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A BOUND FOR THE HILBERT–SCHMIDT NORM OF GENERALIZED COMMUTATORS OF NONSELF–ADJOINT OPERATORS

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Abstract. Let $A,\tilde{A}$ and B be bounded linear operators in a Hilbert space, and $f(z)$ be a function regular on the convex hull of the union of the spectra of A and $\tilde{A}$. Let SN_2 be the ideal of Hilbert-Schmidt operators. In the paper, a sharp estimate for the Hilbert-Schmidt norm of the commutator $f(A)B-Bf(\tilde{A})$ is established, provided $AB-B\tilde{A}\in SN_2$, $A-A^*\in SN_2$ and $\tilde{A}-\tilde{A}^*\in SN_2$. Here the star means the adjointness. Our results are new even in the finite dimensional case.

1. Introduction and statement of the main result

Let $\mathscr{H}$ be a separable complex Hilbert space with a scalar product $(.,.)$, the norm $\|.\|=\sqrt{(.,.)}$ and unit operator I. $\mathscr{B}(\mathscr{H})$ means the algebra of all bounded linear operators in $\mathscr{H}$. For an $A\in\mathscr{B}(\mathscr{H})$, $\sigma(A)$ denotes the spectrum, $R_z(A)=(A-zI)^{-1}$ $(z\notin\sigma(A))$ is the resolvent, $r_s(A)$ is the spectral radius, A^* is the adjoint operator. SN_2 denotes the ideal of Hilbert-Schmidt operators C with the finite norm $N_2(C):=(Trace\ (CC^*))^{1/2}$.

Let $\Omega\supset\sigma(A)$ be an open set whose boundary L consists of a finite number of rectifiable Jordan curves, oriented in the positive sense customary in the theory of complex variables. Suppose that $\Omega\cup L$ is contained in the domain of analyticity of a scalar-valued function f. Then $f(A)$ is defined by

$$f(A)=-\frac{1}{2\pi i}\int_L f(z)R_z(A)dz. \tag{1.1}$$

Furthermore, for $A,B,\tilde{A}\in\mathscr{B}(\mathscr{H})$, $[A,B]:=AB-BA$ is the commutator, $[A,B,\tilde{A}]:=AB-B\tilde{A}$ is the generalized commutator; $[f(A),B]:=f(A)B-Bf(A)$ and $[f(A),B,f(\tilde{A})]:=f(A)B-Bf(\tilde{A})$ will be called the function commutator and the generalized function commutator, respectively.

As it is well-known, the generalized function commutator plays an essential role in the perturbation theory of operators, cf. [2] and references therein. Various estimates for the generalized function commutator and its particular cases have been derived by many mathematicians. For details and further references we recommend the

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papers [1, 3, 4, 11]. Besides, mainly selfadjoint and normal operators have been considered. Recently, for a class of analytic functions, Kittaneh [10] considered the operator $f(A)B - Bf(\tilde{A})$ in symmetric ideals of $\mathscr{H}$. Besides A satisfies the inequality

$$\|R_z(A)\| \leqslant \frac{1}{dist\ (z, \sigma(A))}.$$

The similar condition is satisfied by $\tilde{A}$. In that paper, bounds are established for an arbitrary unitarily invariant norm of the commutator.

In the present paper we consider the generalized function commutator with non-normal operators satisfying the conditions

$$A_I := (A - A^*)/2i \in SN_2, \quad \tilde{A}_I := (\tilde{A} - \tilde{A}^*)/2i \in SN_2, \tag{1.2}$$

and

$$K := AB - B\tilde{A} \in SN_2. \tag{1.3}$$

To formulate the result put

$$g_I(A) := \left[2N_2^2(A_I) - 2\sum_{k=1}^{\infty} |Im\ \lambda_k(A)|^2 \right]^{1/2},$$

where $\lambda_k(A)$ $(k = 1, 2, \ldots)$ are the nonreal eigenvalues of A taken with their multiplicities. If A is normal, then $g_I(A) = 0$, cf. [5, Lemma 7.7.2]. Obviously, $g_I(A) \leqslant \sqrt{2}N_2(A_I)$.

Denote by $co(A, \tilde{A})$ the closed convex hull of $\sigma(A) \cup \sigma(\tilde{A})$. Now we are in a position to formulate the main result of the paper.

THEOREM 1.1. *Let conditions (1.2) and (1.3) hold. Let $f(\lambda)$ be holomorphic on a neighborhood of $co(A, \tilde{A})$. Then with the notations*

$$\psi_{j,k} := \sup_{z \in co\ (A, \tilde{A})} \frac{|f^{(k+j+1)}(z)|}{\sqrt{k!j!}(k+j+1)!} \quad (j, k = 0, 1, 2, \ldots),$$

we have the inequality

$$N_2(f(A)B - Bf(\tilde{A})) \leqslant N_2(K) \sum_{j,k=0}^{\infty} \psi_{j,k} g_I^j(A) g_I^k(\tilde{A}).$$

The proof of this theorem is presented in the next two sections. Theorem 1.1 particularly generalizes the main results from [6, 7].

If A and $\tilde{A}$ are normal operators, then Theorem 1.1 implies the inequality.

$$N_2(f(A)B - Bf(\tilde{A})) \leqslant N_2(K) \sup_{z \in co\ (A, \tilde{A})} |f'(z)|.$$

If $A \in SN_2$, then due to [5, Lemma 6.5.2] we have $g_I(A) = g(A)$, where

$$g(A) := \left[N_2^2(A) - \sum_{k=1}^{\infty} |\hat{\lambda}_k(A)|^2 \right]^{1/2}.$$

Here $\hat{\lambda}_k(A)$ $(k = 1, 2, \ldots)$ are all the eigenvalues of A taken with their multiplicities. In addition,

$$g^2(A) \leqslant N_2^2(A) - |Trace\, A^2|,$$

since

$$\sum_{k=1}^{\infty} |\hat{\lambda}_k(A)|^2 \geqslant |Trace\, A^2|.$$

EXAMPLE 1.2. Let $f(A) = e^{At}, t \geqslant 0$. Then

$$\sup_{z \in co\,(A,\tilde{A})} \left| \frac{d^{k+j+1} e^{zt}}{dz^{k+j+1}} \right| = e^{\alpha t} t^{k+j+1} \quad (j,k = 0,1,2,\ldots;\ t \geqslant 0),$$

where

$$\alpha := \max\{\sup\, Re\sigma(A), \sup\, Re\sigma(\tilde{A})\}.$$

Thus,

$$\psi_{j,k} = \frac{e^{\alpha t} t^{k+j+1}}{\sqrt{k!j!}(k+j+1)!} \quad (j,k = 0,1,2,\ldots).$$

Due to Theorem 1.1

$$N_2(e^{At}B - Be^{\tilde{A}t}) \leqslant e^{\alpha t} N_2(K) \sum_{j,k=0}^{\infty} \frac{t^{k+j+1} g_I^k(A) g_I^j(\tilde{A})}{\sqrt{k!j!}(k+j+1)!} \quad (t \geqslant 0).$$

2. Auxiliary results

In this section our reasonings are valid if $\mathscr{H}$ is considered as a Banach space. Let $A, \tilde{A}, B \in \mathscr{B}(\mathscr{H})$. Then for any $z \notin \sigma(A) \cup \sigma(\tilde{A})$, we have

$$(zI - A)^{-1}B - B(zI - \tilde{A})^{-1} = (Iz - A)^{-1}K(Iz - \tilde{A})^{-1}. \tag{2.1}$$

This well known identity can be checked by multiplying the both sides of (2.1) by $zI - A$ from the left and by $zI - \tilde{A}$ from the right.

Formulas (2.1) and (1.1) imply

$$\begin{aligned} f(A)B - Bf(\tilde{A}) &= -\frac{1}{2\pi i} \int_L f(z)(R_z(A)B - BR_z(\tilde{A}))dz \\ &= \frac{1}{2\pi i} \int_L f(z) R_z(A) K R_z(\tilde{A}) dz \end{aligned} \tag{2.2}$$

for any $f(z)$ regular on a neighborhood of $\sigma(A) \cup \sigma(\tilde{A})$.

Put $\Omega(r) = \{|z| \leqslant r\}$ and $\partial\Omega(r) = \{|z| = r\}$. In the rest of this section it is assumed that $f(z)$ is regular on $\Omega(r)$ with $r > r_s(A,\tilde{A}) := \max\{r_s(A), r_s(\tilde{A})\}$. Take into account that

$$R_\lambda(A) = -\sum_{k=0}^{\infty} \frac{A^k}{\lambda^{k+1}} \ \ (|\lambda| > r_s(A)).$$

Then by the previous lemma

$$f(A)B - Bf(\tilde{A}) = \frac{1}{2\pi i}\int_{\partial\Omega(r)} f(z)R_z(A)KR_z(\tilde{A})dz = \sum_{j,k=0}^{\infty} \frac{1}{2\pi i}\int_{\partial\Omega(r)} \frac{f(z)dz}{z^{k+j+2}} A^j K\tilde{A}^k.$$

Or

$$f(A)B - Bf(\tilde{A}) = \sum_{j,k=0}^{\infty} f_{j+k+1} A^j K\tilde{A}^k, \tag{2.3}$$

where f_j are the Taylor coefficients of f at zero.

If, in particular, $f(z) = z^m$, for an integer $m \geqslant 1$, then $f_{jk} = 0$ for $j+k+1 \neq m$ and $f_{jk} = 1$ for $j+k+1 = m$. Thus we arrive at the identity

$$A^m B - B\tilde{A}^m = \sum_{j=0}^{m-1} A^j K\tilde{A}^{m-j-1}. \tag{2.4}$$

3. Proof of Theorem 1.1

Let A, B and $\tilde{A}$ have n-dimensional ranges $(n < \infty)$. Then by the triangular (Schur) representation

$$A = D + V \ (\sigma(A) = \sigma(D)), \tag{3.1}$$

where D is a normal and V is a nilpotent operators having the joint invariant subspaces. Similarly,

$$\tilde{A} = \tilde{D} + \tilde{V} \ (\sigma(\tilde{A}) = \sigma(\tilde{D})), \tag{3.2}$$

where $\tilde{D}$ is a normal and $\tilde{V}$ is a nilpotent operators having the same invariant subspaces. Let us prove that

$$N_2(f(A)B - Bf(\tilde{A})) \leqslant N_2(K) \sum_{j,k=0}^{n-1} \psi_{j,k} N_2^j(V) N_2^k(\tilde{V}). \tag{3.3}$$

Indeed, by (3.1)

$$R_\lambda(A) = (D + V - I\lambda)^{-1} = (I + R_\lambda(D)V)R_\lambda(D).$$

Note that $R_\lambda(D)V$ is a nilpotent matrix and therefore $(R_\lambda(D)V)^n = 0$. Consequently,

$$R_\lambda(A) = \sum_{k=0}^{n-1} (-1)^k (R_\lambda(D)V)^k R_\lambda(D).$$

Similarly,

$$R_\lambda(\tilde{A}) = \sum_{k=0}^{n-1} (-1)^k (R_\lambda(\tilde{D})\tilde{V})^k R_\lambda(\tilde{D}).$$

So by (2.2) we have

$$f(A)B - Bf(\tilde{A}) = \sum_{m,k=0}^{n-1} C_{mk} \tag{3.4}$$

where

$$C_{mk} = (-1)^{k+m} \frac{1}{2\pi i} \int_L f(\lambda)(R_\lambda(D)V)^m R_\lambda(D) K (R_\lambda(\tilde{D})\tilde{V})^k R_\lambda(\tilde{D}) d\lambda.$$

Since D is a diagonal matrix in the *orthonormal basis of the triangular representations of* A *(the Schur basis)* $\{e_k\}$ *of* A, and $\tilde{D}$ is a diagonal matrix in the *Schur basis* $\{\tilde{e}_k\}$ *of* $\tilde{A}$, we can write out

$$R_\lambda(D) = \sum_{j=1}^{n} \frac{Q_j}{\lambda_j - \lambda}, \quad R_\lambda(\tilde{D}) = \sum_{j=1}^{n} \frac{\tilde{Q}_j}{\tilde{\lambda}_j - \lambda},$$

where $\lambda_j = \lambda_j(A), \tilde{\lambda}_j = \lambda_j(\tilde{A})$, $Q_k = (., e_k)e_k, \tilde{Q}_k = (., \tilde{e}_k)\tilde{e}_k$. Consequently,

$$C_{mk} = \sum_{i_1=1}^{n} Q_{i_1} V \sum_{i_2=1}^{n} Q_{i_2} V \ldots V \sum_{i_{m+1}=1}^{n} Q_{i_{m+1}} K \sum_{j_1=1}^{n} \tilde{Q}_{j_1} \tilde{V} \sum_{j_2=1}^{n} \tilde{Q}_{j_2} \tilde{V} \ldots \tag{3.5}$$

$$\tilde{V} \sum_{j_{k+1}=1}^{n} \tilde{Q}_{j_{k+1}} J_{i_1, i_2, \ldots, i_{m+1}, j_1 j_2 \ldots j_{k+1}}.$$

Here

$$J_{i_1, i_2, \ldots, i_{m+1}, j_1 j_2 \ldots j_{k+1}}$$

$$= \frac{(-1)^{k+m}}{2\pi i} \int_L \frac{f(\lambda) d\lambda}{(\lambda_{i_1} - \lambda) \ldots (\lambda_{i_{m+1}} - \lambda)(\tilde{\lambda}_{j_1} - \lambda) \ldots (\tilde{\lambda}_{j_{k+1}} - \lambda)}.$$

Below the symbol $|V|$ means the operator whose entries are absolute values of the entries of V in the basis $\{e_k\}$ and $|\tilde{V}|$ means the operator whose entries are absolute values of the entries of $\tilde{V}$ in the basis $\{\tilde{e}_k\}$. That is, if

$$Ve_k = \sum_{j=1}^{k-1} a_{jk} e_j, \quad \tilde{V} e_k = \sum_{j=1}^{k-1} \tilde{a}_{jk} \tilde{e}_j,$$

then $|V|$ and $|\tilde{V}|$ are defined by

$$|V| e_k = \sum_{j=1}^{k-1} |a_{jk}| e_j \quad \text{and} \quad \tilde{V} e_k = \sum_{j=1}^{k-1} |\tilde{a}_{jk}| \tilde{e}_j,$$

respectively.

Furthermore, denote $K_{kj} = (K\tilde{e}_j, e_k)$ and $c_{kj}^{(ml)} = (C_{ml}\tilde{e}_j, e_k)$. Then

$$K = \sum_{j,k=1}^{n} K_{kj}(.,\tilde{e}_j)e_k \quad \text{and} \quad C_{ml} = \sum_{j,k=1}^{n} c_{kj}^{(ml)}(.,\tilde{e}_j)e_k.$$

Introduce the operators

$$|K| = \sum_{j,k=1}^{n} |K_{kj}|(.,\tilde{e}_j)e_k \quad \text{and} \quad |C_{ml}| = \sum_{j,k=1}^{n} |c_{kj}^{(ml)}|(.,\tilde{e}_j)e_k.$$

So $|K|$ and $|C_{ml}|$ are defined as the operators whose entries are absolute values of the entries of $|K|$ and $|C_{ml}|$ in the corresponding base.

By [5, Lemma 1.5.1],

$$|J_{i_1,i_2,\ldots,i_{m+1},j_1 j_2 \ldots j_{k+1}}| \leqslant \tilde{\psi}_{m,k} := \sup_{z \in co\,(A,\tilde{A})} \frac{|f^{(k+m+1)}(z)|}{(m+k+1)!}.$$

Now (3.5) implies

$$|C_{mk}| \leqslant \tilde{\psi}_{m,k} \sum_{i_1=1}^{n} Q_{i_1}|V| \sum_{i_2=1}^{n} Q_{i_2}|V| \ldots |V| \sum_{i_{m+1}=1}^{n} Q_{j_{m+1}}|K| \sum_{j_1=1}^{n} \tilde{Q}_{j_2}|\tilde{V}| \ldots |\tilde{V}| \sum_{j_{k+1}=1}^{n} \tilde{Q}_{j_{k+1}}.$$

But

$$\sum_{i=1}^{n} Q_i = I.$$

Thus,

$$|C_{mk}| \leqslant \tilde{\psi}_{m,k}|V|^m|K||\tilde{V}|^k. \tag{3.6}$$

Note that

$$N_2^2(|K|) = \sum_{k=1}^{n} \||K|\tilde{e}_k\|^2 = \sum_{k=1}^{n}\sum_{j=1}^{n} |K_{jk}|^2 = N_2^2(K).$$

Hence (3.6) yields the inequality

$$N_2(C_{mk}) \leqslant \tilde{\psi}_{m,k}\||V|^m\| N_2(K)\||\tilde{V}|^k\|.$$

By [5, Corollary 2.5.2] we have

$$\|\,|V|^m\| \leqslant \frac{N_2^m(|V|)}{\sqrt{m!}}.$$

But obviously,

$$N_2^2(|V|) = \sum_{k=2}^{n}\sum_{j=1}^{k-1} |a_{jk}|^2 = N_2^2(V).$$

Recall that a_{jk} are the entries of V in the Schur basis. Thus,

$$\|\,|V|^m\| \leqslant \frac{N_2^m(V)}{\sqrt{m!}}$$

So

$$N_2(C_{mk}) \leqslant \tilde{\psi}_{m,k} N_2(K) \frac{N_2^m(V) N_2^k(\tilde{V})}{\sqrt{m!}\sqrt{k!}}.$$

Now (3.4) implies the required inequality (3.3).

Furthermore, taking into account the equality $N_2(V) = g_I(A)$, cf. [5, Lemma 7.7.2], from (3.3) we have

$$N_2(f(A)B - Bf(\tilde{A})) \leqslant N_2(K) \sum_{j,k=0}^{n-1} \psi_{j,k} g_I^j(A) g_I^k(\tilde{A}), \tag{3.7}$$

provided A and $\tilde{A}$ are n-dimensional.

Now let us recall that there is a sequence of n-dimensional operators A_n, strongly converging to A, such that $\sigma(A_n) \subseteq \sigma(A)$, cf. [5, Section 7.11]. In addition, $g_I(A_n) \to g_I(A)$ as $n \to \infty$. Indeed, according to [8, Theorem I.5.2], the nonreal spectrum of A under conditions (1.2) consists of no more countable number of points which are normal eigenvalues (that is, isolated and having finite multiplicities). Denote by $\mathscr{E}$ the linear closed convex hall of all the root vectors of A corresponding to non-real eigenvalues. Choice in each root subspace a Jordan basis. Then we obtain vectors ϕ_k for each of which either $A\phi_k = \lambda_k(A)\phi_k$, or $A\phi_k = \lambda_k(A)\phi_k + \phi_{k+1}$. Orthogonalizing the system $\{\phi_k\}$, we obtain the (orthonormal) Schur basis $\{e_k\}$ of the triangular representation:

$$Ae_k = a_{1k}e_1 + a_{2k}e_2 + \ldots + a_{kk}e_k \ \ (k = 1, 2, \ldots) \tag{3.8}$$

with $a_{kk} = \lambda_k(A)$ (see [8, Section II.6])). Besides, $\mathscr{E}$ is an invariant subspace of A. Let P be the orthogonal projection of $\mathscr{H}$ onto $\mathscr{E}$ and $C = AP = PAP$. So $\sigma(C)$ consists of the nonreal spectrum of A. Denote $P_1 = I - P$, $M = P_1AP_1$ and $W = PAP_1$. We have

$$A = (P + P_1)A(P + P_1) = C + M + W,$$

since $P_1AP = P_1PAP = 0$. Take into account that

$$(C + M - \lambda)^{-1} = (C - P\lambda)^{-1} + (M - P_1\lambda)^{-1},$$

where the inverse $(C - P\lambda)^{-1}$ is understood in $P\mathscr{H}$ and the inverse $(M - P_1\lambda)^{-1}$ is understood in $P_1\mathscr{H}$. Hence,

$$(C + M - \lambda)^{-1}W = (C + M - \lambda)^{-1}PAP_1 = P(C - P\lambda)^{-1}PAP_1$$

and therefore, $((C + M - \lambda)^{-1}W)^2 = 0$. Consequently, for sufficiently large λ,

$$(I + (C + M - \lambda)^{-1}W)^{-1} = \sum_{k=0}^{\infty} (-1)^k ((C + M - \lambda)^{-1}W)^k = I - (C + M - \lambda)^{-1}W.$$

So for all regular λ,

$$\begin{aligned}(A - \lambda)^{-1} &= (C + M + W - \lambda)^{-1} = (I + (C + M - \lambda)^{-1}W)^{-1}(C + M - \lambda)^{-1} \\ &= (C + M - \lambda)^{-1} + (C + M - \lambda)^{-1}W(C + M - \lambda)^{-1}.\end{aligned}$$

Consequently,

$$\sigma(A) = \sigma(C) \cup \sigma(M). \tag{3.9}$$

Moreover,

$$(AP_1 - \lambda)^{-1} = (W + M - \lambda)^{-1} = (I + (M-\lambda)^{-1}W)^{-1}(M-\lambda)^{-1}$$
$$= (M-\lambda)^{-1} + (M-\lambda)^{-1}W(M-\lambda)^{-1}.$$

Thus $\sigma(AP_1) = \sigma(M)$ and it is real. Let C_n $(n = 1,2,\dots)$ be the sequence of n-dimensional operators, defined by

$$C_n e_k = a_{1k}e_1 + a_{2k}e_2 + \dots + a_{kk}e_k \ \ (k = 1,\dots,n).$$

Then according to (3.8) $\lambda_k(C_n) = \lambda_k(A)$ $(k = 1,\dots,n)$ and $C_n \to C$ at least strongly. Let Q_n be a sequence of n-dimensional orthogonal projections strongly converging to I. Then $Q_nAP_1 \to AP_1$ strongly. Since $C_n = C_nP$, according to (3.9),

$$\sigma(Q_nAP_1 + C_n) = \sigma(Q_nAP_1) \cup \sigma(C_n).$$

Besides, $A_n := Q_nAP_1 + C_n \to A$ strongly. Due to the upper semicontinuity of the spectrum [9, p. 56, Problem 103], $\lim_{n\to\infty} \sigma(A_n) \subseteq \sigma(A)$. Hence, since $\sigma(AP_1)$ is real, we can write

$$\lim_{n\to\infty} \Im \sigma(Q_nAP_1) = 0$$

and therefore, under condition (1.2),

$$\lim_{n\to\infty} \sum_{k=1}^{n} |\Im \lambda_k(A_n)|^2 = \lim_{n\to\infty} \sum_{k=1}^{n} |\Im \lambda_k(C_n)|^2 = \sum_{k=1}^{\infty} |\Im \lambda_k(C)|^2 = \sum_{k=1}^{\infty} |\Im \lambda_k(A)|^2. \tag{3.10}$$

Since, $A_n \to A$ strongly, we have $A_n - A_n^* \to A - A^*$ strongly, but $A - A^* \in SN_2$, and therefore, $N_2(A_n - A_n^*) \to N_2(A - A^*)$. This and (3.10) implies that really $g_I(A_n) \to g_I(A)$ as $n \to \infty$. Moreover, due to the relation $\sigma(A_n) \subseteq \sigma(A)$ we have $\lim_n co(A_n) \subseteq co(A)$. So replacing in (3.7) A and $\tilde{A}$ by A_n and $\tilde{A}_n$, respectively, we get the required result by passing to the limit $n \to \infty$ in that inequality. $\square$

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