\pi f\left(\sqrt{r^2+s^2+2rs\cos\theta}, x+y\right)\sin^{2\alpha}\theta d\theta. \quad (2.1)$$

For all (r,x) and $(s,y)\in]0,+\infty[\times\mathbb{R}$, and by a standard change of variables, we have

$$\mathscr{T}_{(r,x)}(f)(s,y) = \frac{1}{2^\alpha\Gamma(\alpha+1)}\int_0^{+\infty} f(t,x+y)\mathscr{W}_\alpha(r,s,t)t^{2\alpha+1}dt, \quad (2.2)$$

where the kernel $\mathscr{W}_\alpha$ is given by

$$\mathscr{W}_\alpha(r,s,t) = \frac{\Gamma(\alpha+1)^2}{2^{\alpha-1}\Gamma(\alpha+\frac{1}{2})\sqrt{\pi}}\frac{\left((r+s)^2-t^2\right)^{\alpha-\frac{1}{2}}\left(t^2-(r-s)^2\right)^{\alpha-\frac{1}{2}}}{(rst)^{2\alpha}}\chi_{]|r-s|,r+s[}(t),$$

and $\chi_{]|r-s|,r+s[}$ is the characteristic function of the interval $]|r-s|,r+s[$.

The kernel $\mathscr{W}_\alpha$ satisfies the following properties

- For all $r,s,t>0$, we have

$$\mathscr{W}_\alpha(r,s,t) = \mathscr{W}_\alpha(s,r,t) = \mathscr{W}_\alpha(t,s,r) = \mathscr{W}_\alpha(r,t,s).$$

- For all $r,s>0$, we have

$$\frac{1}{2^\alpha\Gamma(\alpha+1)}\int_0^{+\infty}\mathscr{W}_\alpha(r,s,t)t^{2\alpha+1}dt = 1. \quad (2.3)$$

For every $f \in L^p(d\nu_\alpha)$, $1 \leqslant p \leqslant +\infty$ and $(r,x) \in [0,+\infty[\times\mathbb{R}$, the function $\mathscr{T}_{(r,x)}(f)$ belongs to $L^p(d\nu_\alpha)$ and we have

$$\left\| \mathscr{T}_{(r,x)}(f) \right\|_{p,\nu_\alpha} \leqslant \|f\|_{p,\nu_\alpha}. \tag{2.4}$$

The convolution product of measurable functions f and g is defined, for all $(r,x) \in [0,+\infty[\times\mathbb{R}$, by

$$f * g(r,x) = \int_0^{+\infty} \int_{\mathbb{R}} \mathscr{T}_{(r,-x)}(\check{f})(s,y) g(s,y) d\nu_\alpha(s,y),$$

with $\check{f}(s,y) = f(s,-y)$, whenever the integral of the right hand side is well defined.

Then for p, q and $r \in [1,+\infty]$ such that $1/p + 1/q = 1 + 1/r$, and for all $f \in L^p(d\nu_\alpha)$, $g \in L^q(d\nu_\alpha)$, the function $f * g$ belongs to $L^r(d\nu_\alpha)$ and we have the following Young's inequality

$$\|f * g\|_{r,\nu_\alpha} \leqslant \|f\|_{p,\nu_\alpha} \|g\|_{q,\nu_\alpha}. \tag{2.5}$$

Let $\mathscr{B}_{\Upsilon^+}$ be the σ-algebra defined on Υ^+ by $\mathscr{B}_{\Upsilon^+} = \theta^{-1}(\mathscr{B}_{[0,+\infty[\times\mathbb{R}})$, where γ_α the measure defined by

$$\gamma_\alpha(A) = \nu_\alpha(\theta(A)), \;\; A \in \mathscr{B}_{\Upsilon^+}.$$

If f is a measurable function on $[0,+\infty[\times\mathbb{R}$, then the function $f \circ \theta$ is measurable on Υ^+. Furthermore, if f is a non negative or an integrable function on $[0,+\infty[\times\mathbb{R}$ with respect to the measure ν_α, we have

$$\int\int_{\Upsilon^+} (f \circ \theta)(\mu,\lambda) d\gamma_\alpha(\mu,\lambda) = \int_0^{+\infty} \int_{\mathbb{R}} f(r,x) d\nu_\alpha(r,x).$$

Moreover, the function f belongs to $L^p(d\nu_\alpha)$ if, and only if $f \circ \theta$ belongs to $L^p(d\gamma_\alpha)$ (the Lebesgue space on Υ^+ with respect to the measure γ_α equipped with the L^p- norm denoted by $\|.\|_{p,\gamma_\alpha}$) and we have

$$\|f \circ \theta\|_{p,\gamma_\alpha} = \|f\|_{p,\nu_\alpha}. \tag{2.6}$$

According to these notations, we have

- For $(\mu,\lambda) \in \Upsilon$, we have

$$\mathscr{F}_\alpha(f)(\mu,\lambda) = \widetilde{\mathscr{F}_\alpha}(f) \circ \theta(\mu,\lambda), \tag{2.7}$$

  where $\widetilde{\mathscr{F}_\alpha}$ is the so-called Fourier-Bessel transform defined on $L^1(d\nu_\alpha)$ by

$$\forall(\mu,\lambda) \in \mathbb{R}^2, \widetilde{\mathscr{F}_\alpha}(f)(\mu,\lambda) = \int_0^{+\infty} \int_{\mathbb{R}} f(r,x) j_\alpha(r\mu) e^{-i\lambda x} d\nu_\alpha(r,x).$$

- (Inversion formula) For every function f in $L^1(d\nu_\alpha)$ such that the function $\mathscr{F}_\alpha(f)$ belongs to $L^1(d\gamma_\alpha)$, we have

$$f(r,x) = \int\int_{\Upsilon^+} \mathscr{F}_\alpha(f)(\mu,\lambda) \overline{\varphi_{\mu,\lambda}(r,x)} d\gamma_\alpha(\mu,\lambda); \quad a.e. \tag{2.8}$$

- (Plancherel's theorem) Since the mapping $\widetilde{\mathscr{F}_\alpha}$ is an isometric isomorphism from $L^2(d\nu_\alpha)$ onto itself, then the relations (2.6) and (2.7) show that the Fourier transform $\mathscr{F}_\alpha$ is an isometric isomorphism from $L^2(d\nu_\alpha)$ into $L^2(d\gamma_\alpha)$. Namely, for every $f \in L^2(d\nu_\alpha)$, the function $\mathscr{F}_\alpha(f)$ belongs to the space $L^2(d\gamma_\alpha)$ and we have

$$\|\mathscr{F}_\alpha(f)\|_{2,\gamma_\alpha} = \|f\|_{2,\nu_\alpha}.$$

In addition, we state the following results (and notations) that will be used in subsequent sections.

PROPOSITION 2.1.

1. *For every $f \in L^1(d\nu_\alpha)$ and for all $(r,x),(\mu,\lambda) \in [0,+\infty[\times\mathbb{R}$,*

$$\widetilde{\mathscr{F}_\alpha}(\mathscr{T}_{(r,x)}f)(\mu,\lambda) = j_\alpha(r\mu)e^{-ix\lambda}\widetilde{\mathscr{F}_\alpha}(f)(\mu,\lambda). \tag{2.9}$$

2. *For f in $L^1(d\nu_\alpha)$ and g in $L^2(d\nu_\alpha)$, we have*

$$\widetilde{\mathscr{F}_\alpha}(f*g) = \widetilde{\mathscr{F}_\alpha}(f).\widetilde{\mathscr{F}_\alpha}(g).$$

3. *Let f and g be in $L^2(d\nu_\alpha)$. The function $f*g$ belongs to $L^2(d\nu_\alpha)$ if, and only if $\widetilde{\mathscr{F}_\alpha}(f).\widetilde{\mathscr{F}_\alpha}(g)$ belongs to $L^2(d\nu_\alpha)$ and we have*

$$\widetilde{\mathscr{F}_\alpha}(f*g) = \widetilde{\mathscr{F}_\alpha}(f).\widetilde{\mathscr{F}_\alpha}(g).$$

For every positive real number ε, we denote by $\delta_\varepsilon f$ the dilate of f defined by $\delta_\varepsilon f(r,x) = f(\varepsilon r, \varepsilon x)$, then for every $f \in L^p(d\nu_\alpha)$, $1 \leqslant p < +\infty$,

$$\|\delta_\varepsilon f\|_{p,\nu_\alpha} = \frac{1}{\varepsilon^{\frac{2\alpha+3}{p}}}\|f\|_{p,\nu_\alpha}.$$

In particular for $p=1$, $\delta_\varepsilon f$ belongs to $L^1(d\nu_\alpha)$ and we have

$$\widetilde{\mathscr{F}_\alpha}(\delta_\varepsilon f) = \frac{1}{\varepsilon^{2\alpha+3}}\delta_{\frac{1}{\varepsilon}}\widetilde{\mathscr{F}_\alpha}(f). \tag{2.10}$$

We denote by ℓ_α the Bessel operator defined on $]0,+\infty[$ by

$$\ell_\alpha = \frac{\partial^2}{\partial r^2} + \frac{2\alpha+1}{r}\frac{\partial}{\partial r}.$$

Let $f \in \mathscr{C}_e^\infty(\mathbb{R}^2) \cap L^1(d\nu_\alpha)$ and assume that $\ell_\alpha^{\beta_0} f, \dfrac{\partial^\beta f}{\partial x^\beta} \in L^1(d\nu_\alpha)$, then it's known that for all $(\mu,\lambda) \in [0,+\infty[\times\mathbb{R}$, we have

$$\widetilde{\mathscr{F}_\alpha}\left(\frac{\partial^\beta f}{\partial \lambda^\beta}\right)(\mu,\lambda) = (-i)^\beta \lambda^\beta \widetilde{\mathscr{F}_\alpha}(f)(\mu,\lambda), \tag{2.11}$$

and

$$\widetilde{\mathscr{F}_\alpha}\left((Id - \ell_\alpha f)^{\beta_0}\right)(\mu,\lambda) = (1+\mu^2)^{\beta_0}\widetilde{\mathscr{F}_\alpha}(f)(\mu,\lambda). \tag{2.12}$$

3. Bernstein type inequality

In order to prove the Hörmander-Mikhlin multiplier theorem for the Fourier transform $\mathscr{F}_\alpha$, we will devote this section to establish a Bernstein type inequality for the generalized translation associated with the Riemann-Liouville operator which will play a central role in the final proof of the main theorem. In the sequel, we put $p=\left[\frac{\alpha+1}{2}\right]+1$ and $q=\frac{1}{2}\left[\alpha+\frac{1}{2}\right]+\frac{1}{2}$ and we denote by

$$\mathscr{H}_{a,b}=\{(k,l)\in\mathbb{N}\times\mathbb{N}\mid k\leqslant a,\ l\leqslant b\},\quad a,b\in\mathbb{R}_+.$$

Through this paper, C will denote a nonnegative constant, which is not necessarily the same at each occurrence.

LEMMA 3.1. *Let* $m\in\mathscr{C}_{e,b}^{2p+2q}(\mathbb{R}^2)$. *If*

$$\sup_{R>0}\sum_{(k,\ell)\in\mathscr{H}_{2p,2q}}R^{k+\ell+\frac{1}{2}}\left(\int\int_{B_R^c\cap B_{2R}}\left|\frac{\partial^{k+\ell}m}{\partial r^k\partial x^\ell}(r,x)\right|^2 r^{-3}drdx\right)^{\frac{1}{2}}<+\infty,$$

then for every function $\varphi\in\mathscr{D}_e\left(\mathbb{R}^2\right)$ *with* $supp(\varphi)\subset B_1^c\cap B_2$, *we have*

$$\sup_{j\in\mathbb{Z}}\sum_{(k,\ell)\in\mathscr{H}_{p,2q}}2^{j(2k+\ell-\alpha-\frac{3}{2})}\left(\int_{B_{2^j}^c\cap B_{2^{j+1}}}\left|\ell_\alpha^k\left(\frac{\partial^\ell m_j}{\partial x^\ell}\right)(r,x)\right|^2 d\nu_\alpha(r,x)\right)^{\frac{1}{2}}<+\infty$$

where $m_j=m\delta_{2^{-j}}\varphi$.

Proof. Knowing that for every $\psi\in S_e(\mathbb{R})$, and $d\in\mathbb{N}^*$, we have

$$\forall r\in]0,+\infty[,\ \ell_\alpha^d(\psi)(r)=\sum_{i=1}^{2d}\alpha_i\left(\frac{d^i\psi}{dr^i}\right)(r)\ r^{i-2d},$$

for some constants $\alpha_i\in\mathbb{C}$, then the result follows immediately by using Minkowski's inequality. □

PROPOSITION 3.2. *Let* $m\in\mathscr{C}_{e,b}^{2p+2q}(\mathbb{R}^2)$ *and* $\varphi\in\mathscr{D}_e\left(\mathbb{R}^2\right)$ *with* $supp(\varphi)\subset B_1^c\cap B_2$. *If*

$$\sup_{R>0}\sum_{(k,\ell)\in\mathscr{H}_{2p,2q}}R^{k+\ell+\frac{1}{2}}\left(\int\int_{B_R^c\cap B_{2R}}\left|\frac{\partial^{k+\ell}m}{\partial r^k\partial x^\ell}(r,x)\right|^2 r^{-3}drdx\right)^{\frac{1}{2}}<+\infty,$$

then there is a nonnegative constant C *such that for every* $j\in\mathbb{Z}$, *we have*

$$\left\|(1+2^{2j}\mu^2)^p(1+2^{2j}\lambda^2)^q\widetilde{\mathscr{F}_\alpha}(m_j)\right\|_{2,\nu_\alpha}\leqslant C2^{j(\alpha+\frac{3}{2})},$$

where $m_j=m\delta_{2^{-j}}\varphi$.

Proof. We have,

$$\begin{aligned}&\left\|(1+2^{2j}\mu^2)^p(1+2^{2j}\lambda^2)^q\widetilde{\mathscr{F}}(m_j)\right\|^2_{2,\nu_\alpha}\\&\leqslant\sum_{k=0}^{2q}C^k_{2q}2^{2kj}\int_0^{+\infty}\int_{[0,+\infty[\times\mathbb{R}}\left|\lambda^k(1+(2^j\mu)^2)^p\widetilde{\mathscr{F}}(m_j)(\mu,\lambda)\right|^2d\nu_\alpha(\mu,\lambda).\end{aligned}$$

Using relations (2.11), (2.12) with Plancherel's theorem and Minkowski's inequality, we deduce that

$$\begin{aligned}&\left\|(1+2^{2j}\mu^2)^p(1+2^{2j}\lambda^2)^q\widetilde{\mathscr{F}}(m_j)\right\|_{2,\nu_\alpha}\\&\leqslant\sum_{k=0}^{2q}\sqrt{C^k_{2q}}2^{kj}\left(\int_0^{+\infty}\int_{\mathbb{R}}\left|\widetilde{\mathscr{F}}\left((Id-2^{2j}\ell_\alpha)^p\left(\frac{\partial^k m_j}{\partial\lambda^k}\right)\right)(\mu,\lambda)\right|^2d\nu_\alpha(\mu,\lambda)\right)^{\frac12}\\&\leqslant\sum_{k=0}^{2q}\sqrt{C^k_{2q}}\sum_{l=0}^{p}\sqrt{C^l_p}\,2^{j(k+2l)}\left(\int_0^{+\infty}\int_{\mathbb{R}}\left|\ell^l_\alpha\left(\frac{\partial^k m_j}{\partial\lambda^k}(\mu,\lambda)\right)\right|^2d\nu_\alpha(\mu,\lambda)\right)^{\frac12}.\end{aligned}$$

Hence, by Lemma 3.1

$$\left\|(1+2^{2j}\mu^2)^p(1+2^{2j}\lambda^2)^q\widetilde{\mathscr{F}}(m_j)\right\|^2_{2,\nu_\alpha}\leqslant C2^{j(2\alpha+3)}.\quad\square$$

LEMMA 3.3. *Let $h\in S_e(\mathbb{R}^2)$. Then for every $(s,y),(t,z)\in[0,+\infty[\times\mathbb{R}$, we have*

$$\|\mathscr{T}_{(s,y)}h-\mathscr{T}_{(t,z)}h\|_{1,\nu_\alpha}\leqslant\sup_{s\in\{r,x\}}\left\|\frac{\partial h}{\partial s}\right\|_{1,\nu_\alpha}\rho((s,y),(t,z)),$$

where $\rho((s,y),(t,z))=|s-t|+|y-z|$.

Proof. Following the idea of Stempak [11], let $(r,x),(s,y),(t,z)\in[0,+\infty[\times\mathbb{R}$ and $\psi:[0,1]\longrightarrow[0,+\infty[\times\mathbb{R}$, be the function defined by

$$\psi(\gamma)=(\psi_0(\gamma),\psi_1(\gamma)),$$

where $\psi_0(\gamma)=u_\theta(r,s+\gamma(t-s))$, $\psi_1(\gamma)=x+y+\gamma(z-y)$ and u_θ is the function given by

$$u_\theta(r,s)=\sqrt{r^2+s^2+2rs\cos\theta}.\tag{3.1}$$

Let $h\in S_e(\mathbb{R}^2)$, then we have

$$\begin{aligned}&\left|\mathscr{T}_{(s,y)}h(r,x)-\mathscr{T}_{(t,z)}h(r,x)\right|\\&\qquad\leqslant\frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma\left(\alpha+\frac12\right)}\int_0^\pi|h(u_\theta(r,s),x+y)-h(u_\theta(r,t),x+z)|\sin^{2\alpha}\theta\,d\theta\\&\qquad=\frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma\left(\alpha+\frac12\right)}\int_0^\pi\left|\int_0^1(h\circ\psi)'(\gamma)d\gamma\right|\sin^{2\alpha}\theta\,d\theta\end{aligned}$$

$$\leqslant |s-t|\int_0^1\left(\frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma\left(\alpha+\frac{1}{2}\right)}\int_0^\pi\left|\frac{\partial h}{\partial r}(\psi(\gamma))\right|\sin^{2\alpha}\theta\,d\theta\right)d\gamma$$
$$+|y-z|\int_0^1\left(\frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma\left(\alpha+\frac{1}{2}\right)}\int_0^\pi\left|\frac{\partial h}{\partial x}(\psi(\gamma))\right|\sin^{2\alpha}\theta\,d\theta\right)d\gamma$$

and by relations (3.1) and (2.1), we obtain

$$\left|\mathscr{T}_{(s,y)}h(r,x)-\mathscr{T}_{(t,z)}h(r,x)\right|\leqslant|s-t|\int_0^1\mathscr{T}_{(s+\gamma(t-s),y+\gamma(z-y))}\left|\frac{\partial h}{\partial r}(r,x)\right|d\gamma$$
$$+|y-z|\int_0^1\mathscr{T}_{(s+\gamma(t-s),y+\gamma(z-y))}\left|\frac{\partial h}{\partial x}(r,x)\right|d\gamma. \tag{3.2}$$

Now according to relations (2.4), (3.2) and by applying Fubini's theorem, we get

$$\begin{aligned}
&\|\mathscr{T}_{(s,y)}h-\mathscr{T}_{(t,z)}h\|_{1,\nu_\alpha}\\
&\quad\leqslant|s-t|\int_0^1\left(\int_0^{+\infty}\int_{\mathbb{R}}\mathscr{T}_{(s+\gamma(t-s),y+\gamma(z-y))}\left|\frac{\partial h}{\partial r}(r,x)\right|d\nu_\alpha(r,x)\right)d\gamma\\
&\quad\quad+|y-z|\int_0^1\left(\int_0^{+\infty}\int_{\mathbb{R}}\mathscr{T}_{(s+\gamma(t-s),y+\gamma(z-y))}\left|\frac{\partial h}{\partial x}(r,x)\right|d\nu_\alpha(r,x)\right)d\gamma\\
&\quad\leqslant|s-t|\left\|\frac{\partial h}{\partial r}\right\|_{1,\nu_\alpha}+|y-z|\left\|\frac{\partial h}{\partial x}\right\|_{1,\nu_\alpha}\\
&\quad\leqslant\sup_{s\in\{r,x\}}\left\|\frac{\partial h}{\partial s}\right\|_{1,\nu_\alpha}\rho((s,y),(t,z)).\quad\square
\end{aligned}$$

THEOREM 3.4. (Bernstein type inequality) *There exists a nonnegative constant C such that for every positive real number ε, and for every function $f\in L^1(d\nu_\alpha)$ with supp $\left(\widetilde{\mathscr{F}}_\alpha(f)\right)\subset B_\varepsilon$, we have for all $(s,y),(t,z)\in[0,+\infty[\times\mathbb{R}$*

$$\|\mathscr{T}_{(s,y)}f-\mathscr{T}_{(t,z)}f\|_{1,\nu_\alpha}\leqslant C\varepsilon\|f\|_{1,\nu_\alpha}\rho((s,y),(t,z)).$$

Proof. Let $f\in L^1(d\nu_\alpha)$ and $h\in S_e\left(\mathbb{R}^2\right)$ satisfying $h_{|B_1}=1$, then for every $\varepsilon>0$ the dilate $\delta_{\frac{1}{\varepsilon}}h$ belongs to $S_e\left(\mathbb{R}^2\right)$ and satisfies $\delta_{\frac{1}{\varepsilon}}h\big|_{B_\varepsilon}=1$. Let $(s,y)\in[0,+\infty[\times\mathbb{R}$ then according to the hypothesis and relation (2.9), we deduce that $Supp\left(\widetilde{\mathscr{F}}_\alpha\left(\mathscr{T}_{(s,y)}f\right)\right)\subset B_\varepsilon$, and therefore for every $(\mu,\lambda)\in[0,+\infty[\times\mathbb{R}$, we have

$$\begin{aligned}
\widetilde{\mathscr{F}}_\alpha\left(\mathscr{T}_{(s,y)}f\right)(\mu,\lambda)&=\widetilde{\mathscr{F}}_\alpha\left(\mathscr{T}_{(s,y)}f\right)(\mu,\lambda)\delta_{\frac{1}{\varepsilon}}h(\mu,\lambda)\\
&=j_\alpha(s\mu)e^{i\langle\lambda|y\rangle}\widetilde{\mathscr{F}}_\alpha(f)(\mu,\lambda)\widetilde{\mathscr{F}}_\alpha\left(\widetilde{\mathscr{F}}_\alpha^{-1}(\delta_{\frac{1}{\varepsilon}}h)\right)(\mu,\lambda)\\
&=\widetilde{\mathscr{F}}_\alpha\left(f*\mathscr{T}_{(s,y)}\widetilde{\mathscr{F}}_\alpha^{-1}(\delta_{\frac{1}{\varepsilon}}h)\right)(\mu,\lambda)
\end{aligned}$$

and by inversion formula (2.8), we get

$$\mathscr{T}_{(s,y)} f = f * \mathscr{T}_{(s,y)} \widetilde{\mathscr{F}_\alpha}^{-1}(\delta_{\frac{1}{\varepsilon}} h). \tag{3.3}$$

Using relations (2.10) and (3.3), we get

$$\begin{aligned} \mathscr{T}_{(s,y)} f - \mathscr{T}_{(t,z)} f &= \varepsilon^{2\alpha+3} f * \left(\mathscr{T}_{(s,y)} \delta_\varepsilon \widetilde{\mathscr{F}_\alpha}^{-1}(h) - \mathscr{T}_{(t,z)} \delta_\varepsilon \widetilde{\mathscr{F}_\alpha}^{-1}(h) \right) \\ &= \varepsilon^{2\alpha+3} f * \delta_\varepsilon \left(\mathscr{T}_{(\varepsilon s,\varepsilon y)} \widetilde{\mathscr{F}_\alpha}^{-1}(h) - \mathscr{T}_{(\varepsilon t,\varepsilon z)} \widetilde{\mathscr{F}_\alpha}^{-1}(h) \right) \end{aligned} \tag{3.4}$$

then using relations (2.5), (3.4), Lemma 3.3 and the fact that $\widetilde{\mathscr{F}_\alpha}$ is an isomorphism from $S_e\left(\mathbb{R}^2\right)$ onto itself, we get

$$\begin{aligned} \|\mathscr{T}_{(s,y)} f - \mathscr{T}_{(t,z)} f\|_{1,\nu_\alpha} &\leqslant \varepsilon^{2\alpha+3} \|f\|_{1,\nu_\alpha} \|\delta_\varepsilon \left(\mathscr{T}_{(\varepsilon s,\varepsilon y)} \widetilde{\mathscr{F}_\alpha}^{-1}(h) - \mathscr{T}_{(\varepsilon t,\varepsilon z)} \widetilde{\mathscr{F}_\alpha}^{-1}(h) \right) \|_{1,\nu_\alpha} \\ &= \|f\|_{1,\nu_\alpha} \|\mathscr{T}_{(\varepsilon s,\varepsilon y)} \widetilde{\mathscr{F}_\alpha}^{-1}(h) - \mathscr{T}_{(\varepsilon t,\varepsilon z)} \widetilde{\mathscr{F}_\alpha}^{-1}(h)\|_{1,\nu_\alpha} \\ &\leqslant \varepsilon \sup_{s\in\{r,x\}} \|\frac{\partial \widetilde{\mathscr{F}_\alpha}^{-1}(h)}{\partial s}\|_{1,\nu_\alpha} \|f\|_{1,\nu_\alpha} \rho((s,y),(t,z)) \quad \square \end{aligned}$$

PROPOSITION 3.5. *Let $m \in \mathscr{C}_{e,b}^{2p+2q}(\mathbb{R}^2)$ and $\varphi \in \mathscr{D}_e\left(\mathbb{R}^2\right)$ with $supp(\varphi) \subset B_1^c \cap B_2$. If*

$$\sup_{R>0} \sum_{(k,\ell)\in\mathscr{H}_{2p,2q}} R^{k+\ell+\frac{1}{2}} \left(\int\int_{B_R^c \cap B_{2R}} \left| \frac{\partial^{k+\ell} m}{\partial r^k \partial x^\ell}(r,x) \right|^2 r^{-3} dr dx \right)^{\frac{1}{2}} < +\infty,$$

then there is a nonnegative constant C such that for every $(s,y),(t,z) \in [0,+\infty[\times\mathbb{R}$ and $j \in \mathbb{Z}$, we have

$$\|\mathscr{T}_{(s,-y)} \widetilde{\mathscr{F}_\alpha}(m_j) - \mathscr{T}_{(t,-z)} \widetilde{\mathscr{F}_\alpha}(m_j)\|_{1,\nu_\alpha} \leqslant C 2^{j+1} \rho((s,y),(t,z)), \tag{3.5}$$

where $m_j = m\delta_{2^{-j}}\varphi$.

Proof. Using Proposition 3.2 and Hölder's inequality, we get

$$\begin{aligned} &\|\widetilde{\mathscr{F}_\alpha}(m_j)\|_{1,\nu_\alpha} \\ &\quad \leqslant \|(1+2^{2j}\mu^2)^p (1+2^{2j}\lambda^2)^q \widetilde{\mathscr{F}_\alpha}(m_j)\|_{2,\nu_\alpha} \|(1+2^{2j}\mu^2)^{-p}(1+2^{2j}\lambda^2)^{-q}\|_{2,\nu_\alpha} \\ &\quad \leqslant C 2^{j(\alpha+\frac{3}{2})} \left(\int_0^{+\infty} \frac{\mu^{2\alpha+1} d\mu}{(1+2^{2j}\mu^2)^{2p}} \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \frac{d\lambda}{(1+2^{2j}\lambda^2)^{2q}} \right)^{\frac{1}{2}} \\ &\quad = C \left(\int_0^{+\infty} \frac{s^{2\alpha+1} ds}{(1+s^2)^{2p}} \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \frac{dy}{(1+y^2)^{2q}} \right)^{\frac{1}{2}}. \end{aligned}$$

Then we deduce that there is a nonnegative constant C independent of j such that

$$\|\widetilde{\mathscr{F}_\alpha}(m_j)\|_{1,\nu_\alpha} \leqslant C.$$

In particular $\widetilde{\mathscr{F}_\alpha}(m_j) \in L^1(d\nu_\alpha)$ and knowing that $Supp(m_j) \subset B_{2^{j+1}}$, then by Theorem 3.4, we obtain

$$\|\mathscr{T}_{(s,-y)}\widetilde{\mathscr{F}_\alpha}(m_j) - \mathscr{T}_{(t,-z)}\widetilde{\mathscr{F}_\alpha}(m_j)\|_{1,\nu_\alpha} \leqslant C2^{j+1}\rho((s,y),(t,z)). \quad \square$$

4. Hörmander-Mikhlin multiplier theorem for the Fourier transform $\mathscr{F}_\alpha$

Using the Bernstein inequality showed above, we are now able to prove the main result of this work. For the sake of simplicity, we will express firstly the main theorem in terms of the Fourier transform $\widetilde{\mathscr{F}_\alpha}$. In the following, we introduce the sets defined for every $(s,y),(t,z) \in [0,+\infty[\times\mathbb{R}$, by

$$E_{(s,y),(t,z)} = [0,+\infty[\times\{x \in \mathbb{R} \mid \ |x-y| \geqslant |s-t| + 2|y-z|\},$$

and

$$F_{(s,y),(t,z)} = [0,+\infty[\times\{x \in \mathbb{R} \mid \ |x| \geqslant |s-t| + |y-z|\}.$$

We denote by $\chi_{E_{(s,y),(t,z)}}$ and $\chi_{F_{(s,y),(t,z)}}$ respectively their associated characteristic functions. Notice that

$$\forall (r,x) \in [0,+\infty[\times\mathbb{R},\ \chi_{E_{(s,y),(t,z)}}(r,y-x) \leqslant \chi_{F_{(s,y),(t,z)}}(r,x). \tag{4.1}$$

PROPOSITION 4.1. *Let $m \in \mathscr{C}^{2p+2q}_{e,b}(\mathbb{R}^2)$ and $\varphi \in \mathscr{D}_e\left(\mathbb{R}^2\right)$ with $supp(\varphi) \subset B_1^c \cap B_2$. If*

$$\sup_{R>0} \sum_{(k,\ell)\in\mathscr{H}_{2p,2q}} R^{k+\ell+\frac{1}{2}} \left(\int\int_{B_R^c\cap B_{2R}} \left| \frac{\partial^{k+\ell} m}{\partial r^k \partial x^\ell}(r,x) \right|^2 r^{-3} dr dx \right)^{\frac{1}{2}} < +\infty,$$

then there exists a nonnegative constant C such that, for every $j \in \mathbb{Z}$ and for every $(s,y),(t,z) \in [0,+\infty[\times\mathbb{R}$, we have

$$\|\mathscr{T}_{(s,-y)}\widetilde{\mathscr{F}_\alpha}(m_j)\chi_{E_{(s,y),(t,z)}} - \mathscr{T}_{(t,-z)}\widetilde{\mathscr{F}_\alpha}(m_j)\chi_{E_{(s,y),(t,z)}}\|_{1,\nu_\alpha} \leqslant C\left(2^j\rho((s,y),(t,z))\right)^{1-2q}, \tag{4.2}$$

where $m_j = m\delta_{2^{-j}}\varphi$.

Proof. Assume that $(s,y) \neq (t,z)$, then by relations (2.2), (2.3) and (4.1), we get

$$\begin{aligned}
&\|\mathscr{T}_{(s,-y)}\widetilde{\mathscr{F}_\alpha}(m_j)\chi_{E_{(s,y),(t,z)}}\|_{1,\nu_\alpha}\\
&\leqslant \int_0^{+\infty}\int_{\mathbb{R}} \mathscr{T}_{(s,-y)}\check{\chi}_{E_{(s,y),(t,z)}}(\mu,\lambda)\left|\widetilde{\mathscr{F}_\alpha}^{-1}(m_j)(\mu,\lambda)\right| d\nu_\alpha(\mu,\lambda)\\
&\leqslant \int_0^{+\infty}\int_{\mathbb{R}} \left|\widetilde{\mathscr{F}_\alpha}^{-1}(m_j)(\mu,\lambda)\right|\\
&\quad\times\left(\int_0^{+\infty}\chi_{E_{(s,y),(t,z)}}(r,y-\lambda)\mathscr{W}_\alpha(\mu,s,r)\frac{r^{2\alpha+1}}{2^\alpha\Gamma(\alpha+1)}dr\right)d\nu_\alpha(\mu,\lambda)\\
&\leqslant \int_0^{+\infty}\int_{\mathbb{R}} \left|\widetilde{\mathscr{F}_\alpha}^{-1}(m_j)(\mu,\lambda)\right|\\
&\quad\times\left(\int_0^{+\infty}\chi_{F_{(s,y),(t,z)}}(r,\lambda)\mathscr{W}_\alpha(\mu,s,r)\frac{r^{2\alpha+1}}{2^\alpha\Gamma(\alpha+1)}dr\right)d\nu_\alpha(\mu,\lambda)\\
&=\int\int_{F_{(s,y),(t,z)}}\left|\widetilde{\mathscr{F}_\alpha}(m_j)(\mu,\lambda)\right| d\nu_\alpha(\mu,\lambda).
\end{aligned}\tag{4.3}$$

Now, by Hölder's inequality and Proposition 3.2, we get

$$\begin{aligned}
&\int\int_{F_{(s,y),(t,z)}}\left|\widetilde{\mathscr{F}_\alpha}(m_j)(\mu,\lambda)\right| d\nu_\alpha(\mu,\lambda)\\
&\qquad\leqslant C\|(1+2^{2j}\mu^2)^p(1+2^{2j}\lambda^2)^q\widetilde{\mathscr{F}_\alpha}(m_j)\|_{2,\nu_\alpha}\\
&\qquad\quad\times\left(\int_0^{+\infty}\frac{\mu^{2\alpha+1}d\mu}{(1+2^{2j}\mu^2)^{2p}}\right)^{\frac{1}{2}}\left(\int_{|\lambda|\geqslant\rho((s,y),(t,z))}\frac{d\lambda}{(1+2^{2j}\lambda^2)^{2q}}\right)^{\frac{1}{2}}\\
&\qquad\leqslant C(2^j\rho((s,y),(t,z)))^{\frac{1}{2}-2q}.
\end{aligned}\tag{4.4}$$

Then, inequality (4.2) follows immediately by combining relations (4.3) and (4.4). □

In the following, we introduce for every $(s,y),(t,z)\in[0,+\infty[\times\mathbb{R}$ the sets defined by

$$G_{(s,y),(t,z)}=\{r\in[0,+\infty[\ |\ |r-s|\geqslant 2|s-t|+|y-z|\}\times\mathbb{R},$$

and

$$H_{(s,y),(t,z)}=\{r\in[0,+\infty[\ |\ r\geqslant|s-t|+|y-z|\}\times\mathbb{R}.$$

By a basic calculus, one can see that

$$\mathscr{T}_{(s,0)}\chi_{G_{(s,y),(t,z)}}\leqslant\chi_{H_{(s,y),(t,z)}},\tag{4.5}$$

and

$$\mathscr{T}_{(t,0)}\chi_{G_{(s,y),(t,z)}}\leqslant\chi_{H_{(s,y),(t,z)}}.\tag{4.6}$$

PROPOSITION 4.2. *Let* $m\in\mathscr{C}_{e,b}^{2p+2q}(\mathbb{R}^2)$ *and* $\varphi\in\mathscr{D}_e\left(\mathbb{R}^2\right)$ *with* $supp(\varphi)\subset B_1^c\cap B_2$. *If*

$$\sup_{R>0}\sum_{(k,\ell)\in\mathscr{H}_{2p,2q}}R^{k+\ell+\frac{1}{2}}\left(\int\int_{B_R^c\cap B_{2R}}\left|\frac{\partial^{k+\ell}m}{\partial r^k\partial x^\ell}(r,x)\right|^2 r^{-3}drdx\right)^{\frac{1}{2}}<+\infty,$$

then there exists a nonnegative constant C such that, for every $j \in \mathbb{Z}$ and for all $(s,y),(t,z) \in [0,+\infty[\times\mathbb{R}$, we have

$$\|\mathscr{T}_{(s,-y)}\widetilde{\mathscr{F}_\alpha}(m_j)\chi_{G_{(s,y),(t,z)}} - \mathscr{T}_{(t,-z)}\widetilde{\mathscr{F}_\alpha}(m_j)\chi_{G_{(s,y),(t,z)}}\|_{1,\nu_\alpha} \leqslant C\left(2^j\rho((s,y),(t,z))\right)^{\alpha+1-2p}. \tag{4.7}$$

where $m_j = m\delta_{2^{-j}}\varphi$.

Proof. The proof is similar to that given in the previous proposition, so one can assume that $(s,y) \neq (t,z)$ and a standard change of variables gives

$$\begin{aligned}\|\mathscr{T}_{(s,-y)}\widetilde{\mathscr{F}_\alpha}(m_j)\chi_{G_{(s,y),(t,z)}} &- \mathscr{T}_{(t,-z)}\widetilde{\mathscr{F}_\alpha}(m_j)\chi_{G_{(s,y),(t,z)}}\|_{1,\nu_\alpha}\\ &\leqslant \int_0^{+\infty}\int_{\mathbb{R}} \mathscr{T}_{(s,0)}\chi_{G_{(s,y),(t,z)}}(\mu,\lambda)\left|\widetilde{\mathscr{F}_\alpha}(m_j)(\mu,\lambda)\right| d\nu_\alpha(\mu,\lambda)\\ &\quad + \int_0^{+\infty}\int_{\mathbb{R}} \mathscr{T}_{(t,0)}\chi_{G_{(s,y),(t,z)}}(\mu,\lambda)\left|\widetilde{\mathscr{F}_\alpha}(m_j)(\mu,\lambda)\right| d\nu_\alpha(\mu,\lambda)\end{aligned}$$

so by virtue of relations (4.5), (4.6) and Proposition 3.2, we conclude that

$$\begin{aligned}\|\mathscr{T}_{(s,-y)}\widetilde{\mathscr{F}_\alpha}(m_j)\chi_{G_{(s,y),(t,z)}} - \mathscr{T}_{(t,-z)}\widetilde{\mathscr{F}_\alpha}(m_j)\chi_{G_{(s,y),(t,z)}}\|_{1,\nu_\alpha} &\leqslant 2\|\widetilde{\mathscr{F}_\alpha}(m_j)\chi_{H_{(s,y),(t,z)}}\|_{1,\nu_\alpha}\\ \leqslant 2\int_{\rho((s,y),(t,z))}^{+\infty}\int_{\mathbb{R}}\left|\widetilde{\mathscr{F}_\alpha}(m_j)(\mu,\lambda)\right| d\nu_\alpha(\mu,\lambda)&\\ \leqslant C\|(1+2^{2j}\mu^2)^p(1+2^{2j}\lambda^2)^q\widetilde{\mathscr{F}_\alpha}(m_j)\|_{2,\nu_\alpha}&\\ \times\left(\int_{\rho((s,y),(t,z))}^{+\infty}\frac{\mu^{2\alpha+1}d\mu}{(1+2^{2j}\mu^2)^{2p}}\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}}\frac{d\lambda}{(1+2^{2j}\lambda^2)^{2q}}\right)^{\frac{1}{2}}&\\ \leqslant C\left(2^j\rho((s,y),(t,z))\right)^{\alpha+1-2p} \quad \square&\end{aligned}$$

In order to establish the Hörmander-Mikhlin multiplier theorem for $\mathscr{F}_\alpha$, we need to cite the following useful theorem [8, Theorem 2.4, p. 75] which will be essential in the proof of the main theorem.

THEOREM 4.3. *Let (X,μ) be a measure space and ρ be a metric on X. Let $\mathscr{K}$ be the operator defined on $L^2(X,d\mu)$, by*

$$\mathscr{K}(f)(x) = \int_X k(x,y)f(y)d\mu(y),$$

where $k \in L^2(X\times X, d\mu\otimes d\mu)$. Assume that there is three constants C_1, C_2, C_3 such that

i) $\forall f \in L^2(X,d\mu)$, $\|\mathscr{K}(f)\|_2 \leqslant C_1\|f\|_2$.

ii) $\forall y,z \in X$, $\displaystyle\int_{\rho(x,y)>C_2\rho(y,z)} |k(x,y)-k(x,z)|d\mu(x) < C_3$.

Then, there exists a positive constant C depending only on C_1, C_2 and C_3, such that for every $f \in L^1(X, d\mu) \cap L^2(X, d\mu)$ we have

$$\mu\{x \in X \mid |\mathscr{K}(f)(x)| > \alpha\} \leqslant \frac{C}{\alpha}\|f\|_1.$$

THEOREM 4.4. (Hörmander-Mikhlin) *Let $m \in \mathscr{C}^{2p+2q}_{e,b}(\mathbb{R}^2)$. If*

$$\sup_{R>0} \sum_{(k,\ell)\in\mathscr{H}_{2p,2q}} R^{k+\ell+\frac{1}{2}} \left(\int\int_{B_R^c \cap B_{2R}} \left| \frac{\partial^{k+\ell}}{\partial r^k \partial x^\ell} m(r,x) \right|^2 r^{-3} dr dx \right)^{\frac{1}{2}} < +\infty,$$

then T_m is of a weak type $(1,1)$, and consequently is bounded on $L^r(d\nu_\alpha)$ for $1 < r < +\infty$.

Proof. We follow the idea of Hörmander [18, pp 121], then we know that there is a positive function $\varphi \in \mathscr{D}_e\left(\mathbb{R}^2\right)$ such that $supp(\varphi) \subset B_1^c \cap B_2$ and $\sum_{j=-\infty}^{+\infty} \delta_{2^{-j}}\varphi(\mu,\lambda) = 1$ for every $(\mu,\lambda) \neq (0,0)$ and $m = \sum_{j=-\infty}^{+\infty} m_j$ be the dyadic decomposition of m, where $m_j = m\delta_{2^{-j}}\varphi$. Then $T_m = \sum_{j=-\infty}^{+\infty} T_{m_j}$ and for every $f \in L^1(d\nu_\alpha) \cap L^2(d\nu_\alpha)$, we have for all $(s,y) \in [0,+\infty[\times\mathbb{R}$

$$\begin{aligned} T_{m_j} f(s,y) &= \widetilde{\mathscr{F}_\alpha}^{-1}(m_j) * f(s,y) \\ &= \int_0^{+\infty}\int_{\mathbb{R}} k_j((r,x),(s,y)) f(r,x) d\nu_\alpha(r,x), \end{aligned}$$

where

$$k_j((r,x),(s,y)) = \mathscr{T}_{(s,-y)}(\widetilde{\mathscr{F}_\alpha}^{-1}(m_j))^{\vee}(r,x) = \mathscr{T}_{(s,-y)}(\widetilde{\mathscr{F}_\alpha}(m_j))(r,x),$$

and therefore according to Theorem 4.3, to prove that T_m is of a weak type $(1,1)$, it is sufficient to show that

$$\sum_{j=-\infty}^{+\infty} \int\int_{\rho((r,x),(s,y))>3\rho((s,y),(t,z))} |k_j((r,x),(s,y)) - k_j((r,x),(t,z))| d\nu_\alpha(r,x) < C.$$

However, we remark that

$$\{(r,x) \in [0,+\infty[\times\mathbb{R} \mid \rho((r,x),(s,y)) > 3\rho((s,y),(t,z))\} \subset E_{(s,y),(t,z)} \cup G_{(s,y),(t,z)},$$

and therefore

$$\sum_{j=-\infty}^{+\infty} \int\int_{\rho((r,x),(s,y))>3\rho((s,y),(t,z))} \left| \mathscr{T}_{(s,-y)}(\widetilde{\mathscr{F}_\alpha}(m_j))(r,x) - \mathscr{T}_{(t,-z)}(\widetilde{\mathscr{F}_\alpha}(m_j))(r,x) \right| d\nu_\alpha(r,x)$$

$$\leqslant \sum_{j=-\infty}^{+\infty} \int\int_{E_{(s,y),(t,z)}} \left| \mathscr{T}_{(s,-y)}(\widetilde{\mathscr{F}_\alpha}(m_j))(r,x) - \mathscr{T}_{(t,-z)}(\widetilde{\mathscr{F}_\alpha}(m_j))(r,x) \right| d\nu_\alpha(r,x)$$
$$+ \sum_{j=-\infty}^{+\infty} \int\int_{G_{(s,y),(t,z)}} \left| \mathscr{T}_{(s,-y)}(\widetilde{\mathscr{F}_\alpha}(m_j))(r,x) - \mathscr{T}_{(t,-z)}(\widetilde{\mathscr{F}_\alpha}(m_j))(r,x) \right| d\nu_\alpha(r,x). \tag{4.8}$$

On the other hand, by relations (3.5), (4.2) and (4.7) we deduce that there is a positive constant C such that, for every $j \in \mathbb{Z}$ we have

$$\int\int_{E_{(s,y),(t,z)}} \left| \mathscr{T}_{(s,-y)}(\widetilde{\mathscr{F}_\alpha}(m_j))(r,x) - \mathscr{T}_{(t,-z)}(\widetilde{\mathscr{F}_\alpha}(m_j))(r,x) \right| d\nu_\alpha(r,x)$$
$$\leqslant C \min\left(\left(2^j \rho((s,y),(t,z))\right)^{\frac{1}{2}-2q} ; 2^{j+1} \rho((s,y),(t,z)) \right), \tag{4.9}$$

and

$$\int\int_{G_{(s,y),(t,z)}} \left| \mathscr{T}_{(s,-y)}(\widetilde{\mathscr{F}_\alpha}(m_j))(r,x) - \mathscr{T}_{(t,-z)}(\widetilde{\mathscr{F}_\alpha}(m_j))(r,x) \right| d\nu_\alpha(r,x)$$
$$\leqslant C \min\left(\left(2^j \rho((s,y),(t,z))\right)^{\alpha+1-2p} ; 2^{j+1} \rho((s,y),(t,z)) \right). \tag{4.10}$$

Hence, relations (4.8), (4.9) and (4.10) show that

$$\sum_{j=-\infty}^{+\infty} \int\int_{\rho((r,x),(t,z))>3\rho((s,y),(t,z))} |k_j((r,x),(s,y)) - k_j((r,x),(t,z))| d\nu_\alpha(r,x) < C,$$

and consequently the operator T_m is of a weak type $(1,1)$, but Plancherel's theorem implies that T_m is of a strong type $(2,2)$, and therefore as in the euclidian case, the proof is complete by using the Marcinkiewicz interpolation theorem and duality (see [12, 25]). $\square$

The main result is then given by the following theorem.

THEOREM 4.5. *Let m be a function on Υ satisfying $m \circ \theta^{-1} \in \mathscr{C}_{e,b}^{2p+2q}(\mathbb{R}^2)$ and*

$$\sup_{R>0} \sum_{(k,\ell)\in\mathscr{H}_{2p,2q}} R^{k+\ell+\frac{1}{2}} \left(\int\int_{B_R^c \cap B_{2R}} \left| \frac{\partial^{k+\ell}}{\partial r^k \partial x^\ell} (m \circ \theta^{-1})(r,x) \right|^2 r^{-3} dr dx \right)^{\frac{1}{2}} < +\infty,$$

then the multiplier operator T_m defined by $\mathscr{F}(T_m f) = m\mathscr{F}(f)$, is an L^r-Fourier multiplier for $1 < r < +\infty$.

REFERENCES

[1] L. E. ANDERSSON, *On the determination of a function from spherical averages*, SIAM. J. Math. Anal., **19**: 214–234, 1988.

[2] G. E. ANDREWS, R. ASKEY, AND R. ROY, *Special functions*, volume **71**, Cambridge university press, 1999.

[3] C. BACCAR, N. B. HAMADI, H. HERCH, AND F. MEHERZI, *Inversion of the Riemann-Liouville operator and its dual using wavelets*, Opuscula Math., **35** (6): 867–887, 2015.

[4] C. BACCAR, N. B. HAMADI, AND L. T. RACHDI, *Inversion formulas for the Riemann-Liouville transform and its dual associated with singular partial differential operators*, Int. J. Math. Math. Sci., **2006**: 1–26, 2006.

[5] C. BACCAR, N. B. HAMADI, AND L. T. RACHDI, *Best approximation for Weierstrass transform connected with Riemann-Liouville operator*, Commun. Math. Anal., **5** (1), 2008.

[6] C. BACCAR AND L. RACHDI, *Spaces of DL^p-type and a convolution product associated with the Riemann-Liouville operator*, Bull. Math. Anal. Appl., **1** (3): 16–41, 2009.

[7] W. R. BLOOM AND Z. XU, *Fourier multipliers for L^p on Chébli–Trimèche hypergroups*, Proc. London Math. Soc., **80** (03): 643–664, 2000.

[8] R. COIFMAN AND G. WEISS, *Analyse harmonique non-commutative sur certains espaces homogènes*, Lecture Notes in Math., **242**: 569–645, 1977.

[9] J. A. FAWCETT, *Inversion of N-dimensional spherical averages*, SIAM J. Appl. Math., **45** (2): 336–341, 1985.

[10] V. FISCHER AND M. RUZHANSKY, *Quantization on nilpotent Lie groups*, Progress in Mathematics, volume **314**, Birkhauser, 2016.

[11] J. GOSSELIN AND K. STEMPAK, *A weak-type estimate for Fourier-Bessel multipliers*, Proc. Amer. Math. Sec., **106** (3): 655–662, 1989.

[12] L. GRAFAKOS, *Classical Fourier Analysis*, volume **2**, Springer, 2008.

[13] N. B. HAMADI, *Generalized homogeneous Besov spaces associated with the Riemann–Liouville operator*, Internat. J. Math., **26** (02): 21, 2015.

[14] N. B. HAMADI AND L. RACHDI, *Fock spaces and associated operators for singular partial differential operators*, Int. J. Math. Anal., **1** (18): 873–895, 2007.

[15] N. B. HAMADI AND L. T. RACHDI, *Weyl transforms associated with the Riemann-Liouville operator*, Int. J. Math. Math. Sci., **2006**, 2006.

[16] H. HELLSTEN AND L.-E. ANDERSSON, *An inverse method for the processing of synthetic aperture radar data*, Inverse problems, **3** (1): 111, 1987.

[17] M. HERBERTHSON, *A numerical implementation of an inverse formula for carabas raw data*, National Defense Research Institute, Internal Report D, pages 303–304, 1986.

[18] L. HÖRMANDER, *Estimates for translation invariant operators in L^p spaces*, Acta Math., **104** (1): 93–140, 1960.

[19] R. KAPELKO, *A multiplier theorem for the Hankel transform*, Rev. Mat. Complut., **11** (2): 281–288, 1998.

[20] N. N. LEBEDEV, *Special functions and their applications*, Courier Corporation, 1972.

[21] S. G. MIKHLIN, *The multipliers of Fourier integrals*, *Dokl. Akad. Naulc SSSR (N. S.)*, **109** (4): 701–703, 1956.

[22] S. OMRI AND L. RACHDI, *Heisenberg-Pauli-Weyl uncertainty principle for the Riemann-Liouville operator*, J. Inequal. Pure Appl. Math, **9** (3): 1–23, 2008.

[23] M. RUZHANSKY, M. VLADIMIROVICH AND J. WIRTH, *On multipliers on compact Lie groups*, Funct. Anal. Appl, **47** (1): 72–75, 2013.

[24] M. RUZHANSKY AND J. WIRTH, *Lp Fourier multipliers on compact Lie groups*, Math. Z, **280**: 621–642, 2015.

[25] E. M. STEIN, *Singular integrals and differentiability properties of functions*, volume **2**, Princeton university press, 1970.

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