Operators and Matrices
Volume 11, Number 1 (2017), 101–113 doi:10.7153/oam-11-07

ANALYTIC MODEL OF DOUBLY COMMUTING CONTRACTIONS

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(*Communicated by R. Curto*)

Abstract. An n-tuple $(n \geqslant 2)$, $T = (T_1, \ldots, T_n)$, of commuting bounded linear operators on a Hilbert space $\mathscr{H}$ is doubly commuting if $T_i T_j^* = T_j^* T_i$ for all $1 \leqslant i < j \leqslant n$. If in addition, each $T_i \in C_{\cdot 0}$, then we say that T is a doubly commuting pure tuple. In this paper we prove that a doubly commuting pure tuple T can be dilated to a tuple of shift operators on some suitable vector-valued Hardy space $H^2_{\mathscr{D}_{T^*}}(\mathbb{D}^n)$. As a consequence of the dilation theorem, we prove that there exists a closed subspace $\mathscr{S}_T$ of the form

$$\mathscr{S}_T := \sum_{i=1}^{n} \Phi_{T_i} H^2_{\mathscr{E}_{T_i}}(\mathbb{D}^n),$$

such that $\mathscr{H} \cong \mathscr{S}_T^{\perp}$ and

$$(T_1, \ldots, T_n) \cong P_{\mathscr{S}_T^{\perp}}(M_{z_1}, \ldots, M_{z_n})|_{\mathscr{S}_T^{\perp}}$$

where $\{\mathscr{E}_{T_i}\}_{i=1}^n$ are Hilbert spaces and each $\Phi_{T_i} \in H^\infty_{\mathscr{B}(\mathscr{E}_{T_i}, \mathscr{D}_{T^*})}(\mathbb{D}^n)$, $1 \leqslant i \leqslant n$ is either a one variable either a one variable inner function in z_i, or the zero function.

1. Introduction

Consider a complex separable Hilbert space $\mathscr{E}$ and a closed subspace $\mathscr{S}$ of $H^2_{\mathscr{E}}(\mathbb{D})$ that is invariant under the operator M_z on $H^2_{\mathscr{E}}(\mathbb{D})$, i.e.,

$$M_z \mathscr{S} \subseteq \mathscr{S}.$$

Clearly, $T = P_{\mathscr{S}^\perp} M_z|_{\mathscr{S}^\perp}$ is a contraction. But, moreover, T^{*m} converges to 0 in strong operator topology as $m \to \infty$. This is the so called $C_{\cdot 0}$ property that T inherits from M_z.

In their pioneering work in the late 1960's, Sz.-Nagy and Foias showed that for a contraction to qualify as $C_{\cdot 0}$, it must be of the above form. See [18]. More precisely, if T is a $C_{\cdot 0}$ contraction on a Hilbert space $\mathscr{H}$, then there is an $\mathscr{E}$ as above and a subspace $\mathscr{S}_T$ of $H^2_{\mathscr{E}}(\mathbb{D})$ such that $\mathscr{S}_T$ is invariant under M_z and T is unitarily equivalent to

Mathematics subject classification (2010): 30H05, 46E22, 46M05, 46N99, 47A20, 47A45, 47B32, 47B38.

Keywords and phrases: Hardy space over polydisc, doubly commuting contractions, shift operators, Sz.-Nagy and Foias model, isometric dilation.

This research was supported in part by Department of Science and Technology, India through the project numbered SR/S4/MS:766/12 and University Grants Commission, India via DSA-SAP. Research of JS was also partially supported by NBHM, India through the research grant NBHM/R.P.64/2014.

$P_{\mathscr{S}_T^\perp} M_z|_{\mathscr{S}_T^\perp}$. Here $\mathscr{E}$ is explicit. Indeed, if we denote by D_{T^*} the defect operator $(I - TT^*)^{1/2}$, then $\mathscr{E}$ is nothing but $\mathscr{D}_{T^*}$, the closure of the range of D_{T^*}. This result was just one part of the revelation. The technique through which it was achieved was equally revealing. They produced $\mathscr{S}_T$ as the range of the multiplier M_{θ_T} where θ_T is the characteristic function of T. Thus, they gave a Beurling-Lax-Halmos form of $\mathscr{S}_T$.

Recall that the characteristic function of a contraction $T \in \mathscr{B}(\mathscr{H})$ is defined by

$$\theta_T(z) = -T + D_{T^*}(I - zT^*)^{-1} z D_T. \qquad (z \in \mathbb{D}) \tag{1.1}$$

We refer to [18] for more properties of this function.

Such an elegant characterization of all $C_{\cdot 0}$ contractions obviously led to a search for such a phenomenon in the polydisk and the Euclidean unit ball. The challenges in a several variables situation are manifold. One first had to identify the space that would play the role of the Hardy space. For the ball, it became clear only in the 1990's with works of Drury [7], Pott [13], Popescu [12] and Arveson [4] that the natural space for this purpose on the Euclidean unit ball is the one with reproducing kernel $\frac{1}{1-\langle z,w\rangle}$. It was shown in [5] that the above mentioned result of Sz.-Nagy and Foias can be generalized to the Euclidean unit ball.

The case of the polydisk is more interesting. There is no generalization of the Sz.-Nagy and Foias result mentioned above to this situation. There are invariant subspace results though due to Ahern and Clark [1], Mandrekar [9], Rudin [14] and Izuchi, Nakazi and Seto [10]. As far as the model theory results are concerned, there is a general framework due to Ambrozie, Englis and Muller [2]. They do have a generalization of the $C_{\cdot 0}$ condition which although pretty natural when stated in an abstract setting, is quite intractable after specializing to the polydisk.

This brings us to what we are doing in this note. We consider a commuting tuple of contractions $T = (T_1, T_2, \ldots, T_n)$ such that $T_i^* T_j = T_j T_i^*$ for $i \neq j$ (double commutativity) and $T_i^{*m} \to 0$ strongly for each i. Under these assumptions, we give a generalization of the Sz.-Nagy Foias result involving characteristic functions of the individual contractions. En route, we produce a new proof of the model.

The paper is organized as follows. In section 2, we review and collect some of the preliminary concepts that will be useful. In section 3, we obtain a dilation result for pure doubly commuting tuple of contractions. In section 4, we obtain a functional model for the class of pure doubly commuting tuples of contractions. In the final section, section 5, we establish a relationship between the class of pure doubly commuting tuples of contractions and one variable inner functions defined on the unit polydisc.

2. Preliminaries

Before we introduce a tuple of doubly commuting contractions, let us briefly review the case of a single contraction $T \in \mathscr{B}(\mathscr{H})$ which is $C_{\cdot 0}$. Consider the vector valued Hardy space $H^2_{\mathscr{D}_{T^*}}(\mathbb{D})$. The contraction T is then realized as $P_{\mathscr{Q}_T} M_z|_{\mathscr{Q}_T}$, where $\mathscr{Q}_T$ is the orthogonal complement of $M_{\theta_T} H^2_{\mathscr{D}_T}(\mathbb{D})$. A key ingredient in this theory is the map $L_T : \mathscr{H} \to H^2_{\mathscr{D}_{T^*}}(\mathbb{D})$ defined by

$$L_T h := D_{T^*}(I - zT^*)^{-1}h = \sum_{m=0}^{\infty} z^m D_{T^*} T^{*m} h. \qquad (h \in \mathscr{H}). \tag{2.1}$$

Then L_T is an isometry and

$$L_T T^* = M_z^* L_T. \tag{2.2}$$

Moreover,

$$L_T^*(\mathbb{S}_w \otimes \eta) = (I - \bar{w}T)^{-1} D_{T^*}\eta, \qquad (w \in \mathbb{D}, \eta \in \mathscr{D}_{T^*})$$

and

$$\mathbb{S}(\lambda, w)(I - \theta_T(\lambda)\theta_T(w)^*) = D_{T^*}(I - \lambda T^*)^{-1}(I - \bar{w}T)^{-1} D_{T^*}, \qquad (\lambda, w \in \mathbb{D}) \tag{2.3}$$

where $\mathbb{S}$ is the Szego kernel on the unit disk defined by $\mathbb{S}(z,w) = (1 - z\bar{w})^{-1}$ for all $z, w \in \mathbb{D}$.

The above two equalities and the definition of the characteristic function (1.1) yield (cf. Lemmas 2.2 and 3.6 in [5])

$$L_T^* L_T = I_{H^2_{\mathscr{D}_{T^*}}(\mathbb{D})} - M_{\theta_T} M_{\theta_T}^*, \tag{2.4}$$

where M_{θ_T} is the multiplication operator defined by

$$(M_{\theta_T} f)(w) = \theta_T(w) f(w)$$

for all $f \in H^2_{\mathscr{D}_T}(\mathbb{D})$ and $w \in \mathbb{D}$. See [5] for more details, where this is carried out for a tuple of operators satisfying a *ball type condition*.

Now we can focus on n tuples of commuting operators. From this point on, we shall assume that n is an integer and $n \geqslant 2$. We shall denote by $\mathbb{N}^n$ the set of all multi-indices $\boldsymbol{k} := (k_1, \ldots, k_n)$ where $k_i \in \mathbb{N}$ for $i = 1, \ldots, n$. For a multi-index $\boldsymbol{k} \in \mathbb{N}^n$ we denote $z^{\boldsymbol{k}} = z_1^{k_1} \cdots z_n^{k_n}$ and $T^{\boldsymbol{k}} = T_1^{k_1} \cdots T_n^{k_n}$ where $\boldsymbol{z} := (z_1, \ldots, z_n) \in \mathbb{C}^n$ and $T = (T_1, \ldots, T_n)$ a commuting tuple (that is, $T_i T_j = T_j T_i$ for $i, j = 1, \ldots, n$) of operators on some Hilbert space $\mathscr{H}$.

Now, we introduce the notion of isometric dilation of an n-tuple operators (cf. [15]). Let T and V be n-tuples of operators on Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$, respectively. Then V is said to be a *dilation* of T if there exists an isometry $\Pi : \mathscr{H} \to \mathscr{K}$ such that

$$\Pi T_i^* = V_i^* \Pi. \qquad (1 \leqslant i \leqslant n)$$

The dilation is said to be *minimal* if

$$\mathscr{K} = \overline{\operatorname{span}}\{V^{\boldsymbol{k}}(\Pi\mathscr{H}) : \boldsymbol{k} \in \mathbb{N}^n\}.$$

Note that V on $\mathscr{K}$ is a dilation of T on $\mathscr{H}$ if and only if

$$T_i \cong P_{\mathscr{Q}} V_i|_{\mathscr{Q}}, \qquad (1 \leqslant i \leqslant n)$$

where $\mathscr{Q}$ is a joint $(V_1^*,\dots,V_n^*)$-invariant subspace of $\mathscr{K}$ (see Section 2 of [15] for more details).

Let $T=(T_1,\dots,T_n)$ be an n-tuple of doubly commuting contractions on $\mathscr{H}$. That is, T is a commuting tuple and $T_iT_j^*=T_j^*T_i$ for $i\neq j$. Define the defect operator D_{T^*} by

$$D_{T^*}:=\prod_{i=1}^n D_{T_i^*}=\big(\prod_{i=1}^n(I_{\mathscr{H}}-T_iT_i^*)\big)^{\frac{1}{2}}.$$

and the defect space $\mathscr{D}_{T^*}$ by

$$\mathscr{D}_{T^*}:=\overline{\operatorname{ran}}D_{T^*}=\overline{\operatorname{ran}}\prod_{i=1}^n D_{T_i^*}.$$

The *Hardy space* $H^2(\mathbb{D}^n)$ over the unit polydisc $\mathbb{D}^n$ is the Hilbert space of all holomorphic functions f on $\mathbb{D}^n$ such that

$$\|f\|_{H^2(\mathbb{D}^n)}:=\left(\sup_{0\leqslant r<1}\int_{\mathbb{T}^n}|f(r\boldsymbol{z})|^2d\boldsymbol{\theta}\right)^{\frac{1}{2}}<\infty,$$

where $d\boldsymbol{\theta}$ is the normalized Lebesgue measure on the torus $\mathbb{T}^n$, the distinguished boundary of $\mathbb{D}^n$, and $r\boldsymbol{z}:=(rz_1,\dots,rz_n)$ (cf. [14], [8]). Note also that $H^2(\mathbb{D}^n)$ is a reproducing kernel Hilbert space [3] corresponding to the Szego kernel $\mathbb{S}:\mathbb{D}^n\times\mathbb{D}^n\to\mathbb{C}$, where

$$\mathbb{S}(\boldsymbol{z},\boldsymbol{w})=\prod_{i=1}^n(1-z_i\bar{w}_i)^{-1}. \qquad (\boldsymbol{z},\boldsymbol{w}\in\mathbb{D}^n)$$

We denote the Banach algebra of all bounded holomorphic functions on $\mathbb{D}^n$ by $H^\infty(\mathbb{D}^n)$ equipped with the supremum norm.

Given a Hilbert space $\mathscr{E}$ we identify $H^2(\mathbb{D}^n)\otimes\mathscr{E}$ with $H^2_{\mathscr{E}}(\mathbb{D}^n)$ via the unitary map $\boldsymbol{z}^{\boldsymbol{k}}\otimes\eta\mapsto\boldsymbol{z}^{\boldsymbol{k}}\eta$ for all $\boldsymbol{k}\in\mathbb{N}^n$ and $\eta\in\mathscr{E}$. Moreover, it is easy to see that the corresponding multiplication operators by the coordinate functions are intertwined by this unitary map.

DEFINITION 2.1. Let T be an n-tuple ($n>1$) of doubly commuting contractions on a Hilbert space $\mathscr{H}$. The tuple is said to be a *doubly commuting pure tuple* if $T_i\in C_{\cdot 0}$ for all $1\leqslant i\leqslant n$.

The tuple of shift operators $(M_{z_1},\dots,M_{z_n})$ on $H^2_{\mathscr{E}}(\mathbb{D}^n)$ is a natural example of a doubly commuting pure tuple of operators.

3. Isometric dilation

In this section we will be concerned with the isometric dilation of a doubly commuting pure tuple on a Hilbert space $\mathscr{H}$. Suppose that $T=(T_1,\dots,T_n)$ is a doubly commuting tuple. Then

$$T_iD_{T_j^*}=D_{T_j^*}T_i \tag{3.1}$$

for $1 \leqslant i, j \leqslant n$ and $i \neq j$ and

$$D_{T_i^*} D_{T_j^*} = D_{T_j^*} D_{T_i^*}. \qquad (1 \leqslant i < j \leqslant n) \tag{3.2}$$

THEOREM 3.1. *Let T be a doubly commuting pure tuple on $\mathscr{H}$. Then the bounded linear operator $L_T : \mathscr{H} \to H^2_{\mathscr{D}_{T^*}}(\mathbb{D}^n)$ defined by*

$$(L_T h)(\boldsymbol{z}) = D_{T^*} \prod_{i=1}^{n} (I - z_i T_i^*)^{-1} h$$

for $h \in \mathscr{H}$ and $z \in \mathbb{D}^n$, is an isometry, and

$$L_T T_i^* = M_{z_i}^* L_T,$$

for $i = 1, \ldots, n$. Moreover,

$$L_T^* (\mathbb{S}(\cdot, \boldsymbol{w})\eta) = \prod_{i=1}^{n} (I - \bar{w}_i T_i)^{-1} D_{T^*} \eta,$$

for all $\boldsymbol{w} \in \mathbb{D}^n$ and $\eta \in \mathscr{D}_{T^*}$, and

$$H^2_{\mathscr{D}_{T^*}}(\mathbb{D}^n) = \overline{span}\{ z^{\boldsymbol{k}} (L_T \mathscr{H}) : \boldsymbol{k} \in \mathbb{N}^n \}.$$

Proof. First identify $H^2_{\mathscr{D}_{T_i}}(\mathbb{D}^n)$ with $H^2(\mathbb{D}) \otimes \cdots \otimes (H^2(\mathbb{D}) \otimes \mathscr{D}_{T_i}) \otimes \cdots \otimes H^2(\mathbb{D})$ and $H^2_{\mathscr{D}_{T_i^*}}(\mathbb{D}^n)$ with $H^2(\mathbb{D}) \otimes \cdots \otimes (H^2(\mathbb{D}) \otimes \mathscr{D}_{T_i^*}) \otimes \cdots \otimes H^2(\mathbb{D})$. Then (2.1) shows that the operator $L_{T_i} : \mathscr{H} \to H^2_{\mathscr{D}_{T_i^*}}(\mathbb{D}^n)$ defined by

$$(L_{T_i} h)(\boldsymbol{z}) = D_{T_i^*} (I - z_i T_i^*)^{-1} h, \qquad (h \in \mathscr{H}, \boldsymbol{z} \in \mathbb{D}^n)$$

is an isometry for $i = 1, \ldots, n$. We now calculate

$$\begin{aligned}
\|h\|^2_{\mathscr{H}} &= \|L_{T_1} h\|^2_{H^2(\mathbb{D}^n) \otimes \mathscr{D}_{T_1^*}} = \| \sum_{k_1 \in \mathbb{N}} z_1^{k_1} D_{T_1^*} T_1^{*k_1} h \|^2_{H^2(\mathbb{D}^n) \otimes \mathscr{D}_{T_1^*}} \\
&= \sum_{k_1 \in \mathbb{N}} \|D_{T_1^*} T_1^{*k_1} h\|^2_{\mathscr{D}_{T_1^*}} = \sum_{k_1 \in \mathbb{N}} \|L_{T_2} (D_{T_1^*} T_1^{*k_1} h)\|^2_{H^2(\mathbb{D}^n) \otimes \mathscr{D}_{T_2^*}} \\
&= \sum_{k_1 \in \mathbb{N}} \| \sum_{k_2 \in \mathbb{N}} z_2^{k_2} D_{T_2^*} T_2^{*k_2} D_{T_1^*} T_1^{*k_1} h \|^2_{H^2(\mathbb{D}^n) \otimes \mathscr{D}_{T_2^*}} \\
&= \sum_{k_1, k_2 \in \mathbb{N}} \|D_{T_2^*} D_{T_1^*} T_1^{*k_1} T_2^{*k_2} h\|^2_{\mathscr{D}_{T_2^*}} \\
&= \sum_{k_1, k_2 \in \mathbb{N}} \|D_{T_1^*} D_{T_2^*} T_1^{*k_1} T_2^{*k_2} h\|^2_{\overline{\text{ran}}(D_{T_1^*} D_{T_2^*})}. \qquad (h \in \mathscr{H})
\end{aligned}$$

Continuing this process we obtain

$$\|h\|^2_{\mathscr{H}} = \sum_{\boldsymbol{k} \in \mathbb{N}^n} \| \prod_{i=1}^{n} D_{T_i^*} T^{*\boldsymbol{k}} h \|^2_{\overline{\text{ran}}(D_{T_1^*} \cdots D_{T_n^*})} = \sum_{\boldsymbol{k} \in \mathbb{N}^n} \|D_{T^*} T^{*\boldsymbol{k}} h\|^2_{\mathscr{D}_{T^*}}.$$

Hence it follows that

$$\|h\|^2_{\mathscr{H}} = \|\sum_{\boldsymbol{k}\in\mathbb{N}^n} z^{\boldsymbol{k}} D_{T^*} T^{*\boldsymbol{k}} h\|^2_{H^2_{\mathscr{D}_{T^*}}(\mathbb{D}^n)} = \|L_T h\|^2_{H^2_{\mathscr{D}_{T^*}}(\mathbb{D}^n)}. \qquad (h\in\mathscr{H})$$

This implies that L_T is an isometry. Moreover

$$L_T T_i^* h = D_{T^*} \sum_{\boldsymbol{k}\in\mathbb{N}^n} z^{\boldsymbol{k}} T^{*(\boldsymbol{k}+e_i)} h = M_{z_i}^* D_{T^*} \sum_{\boldsymbol{k}\in\mathbb{N}^n} z^{\boldsymbol{k}} T^{*\boldsymbol{k}} h = M_{z_i}^* L_T h, \qquad (h\in\mathscr{H}, 1\leqslant i\leqslant n)$$

and consequently

$$L_T T_i^* = M_{z_i}^* L_T. \qquad (1\leqslant i\leqslant n)$$

Also for all $h\in\mathscr{H}$, $\eta\in\mathscr{D}_{T^*}$ and $\boldsymbol{w}\in\mathbb{D}^n$, it follows that

$$\begin{aligned}\langle L_T^*(\mathbb{S}(\cdot,\boldsymbol{w})\eta), h\rangle_{\mathscr{H}} &= \langle \mathbb{S}(\cdot,\boldsymbol{w})\eta, L_{\boldsymbol{T}} h\rangle_{H^2_{\mathscr{D}_{T^*}}(\mathbb{D}^n)}\\ &= \langle \sum_{\boldsymbol{k}\in\mathbb{N}^n} z^{\boldsymbol{k}} \bar{w}^{\boldsymbol{k}}\eta, \sum_{\boldsymbol{l}\in\mathbb{N}^n} z^{\boldsymbol{l}} D_{T^*} T^{*\boldsymbol{l}} h\rangle_{H^2_{\mathscr{D}_{T^*}}(\mathbb{D}^n)}\\ &= \sum_{\boldsymbol{k}\in\mathbb{N}^n} \langle \bar{w}^{\boldsymbol{k}}\eta, D_{T^*} T^{*\boldsymbol{k}} h\rangle_{\mathscr{H}},\end{aligned}$$

and so

$$\langle L_T^*(\mathbb{S}(\cdot,\boldsymbol{w})\eta), h\rangle_{\mathscr{H}} = \langle \prod_{i=1}^n (I-\bar{w}_i T_i)^{-1} D_{T^*}\eta, h\rangle_{\mathscr{H}}.$$

We complete the proof by showing that the dilation $(M_{z_1},\ldots,M_{z_n})$ on $H^2_{\mathscr{D}_{T^*}}(\mathbb{D}^n)$ is minimal, that is,

$$H^2_{\mathscr{D}_{T^*}}(\mathbb{D}^n) = \overline{\text{span}}\{z^{\boldsymbol{k}}(L_T\mathscr{H}) : \boldsymbol{k}\in\mathbb{N}^n\}.$$

But since $\overline{\text{span}}\{z^{\boldsymbol{k}}(L_T\mathscr{H}) : \boldsymbol{k}\in\mathbb{N}^n\}$ is a joint $(M_{z_1},\ldots,M_{z_n})$-reducing closed subspace of $H^2_{\mathscr{D}_{T^*}}(\mathbb{D}^n)$, it follows from Proposition 2.2 in [17] that

$$\overline{\text{span}}\{z^{\boldsymbol{k}}(L_T\mathscr{H}) : \boldsymbol{k}\in\mathbb{N}^n\} = H^2_{\mathscr{E}}(\mathbb{D}^n),$$

for some $\mathscr{E}\subseteq\mathscr{D}_{T^*}$. We claim that $\mathscr{E}=\mathscr{D}_{T^*}$. To see that, first we note that for $(M_{z_1},\ldots,M_{z_n})$ on $H^2_{\mathscr{D}_{T^*}}(\mathbb{D}^n)$ we have (cf. [17])

$$\sum_{0\leqslant i_1<\ldots<i_l\leqslant n} (-1)^l M_{z_{i_1}}\cdots M_{z_{i_l}} M^*_{z_{i_1}}\cdots M^*_{z_{i_l}} = P_{\mathscr{D}_{T^*}},$$

where $P_{\mathscr{D}_{T^*}}$ is the projection to the space of constant functions. We then have

$$\Big(\sum_{0\leqslant i_1<\ldots<i_l\leqslant n} (-1)^l M_{z_{i_1}}\cdots M_{z_{i_l}} M^*_{z_{i_1}}\cdots M^*_{z_{i_l}}\Big)(L_T h) = P_{\mathscr{D}_{T^*}}(L_T h) = (L_T h)(0). \quad (h\in\mathscr{H})$$

On the other hand,

$$(L_T h)(0) = (D_{T^*}\prod_{i=1}^n (I - z_i T_i^*)^{-1} h)(0) = D_{T^*} h.$$

It now follows that $\mathscr{E}=\mathscr{D}_{T^*}$ and the proof is complete. □

The following corollary is a rephrasing of the definition of isometric dilation and Theorem 3.1.

COROLLARY 3.2. *Let T be a doubly commuting pure tuple on $\mathcal{H}$. Then $(M_{z_1},\dots,M_{z_n})$ on $H^2_{\mathcal{D}_{T^*}}(\mathbb{D}^n)$ is the minimal isometric dilation of T, that is, there exists a joint $(M^*_{z_1},\dots,M^*_{z_n})$-invariant subspace $\mathcal{Q}$ of $H^2_{\mathcal{D}_{T^*}}(\mathbb{D}^n)$ such that*

$$T_i \cong P_{\mathcal{Q}} M_{z_i}|_{\mathcal{Q}},$$

for all $1 \leqslant i \leqslant n$, and

$$H^2_{\mathcal{D}_{T^*}}(\mathbb{D}^n) = \overline{span}\{z^{\boldsymbol{k}}\mathcal{Q} : \boldsymbol{k} \in \mathbb{N}^n\}.$$

The proofs of the dilation theorem obtained in this way are quite different from any earlier proofs (cf. [11], [6], [4], [15]).

REMARK. An anonymous referee of an earlier version of this paper pointed out that most of Theorem 3.1 can be obtained using results from [2]. But, our proofs are essentially arguments based on the case of a single contraction (unlike that of [2]), because the deflect operator splits into a product of individual defect operators. Hence, our techniques demonstrate the importance of the dilation theory of a single contraction in the dilation theory of a tuple of doubly commuting contractions.

4. Canonical model

In this section, we study the analytic structure of the backward shift invariant subspace $\mathcal{Q}$ in Corollary 3.2. We begin with a few definitions.

Let $T = (T_1,\dots,T_n)$ be an n-tuple of commuting contractions on $\mathcal{H}$. Define a one variable multiplier $\Theta_{T_i} \in H^\infty_{\mathcal{B}(\mathcal{D}_{T_i},\mathcal{D}_{T_i^*})}(\mathbb{D}^n)$ by

$$\Theta_{T_i}(\boldsymbol{z}) = \theta_{T_i}(z_i), \qquad (\boldsymbol{z} \in \mathbb{D}^n)$$

where θ_{T_i} is the characteristic function of the contraction T_i and $i = 1,\dots,n$ (see the definition in (1.1)). Therefore, $M_{\Theta_{T_i}} : H^2_{\mathcal{D}_{T_i}}(\mathbb{D}^n) \to H^2_{\mathcal{D}_{T_i^*}}(\mathbb{D}^n)$ is a bounded linear operator defined by

$$(M_{\Theta_{T_i}} f)(\boldsymbol{z}) = (\Theta_{T_i} f)(\boldsymbol{z}) = \theta_{T_i}(z_i) f(\boldsymbol{z}), \qquad (\boldsymbol{z} \in \mathbb{D}^n, f \in H^2_{\mathcal{D}_{T_i}}(\mathbb{D}^n)) \tag{4.1}$$

for $i = 1,\dots,n$. It is easy to see that

$$M_{\Theta_{T_i}} M_{z_j} = M_{z_j} M_{\Theta_{T_i}},$$

for all $i,j = 1,\dots,n$, and

$$M_{\Theta_{T_i}} M^*_{z_j} = M^*_{z_j} M_{\Theta_{T_i}},$$

for all $i,j = 1,\dots,n$, and $i \neq j$. We have,

$$M_{\Theta_{T_i}} = I_{H^2(\mathbb{D})} \otimes \cdots \otimes M_{\theta_{T_i}} \otimes \cdots \otimes I_{H^2(\mathbb{D})},$$

for $i = 1,\dots,n$. We have also by virtue of (2.3)

$$\mathbb{S}(z_i,w_i)(I_{\mathcal{D}_{T_i^*}} - \Theta_{T_i}(\boldsymbol{z})\Theta_{T_i}(\boldsymbol{w})^*) = D_{T_i^*}(I - z_i T_i^*)^{-1}(I - \bar{w}_i T_i)^{-1} D_{T_i^*}, \tag{4.2}$$

for $i = 1,\dots,n$. Here we record the following simple observation.

LEMMA 4.1. *Let $T = (T_1,\dots,T_n)$ be an n-tuple of bounded linear operators on a Hilbert space $\mathcal{H}$ and T_i be of class $C_{\cdot 0}$ for all $1 = 1,\dots,n$. Then $\Theta_{T_i} \in H^\infty_{\mathcal{B}(\mathcal{D}_{T_i},\mathcal{D}_{T^*})}(\mathbb{D}^n)$ is a one variable inner function for $i = 1,\dots,n$.*

Now suppose that T is a doubly commuting tuple. Let $\boldsymbol{z},\boldsymbol{w} \in \mathbb{D}^n$. The equalities (3.1) and (3.2) imply that

$$[D_{T_i^*}(I-z_iT_i^*)^{-1}(I-\bar{w}_iT_i)^{-1}D_{T_i^*}]\big(\prod_{j=1}^n D_{T_j^*}\big) = \big(\prod_{j=1}^n D_{T_j^*}\big)[(I-z_iT_i^*)^{-1}(I-\bar{w}_iT_i)^{-1}D^2_{T_i^*}],$$

and hence

$$[D_{T_i^*}(I-z_iT_i^*)^{-1}(I-\bar{w}_iT_i)^{-1}D_{T_i^*}]\mathcal{D}_{T^*} \subseteq \mathcal{D}_{T^*}, \tag{4.3}$$

for $i = 1,\dots,n$. This observation, together with (4.2) imply that

$$(\Theta_{T_i}(\boldsymbol{z})\Theta_{T_i}(\boldsymbol{w})^*)\mathcal{D}_{T^*} \subseteq \mathcal{D}_{T^*}. \tag{4.4}$$

In particular,

$$(M_{\Theta_{T_i}}M^*_{\Theta_{T_i}})H^2_{\mathcal{D}_{T^*}}(\mathbb{D}^n) \subseteq H^2_{\mathcal{D}_{T^*}}(\mathbb{D}^n). \qquad (1 \leqslant i \leqslant n) \tag{4.5}$$

Moreover, it follows from (4.2), (4.3) and (4.4) that

$$\prod_{i=1}^n[D_{T_i^*}(I-z_iT_i^*)^{-1}(I-\bar{w}_iT_i)^{-1}D_{T_i^*}]|_{\mathcal{D}_{T^*}} = \mathbb{S}(\boldsymbol{z},\boldsymbol{w})\prod_{i=1}^n(I_{\mathcal{D}_{T_i^*}} - \Theta_{T_i}(\boldsymbol{z})\Theta_{T_i}(\boldsymbol{w})^*)|_{\mathcal{D}_{T^*}}. \tag{4.6}$$

The following result relates the characteristic functions of the coordinate operators and the isometric dilation of a doubly commuting pure tuple T.

PROPOSITION 4.2. *Let T be a doubly commuting pure tuple of operators on $\mathcal{H}$. Then*

$$L_TL_T^* = \prod_{i=1}^n(I_{H^2_{\mathcal{D}_{T^*}}(\mathbb{D}^n)} - M_{\Theta_{T_i}}M^*_{\Theta_{T_i}}|_{H^2_{\mathcal{D}_{T^*}(\mathbb{D}^n)}}).$$

Proof. Let $\boldsymbol{z},\boldsymbol{w} \in \mathbb{D}^n$ and $\eta,\zeta \in \mathcal{D}_{T^*}$ so that

$$\begin{aligned}\langle L_TL_T^*(\mathbb{S}(\cdot,\boldsymbol{w})\eta),\mathbb{S}(\cdot,\boldsymbol{z})\zeta\rangle_{H^2_{\mathcal{D}_{T^*}}(\mathbb{D}^n)} &= \langle\prod_{i=1}^n(I-\bar{w}_iT_i)^{-1}D_{T^*}\eta,\prod_{j=1}^n(I-\bar{z}_jT_j)^{-1}D_{T^*}\zeta\rangle_{\mathcal{H}}\\ &= \langle\prod_{i=1}^n D_{T^*}(1-z_iT_i^*)^{-1}(I-\bar{w}_iT_i)^{-1}D_{T^*}\eta,\zeta\rangle_{\mathcal{H}}.\end{aligned}$$

By virtue of (4.6), it follows that

$$\begin{aligned}\langle L_TL_T^*(\mathbb{S}(\cdot,\boldsymbol{w})\eta),\mathbb{S}(\cdot,\boldsymbol{z})\eta\rangle_{H^2_{\mathcal{D}_{T^*}}(\mathbb{D}^n)} &= \mathbb{S}(\boldsymbol{z},\boldsymbol{w})\langle\prod_{i=1}^n(I-\Theta_{T_i}(\boldsymbol{z})\Theta_{T_i}(\boldsymbol{w})^*)\eta,\zeta\rangle\\ &= \langle\prod_{i=1}^n(I_{H^2_{\mathcal{D}_{T^*}}(\mathbb{D}^n)} - M_{\Theta_{T_i}}M^*_{\Theta_{T_i}})(\mathbb{S}(\cdot,\boldsymbol{w})\eta),\mathbb{S}(\cdot,\boldsymbol{z})\eta\rangle,\end{aligned}$$

which completes the proof of the proposition. □

The following well known result (cf. [16]), concerning the range of the sum of a finite family of commuting orthogonal projections, will play a key role in the model theory for doubly commuting pure tuples.

LEMMA 4.3. *Let $\{P_i\}_{i=1}^n$ be a collection of commuting orthogonal projections on a Hilbert space $\mathscr{H}$. Then $\mathscr{L} := \sum_{i=1}^n ranP_i$ is closed and the orthogonal projection of $\mathscr{H}$ onto $\mathscr{L}$ is given by*

$$P_{\mathscr{L}} = I_{\mathscr{H}} - \prod_{i=1}^{n}(I_{\mathscr{H}} - P_i).$$

Proof. We set $X_i = P_i(I_{\mathscr{H}} - P_{i+1})\cdots(I_{\mathscr{H}} - P_{n-1})(I_{\mathscr{H}} - P_n)$ for all $i = 1,\ldots,n-1$, and $X_n = P_n$. Since

$$\sum_{i=1}^{n} X_i = I_{\mathscr{H}} - \prod_{i=1}^{n}(I_{\mathscr{H}} - P_i),$$

and $\{X_i\}_{i=1}^n$ is a family of orthogonal projections with orthogonal ranges, we have

$$\mathscr{L} = \mathrm{ran}X_1 \oplus \cdots \oplus \mathrm{ran}X_n.$$

This completes the proof of the lemma. □

We now have the following key corollary to the main result of this section.

COROLLARY 4.4. *Let T be a doubly commuting pure tuple on $\mathscr{H}$. Then*

$$\mathscr{S}_T := \sum_{i=1}^{n}\left(H^2_{\mathscr{D}_{T^*}}(\mathbb{D}^n)\bigcap \Theta_{T_i} H^2_{\mathscr{D}_{T_i}}(\mathbb{D}^n)\right)$$

is a closed subspace of $H^2_{\mathscr{D}_{T^*}}(\mathbb{D}^n)$ and

$$I_{H^2_{\mathscr{D}_{T^*}}(\mathbb{D}^n)} - P_{\mathscr{S}_T} = \prod_{i=1}^{n}(I_{H^2_{\mathscr{D}_{T^*}}(\mathbb{D}^n)} - M_{\Theta_{T_i}} M^*_{\Theta_{T_i}})|_{H^2_{\mathscr{D}_{T^*}}(\mathbb{D}^n)}.$$

Proof. It follows from the definition of $M_{\Theta_{T_i}}$ and the fact that T_i is pure, that $M_{\Theta_{T_i}}$ is an isometry and hence $M_{\Theta_{T_i}} M^*_{\Theta_{T_i}}$ is an orthogonal projection for $i = 1,\ldots,n$. Also by (4.5), we have

$$P_{H^2_{\mathscr{D}_{T^*}}(\mathbb{D}^n)}(M_{\Theta_{T_i}} M^*_{\Theta_{T_i}})P_{H^2_{\mathscr{D}_{T^*}}(\mathbb{D}^n)} = (M_{\Theta_{T_i}} M^*_{\Theta_{T_i}})P_{H^2_{\mathscr{D}_{T^*}}(\mathbb{D}^n)}.$$

Let $P_i = (M_{\Theta_{T_i}} M^*_{\Theta_{T_i}})|_{H^2_{\mathscr{D}_{T^*}}(\mathbb{D}^n)} \in \mathscr{B}(H^2_{\mathscr{D}_{T^*}}(\mathbb{D}^n))$. Then P_i, for each $i = 1,\ldots,n$, is an orthogonal projection and

$$\mathrm{ran}P_i = \mathrm{ran}M_{\Theta_{T_i}}\bigcap H^2_{\mathscr{D}_{T^*}}(\mathbb{D}^n) = \Theta_{T_i} H^2_{\mathscr{D}_{T_i}}(\mathbb{D}^n)\bigcap H^2_{\mathscr{D}_{T^*}}(\mathbb{D}^n). \tag{4.7}$$

Further,

$$P_iP_j = P_jP_i. \qquad (1 \leqslant i < j \leqslant n)$$

By Lemma 4.3 and (4.7), we have

$$\mathscr{S}_T = \sum_{i=1}^{n} \operatorname{ran} P_i = \sum_{i=1}^{n} \left(H^2_{\mathscr{D}_{T^*}}(\mathbb{D}^n) \bigcap \Theta_{T_i} H^2_{\mathscr{D}_{T_i}}(\mathbb{D}^n)\right),$$

is a closed subspace of $H^2_{\mathscr{D}_{T^*}}(\mathbb{D}^n)$. Again by Lemma 4.3, we have

$$P_{\mathscr{S}_T} = I_{H^2_{\mathscr{D}_{T^*}}(\mathbb{D}^n)} - \prod_{i=1}^{n}(I_{H^2_{\mathscr{D}_{T^*}}(\mathbb{D}^n)} - P_i) = I_{H^2_{\mathscr{D}_{T^*}}(\mathbb{D}^n)} - \prod_{i=1}^{n}(I_{H^2_{\mathscr{D}_{T^*}}(\mathbb{D}^n)} - M_{\Theta_{T_i}} M^*_{\Theta_{T_i}})|_{H^2_{\mathscr{D}_{T^*}}(\mathbb{D}^n)}.$$

This completes the proof. □

THEOREM 4.5. *Let T be a doubly commuting pure tuple on $\mathscr{H}$. Then for all $i = 1, \ldots, n$,*

$$T_i \cong P_{\mathscr{Q}_T} M_{z_i}|_{\mathscr{Q}_T},$$

where

$$\mathscr{Q}_T = \mathscr{S}_T^{\perp} \cong H^2_{\mathscr{D}_{T^*}}(\mathbb{D}^n)/\mathscr{S}_T,$$

is a joint $(M^*_{z_1}, \ldots, M^*_{z_n})$-invariant subspace of $H^2_{\mathscr{D}_{T^*}}(\mathbb{D}^n)$ corresponding to the joint $(M_{z_1}, \ldots, M_{z_n})$-invariant subspace

$$\mathscr{S}_T = \sum_{i=1}^{n} \left(H^2_{\mathscr{D}_{T^*}}(\mathbb{D}^n) \bigcap \Theta_{T_i} H^2_{\mathscr{D}_{T_i}}(\mathbb{D}^n)\right).$$

Proof. Let T be a doubly commuting pure tuple on $\mathscr{H}$. By Proposition 4.2, we have

$$L_T L_T^* = \prod_{i=1}^{n}(I_{H^2_{\mathscr{D}_{T^*}}(\mathbb{D}^n)} - M_{\Theta_{T_i}} M^*_{\Theta_{T_i}})|_{H^2_{\mathscr{D}_{T^*}}(\mathbb{D}^n)}.$$

This along with Corollary 4.4 yields

$$\begin{aligned} L_T L_T^* &= I_{H^2_{\mathscr{D}_{T^*}}(\mathbb{D}^n)} - [I_{H^2_{\mathscr{D}_{T^*}}(\mathbb{D}^n)} - \prod_{i=1}^{n}(I_{H^2_{\mathscr{D}_{T^*}}(\mathbb{D}^n)} - M_{\Theta_{T_i}} M^*_{\Theta_{T_i}})]|_{H^2_{\mathscr{D}_{T^*}}(\mathbb{D}^n)} \\ &= I_{H^2_{\mathscr{D}_{T^*}}(\mathbb{D}^n)} - P_{\mathscr{S}_T}. \end{aligned}$$

Consequently,

$$\operatorname{ran} L_T \cong \mathscr{S}_T^{\perp} \cong H^2_{\mathscr{D}_{T^*}}(\mathbb{D}^n)/\mathscr{S}_T,$$

and

$$T_i \cong P_{\mathscr{Q}_T} M_{z_i}|_{\mathscr{Q}_T},$$

for $i = 1, \ldots, n$. This completes the proof. □

5. One variable inner functions

The purpose of this section is to obtain a concrete realization of the joint $(M_{z_1},\dots,M_{z_n})$-invariant subspace $\mathcal{S}_T$, in Theorem 4.5, in terms of one variable inner functions on the polydisc.

Let T be a doubly commuting pure tuple of operators on $\mathcal{H}$. By Theorem 4.5, we get

$$\mathcal{H} \cong \mathcal{S}_T^{\perp}, \quad \text{and} \quad T_i \cong P_{\mathcal{S}_T^{\perp}} M_{z_i}|_{\mathcal{S}_T^{\perp}},$$

where

$$\mathcal{S}_T = \sum_{i=1}^{n} \mathcal{S}_{T_i},$$

is a joint $(M_{z_1},\dots,M_{z_n})$-invariant subspace of $H^2_{\mathcal{D}_{T^*}}(\mathbb{D}^n)$ and

$$\mathcal{S}_{T_i} := H^2_{\mathcal{D}_{T^*}}(\mathbb{D}^n) \bigcap \Theta_{T_i} H^2_{\mathcal{D}_{T_i}}(\mathbb{D}^n). \qquad (1 \leqslant i \leqslant n)$$

Recall that $H^2_{\mathcal{D}_{T^*}}(\mathbb{D}^n)$ and $\Theta_{T_i} H^2_{\mathcal{D}_{T_i}}(\mathbb{D}^n)$ can be identified with $H^2(\mathbb{D}) \otimes \cdots \otimes H^2_{\mathcal{D}_{T^*}}(\mathbb{D}) \otimes \cdots \otimes H^2(\mathbb{D})$ and $H^2(\mathbb{D}) \otimes \cdots \otimes \big(\theta_{T_i} H^2_{\mathcal{D}_{T_i}}(\mathbb{D})\big) \otimes \cdots \otimes H^2(\mathbb{D})$, respectively. Also

$$\mathcal{S}_{T_i} \cong H^2(\mathbb{D}) \otimes \cdots \otimes \tilde{\mathcal{S}}_{T_i} \otimes \cdots \otimes H^2(\mathbb{D}),$$

for some M_z-invariant subspace $\tilde{\mathcal{S}}_{T_i}$ of $H^2_{\mathcal{D}_{T_i^*}}(\mathbb{D})$.

Let $1 \leqslant i \leqslant n$ and assume that $\mathcal{S}_{T_i} \neq \{0\}$. Then by the Beurling-Lax-Halmos theorem, on shift invariant subspaces of vector-valued Hardy spaces ([18]), there exist a Hilbert space $\mathcal{E}_{T_i}$ and an inner multiplier $\phi_{T_i} \in H^\infty_{\mathcal{B}(\mathcal{E}_{T_i},\mathcal{D}_{T^*})}(\mathbb{D})$, such that

$$\tilde{\mathcal{S}}_{T_i} = \phi_{T_i} H^2_{\mathcal{E}_{T_i}}(\mathbb{D}).$$

Thus

$$\mathcal{S}_{T_i} \cong H^2(\mathbb{D}) \otimes \cdots \otimes \big(\phi_{T_i} H^2_{\mathcal{E}_{T_i}}(\mathbb{D})\big) \otimes \cdots \otimes H^2(\mathbb{D}).$$

Let

$$(\Phi_{T_i} f)(\boldsymbol{z}) = \phi_{T_i}(z_i) f(\boldsymbol{z}). \qquad (\boldsymbol{z} \in \mathbb{D}^n, f \in H^2_{\mathcal{E}_{T_i}}(\mathbb{D}^n))$$

Certainly $\Phi_{T_i} \in H^\infty_{\mathcal{B}(\mathcal{E}_{T_i},\mathcal{D}_{T^*})}(\mathbb{D}^n)$ is a one variable inner function. Moreover, $H^2(\mathbb{D}) \otimes \cdots \otimes \big(\phi_{T_i} H^2_{\mathcal{E}_{T_i}}(\mathbb{D})\big) \otimes \cdots \otimes H^2(\mathbb{D})$ can be identified to $\Phi_{T_i} H^2_{\mathcal{E}_{T_i}}(\mathbb{D}^n)$, via the same identification map, and

$$\tilde{\mathcal{S}}_{T_i} = \Phi_{T_i} H^2_{\mathcal{E}_{T_i}}(\mathbb{D}^n).$$

Consequently,

$$\mathcal{S}_T = \sum_{i=1}^{n} \Phi_{T_i} H^2_{\mathcal{E}_{T_i}}(\mathbb{D}^n),$$

where each $\Phi_{T_i} \in H^\infty_{\mathcal{B}(\mathcal{E}_{T_i},\mathcal{D}_{T^*})}(\mathbb{D}^n)$ is either a one variable inner function in z_i, or the zero function and $i = 1,\dots,n$.

This along with Theorem 4.5 proves the following result.

THEOREM 5.1. *Let T be a doubly commuting pure tuple on $\mathcal{H}$. Then there exists a joint $(M_{z_1}^*, \dots, M_{z_n}^*)$-invariant subspace $\mathcal{Q}_T$ of $H^2_{\mathcal{D}_{T^*}}(\mathbb{D}^n)$ such that*

$$\mathcal{H} \cong \mathcal{Q}_T, \quad \text{and} \quad T_i \cong P_{\mathcal{Q}_T} M_{z_i}|_{\mathcal{Q}_T},$$

for $i = 1, \dots, n$. Moreover, there exist Hilbert spaces $\{\mathcal{E}_{T_i}\}_{i=1}^n$ and $\Phi_{T_i} \in H^\infty_{\mathcal{B}(\mathcal{E}_{T_i}, \mathcal{D}_{T^*})}(\mathbb{D}^n)$, such that each Φ_{T_i} $(1 \leqslant i \leqslant n)$ is either a one variable inner function in z_i, or the zero function and

$$\mathcal{S}_T := \sum_{i=1}^n \Phi_{T_i} H^2_{\mathcal{E}_{T_i}}(\mathbb{D}^n)$$

is closed in $H^2_{\mathcal{D}_{T^*}}(\mathbb{D}^n)$, and

$$\mathcal{Q}_T = \mathcal{S}_T^\perp.$$

In particular, Theorem 5.1 says that the class of all doubly commuting pure tuples on separable Hilbert spaces is equal, to the class of all doubly commuting $(M_{z_1}^*, \dots, M_{z_n}^*)$-invariant subspaces of vector-valued Hardy spaces over the polydisc.

As a special case of Theorem 5.1 we obtain the following corollary.

COROLLARY 5.2. *Let $\mathcal{Q}$ be a joint $(M_{z_1}^*, \dots, M_{z_n}^*)$-invariant closed proper subspace of $H^2(\mathbb{D}^n)$ and let $C_{z_i} := P_{\mathcal{Q}} M_{z_i}|_{\mathcal{Q}}$ for $i = 1, \dots, n$. Then $(C_{z_1}, \dots, C_{z_n})$ is doubly commuting if and only if there exists $\{\theta_i\}_{i=1}^n \subseteq H^\infty(\mathbb{D})$ such that each θ_i is either inner or the zero function for $i = 1, \dots, n$ and*

$$\mathcal{Q} = \Big(\sum_{i=1}^n \Theta_i H^2(\mathbb{D}^n) \Big)^\perp,$$

where $\Theta_i(\boldsymbol{z}) = \theta_i(z_i)$ for all $\boldsymbol{z} \in \mathbb{D}^n$ and $i = 1, \dots, n$.

Proof. If $T := (C_{z_1}, \dots, C_{z_n})$, then

$$D_{T^*}^2 = \prod_{i=1}^n (I_{\mathcal{Q}} - C_{z_i} C_{z_i}^*) = P_{\mathcal{Q}} \Big(\prod_{i=1}^n (I_{H^2(\mathbb{D}^n)} - M_{z_i} M_{z_i}^*) \Big)|_{\mathcal{Q}} = P_{\mathcal{Q}} P_{\mathbb{C}}|_{\mathcal{Q}}.$$

Thus the rank of D_{T^*} is one. Now the result follows from Theorem 5.1. □

This result was proved by the third author in [16]. See also the work by Izuchi, Nakazi and Seto [10] for the base case $n = 2$.

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(Received October 6, 2015)

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