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INVERSE PROBLEMS FOR A CLASS OF STURM–LIOUVILLE OPERATORS WITH THE MIXED SPECTRAL DATA

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Abstract. In this paper, we study the inverse spectral problem for Sturm-Liouville equations with boundary conditions polynomially dependent on the spectral parameter and establish a uniqueness theorem with the mixed spectral data. In addition, we obtain three corollaries of the uniqueness theorem for the above boundary value problem.

1. Introduction

Consider the following boundary value problem $L := L(q, U_0, U_1)$ defined by

$$lu := -u'' + q(x)u = \lambda u, \quad x \in (0,1) \tag{1.1}$$

with boundary conditions

$$U_0(u) := R_{01}(\lambda)u'(0,\lambda) + R_{00}(\lambda)u(0,\lambda) = 0, \tag{1.2}$$
$$U_1(u) := R_{11}(\lambda)u'(1,\lambda) + R_{10}(\lambda)u(1,\lambda) = 0, \tag{1.3}$$

where λ is the spectral parameter, q is a real-valued function and $q \in L^2(0,1)$,

$$R_{\xi k}(\lambda) = \sum_{l=0}^{r_{\xi k}} a_{\xi kl} \lambda^{r_{\xi k} - l}, \quad r_{\xi 1} = r_{\xi 0} = r_\xi \geqslant 0, \quad a_{\xi 10} = 1, \quad \xi, k = 0, 1,$$

are arbitrary polynomials of degree r_ξ with real coefficients such that $R_{\xi 1}(\lambda)$ and $R_{\xi 0}(\lambda)$, $\xi = 0, 1$, have no common zeros.

Freiling and Yurko [4] discussed three inverse problems for the BVP L where coefficients $q(x), R_{\xi 1}(\lambda)$ are complex, either from the Weyl function, or from discrete spectral data, or from two spectra and provided procedures for reconstructing this differential operator from the above spectral data, respectively. More related results for Sturm-Liouville equations with boundary conditions linearly or polynomially dependent on the spectral parameter can be found in [1, 3, 4, 5, 11, 13, 15].

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It is well known that two spectra $\{\lambda_n, \mu_n\}$ (see below) of the classical Sturm-Liouville problem (1.1)–(1.3), where $R_{01}(\lambda)=1$, $R_{00}(\lambda)=-h_0$, $R_{11}(\lambda)=1$, $R_{10}(\lambda)=h_1$, is sufficient to determine the potential q and coefficients h_0, h_1, of the boundary conditions (see [2, 16]). Later, Gesztesy and Simon [6] studied the inverse spectral problem for the Sturm-Liouville operator by the Weyl function and showed that if q is prescribed on the interval $[0, \frac{1+\varepsilon}{2}]$ for some ε, $0 \leqslant \varepsilon < 1$, and coefficient h_0 of the boundary condition is given a priori, then parts of one spectrum are sufficient to determine the potential q on the interval $[0,1]$ and coefficient h_1 of the boundary condition (see [6, Theorem 1.3]), which is a generalization of the Hochstadt-Lieberman theorem [7]. More related results were obtained by this approach (see [3, 6, 8, 9, 13, 14]), or the method of spectral mappings (see [4, 10, 16]). Suzuki [12] verified that if q is given on $[0, \frac{1-\varepsilon}{2}]$ for $0 < \varepsilon < 1$, then one spectrum cannot uniquely determine the potential q by a counterexample. Therefore, it is interesting to study the inverse spectral problem for the BVP L with partial information on the potential and parts of two spectra together with the given boundary condition at $x=0$, which is called the mixed spectral data.

The aim of this article is to investigate the inverse spectral problem for Sturm-Liouville equations with boundary conditions polynomially dependent on the spectral parameter. We show that if q is prescribed on $[0, \alpha_1]$ for some α_1, $0 \leqslant \alpha_1 < 1$, and functions $R_{0\xi}(\lambda)$, $\xi = 0,1$, of the boundary condition are known a priori, then the potential q on the interval $[0,1]$ and functions $R_{1\xi}(\lambda)$, $\xi = 0,1$, of the boundary condition can be uniquely determined by parts of two spectra. In particular, we still establish either the Borg type theorem, or the Gesztesy-Simon type theorem, or the Hochstadt-Lieberman type theorem for the BVP L. The techniques used here are based on the methods developed in [3, 4, 6].

This article is organized as follows. In Section 2, we present preliminaries. In Section 3, we prove our main results.

2. Preliminaries

Let $S_1(x,\lambda)$, $S_2(x,\lambda)$, $u_-(x,\lambda)$ and $u_+(x,\lambda)$ be solutions of Equation (1.1) under the initial conditions

$$\begin{array}{ll} S_1(0,\lambda) = S_2'(0,\lambda) = 0, & S_1'(0,\lambda) = S_2(0,\lambda) = 1 \\ u_-(0,\lambda) = R_{01}(\lambda), & u_-'(0,\lambda) = -R_{00}(\lambda), \\ u_+(1,\lambda) = R_{11}(\lambda), & u_+'(1,\lambda) = -R_{10}(\lambda). \end{array}$$

Denote $\Delta_j(\lambda) = U_1(S_j)$. Clearly, $U_0(u_-) = U_1(u_+) = 0$, and

$$\begin{aligned} u_-(x,\lambda) &= R_{01}(\lambda)S_2(x,\lambda) - R_{00}(\lambda)S_1(x,\lambda), \\ u_+(x,\lambda) &= \Delta_1(\lambda)S_2(x,\lambda) - \Delta_2(\lambda)S_1(x,\lambda). \end{aligned}$$

The following formula is called as the Green formula

$$\int_0^1 (yl(z) - zl(y)) = [y,z](1) - [y,z](0),$$

where $[y,z](x) := y(x)z'(x) - y'(x)z(x)$ is the Wronskian of y and z.

Let

$$\Delta(\lambda) := [u_+, u_-](x, \lambda).$$

Then

$$\Delta(\lambda) = R_{01}(\lambda)\Delta_2(\lambda) - R_{00}(\lambda)\Delta_1(\lambda) = U_1(u_-) = -U_0(u_+), \tag{2.1}$$

which is called the characteristic function of L.

Denote $\lambda = \rho^2$, $\tau = |\mathrm{Im}\rho|$, for sufficiently large $|\lambda|$, we have

$$S_1(x,\lambda) = \frac{\sin\rho x}{\rho} + O\left(\frac{e^{\tau x}}{\rho^2}\right), \quad S_1'(x,\lambda) = \cos\rho x + O\left(\frac{e^{\tau x}}{\rho}\right) \tag{2.2}$$

$$S_2(x,\lambda) = \cos\rho x + O\left(\frac{e^{\tau x}}{\rho}\right), \quad S_2'(x,\lambda) = -\rho\sin\rho x + O(e^{\tau x}), \tag{2.3}$$

$$u_-(x,\lambda) = \lambda^{r_0}\left(\cos\rho x + O\left(\frac{e^{\tau x}}{\rho}\right)\right), \tag{2.4}$$

$$u_-'(x,\lambda) = \lambda^{r_0}(-\rho\sin\rho x + O(e^{\tau x})), \tag{2.5}$$

$$u_+(x,\lambda) = \lambda^{r_1}\left(\cos\rho(1-x) + O\left(\frac{e^{\tau(1-x)}}{\rho}\right)\right), \tag{2.6}$$

$$u_+'(x,\lambda) = \lambda^{r_1}(\rho\sin\rho(1-x) + O(e^{\tau(1-x)})). \tag{2.7}$$

By calculating, we get

$$\Delta(\lambda) = \lambda^{r_0+r_1}(-\rho\sin\rho + \omega\cos\rho + \kappa(\rho)), \tag{2.8}$$

$$\Delta_0(\lambda) = u_+(0,\lambda) = \lambda^{r_1}\left(\cos\rho + \omega_0\frac{\sin\rho}{\rho} + \frac{\kappa_0(\rho)}{\rho}\right), \tag{2.9}$$

where

$$\kappa(\rho) = \textstyle\int_0^1 f(t)\cos(\rho t)dt + O\left(\frac{e^\tau}{\rho}\right), \quad f \in L^2(0,1), \quad \omega = q_0 - a_{000} + a_{100},$$

$$q_0 = \tfrac{1}{2}\textstyle\int_0^1 q(t)dt,$$

$$\kappa_0(\rho) = \textstyle\int_0^1 f_0(t)\frac{\sin(\rho t)}{\rho}dt + O\left(\frac{e^\tau}{\rho^2}\right), \quad f_0 \in L^2(0,1), \quad \omega_0 = q_0 + a_{100}.$$

Let $\sigma(L) := \{\lambda_n\}_{n\geqslant 0}$ be the zeros (counting with multiplicities) of the entire function $\Delta(\rho)$. The numbers $\{\lambda_n\}_{n\geqslant 0}$ coincide with the eigenvalues of the BVP L. When n sufficiently large, then λ_n are real and simple and satisfy the asymptotic formula [4]

$$\rho_n := \sqrt{\lambda_n} = (n - r_0 - r_1)\pi + \frac{\omega}{n\pi} + \frac{\kappa_n}{n}, \quad \{\kappa_n\} \in l^2. \tag{2.10}$$

Indeed, according to (2.8) and (2.10) for sufficiently large n in the domain $D_n := \{\lambda : |\lambda - (n-r_0-r_1)^2\pi^2| < \frac{1}{2}\}$, there is exactly one eigenvalue λ_n. Taking into account the realvaluedness of $q(x)$ and $R_{\xi k}(\lambda)$, we conclude that there is also an eigenvalue $\overline{\lambda}_n \in D_n$, and hence $\lambda_n = \overline{\lambda}_n$. Therefore, the functions $u_-(x,\lambda_n)$ are real-valued for sufficiently large n. Denote $G_\delta := \{\rho : |\rho - k\pi| \geqslant \delta, k \in \mathbb{Z}\}$ for fixed $\delta > 0$, then

$$|\Delta(\lambda)| \geqslant C_\delta|\rho||\lambda|^{(r_0+r_1)}e^\tau, \quad \lambda \in G_\delta, \quad |\lambda| \quad \text{sufficiently large.} \tag{2.11}$$

Let m_n be the multiplicity of λ_n (i.e., $\lambda_n = \lambda_{n+1} = \cdots = \lambda_{n+m_n-1}$), and put $\overline{N}_0 := \{n : \lambda_{n-1} \neq \lambda_n\}$. Denote

$$u_-^{(\nu)}(x,\lambda) := \frac{1}{\nu!}\frac{\partial^\nu}{\partial\lambda^\nu}u_-(x,\lambda),$$
$$u_-(x,\lambda_{n+\nu}) := u_-^{(\nu)}(x,\lambda_n), \quad n \in \overline{N}_0, \quad \nu = 1,2,\cdots,m_n-1.$$

Therefore, $u_-(x,\lambda_n)$ is the eigenfunction corresponding to the eigenvalue λ_n.

Let $\sigma(L_0) := \{\mu_n\}_{n\geqslant 0}$ be the zeros (counting with multiplicities) of the entire function $\Delta_0(\rho)$. The numbers $\{\mu_n\}_{n\geqslant 0}$ coincide with the eigenvalues of the BVP L_0, which is defined by Equation (1.1), $u_+(0,\lambda) = 0$ and (1.3). When n sufficiently large, then μ_n are real and simple and satisfy the asymptotic formula [4]

$$\sqrt{\mu_n} = \left(n - r_1 + \frac{1}{2}\right)\pi + \frac{\omega_0}{n\pi} + \frac{\kappa_{n0}}{n}, \quad \{\kappa_{n0}\} \in l^2. \tag{2.12}$$

Let $\Phi(x,\lambda)$ be the solution of Equation (1.1) satisfying the boundary conditions $U_0(\Phi) = 1$ and $U_1(\Phi) = 0$. Then

$$\Phi(x,\lambda) = -\frac{u_+(x,\lambda)}{\Delta(\lambda)}. \tag{2.13}$$

Denote

$$M(\lambda) := \Phi(0,\lambda) = -\frac{u_+(0,\lambda)}{\Delta(\lambda)}, \tag{2.14}$$

which is called the Weyl function of the BVP L. For sufficiently large $|\rho|$, the asymptotic formula of the Weyl function $M(\lambda)$ is as follows

$$M(\lambda) = \frac{1}{i\rho\lambda^{r_0}}\left(1 - \frac{a_{000}}{i\rho} + o\left(\frac{1}{\rho}\right)\right), \quad \rho \in \Lambda_{\delta'}, \tag{2.15}$$

where $\Lambda_{\delta'} := \{\rho : \arg\rho \in [\delta', \pi - \delta']\}$ for some $\delta' > 0$.

The following two lemmas are important for proofs of the main results.

LEMMA 2.1. [3, 4] *Let $M(\lambda)$ be the Weyl function of the BVP L. Then $M(\lambda)$ and functions $R_{0\xi}(\lambda)$, $\xi = 0,1$, of the boundary condition can uniquely determines functions $R_{1\xi}(\lambda)$, $\xi = 0,1$, of the boundary condition as well as q (a.e.) on the interval* $[0,1]$.

LEMMA 2.2. ([6, Proposition B.6]) *Let $f(z)$ be an entire function such that*

1. $\sup_{|z|=R_k}|f(z)| \leqslant C_1\exp(C_2R_k^\beta)$ *for some β, $0 < \beta < 1$, some sequence $R_k \to \infty$ as $k \to \infty$ and $C_1, C_2 > 0$;*
2. $\lim_{|x|\to\infty}|f(ix)| = 0$, $x \in \mathbb{R}$.

Then $f \equiv 0$.

3. Main results and Proofs

In this section, we discuss the inverse spectral problem for the BVP L with the mixed spectral data. We agree that together with L we consider here and in the sequel a boundary value problem $\widetilde{L} = L(\widetilde{q}, \widetilde{U}_0, \widetilde{U}_1)$ of the same form but with different coefficients. If a certain symbol γ denotes an object related to L, then the corresponding symbol $\widetilde{\gamma}$ with tilde denotes the analogous object related to $\widetilde{L}$, and $\widehat{\gamma} = \gamma - \widetilde{\gamma}$.

Without loss of generality, we assume all eigenvalues $\lambda_n \neq 0$, and $\mu_n \neq 0$, in this paper. For any $M = \{\beta_n : \beta_n \in \mathbb{C}\}_{n=0}^{\infty}$, denote

$$N_M(t) = \sharp\{n \in \mathbb{N}_0 : |\beta_n| \leqslant t, \beta_n \in M\}$$

for all sufficiently large $t \in \mathbb{R}^+$, where $\mathbb{N}_0 = \{0, 1, 2, \cdots\}$. Here and everywhere below we might assume if $\lambda_n = \widetilde{\lambda}_{\widetilde{n}}$ (resp. $\mu_n = \widetilde{\mu}_{\widetilde{n}}$), then $m_n = \widetilde{m}_{\widetilde{n}}$ (resp. $m'_n = \widetilde{m}'_{\widetilde{n}}$, where m'_n is the multiplicity of μ_n).

Using partial information on the potential and parts of two spectra as mixed spectral data, we have the following uniqueness theorem.

THEOREM 3.1. *Let* $S = \{\lambda_n\}_{n\in\Lambda} \subseteq \sigma(L) \bigcap \sigma(\widetilde{L})$ *and* $S_0 = \{\mu_n\}_{n\in\Lambda_0} \subseteq \sigma(L_0) \bigcap \sigma(\widetilde{L}_0)$, *where* $\Lambda, \Lambda_0 \subseteq \mathbb{N}_0$. *If the following three conditions are satisfied*

1. $R_{0\xi}(\lambda) = \widetilde{R}_{0\xi}(\lambda)$, $\xi = 1, 2$, $q = \widetilde{q}$ *on the interval* $[0, \alpha_1]$ *for some* $\alpha_1 \in [0, 1)$

2. $\alpha_1 - \alpha_0 - \alpha \geqslant 0$, *where* $(\alpha_0, \alpha) \in [0, \frac{1}{2}] \times [0, \frac{1}{2}]$.

3. *The inequalities*

$$N_S(t) \geqslant (1 - 2\alpha) N_{\sigma(L)}(t) + (2r_1 + 2r_0 + 1)\alpha - r_0 + r_1, \tag{3.1}$$
$$N_{S_0}(t) \geqslant (1 - 2\alpha_0) N_{\sigma(L_0)}(t) + (2\alpha_0 - 1) r_1 \tag{3.2}$$

hold for all sufficiently large $t \in \mathbb{R}^+$,

then

$$q(x) \overset{a.e.}{=} \widetilde{q}(x) \ \text{ on } \ [0,1], \ \text{ and } \ R_{1\xi}(\lambda) = \widetilde{R}_{1\xi}(\lambda), \quad \xi = 0, 1.$$

Moreover, we obtain the following Borg type theorem for the BVP L except for at most r_0 eigenvalues. i.e.,

THEOREM 3.2. *If functions* $R_{0\xi}(\lambda)$, $\xi = 0, 1$, *of the boundary condition are given a priori, then* $q(x)$ *and functions* $R_{1\xi}(\lambda)$, $\xi = 0, 1$, *of the boundary condition can be uniquely determined by two spectra except for at most* r_0 *eigenvalues.*

If $(\alpha_1, \alpha_0, \alpha) = (\frac{1+\varepsilon}{2}, \frac{1}{2}, \frac{\varepsilon}{2})$ for $0 \leqslant \varepsilon < 1$ in Theorem 3.1, we have the following Gesztesy-Simon type theorem for the BVP L.

COROLLARY 3.3. *If functions $R_{0\xi}(\lambda)$, $\xi=0,1$, of the boundary condition are given a priori, and $q(x)$ is proscribed on the interval $[0,\frac{1+\varepsilon}{2}]$ for some ε, $0\leqslant\varepsilon<1$, and the inequality*

$$N_S(t)\geqslant(1-\varepsilon)N_{\sigma(L)}(t)+\frac{(2r_1+2r_0+1)\varepsilon}{2}-r_0+r_1 \tag{3.3}$$

holds for all sufficiently large $t\in\mathbb{R}^+$, then

$$q(x)\overset{a.e.}{=}\widetilde{q}(x)\ \text{ on }\ [0,1],\ \text{ and }\ R_{1\xi}(\lambda)=\widetilde{R}_{1\xi}(\lambda),\quad \xi=0,1.$$

In particular, let $\varepsilon=0$ in Corollary 3.3, we get the Hochstadt-Lieberman type theorem for the BVP L.

Next, we prove Theorem 3.1.

Proof of Theorem 3.1. Let $u_+(x,\lambda)$ be the solution of Equation (1.1) under the initial conditions $u_+(1,\lambda)=R_{11}(\lambda)$ and $u_+'(1,\lambda)=-R_{10}(\lambda)$. By the Green formula, we have

$$\begin{aligned}\int_0^1\widehat{q}u_+(x,\lambda)\widetilde{u}_+(x,\lambda)dx&=[u_+,\widetilde{u}_+](0,\lambda)-[u_+,\widetilde{u}_+](1,\lambda)\\&=F(0,\lambda)-F(1,\lambda),\end{aligned}$$

where $F(x,\lambda)=[u_+,\widetilde{u}_+](x,\lambda)$.

From $\widehat{q}(x)=0$ on $[0,\alpha_1]$, we get

$$F(0,\lambda)=F(\alpha_1,\lambda)=F(1,\lambda)+\int_{\alpha_1}^1\widehat{q}(x)u_+(x,\lambda)\widetilde{u}_+(x,\lambda)dx. \tag{3.4}$$

Note that

$$F(0,\lambda)=u_+(0,\lambda)\widetilde{u}_+'(0,\lambda)-u_+'(0,\lambda)\widetilde{u}_+(0,\lambda) \tag{3.5}$$

$$=\frac{1}{R_{00}(\lambda)}\big(u_+'(0,\lambda)U_0(\widetilde{u}_+)-\widetilde{u}_+'(0,\lambda)U_0(u_+)\big), \tag{3.6}$$

$$=\frac{1}{R_{01}(\lambda)}\big(\widetilde{u}_+(0,\lambda)U_0(u_+)-u_+(0,\lambda)U_0(\widetilde{u}_+)\big), \tag{3.7}$$

By virtue of (3.4), (3.6) and (3.7) together with Lemma 1 in [4], we have

$$F(0,\lambda_n)=F(\alpha_1,\lambda_n)=0,\quad \forall\lambda_n\in S, \tag{3.8}$$

$$F(0,\mu_n)=F(\alpha_1,\mu_n)=0,\quad \forall\mu_n\in S_0. \tag{3.9}$$

If λ_n is the repeated root of $\Delta(\lambda)$ and the multiplicity of λ_n is m_n, by virtue of (3.6) and (3.7) together with the assumption of Theorem 3.1, we see that λ_n is also the repeated root of $F(0,\lambda_n)$ and the multiplicity of λ_n is m_n. In addition, if $\lambda^0\in\sigma(L)\cap\sigma(L_0)$, we obtain

$$R_{00}(\lambda^0)=0.$$

This implies

$$R_{01}(\lambda^0) \neq 0.$$

Therefore, from (3.7), we see that λ^0 is a zero of at least order two of the function $F(0,\lambda)$.

Since one of the sets S and S_0 is an infinite set, then (2.10) together with the known r_0, or (2.12) shows that

$$r_1 = \tilde{r}_1.$$

By virtue of (3.4) and (2.6) together with Schwarz inequality, this yields

$$\begin{aligned} |F(\alpha_1,\lambda)| &\leqslant |F(1,\lambda)| + |\textstyle\int_{\alpha_1}^1 \widehat{q}(x)u_+(x,\lambda)\widetilde{u}_+(x,\lambda)dx| \\ &\leqslant |F(1,\lambda)| + ||\widehat{q}||_2(\textstyle\int_{\alpha_1}^1 |u_+(x,\lambda)\widetilde{u}_+(x,\lambda)|^2 dx)^{\frac{1}{2}} \\ &\leqslant c_1|\lambda|^{2r_1} + c_2||\widehat{q}||_2\sqrt{1-\alpha_1}|\lambda|^{2r_1}e^{2\tau(1-\alpha_1)} \\ &= O(|\lambda|^{2r_1}e^{2\tau(1-\alpha_1)}), \end{aligned} \tag{3.10}$$

where c_1, c_2 are constant.

Define the functions $G(\lambda)$, $G_0(\lambda)$, and $K_1(\lambda)$ by

$$G(\lambda) = \prod_{\lambda_n \in S}\left(1 - \frac{\lambda}{\lambda_n}\right), \quad G_0(\lambda) = \prod_{\mu_n \in S_0}\left(1 - \frac{\lambda}{\mu_n}\right) \tag{3.11}$$

and

$$K_1(\lambda) = \frac{F(\alpha_1,\lambda)}{G(\lambda)G_0(\lambda)}. \tag{3.12}$$

Then $K_1(\lambda)$ is an entire function in λ.

Since $\Delta(\lambda)$ is an entire functions in λ of order $\frac{1}{2}$, there exists a positive constant c such that

$$N_S(t) \leqslant N_{\sigma(L)}(t) \leqslant ct^{\frac{1}{2}}. \tag{3.13}$$

Next, we prove (3.14) by two steps

$$|G(iy)| \geqslant C|y|^{(2r_1+\frac{1}{2})}e^{\mathrm{Im}\sqrt{i}(1-2\alpha)\sqrt{|y|}} \tag{3.14}$$

for sufficiently large $|y|$, $y \in \mathbb{R}$, C is constant.

Step 1: Let all eigenvalues $\lambda_n \in S$ be real. Denote $t_0 := \inf_{n\in\mathbb{N}_0}\{|\lambda_n|, |\mu_n|\}$, then $t_0 > 0$ and $N_S(t_0) = N_{\sigma(L)}(t_0) = 0$. By the assumption (3.1) on S of Theorem 3.1, there exists a constant $t_1 \geqslant t_0$ and C_1 such that

$$\begin{cases} N_S(t) \geqslant (1-2\alpha)N_{\sigma(L)}(t) + (2r_1 + 2r_0 + 1)\alpha - r_0 + r_1, & t \geqslant t_1, \\ N_S(t) \geqslant (1-2\alpha)N_{\sigma(L)}(t) - C_1, & t < t_1. \end{cases} \tag{3.15}$$

For each fixed $y \in \mathbb{R}$, and $|y|$ sufficiently large, we get

$$\begin{aligned} \ln|G(\mathrm{i}y)| &= \tfrac{1}{2}\ln G(\mathrm{i}y)\overline{G(\mathrm{i}y)} = \tfrac{1}{2}\sum_{\lambda_n\in S}\ln\left|\left(1 - \tfrac{iy}{\lambda_n}\right)\left(1 + \tfrac{\mathrm{i}y}{\lambda_n}\right)\right| \\ &= \tfrac{1}{2}\sum_{\lambda_n\in S}\ln\left(1 + \tfrac{y^2}{\lambda_n^2}\right) = \tfrac{1}{2}\textstyle\int_{t_0}^\infty \ln\left(1 + \tfrac{y^2}{t^2}\right)\mathrm{d}N_S(t) \\ &= \tfrac{1}{2}\ln\left(1 + \tfrac{y^2}{t^2}\right)N_S(t)\Big|_{t_0}^\infty - \tfrac{1}{2}\textstyle\int_{t_0}^\infty N_S(t)\mathrm{d}\left(\ln\left(1 + \tfrac{y^2}{t^2}\right)\right). \end{aligned} \tag{3.16}$$

For sufficiently large t, since

$$\ln\left(1+\frac{y^2}{t^2}\right)=O\left(\frac{1}{t^2}\right),$$

then

$$\lim_{t\to\infty}\ln\left(1+\frac{y^2}{t^2}\right)N_S(t)=0 \text{ and } \lim_{t\to\infty}\ln\left(1+\frac{y^2}{t^2}\right)N_{\sigma(L)}(t)=0.$$

By virtue of (3.15) and (3.16) together with the following relation

$$\frac{y^2}{t^3+ty^2}=-\frac{d}{dt}\left(\frac{1}{2}\ln\left(1+\frac{y^2}{t^2}\right)\right),$$

we obtain

$$\begin{aligned}
\ln|G(\mathrm{i}y)| &= \int_{t_0}^{\infty}\frac{y^2}{t^3+ty^2}N_S(t)\mathrm{d}t \\
&= \int_{t_0}^{t_1}\frac{y^2}{t^3+ty^2}N_S(t)\mathrm{d}t+\int_{t_1}^{\infty}\frac{y^2}{t^3+ty^2}N_S(t)\mathrm{d}t \\
&\geqslant (1-2\alpha)\int_{t_0}^{\infty}\frac{y^2}{t^3+ty^2}N_{\sigma(L)}(t)\mathrm{d}t+[(2r_1+2r_0+1)\,\alpha]\int_{t_0}^{\infty}\frac{y^2}{t^3+ty^2}\mathrm{d}t \\
&\quad -\left[(2r_1+2r_0+1)\,\alpha-r_0+r_1+C_1\right]\int_{t_0}^{t_1}\frac{y^2}{t^3+ty^2}dt \\
&= (1-2\alpha)\ln|\Delta(\mathrm{i}y)|+\frac{(2r_1+2r_0+1)\alpha-r_0+r_1}{2}\ln\left(1+\frac{y^2}{t_0^2}\right) \\
&\quad +\frac{(2r_1+2r_0+1)\alpha-r_0+r_1+C_1}{2}\left(\ln\left(\frac{t_0^2+y^2}{t_1^2+y^2}\right)+\ln\left(\frac{t_1^2}{t_0^2}\right)\right).
\end{aligned}\tag{3.17}$$

By virtue of (3.17) together with (2.8), for sufficiently large $y\in\mathbb{R}$, we have

$$\begin{aligned}
|G(\mathrm{i}y)| &\geqslant C_{01}|\Delta(iy)|^{1-2\alpha}|y|^{\frac{[2(r_1+2r_0)+1]\alpha-r_0+r_1}{2}} \\
&= C_{01}|y|^{(2r_1+\frac{1}{2})}e^{\mathrm{Im}\sqrt{i}(1-2\alpha)\sqrt{|y|}},
\end{aligned}\tag{3.18}$$

where C_{01} is constant.

Step 2: Let $S_1:=\{\lambda_{n_k}\}_{k=1}^{k_0}\subseteq S$, be the set of all imaginary numbers of S, where $\{\lambda_{n_k}\}_{k=1}^{k_0}$ may be $\emptyset$. Choose real numbers $\beta_{n_k}\neq 0$ such that $\beta_{n_1}<\beta_{n_2}<\cdots<\beta_{n_{k_0}}$. Note that the following identity:

$$G(\lambda)=\left[\prod_{k=1}^{k_0}\left(1-\frac{\lambda}{\beta_{n_k}}\right)\times\prod_{\lambda_n\in S\setminus S_1}\left(1-\frac{\lambda}{\lambda_n}\right)\right]\times\prod_{k=1}^{k_0}\frac{\beta_{n_k}(\lambda_{n_k}-\lambda)}{\lambda_{n_k}(\beta_{n_k}-\lambda)}.\tag{3.19}$$

Analogues to the proof in *Step* 1, we have

$$\left|\prod_{k=1}^{k_0}\left(1-\frac{\lambda}{\beta_{n_k}}\right)\times\prod_{\lambda_n\in S\setminus S_1}\left(1-\frac{\lambda}{\lambda_n}\right)\right|\geqslant C_{02}|y|^{(2r_1+\frac{1}{2})}e^{\mathrm{Im}\sqrt{i}(1-2\alpha)\sqrt{|y|}}.\tag{3.20}$$

Thus, for each fixed $y \in \mathbb{R}$, and $|y|$ sufficiently large, we obtain

$$\left|G(iy)\right| \geqslant C|y|^{(2r_1+\frac{1}{2})}e^{\operatorname{Im}\sqrt{i}(1-2\alpha)\sqrt{|y|}}.$$

Applying the same arguments as the above calculation of $|G(iy)|$, we get

$$\left|G_0(iy)\right| \geqslant C_0 e^{\operatorname{Im}\sqrt{i}(1-2\alpha_0)\sqrt{|y|}}. \tag{3.21}$$

Therefore,

$$|G(iy)G_0(iy)| \geqslant CC_0|y|^{(2r_1+\frac{1}{2})}e^{\operatorname{Im}\sqrt{i}(2-2\alpha_0-2\alpha)\sqrt{|y|}}. \tag{3.22}$$

By virtue of (3.9), (3.12) and (3.22), for sufficiently large $y \in \mathbb{R}$, we have

$$|K_1(iy)| = \left|\frac{F(\alpha_1, iy)}{G(iy)G_0(iy)}\right| = O\left(\frac{1}{|y|^{\frac{1}{2}}}e^{-2\operatorname{Im}\sqrt{i}(\alpha_1-\alpha_0-\alpha)\sqrt{|y|}}\right). \tag{3.23}$$

From Lemma 2.2 together with (3.23), we obtain

$$K_1(\lambda) = 0, \quad \forall \lambda \in \mathbb{C}.$$

Therefore,

$$F(\alpha_1, \lambda) = F(0, \lambda) = 0, \quad \forall \lambda \in \mathbb{C}.$$

This implies

$$u_+(0,\lambda)\widetilde{u}'_+(0,\lambda) - u'_+(0,\lambda)\widetilde{u}_+(0,\lambda) = 0, \quad \forall \lambda \in \mathbb{C}. \tag{3.24}$$

From (3.24), we get

$$\begin{aligned} &u_+(0,\lambda)(R_{01}(\lambda)\widetilde{u}'_+(0,\lambda) + R_{00}(\lambda)\widetilde{u}_+(0,\lambda)) \\ &= (R_{01}(\lambda)u'_+(0,\lambda) + R_{00}(\lambda)u_+(0,\lambda))\widetilde{u}_+(0,\lambda), \quad \forall \lambda \in \mathbb{C}. \end{aligned} \tag{3.25}$$

Hence,

$$M(\lambda) = \widetilde{M}(\lambda), \quad \forall \lambda \in \mathbb{C}. \tag{3.26}$$

From Lemma 2.1 together with (3.26), we have

$$q(x) \overset{a.e.}{=} \widetilde{q}(x) \text{ a.e. on } [0,1] \text{ and } R_{1\xi}(\lambda) = \widetilde{R}_{1\xi}(\lambda), \quad \xi = 0, 1.$$

Therefore, the proof of Theorem 3.1 is now completed. □

In the remaining of this paper, we show that Theorem 3.2 holds.

Proof of Theorem 3.2. Define the function $K_2(\lambda)$ by

$$K_2(\lambda) = \frac{F(0,\lambda)}{\frac{\Delta(\lambda)}{\prod_{k=1}^{N_1}(\lambda-\lambda_{n_k})} \times \frac{\Delta_0(\lambda)}{\prod_{k=1}^{N_2}(\lambda-\mu_{n_k})}}, \tag{3.27}$$

where $N_j \in \mathbb{N}_0,\ j=1,2$ and $N_1+N_2=r_0$. Then $K_2(\lambda)$ is an entire function in λ. It is easy to prove

$$K_2(\lambda)=0, \quad \forall\lambda\in\mathbb{C}.$$

This implies

$$M(\lambda)=\widetilde{M}(\lambda), \quad \forall\lambda\in\mathbb{C}. \tag{3.28}$$

Thus, we get

$$q(x) \overset{a.e.}{=} \widetilde{q}(x) \text{ on } [0,1] \text{ and } R_{1\xi}(\lambda)=\widetilde{R}_{1\xi}(\lambda), \quad \xi=0,1.$$

Therefore, this completes the proof of Theorem 3.2. □

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REFERENCES

[1] P. A. BINDING, P. J. BROWNE AND K. SEDDIGHI, *Sturm-Liouville problems with eigenparameter dependent boundary conditions*, Proc. Roy. Soc. Edinburgh, **37** (1993), 57–72.

[2] G. BORG, *Eine Umkehrung der Sturm-Liouvilleschen Eigenwertaufgabe*, Acta Math., **78**(1946), 1–96.

[3] A. Y. CHERNOZHUKOVA AND G. FREILING, *A uniqueness theorem for inverse spectral problems depending nonlinearly on the spectral parameter*, Inverse Probl. Sci. Eng., **17** (2009), 777–785.

[4] G. FREILING AND V. A. YURKO, *Inverse problems for Sturm-Liouville equations with boundary conditions polynomially dependent on the spectral parameter*, Inverse Problems, **26** (2010), 055003 (17 pp).

[5] C. T. FULTON, *Two-point boundary value problems with eigenvalue parameter contained in the boundary conditions*, Proc. Roy. Soc. Edinburgh, **77a** (1977), 293–308.

[6] F. GESZTESY AND B. SIMON, *Inverse spectral analysis with partial information on the potential, II. The case of discrete spectrum*, Trans. Amer. Math. Soc., **352** (2000), 2765–2787.

[7] H. HOCHSTADT AND B. LIEBERMAN, *An inverse Sturm-Liouville problem wity mixed given data*, SIAM J. Appl. Math., **34** (1978), 676–680.

[8] M. HORVATH, *On the inverse spectral theory of Schrödinger and Dirac operators*, Trans. Amer. Math. Soc., **353** (2001), 4155–4171.

[9] V. A. MARCHENKO, *Some questions in the theory of one-dimensional linear differential operators of the second order, I*, Trudy Moskov. Mat. Obsc. **1** (1952), 327–420 (Russian); English transl. in Amer. Math. Soc. Transl., **2**, 101 (1973), 1–104.

[10] V. PIVOVARCHIK, *On the Hald-Gesztesy-Simon theorem*, Integral Equations and Operator Theory, **73** (2012), 383–393.

[11] C. T. SHIEH, S. A. BUTERIN, AND M. IGNATIEV, *On Hochstadt-Liberman theorem for Sturm-Liouville operators*, Far East J. Appl. Math., **52** (2011), 131–146.

[12] T. SUZUKI, *Inverse problems for heat equations on compact intervals and on circles, I*, J. Math. Soc. Japan., **38** (1986), 39–65.

[13] Y. P. WANG, *Uniqueness Theorems for Sturm-Liouville Operators with Boundary Conditions Polynomially Dependent on the Eigenparameter from Spectral Data*, Results in Mathethmatics, **63** (2013), 1131–1144.

[14] G. S. WEI AND H. K. XU, *Inverse spectral problem with partial information given on the potential and norming constants*, Trans. Amer. Math. Soc., **364** (2012), 3265–3288.

[15] C. F. YANG AND X. P. YANG, *Inverse nodal problem for the Sturm-Liouville equation with polynomially dependent on the eigenparameter*, Inverse Probl. Sci. Eng., **19** (2011), 951–961.
[16] V. A. YURKO, *Method of Spectral Mappings in the Inverse Problem Theory (Inverse and Ill-posed Problems Series)*, Utrecht: VSP, 2002.

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