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MAPS PRESERVING THE LOCAL SPECTRUM OF THE SKEW JORDAN PRODUCT OF OPERATORS

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Abstract. Let $\mathscr{H}$ and $\mathscr{K}$ be two infinite-dimensional complex Hilbert spaces, and fix two nonzero vectors $h_0 \in \mathscr{H}$ and $k_0 \in \mathscr{K}$. Let $\mathscr{L}(\mathscr{H})$ (resp. $\mathscr{L}(\mathscr{K})$) denote the algebra of all bounded linear operators on $\mathscr{H}$ (resp. on $\mathscr{K}$), and let $\mathscr{F}_2(\mathscr{K})$ be the set of all operators in $\mathscr{L}(\mathscr{K})$ of rank at most two. We show that a map φ from $\mathscr{L}(\mathscr{H})$ into $\mathscr{L}(\mathscr{K})$ such that its range contains $\mathscr{F}_2(\mathscr{K})$ satisfies

$$\sigma_{\varphi(T)\varphi(S)^*+\varphi(S)^*\varphi(T)}(k_0) = \sigma_{TS^*+S^*T}(h_0), \quad (T,\, S \in \mathscr{L}(\mathscr{H})),$$

if and only if there exist a unitary operator U from $\mathscr{H}$ into $\mathscr{K}$ and a scalar $\alpha \in \mathbb{C}$ such that $Uh_0 = \alpha k_0$ and $\varphi(T) = \lambda UTU^*$ for all $T \in \mathscr{L}(\mathscr{H})$, where λ is a scalar of modulus 1.

1. Introduction and statement of the main result

Let X and Y be two infinite-dimensional complex Banach spaces, and let $\mathscr{B}(X,Y)$ denote the space of all bounded linear maps from X into Y. When $X = Y$, we simply write $\mathscr{B}(X)$ instead of $\mathscr{B}(X,X)$ and denote its identity operator by $\mathbf{1}$. The local resolvent set, $\rho_T(x)$, of an operator $T \in \mathscr{B}(X)$ at a point $x \in X$ is the union of all open subsets U of $\mathbb{C}$ for which there is an analytic function $\phi : U \to X$ such that $(T-\lambda)\phi(\lambda) = x$, $(\lambda \in U)$. The local spectrum of T at x is defined by $\sigma_T(x) := \mathbb{C} \setminus \rho_T(x)$, and is obviously a closed subset (possibly empty) of $\sigma(T)$, the spectrum of T. In fact, $\sigma_T(x) \neq \emptyset$ for all nonzero vectors x in X precisely when T has the single-valued extension property (SVEP). Recall that T is said to have SVEP provided that for every open subset U of $\mathbb{C}$, the equation $(T-\lambda)\phi(\lambda) = 0$, $(\lambda \in U)$, has no nontrivial analytic solution ϕ. Every operator $T \in \mathscr{B}(X)$ for which the interior of its point spectrum, $\sigma_p(T)$, is empty enjoys this property. Our references are the books by P. Aiena [2] and by K. B. Laursen, M. M. Neumann [14] which provide an excellent exposition as well as a rich bibliography of the local spectral theory.

In recent years, there has been considerable interest in studying preserver problems of local spectra; see [1, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12] and the references therein. In [4], A. Bourhim and M. Mabrouk described maps from $\mathscr{B}(X)$ onto $\mathscr{B}(Y)$ preserving the local spectrum of the Jordan product of operators, and established the following result.

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THEOREM 1.1. ([4]) *Let* $x_0 \in X$ *and* $y_0 \in Y$ *be two nonzero vectors. A map* φ *from* $\mathscr{B}(X)$ *onto* $\mathscr{B}(Y)$ *satisfies*

$$\sigma_{\varphi(T)\varphi(S)+\varphi(S)\varphi(T)}(y_0) = \sigma_{TS+ST}(x_0), \; (T, \, S \in \mathscr{B}(X)), \tag{1.1}$$

if and only if there exists a bijective bounded linear mapping A *from* X *into* Y *such that* $Ax_0 = y_0$ *and* $\varphi(T) = \pm ATA^{-1}$ *for all* $T \in \mathscr{B}(X)$.

In the sequel, let $\mathscr{H}$ and $\mathscr{K}$ be two infinite-dimensional complex Hilbert spaces, and let T^* denote as usual the adjoint of any operator $T \in \mathscr{L}(\mathscr{H})$. The purpose of this note is to characterize all maps φ from $\mathscr{L}(\mathscr{H})$ into $\mathscr{L}(\mathscr{K})$ which preserve the local spectrum of the skew-Jordan product "$TS^* + S^*T$" of operators.

The main result of this article is the following

THEOREM 1.2. *Let* $h_0 \in \mathscr{H}$ *and* $k_0 \in \mathscr{K}$ *be two nonzero vectors, and let* φ *be a map from* $\mathscr{L}(\mathscr{H})$ *into* $\mathscr{L}(\mathscr{K})$ *such that its range contains* $\mathscr{F}_2(\mathscr{K})$*, the ideal of all operators in* $\mathscr{L}(\mathscr{K})$ *of rank at most two. Then* φ *satisfies*

$$\sigma_{TS^*+S^*T}(h_0) = \sigma_{\varphi(T)\varphi(S)^*+\varphi(S)^*\varphi(T)}(k_0), \; (T, \, S \in \mathscr{L}(\mathscr{H})), \tag{1.2}$$

if and only if there exist a scalar α *and a unitary operator* $U \in \mathscr{L}(\mathscr{H}, \mathscr{K})$ *such that* $Uh_0 = \alpha k_0$ *and* $\varphi(T) = \lambda UTU^*$ *for all* $T \in \mathscr{L}(\mathscr{H})$*, where* λ *is a scalar of modulus* 1.

Our arguments are influenced by ideas from [4] and the approach given therein, but, besides some known results quoted from [4], the proof of the above result requires new ingredients which will be established in Section 3. We also would like to mention that without any restriction on the range of the map φ, our result does not hold as $\sigma_{T\oplus T}(h \oplus h) = \sigma_T(h)$ for all $T \in \mathscr{L}(\mathscr{H})$ and all $h \in \mathscr{H}$. We also would like to point out that if $\mathscr{H}$ and $\mathscr{K}$ are isomorphic Hilbert spaces, then the statements of our result can be reduced to the case when $\mathscr{H} = \mathscr{K}$ and $h_0 = k_0$. But the fact that "$\mathscr{H}$ and $\mathscr{K}$ are isomorphic" is a part of the conclusion of this result rather being a part of its hypothesis. Finally, we would like to point out that the restriction to infinite-dimensional Hilbert spaces in the statement of A. Bourhim and M. Mabrouk's result [4] and our main result are just for the sake of simplicity. These results and their proofs remain valid for finite-dimensional case. In fact, A. Bourhim and M. Mabrouk showed in [3] that Theorem 1.1 remains valid when $X = \mathbb{C}^n$ is a finite-dimensional Banach space and without the surjectivity of the map φ or any restriction on its range.

2. Preliminaries

In this section, we collect some elementary properties of the local spectra together with useful results from [4] that will be used in the proof of the main theorem.

2.1. Elementary properties of the local spectrum

The following lemma summarizes some basic properties of the local spectrum; see for instance [2, 14].

LEMMA 2.1. *For an operator $T \in \mathscr{L}(\mathscr{H})$, two vectors $x,\ y \in \mathscr{H}$ and a nonzero scalar $\alpha \in \mathbb{C}$, the following statements hold.*

(a) *If T has SVEP, then $\sigma_T(x) \neq \emptyset$ provided that $x \neq 0$.*

(b) $\sigma_T(\alpha x) = \sigma_T(x)$ *and* $\sigma_{\alpha T}(x) = \alpha\sigma_T(x)$.

(c) $\sigma_T(x+y) \subset \sigma_T(x) \cup \sigma_T(y)$. *The equality holds if* $\sigma_T(x) \cap \sigma_T(y) = \emptyset$.

(d) *If T has SVEP, $x \neq 0$ and $Tx = \lambda x$ for some $\lambda \in \mathbb{C}$, then $\sigma_T(x) = \{\lambda\}$.*

(e) *If T has SVEP and $Tx = \alpha y$, then $\sigma_T(y) \subset \sigma_T(x) \subset \sigma_T(y) \cup \{0\}$.*

(f) *If $R \in \mathscr{L}(\mathscr{H})$ commutes with T, then $\sigma_T(Rx) \subset \sigma_T(x)$.*

(g) $\sigma_{T^n}(x) = \{\sigma_T(x)\}^n$ *for all $x \in \mathscr{H}$ and $n \geqslant 1$.*

In the sequel, for two vectors x and y in $\mathscr{H}$, let $x \otimes y$ stand for the operator of rank at most one defined by

$$(x \otimes y)z := \langle z, y\rangle x,\ (z \in \mathscr{H}).$$

Note that every rank one operator in $\mathscr{L}(\mathscr{H})$ has such a form and that every finite rank operator $T \in \mathscr{L}(\mathscr{H})$ can be written as a finite sum of rank one operators in $\mathscr{L}(\mathscr{H})$. Let $\mathscr{F}(\mathscr{H})$ denote the ideal of all finite rank operators on $\mathscr{H}$. For a positive integer n, let $\mathscr{F}_n(\mathscr{H})$ be the set of all operators of $\mathscr{L}(\mathscr{H})$ of rank at most n. Denote by $\mathscr{N}_1(\mathscr{H})$ the set of all rank one nilpotent operators on $\mathscr{H}$, and observe that $x \otimes y \in \mathscr{N}_1(\mathscr{H})$ if and only if $\langle x, y\rangle = 0$. For a nonzero $h_0 \in \mathscr{H}$ and an operator $T \in \mathscr{L}(\mathscr{H})$, we use a useful notation defined by A. Bourhim and J. Mashreghi in [5, 6] by

$$\sigma_T^*(h_0) := \begin{cases} \{0\} & \text{if } \sigma_T(h_0) = \{0\} \\ \sigma_T(h_0) \setminus \{0\} & \text{if } \sigma_T(h_0) \neq \{0\}. \end{cases} \tag{2.3}$$

The second lemma, quoted from [4], gives a complete description of the local spectrum at a fixed vector of Jordan product of every rank one operator and arbitrary operator in $\mathscr{L}(\mathscr{H})$.

LEMMA 2.2. ([4, Lemma 3.4]) *Let $x,\ h_o$ and y be any nonzero vectors in $\mathscr{H}$. Then, for every operator $T \in \mathscr{L}(\mathscr{H})$, the following statements hold.*

1. *If $\langle h_0, y\rangle = \langle Th_0, y\rangle = 0$, then*

$$\sigma^*_{T(x\otimes y)+(x\otimes y)T}(h_0) = \{0\}.$$

2. *If* $\langle h_0, y\rangle \neq 0$ *or* $\langle Th_0, y\rangle \neq 0$ *and* $\langle x, y\rangle = 0$ *or* $\langle T^2x, y\rangle = 0$*, then*

$$\sigma^*_{T(x\otimes y)+(x\otimes y)T}(h_0) = \{\langle Tx, y\rangle\}.$$

The following lemma, quoted from [4], is a useful observation, and together with a local spectral identity principal and local spectral characterization of rank one nilpotent operators given below allow us to show that if a map φ from $\mathscr{L}(\mathscr{H})$ into $\mathscr{L}(\mathscr{K})$ preserves the local spectrum at a fixed nonzero vector of skew Jordan product of operators and its range contains $\mathscr{F}_2(\mathscr{K})$, then φ is automatically a bijective linear map from $\mathscr{N}_1(\mathscr{H})$ into $\mathscr{N}_1(\mathscr{K})$.

LEMMA 2.3. ([4, Lemma 3.5]) *Let* h_0 *be a nonzero vector in* $\mathscr{H}$. *For every* $N \in N_1(\mathscr{X})$, *we have*

$$\sigma^*_{(T+S)N^*+N^*(T+S)}(h_0) = \sigma^*_{TN^*+N^*T}(h_0) + \sigma^*_{SN^*+N^*S}(h_0)$$

for all $T,\ S \in \mathscr{L}(\mathscr{H})$.

2.2. Local spectral identity principles

In this section, we state two *local spectral identity principles* that will be exploited in the proof of Theorem 1.2. The first principle provides necessary and sufficient conditions for two operators to be the same modulo a scalar operator.

LEMMA 2.4. ([4, Theorem 4.1]) *Let* h_0 *be a nonzero vector in* $\mathscr{H}$. *For two operators* $A,\ B \in \mathscr{L}(\mathscr{H})$, *the following statements are equivalent.*

1. $A = B + \delta \mathbf{1}$ *for some* $\delta \in \mathbb{C}$.

2. $\sigma^*_{AN+NA}(h_0) = \sigma^*_{BN+NB}(h_0)$ *for all* $N \in N_1(\mathscr{H})$.

The following result is the second promised principle that gives necessary and sufficient conditions for two operators to be the same.

LEMMA 2.5. ([4, Theorem 4.3]) *For a nonzero vector* h_0 *in* $\mathscr{H}$ *and two operators* A *and* B *in* $\mathscr{L}(\mathscr{H})$, *the following statements are equivalent.*

1. $A = B$.

2. $\sigma^*_{AT^*+T^*A}(h_0) = \sigma^*_{BT^*+T^*B}(h_0)$ *for all* $T \in \mathscr{L}(\mathscr{H})$.

3. $\sigma^*_{AT^*+T^*A}(h_0) = \sigma^*_{BT^*+T^*B}(h_0)$ *for all* $T \in \mathscr{F}_1(\mathscr{H})$.

2.3. Characterization of rank one nilpotent operators

In term of local spectrum at fixed nonzero vector $h_0 \in \mathscr{H}$ of operator Jordan skew product, the following result characterizes all rank one nilpotent operators $N = x \otimes y$ for which x is linearly independent with h_0.

LEMMA 2.6. ([4, Theorem 5.3]) *For a nonzero vector h_0 in $\mathscr{H}$ and a nonzero operator $N \in \mathscr{L}(\mathscr{H})$ not of the form $\gamma I + h_0 \otimes y$, where $\gamma \in \mathbb{C}$ and $y \in \mathscr{H}$, the following statements are equivalent.*

1. *N is a rank one nilpotent operator.*
2. *$\sigma^*_{NT^*+T^*N}(h_0)$ is a singleton for all $T \in \mathscr{L}(\mathscr{H})$.*
3. *$\sigma^*_{NT^*+T^*N}(h_0)$ is a singleton for all $T \in \mathscr{F}_2(\mathscr{H})$.*

The next result shows that there are operators $N \in \mathscr{L}(\mathscr{H})$ other than rank one nilpotent operators for which $\sigma^*_{NT^*+T^*N}(h_0)$ is a singleton.

LEMMA 2.7. ([4, Theorem 5.4]) *For a nonzero vector h_0 in $\mathscr{H}$ and a nonzero operator $N \in \mathscr{L}(\mathscr{H})$, the following statements are equivalent.*

1. *$N = \gamma I + h_0 \otimes y$ for some $\gamma \in \mathbb{C}$ and $y \in \mathscr{H}$ for which $2\gamma + \langle h_0, y\rangle = 0$.*
2. *$\sigma^*_{NT^*+T^*N}(h_0)$ is a singleton for all $T \in \mathscr{L}(\mathscr{H})$.*
3. *$\sigma^*_{NT^*+T^*N}(h_0)$ is a singleton for all $T \in \mathscr{F}_2(\mathscr{H})$.*

The following lemma describes, in terms of the local spectrum at fixed nonzero vector $h_0 \in \mathscr{H}$ of operator Jordan product, all rank one nilpotent operators of the form $h_0 \otimes y$ for which $\langle h_0, y\rangle = 0$.

LEMMA 2.8. ([4, Corollary 5.5]) *Let h_0 be a nonzero vector in $\mathscr{H}$, and $N \in \mathscr{L}(\mathscr{H})$ be a nonzero operator. Then the following statements are equivalent.*

1. *$N = h_0 \otimes y$ for some $y \in \mathscr{H}$ for which $\langle h_0, y\rangle = 0$.*
2. *$\sigma_N(h_0) = \{0\}$, and $\sigma_{NT+TN}(h_0)$ is a singleton for all $T \in \mathscr{L}(\mathscr{H})$.*
3. *$\sigma_N(h_0) = \{0\}$, and $\sigma_{NT+TN}(h_0)$ is a singleton for all $T \in \mathscr{F}_2(\mathscr{H})$.*

3. Auxiliary results

In this section, we establish two more lemmas needed for the proof of our main result.

LEMMA 3.1. *Let $h_0 \in \mathscr{H}$ and $k_0 \in \mathscr{K}$ be two nonzero fixed vectors and A be a bijective linear operator form $\mathscr{H}$ to $\mathscr{K}$, and $\varphi : \mathscr{N}_1(\mathscr{H}) \longrightarrow \mathscr{N}_1(\mathscr{K})$ be the map defined by $\varphi(N) := \lambda ANA^{-1}$ for all $N \in \mathscr{N}_1(\mathscr{H})$, for some nonzero scalar $\lambda \in \mathbb{C}$. If φ satisfies Eq. (1.2) for all S and T in $\mathscr{N}_1(\mathscr{H})$, then $A = \beta U$ where U is a unitary operator and $\beta > 0$ and there exists a nonzero scalar $\alpha \in \mathbb{C}$ such that $Ah_0 = \alpha k_0$.*

Proof. First, we will show that A is a scalar multiple of a unitary operator. Since A is invertible, it suffices to show that the operator $V := A^*A$ is a scalar multiple of the identity $\mathbf{1}$. Observe that, for any orthogonal vectors $x,\ y \in \mathscr{H}$, we have

$$\begin{aligned}
\{0\} &= \sigma^*_{(x\otimes y)(x\otimes y)+(x\otimes y)(x\otimes y)}(h_0) \\
&= \sigma^*_{(x\otimes y)(y\otimes x)^*+(y\otimes x)^*(x\otimes y)}(h_0) \\
&= \sigma^*_{\varphi(x\otimes y)\varphi(y\otimes x)^*+\varphi(y\otimes x)^*\varphi(x\otimes y)}(k_0) \\
&= |\lambda|^2\sigma^*_{(Ax\otimes yA^{-1}(A^{-1})^*x\otimes yA^*+(A^{-1})^*x\otimes yA^*Ax\otimes yA^{-1})}(k_0) \\
&= |\lambda|^2\sigma^*_{A[(x\otimes y)(A^*A)^{-1}x\otimes yA^*A+(A^*A)^{-1}x\otimes yA^*A(x\otimes y)]A^{-1}}(k_0) \\
&= |\lambda|^2\sigma^*_{(x\otimes y).(A^*A)^{-1}x\otimes y(A^*A)+(A^*A)^{-1}x\otimes y(A^*A).(x\otimes y)}(A^{-1}k_0) \\
&= |\lambda|^2\sigma^*_{(x\otimes y).V^{-1}x\otimes yV+V^{-1}x\otimes yV.(x\otimes y)}(A^{-1}k_0).
\end{aligned}$$

It follows that

$$\sigma^*_{(x\otimes y).V^{-1}x\otimes yV+V^{-1}x\otimes yV.(x\otimes y)}(A^{-1}k_0) = \{0\}. \tag{3.4}$$

Now, suppose to the contrary that V is not a scalar multiple of the identity. Then there exists $x \in \mathscr{H}$ such that Vx and x are linearly independent. Observe, without loss of generality, that we may and shall assume that x and $A^{-1}k_0$ are linearly independent. By keeping in mind that $V^{-1}x$ and x are also linearly independent, then there exists $a \in \mathscr{H}$ so that

$$\langle x,a\rangle = 0,\ \langle Vx,a\rangle = 1 \text{ and } \langle V^{-1}x,a\rangle \neq 0.$$

Pick up a vector $b \in \mathscr{H}$ so that

$$\langle x,b\rangle = 0,\ \langle A^{-1}k_0,b\rangle \neq 0 \text{ and } \max(|\langle Vx,b\rangle|,|\langle V^{-1}x,b\rangle|) < \min(1,|\langle V^{-1}x,a\rangle|).$$

Set

$$y = \begin{cases} a+b \text{ if } \langle A^{-1}k_0,a\rangle + \langle A^{-1}k_0,b\rangle \neq 0 \\ a-b \text{ if } \langle A^{-1}k_0,a\rangle + \langle A^{-1}k_0,b\rangle = 0 \end{cases}$$

We have $\langle A^{-1}k_0,y\rangle = \langle A^{-1}k_0,a\rangle \pm \langle A^{-1}k_0,b\rangle \neq 0$. Then Lemma 2.2 and Eq. (3.4) entail that $\{0\} = \{|\lambda|^2\langle V^{-1}x,y\rangle\}$. Which is impossible since $|\lambda|^2\langle V^{-1}x,y\rangle$ is non zero. This contradiction shows that $V = A^*A$ is a scalar multiple of the identity; as claimed. Accordingly, $A^*A = \beta^2\mathbf{1}$ where $\beta > 0$. In the sequel, we may and shall assume that A is a unitary operator.

Next, we show that $Ah_0 = \alpha k_0$ for some nonzero scalar α. To that end, suppose to the contrary that h_0 and $A^{-1}k_0$ are linearly independent and take two vectors $h_1, h_2 \in \mathscr{H}$ such that h_1, h_2, h_0 and $A^{-1}k_0$ are linearly independent. Pick up two vectors h_3 and h_4 in $\mathscr{H}$ such that

$$\langle h_0,h_3\rangle = \langle h_2,h_3\rangle = 1,\ \langle h_1,h_3\rangle = \langle A^{-1}k_0,h_3\rangle = 0,$$

and

$$\langle h_1,h_4\rangle = 1,\ \langle h_2,h_4\rangle = \langle A^{-1}k_0,h_4\rangle = 0.$$

By Lemma 2.2, we have

$$\begin{aligned}\{1\}=\{\langle h_2,h_3\rangle\langle h_1,h_4\rangle\} &= \sigma^*_{(h_2\otimes h_4)(h_3\otimes h_1)^*+(h_3\otimes h_1)^*(h_2\otimes h_4)}(h_0)\\ &= \sigma^*_{\varphi(h_2\otimes h_4)\varphi(h_3\otimes h_1)^*+\varphi(h_3\otimes h_1)^*\varphi(h_2\otimes h_4)}(k_0)\\ &= |\lambda|^2\sigma^*_{A(h_2\otimes h_4)(h_1\otimes h_3)A^*+A(h_1\otimes h_3)(h_2\otimes h_4)A^*}(k_0)\\ &= |\lambda|^2\sigma^*_{(h_2\otimes h_4)(h_1\otimes h_3)+(h_1\otimes h_3)(h_2\otimes h_4)}(A^*k_0)=\{0\}.\end{aligned}$$

This contradiction shows that $A^*k_0=\alpha h_0$ for some nonzero scalar $\alpha\in\mathbb{C}$.

The proof is thus complete. □

LEMMA 3.2. *Let $h_0\in\mathscr{H}$ and $k_0\in\mathscr{K}$ be two nonzero fixed vectors and $A:\mathscr{H}\to\mathscr{K}$ be a bijective bounded linear transformation, and φ be a map on $\mathscr{N}_1(\mathscr{H})$ defined for all $N\in\mathscr{N}_1(\mathscr{H})$ by*

$$\varphi(N):=\lambda AN^*A^{-1},$$

for some fixed $\lambda\in\mathbb{C}$. Then there are rank one nilpotent operators $S,T\in\mathscr{N}_1(\mathscr{H})$ such that

$$\sigma_{TS^*+S^*T}(h_0)\neq\sigma_{\varphi(T)\varphi(S)^*+\varphi(S)^*\varphi(T)}(k_0).$$

Proof. Assume by the way of contradiction that

$$\sigma_{TS^*+S^*T}(h_0)=\sigma_{\varphi(T)\varphi(S)^*+\varphi(S)^*\varphi(T)}(k_0)$$

for all $S,\ T\in N_1(\mathscr{H})$. Just as in the proof of the previous lemma, one can show that A is unitary operator (Otherwise we could just take the function $\psi(N)=\dfrac{\lambda}{\beta^2}AN^*A^*$ instead of φ where β is such that $A=\beta U$). So, φ reads as $\varphi(N)=\lambda AN^*A^*$ for all $N\in N_1(\mathscr{H})$. Now take two nonzero vectors x_1 and $x_2\in\{A^*(k_0)\}^{\perp}$ such that h_0,x_1 and x_2 are linearly independent. Then there exists two vectors y_1 and $y_2\in\mathscr{H}$ such that

$$\langle x_2,y_1\rangle=\langle h_0,y_1\rangle=1,\langle x_1,y_1\rangle=0 \text{ and } \langle x_2,y_2\rangle=0,\langle x_1,y_2\rangle=1.$$

For $N_1:=x_1\otimes y_1$ and $N_2:=x_2\otimes y_2$, we have $(N_1^*N_2^*+N_2^*N_1^*)(A^*(k_0))=0$ and

$$\begin{aligned}\{1\}&=\{\langle x_1,y_2\rangle.\langle x_2,y_1\rangle\}\\ &=\sigma^*_{(x_2\otimes y_2)(x_1\otimes y_1)+(x_1\otimes y_1)(x_2\otimes y_2)}(h_0)\\ &=\sigma^*_{\varphi(x_2\otimes y_2)\varphi(y_1\otimes x_1)^*+\varphi(y_1\otimes x_1)^*\varphi(x_2\otimes y_2)}(k_0)\\ &=|\lambda|^2\sigma^*_{A(x_2\otimes y_2)^*A^*A(x_1\otimes y_1)^*A^*+A(x_1\otimes y_1)^*A^*A(x_2\otimes y_2)^*A^*}(k_0)\\ &=|\lambda|^2\sigma^*_{A[N_2^*N_1^*+N_1^*N_2^*]A^*}(k_0)\\ &=|\lambda|^2\sigma^*_{N_2^*N_1^*+N_1^*N_2^*}(A^*k_0)=\{0\}.\end{aligned}$$

This is a contradiction, and thus the proof of this lemma is complete. □

4. Proof of the main result 'Theorem 1.2'

Assume that there exists unitary operator U from $\mathscr{H}$ into $\mathscr{K}$ such that $Uh_0 = \alpha k_0$ for some scalar $\alpha \in \mathbb{C}$ and $\varphi(T) = \lambda UTU^*$ for all $T \in \mathscr{L}(\mathscr{H})$ where $|\lambda| = 1$. We have

$$\sigma_{\varphi(S)\varphi(T)^*+\varphi(T)^*\varphi(S)}(k_0) = \lambda\overline{\lambda}\ \sigma_{USU^*UT^*U^*+UT^*U^*USU^*}(k_0) = \sigma_{ST^*+T^*S}(x_0)$$

for all operators T and S in $\mathscr{L}(\mathscr{H})$. This establishes the 'if' part, and we now move to prove the 'only if' part. So, assume that φ satisfies (1.2), and let us proceed to show that φ takes the desired form. The proof breaks down into several claims.

CLAIM 1. *φ is injective and $\varphi(0) = 0$.*

If $\varphi(A) = \varphi(B)$ for some operators $A,\ B \in \mathscr{L}(\mathscr{H})$, then by (1.2) we get

$$\begin{aligned} \sigma_{AT^*+T^*A}(h_0) &= \sigma_{\varphi(A)\varphi(T)^*+\varphi(T)^*\varphi(A)}(k_0) \\ &= \sigma_{\varphi(B)\varphi(T)^*+\varphi(T)^*\varphi(B)}(k_0) \\ &= \sigma_{BT^*+T^*B}(h_0) \end{aligned}$$

for all $T \in \mathscr{L}(\mathscr{H})$. By Lemma 2.5, we see that $A = B$ and thus φ is injective. In a similar way, we show that $\varphi(0) = 0$. Indeed, let $S \in \mathscr{L}(\mathscr{K})$ be a rank one operator and note that, since the range of φ contains all rank one operators, there is $T \in \mathscr{L}(\mathscr{H})$ such that $\varphi(T) = S$. We have

$$\begin{aligned} \sigma_{\varphi(0)S^*+S^*\varphi(0)}(k_0) &= \sigma_{\varphi(0)\varphi(T)^*+\varphi(T)^*\varphi(0)}(k_0) \\ &= \sigma_{0.T^*+T^*.0}(h_0) = \{0\} \\ &= \sigma_{0.\varphi(T)^*+\varphi(T)^*.0}(k_0) \\ &= \sigma_{0.S^*+S^*.0}(k_0). \end{aligned}$$

Since S is an arbitrary rank one operator in $\mathscr{L}(\mathscr{K})$, Lemma 2.5 entails that $\varphi(0) = 0$.

CLAIM 2. *φ preserves rank one nilpotent operators in both directions.*

Let $N = x \otimes y \in \mathscr{L}(\mathscr{H})$ be a nonzero rank one nilpotent operator, where x and y are two vectors in $\mathscr{H}$. We distinguish two cases.

Case 1: If x and h_0 are linearly independent, then obviously N is not of the form $\gamma\mathbf{1} + h_0 \otimes y$ with $2\gamma + \langle h_0, y\rangle = 0$. Thus Lemma 2.7 tells us that $\varphi(N)$ is not of this form either. Now, note that $\sigma^*_{NT^*+T^*N}(h_0)$ is a singleton for all $T \in \mathscr{L}(\mathscr{H})$, and so does $\sigma^*_{\varphi(N)\varphi(T)^*+\varphi(T)^*\varphi(N)}(k_0)$ for all $T \in \mathscr{L}(\mathscr{H})$. Since $\varphi(N) \neq 0$, by Lemma 2.6 and the fact that the range of φ contains $\mathscr{F}_2(\mathscr{K})$, we see that $\varphi(N)$ is a rank one nilpotent operator too.

Case 2: If $x = \alpha h_0$ for some nonzero scalar $\alpha \in \mathbb{C}$, then $N = h_0 \otimes (\alpha y)$. So $N^* = (\alpha y) \otimes h_0$ is a nonzero rank one nilpotent operator. Since y and h_0 are linearly independent, then N^* is not of the form $\gamma\mathbf{1} + h_0 \otimes \alpha h_0$ with $2\gamma + \alpha\|h_0\|^2 = 0$. By Lemma 2.7, we see that $\varphi(N)^*$ is not of this form either. Since $\sigma_{TN^*+N^*T}(h_0) =$

$\sigma_{\varphi(T)\varphi(N)^*+\varphi(N)^*\varphi(T)}(k_0)$ is a singleton for all $T\in\mathscr{L}(\mathscr{H})$ and since $\varphi(N)^*\neq 0$, and the range of φ contains all operators of rank at most 2, Lemma 2.6 tells us that $\varphi(N)^*$ is a rank one nilpotent operator and so is $\varphi(N)$.

Conversely let $\varphi(N)=y\otimes z$ be a nonzero rank one nilpotent operator of $\mathscr{L}(\mathscr{K})$, where y and z are two vectors in $\mathscr{K}$. Firstly, let us suppose that y and k_0 are linearly independent, and then obviously $\varphi(N)$ is not of the form $\gamma\mathbf{1}+k_0\otimes z$ with $2\gamma+\langle k_o,z\rangle=0$. Thus, by Lemma 2.7, N is not of this form either. Now, note that $\sigma^*_{\varphi(N)\varphi(T)^*+\varphi(T)^*\varphi(N)}(k_0)$ is a singleton for all $T\in\mathscr{L}(\mathscr{H})$, and so is $\sigma^*_{NT^*+T^*N}(h_0)$ for all $T\in\mathscr{L}(\mathscr{H})$. Since $N\neq 0$, by Lemma 2.6 and the fact that the range of φ contains $\mathscr{F}_2(\mathscr{K})$, we see that N is a rank one nilpotent operator. In order to finish the proof of claim, let us consider the case where y and k_0 are linearly dependent. That is $\varphi(N)=k_0\otimes z$ for some nonzero vector $z\in\mathscr{K}$. Then by Eq. (1.2) one can sees easily that $\sigma_N(h_0)=\{0\}$, and $\sigma_{NT+TN}(h_0)$ is a singleton for all $T\in\mathscr{L}(\mathscr{H})$. By Lemma 2.8, we infer that N is of the form $h_0\otimes y$ for some nonzero vector $y\in\mathscr{H}$ such that $\langle h_0,y\rangle=0$. This ends the proof of Claim 2.

CLAIM 3. *φ is homogeneous.*

For every $\lambda\in\mathbb{C}$ and $S,\ T\in\mathscr{L}(\mathscr{H})$, we have

$$\begin{aligned}\sigma_{\varphi(T)(\lambda\varphi(S))^*+(\lambda\varphi(S))^*\varphi(T)}(k_0) &= \overline{\lambda}\ \sigma_{\varphi(T)\varphi(S)^*+\varphi(S)^*\varphi(T)}(k_0)\\ &= \overline{\lambda}\ \sigma_{TS^*+S^*T}(h_0)\\ &= \sigma_{T(\lambda S)^*+(\lambda S)^*T}(h_0)\\ &= \sigma_{\varphi(T)\varphi(\lambda S)^*+\varphi(\lambda S)^*\varphi(T)}(k_0).\end{aligned}$$

Since the range of φ contains all rank one operators, Lemma 2.5 shows that $\varphi(\lambda S)^*=(\lambda\varphi(S))^*$ for all $\lambda\in\mathbb{C}$ and $S\in\mathscr{L}(\mathscr{H})$. Accordingly, $\varphi(\lambda S)=\lambda\varphi(S)$ for all $\lambda\in\mathbb{C}$ and $S\in\mathscr{L}(\mathscr{H})$.

CLAIM 4. *$\varphi(\mathbf{1})=\alpha\mathbf{1}$ where $|\alpha|=1$, and thus upon replacing φ by $\overline{\alpha}\varphi$, there is no loss of generality in assuming that $\alpha=1$.*

For any rank one nilpotent operator N, we have from (1.2) that

$$\sigma_{\varphi(N)\varphi(\mathbf{1})^*+\varphi(\mathbf{1})^*\varphi(N)}(k_0)=2\sigma_N(h_0)=\{0\}=\sigma_{\varphi(N)0^*+0^*\varphi(N)}(k_0).$$

Since φ preserves nilpotent rank one operators in both directions, Lemma 2.4 implies that $\varphi(\mathbf{1})=\alpha\mathbf{1}$ for some scalar $\alpha\in\mathbb{C}$. Since

$$\{2\}=\sigma_{\mathbf{1}+\mathbf{1}}(h_0)=\sigma_{\varphi(\mathbf{1})\varphi(\mathbf{1})^*+\varphi(\mathbf{1})^*\varphi(\mathbf{1})}(k_0)=\{2\alpha\overline{\alpha}\},$$

we see that $|\alpha|=1$ and thus $\varphi(\mathbf{1})=\alpha\mathbf{1}$ with $|\alpha|=1$.

CLAIM 5. *When restricted on $\mathscr{N}_1(\mathscr{H})$, the map φ takes one of the following forms:*

(1) There exists a bijective bounded linear or conjugate linear transformation $A: \mathscr{H} \to \mathscr{K}$ such that

$$\varphi(N) = \tau_N ANA^{-1} \tag{4.5}$$

for all $N \in \mathscr{N}_1(\mathscr{H})$, where τ_N is a scalar depending on N.

(2) There exists a bijective bounded linear or conjugate linear transformation $A: \mathscr{H} \to \mathscr{K}$ such that

$$\varphi(N) = \tau_N AN^*A^{-1} \tag{4.6}$$

for all $N \in \mathscr{N}_1(\mathscr{H})$, where τ_N is a scalar depending on N.

Let us begin by showing that for $N_1, N_2 \in \mathscr{N}_1(\mathscr{H})$ for which $N_1 + N_2 \in \mathscr{N}_1(\mathscr{H})$, we have

$$\varphi(N_1 + N_2) = \varphi(N_1) + \varphi(N_2). \tag{4.7}$$

Since φ preserves rank one nilpotent operators in both directions, it suffices to show that for every operators $S, T \in \mathscr{L}(\mathscr{H})$, there is $\delta_{S,\,T}$ such that

$$\varphi(S+T) = \varphi(S) + \varphi(T) + \delta_{S,\,T}\mathbf{1}. \tag{4.8}$$

To that end, pick two operator $S, T \in \mathscr{L}(\mathscr{H})$ and $N \in \mathscr{N}_1(\mathscr{H})$. By Lemma 2.3 and condition (1.2), we have

$$\begin{aligned}
\sigma^*_{\varphi(T+S)\varphi(N)^*+\varphi(N)^*\varphi(S+T)}(k_0) &= \sigma^*_{(S+T)N^*+N^*(S+T)}(h_0) \\
&= \sigma^*_{TN^*+N^*T}(h_0) + \sigma^*_{SN^*+N^*S}(h_0) \\
&= \sigma^*_{\varphi(T)\varphi(N)^*+\varphi(N)^*\varphi(T)}(k_0) + \sigma^*_{\varphi(S)\varphi(N)^*+\varphi(N)^*\varphi(S)}(k_0) \\
&= \sigma^*_{(\varphi(T)+\varphi(S))\varphi(N)^*+\varphi(N)^*(\varphi(T)+\varphi(S))}(k_0).
\end{aligned}$$

Since φ preserves rank one nilpotent operators, Lemma 2.4 entails that there is $\delta_{S,\,T}$ such that $\varphi(T+S) = \varphi(T) + \varphi(S) + \delta_{S,\,T}\mathbf{1}$; as desired.

So far it has been shown that φ is a bijective map from $\mathscr{N}_1(\mathscr{H})$ into $\mathscr{N}_1(\mathscr{K})$, and thus (4.7) applied to both φ and φ^{-1} shows that

$$N_1 + N_2 \in \mathscr{N}_1(\mathscr{H}) \iff \varphi(N_1 + N_2) \in \mathscr{N}_1(\mathscr{K})$$

for all $N_1, N_2 \in \mathscr{N}_1(\mathscr{H})$. By [13, Lemma 2.2], φ when restricted on $\mathscr{N}_1(\mathscr{H})$ takes either the form (4.5) or the form (4.6); as claimed.

CLAIM 6. *There exist a unitary operator $U \in \mathscr{L}(\mathscr{H}, \mathscr{K})$ and two scalars α and λ in $\mathbb{C}$ such that $|\lambda| = 1$, $Uh_0 = \alpha k_0$ and*

$$\varphi(N) = \lambda UNU^*$$

for all $N \in \mathscr{N}_1(\mathscr{H})$).

Based on Claim 5, we know that φ takes either the form (4.5) or the (4.6). By Lemma 3.2, we have that φ can not take the second form(4.6) and therefore φ takes only the form (4.5). Accordingly there exists a bijective bounded linear or conjugate linear transformation $A:\mathscr{H}\to\mathscr{K}$ such that $\varphi(N)=\tau_N ANA^{-1}$ for all $N\in N_1(\mathscr{H})$. Firstly, we show that A must be linear and τ_N is a nonzero scalar free of $N\in\mathscr{N}_1(\mathscr{H})$, say λ. Indeed, for any $x\in\mathscr{H}$,$y\in\mathscr{H}$ and $z\in\mathscr{H}$ with $\langle x,y\rangle=0$ and $\langle x,z\rangle=0$, there exists a scalars $\tau_{x,y}$, $\tau_{x,z}$ and $\tau_{x,y,z}$ such that

$$\varphi(x\otimes y)=\tau_{x,y}A(x\otimes y)A^{-1}$$

$$\varphi(x\otimes z)=\tau_{x,z}A(x\otimes z)A^{-1}$$

and

$$\varphi(x\otimes(y+z))=\tau_{x,y,z}A(x\otimes(y+z))A^{-1}.$$

Since

$$\varphi(x\otimes(y+z))=\varphi(x\otimes y)+\varphi(x\otimes z),$$

we infer that that $\tau_{x,y}=\tau_{x,z}=\tau_{x,y,z}=\lambda$ is a nonzero constant. That is $\tau_{x,y}$ is constant with respect to y. A similar reasoning shows that $\tau_{x,y}$ is also constant with respect to x and therefore τ_N is a nonzero scalar free of $N\in\mathscr{N}_1(\mathscr{H})$. Next let us show that A must be linear. To that end, take an arbitrary vectors x, z and $y\in\mathscr{H}$ such that

$$\langle x,y\rangle=0 \text{ and } \langle z,y\rangle=1.$$

Since φ is homogeneous we have:

$$\begin{aligned}\alpha\lambda A(x\otimes y)A^{-1}&=\alpha\varphi(x\otimes y)\\&=\varphi(\alpha(x\otimes y))\\&=\lambda A(\alpha(x\otimes y))A^{-1}.\end{aligned}$$

It follows that

$$\alpha Ax=\alpha A(x\otimes y)A^{-1}Az=A(\alpha(x\otimes y))A^{-1}Az=A(\alpha x)$$

and A is linear; as desired. Therefore we have

$$\varphi(N)=\lambda ANA^{-1}$$

for all $N\in N_1(\mathscr{H})$. Since φ satisfies (1.2), Lemma 3.1 shows that we may assume and shall that $A:=U$ is a unitary operator and $Uh_0=\alpha k_0$.

Finally, let us verify that $|\lambda|=1$. Take two vectors x_3, $x_4\in\mathscr{H}$ such that x_3, x_4 and h_0 are linearly independent. Then there exist two vectors y_3 and y_4 in $\mathscr{H}$ such that

$$\langle h_0,y_3\rangle=\langle x_4,y_3\rangle=1,\langle x_3,y_3\rangle=0$$

and

$$\langle x_3,y_4\rangle=1,\langle x_4,y_4\rangle=\langle h_0,y_4\rangle=0.$$

By Lemma 2.2 and (1.2), we have

$$
\begin{aligned}
\{1\} &= \{\langle x_3, y_4\rangle . \langle x_4, y_3\rangle\} \\
&= \sigma^*_{(x_4\otimes y_4)(x_3\otimes y_3)+(x_3\otimes y_3)(x_4\otimes y_4)}(h_0) \\
&= \sigma^*_{\varphi(x_4\otimes y_4)\varphi(y_3\otimes x_3)^*+\varphi(y_3\otimes x_3)^*\varphi(x_4\otimes y_4)}(k_0) \\
&= |\lambda|^2\sigma^*_{U[(x_4\otimes y_4)(x_3\otimes y_3)+(x_3\otimes y_3)(x_4\otimes y_4)]U^*}(k_0) \\
&= |\lambda|^2\sigma^*_{(x_4\otimes y_4)(x_3\otimes y_3)+(x_3\otimes y_3)(x_4\otimes y_4)}(U^*k_0).
\end{aligned}
$$

Since $U^*k_0 = \frac{1}{\alpha}h_0$, Lemma (2.1)-(b) entails that

$$
\{1\} = |\lambda|^2\sigma^*_{(x_4\otimes y_4)(x_3\otimes y_3)+(x_3\otimes y_3)(x_4\otimes y_4)}(h_0) = \{|\lambda|^2\}.
$$

Accordingly $|\lambda| = 1$. This completes the proof of this claim.

CLAIM 7. *φ takes the desired form.*

Observe first that the map $\bar{\lambda}\varphi$ satisfies (1.2), and thus we may and shall assume that $\varphi(N) = UNU^*$ for all $N \in \mathcal{N}_1(\mathcal{H})$. Now, for every $N \in \mathcal{N}_1(\mathcal{H})$ and $T \in \mathcal{L}(\mathcal{H})$ we have

$$
\begin{aligned}
\sigma_{UTU^*\varphi(N)^*+\varphi(N)^*UTU^*}(k_0) &= \sigma_{UTU^*UN^*U^*+UN^*U^*UTU^*}(k_0) \\
&= \sigma_{U(TN^*+N^*T)U^*}(k_0) \\
&= \sigma_{TN^*+N^*T}(U^*k_0) \\
&= \sigma_{TN^*+N^*T}(h_0) \\
&= \sigma_{\varphi(T)\varphi(N)^*+\varphi(N)^*\varphi(T)}(k_0).
\end{aligned}
$$

By Lemma 2.4, we deduce that $\varphi(T) = UTU^* + \alpha_T\mathbf{1}$, where α_T is scalar depending on T.

Now, we show that $\varphi(R) = URU^*$ for all rank one operator $R \in \mathcal{L}(\mathcal{H})$. By keeping in mind that $\varphi(\mathbf{1}) = \mathbf{1}$ and using (1.2), we have

$$
\begin{aligned}
\sigma^*_R(h_0) &= \sigma^*_{\frac{R}{2}\mathbf{1}^*+\mathbf{1}^*\frac{R}{2}}(h_0) \\
&= \sigma^*_{\varphi(\frac{R}{2})\varphi(\mathbf{1})^*+\varphi(\mathbf{1})^*\varphi(\frac{R}{2})}(k_0) \\
&= \sigma^*_{\varphi(R)}(k_0) \\
&= \sigma^*_{URU^*+\alpha_R}(k_0) \\
&= \sigma^*_{URU^*}(k_0) + \{\alpha_R\} \\
&= \sigma^*_R(h_0) + \{\alpha_R\}
\end{aligned}
$$

As $\sigma_R^*(h_0)$ is a nonempty set containing at most one nonzero number, we have $\alpha_R = 0$. To finish the proof, take an arbitrary operator $T \in \mathscr{L}(\mathscr{H})$ and observe that

$$\begin{aligned}
\sigma^*_{\varphi(R)UT^*U^*+UT^*U^*\varphi(R)}(k_0) &= \sigma^*_{URU^*UT^*U^*+UT^*U^*URU^*}(k_0) \\
&= \sigma^*_{URT^*U^*+UT^*RU^*}(k_0) \\
&= \sigma^*_{U(RT^*+T^*R)U^*}(k_0) \\
&= \sigma^*_{RT^*+T^*R}(U^*k_0) \\
&= \sigma^*_{RT^*+T^*R}(h_0) \\
&= \sigma^*_{\varphi(R)\varphi(T)^*+\varphi(T)^*\varphi(R)}(k_0).
\end{aligned}$$

By Lemma 2.5, we have $\varphi(T)^* = UT^*U^*$ for all $T \in \mathscr{L}(\mathscr{H})$, and the proof is therefore complete.

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