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FACTORIZATION OF THE CHARACTERISTIC FUNCTION OF A JACOBI MATRIX

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Abstract. In a recent paper, a class of infinite Jacobi matrices with discrete character of spectra has been introduced. With each Jacobi matrix from this class an analytic function is associated, called the characteristic function, whose zero set coincides with the point spectrum of the corresponding Jacobi operator. Here it is shown that the characteristic function admits Hadamard's factorization in two possible ways – either in the spectral parameter or in an auxiliary parameter which may be called the coupling constant. As an intermediate result, a formula for the logarithm of the characteristic function is obtained which is then used to handle the spectral zeta function of the Jacobi matrix. In a number of examples the characteristic function coincides with a special function, and hence to those special functions these general results can be directly applied.

1. Introduction

In [33] we have introduced a class of infinite Jacobi matrices characterized by a simple convergence condition. Each Jacobi matrix from this class unambiguously determines a closed operator on $\ell^2(\mathbb{N})$ having a discrete spectrum. Moreover, with such a matrix one associates a complex function, called the characteristic function, which is analytic on the complex plane with the closure of the range of the diagonal sequence being excluded, and meromorphic on the complex plane with the set of accumulation points of the diagonal sequence being excluded. It turns out that the zero set of the characteristic function actually coincides with the point spectrum of the corresponding Jacobi operator on the domain of definition (with some subtleties when handling the poles; see Theorem 2.1 below).

A number of examples are now known where this characteristic function can be expressed in terms of special functions [33, 35]. Earlier, in several papers, this relationship between the zeros of certain special function and the eigenvalues of an appropriate Jacobi matrix was proposed as a useful method for numerical computation of the zeros [17, 20]. On the theoretical level, the study of the zeros of a special function belongs to principal tasks. Many special functions (or their reciprocals) are entire functions, and under various circumstances it is convenient to represent such a function as an infinite product by applying to it the Hadamard (or Weierstrass) factorization.

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As is commonly known, the infinite product representation is available for some elementary functions like trigonometric functions, Bessel functions and the Gamma function. But the range of special functions admitting this factorization is much wider including the Kummer function, Struve functions, Lommel functions of the first kind and q-Bessel functions, as shown comparatively recently in [4, 3, 26]. Another special function for which Hadamard's factorization is known to work is the regular Coulomb wave function [34]. This particular example demonstrates that the infinite product representation may have interesting consequences. By applying logarithmic derivative and making use of a recurrence relation one can derive the Mittag-Leffler expansion [2].

Hadamard's factorization of an entire function f is closely related to the values of the zeta function associated with f for integer arguments. Note that such a zeta function is defined in terms of the zeros of f. If f coincides with the characteristic function of an operator T with a discrete spectrum one also speaks about the spectral zeta function of T.

A particularly widely known and thoroughly studied example is that of the zeta function associated with the Bessel function which is usually called the Rayleigh function. Its history goes as far back as the second half of the 19th century but since then its study continued for many decades with quite a few cases of rediscovering old results; see [23] on the account of the history as well as of the recent state of the art. The primary motivation for these studies was localization of the zeros. This old method due to Euler and Rayleigh and others can be of interest even nowadays [22, 26] and finds its applications in mathematical physics (see, for instance, [12]).

What is perhaps less expected are some combinatorial aspects of the Rayleigh function. This feature is related to a recurrence the Rayleigh function obeys [24, 5, 19]. In this connection it is worthwhile pointing out a remarkable property of the Rayleigh function as well as of other zeta functions associated with various special functions. They can be explicitly evaluated at integer arguments although the zeros of those special functions can be evaluated only numerically. As an example one can mention the zeta function for the q-Bessel equation, the Airy zeta function, the zeta function associated with the Coulomb wave function and the hypergeometric zeta function [26, 10, 34, 4]. In all these cases the sequence of values of the zeta function at integer arguments is defined by a linear or quadratic recurrence, and Hadamard's factorization of the special function in question plays a crucial role in the derivation of this recurrence rule. As regards the Rayleigh function itself, the recurrence rules have been studied in great detail [28, 24, 25, 6, 27].

Special functions frequently depend on a parameter, called the order, and obey a three-term recurrence with respect to it. Moreover, the dependence of a special function on the order can be of interest in various applications. For instance, the zeros of $J_\nu(z)$ in the order ν, with $z > 0$ being fixed, are of importance in some combinatorial problems [15, 13] as well as in the analysis of some birth-and-death processes [31]. Let us note that the equation $J_\nu(z) = 0$ in the order ν started to be studied much later than the same equation in the variable z [8]. The problem of the asymptotic behavior of the zeros in ν for z small is addressed in [33, Prop. 30] where an explicit error estimate for the leading asymptotic term is derived. Furthermore, in [31, Prop. 6], the zeros of the Bessel function in the order are studied perturbatively while recognizing them as the eigenvalues of an appropriate Jacobi matrix. This is basically the same approach as we are advocating here. In particular, in Example 4.1 below, in equation (39), we derive a Hadamard-type factorization of $J_\nu(w)$ in ν as a straightforward application of a more general result (Theorem 4.3), and to the best of our knowledge this factorization has not yet been presented elsewhere.

In the current paper we aim to propose a unifying and more abstract approach to the Hadamard factorization of special functions in those cases when the addressed special function can be identified with the characteristic function of a suitable Jacobi matrix. As the main result we show that the characteristic function of a Jacobi matrix from a rather broad class, as introduced in [33], admits Hadamard's factorization in two possible ways – either in the spectral parameter or in an auxiliary parameter which may be called the coupling constant (see below Theorem 4.3 and 5.2, respectively). Regarding the Bessel function as a typical example these two factorizations correspond to its Hadamard factorization in the order and in the variable, respectively (see Example 4.1 and 5.1). As an additional auxiliary result which may be of independent interest we derive a formula for the logarithm of the characteristic function. This formula is then applied when dealing with the spectral zeta function of the corresponding Jacobi matrix (or, equivalently, with the zeta function associated with the characteristic function). A possible recurrence for the values of the zeta function at integer arguments is addressed as well.

These general results cover most of the examples discussed above. But the authors believe that the list of special functions which can be treated in this unified way can be extended notably in the near future.

Let us outline the organization of the paper and indicate its main results. In Section 2, some basic notions and preliminary results needed for the analysis to follow are summarized. In Section 3, in Theorem 3.1, a formula for the logarithm of the characteristic function is derived. Section 4 deals with Hadamard's factorization of the characteristic function in the spectral parameter. As stated in Theorem 4.3, such a factorization actually exists under plausible assumptions. The focus of Section 5 is on the factorization in the coupling constant and on consequent applications to the spectral zeta function. In Theorem 5.1, the algebraic multiplicity of eigenvalues of the Jacobi matrix in question is examined. Then, in Theorem 5.2, the desired factorization is formulated and proved.

2. Preliminaries

Many formulas throughout the paper are expressed in terms of a function, called $\mathfrak{F}$, which is defined on a suitable subset of the linear space of all complex sequences; see [32] for its original definition. Here we recall the definition and several basic properties that are referred to in what follows.

Define $\mathfrak{F}: D \to \mathbb{C}$,

$$\mathfrak{F}(x) = 1 + \sum_{m=1}^{\infty} (-1)^m \sum_{k_1=1}^{\infty} \sum_{k_2=k_1+2}^{\infty} \cdots \sum_{k_m=k_{m-1}+2}^{\infty} x_{k_1} x_{k_1+1} x_{k_2} x_{k_2+1} \cdots x_{k_m} x_{k_m+1}, \tag{1}$$

where

$$D=\left\{\{x_k\}_{k=1}^{\infty}\subset\mathbb{C};\ \sum_{k=1}^{\infty}|x_kx_{k+1}|<\infty\right\}. \tag{2}$$

For a finite number of complex variables we identify $\mathfrak{F}(x_1,x_2,\dots,x_n)$ with $\mathfrak{F}(x)$ where $x=(x_1,x_2,\dots,x_n,0,0,0,\dots)$. By convention, let $\mathfrak{F}(\emptyset)=1$ where $\emptyset$ is the empty sequence.

Notice that $\ell^2(\mathbb{N})\subset D$. For $x\in D$, one has the estimates

$$|\mathfrak{F}(x)|\leqslant\exp\left(\sum_{k=1}^{\infty}|x_kx_{k+1}|\right),\quad |\mathfrak{F}(x)-1|\leqslant\exp\left(\sum_{k=1}^{\infty}|x_kx_{k+1}|\right)-1, \tag{3}$$

and it is true that

$$\mathfrak{F}(x)=\lim_{n\to\infty}\mathfrak{F}(x_1,x_2,\dots,x_n). \tag{4}$$

Let us also point out a simple invariance property. For $x\in D$ and $s\in\mathbb{C}$, $s\neq 0$, it is true that $y\in D$ and

$$\mathfrak{F}(x)=\mathfrak{F}(y),\ \text{where}\ \ y_{2k-1}=sx_{2k-1},\ y_{2k}=x_{2k}/s,\ k\in\mathbb{N}. \tag{5}$$

We shall deal with symmetric Jacobi matrices

$$J=\begin{bmatrix}\lambda_1 & w_1 & & & \\ w_1 & \lambda_2 & w_2 & & \\ & w_2 & \lambda_3 & w_3 & \\ & & \ddots & \ddots & \ddots\end{bmatrix}, \tag{6}$$

where $\lambda=\{\lambda_n\}_{n=1}^{\infty}\subset\mathbb{C}$ and $w=\{w_n\}_{n=1}^{\infty}\subset\mathbb{C}\setminus\{0\}$. Let us put

$$\gamma_{2k-1}=\prod_{j=1}^{k-1}\frac{w_{2j}}{w_{2j-1}},\quad \gamma_{2k}=w_1\prod_{j=1}^{k-1}\frac{w_{2j+1}}{w_{2j}},\quad k=1,2,3,\dots. \tag{7}$$

Then $\gamma_k\gamma_{k+1}=w_k$.

For $n\in\mathbb{N}$, let J_n be the $n\times n$ Jacobi matrix: $(J_n)_{j,k}=J_{j,k}$ for $1\leqslant j,k\leqslant n$, and I_n be the $n\times n$ unit matrix. Then the formula

$$\det(J_n-zI_n)=\left(\prod_{k=1}^{n}(\lambda_k-z)\right)\mathfrak{F}\left(\frac{\gamma_1^2}{\lambda_1-z},\frac{\gamma_2^2}{\lambda_2-z},\dots,\frac{\gamma_n^2}{\lambda_n-z}\right). \tag{8}$$

holds true for all $z\in\mathbb{C}$ (after obvious cancellations, the RHS is well defined even for $z=\lambda_k$; here and throughout RHS means "right-hand side", and similarly for LHS).

Let us denote

$$\mathbb{C}_0^{\lambda}:=\mathbb{C}\setminus\overline{\{\lambda_n;n\in\mathbb{N}\}}.$$

Moreover, $\mathrm{der}(\lambda)$ designates the set of all accumulation points of the sequence λ. The following theorem is a compilation of several results from [33, Subsec. 3.3].

THEOREM 2.1. *Let a Jacobi matrix J, as introduced in (6), be real and suppose that*
$$\sum_{n=1}^{\infty}\left|\frac{w_n^2}{(\lambda_n-z)(\lambda_{n+1}-z)}\right|<\infty \tag{9}$$
for at least one $z\in\mathbb{C}_0^{\lambda}$. Then

(i) J represents a unique self-adjoint operator on $\ell^2(\mathbb{N})$,

(ii) $\operatorname{spec}(J)\cap(\mathbb{C}\setminus\operatorname{der}(\lambda))$ consists of simple real eigenvalues with no accumulation points in $\mathbb{C}\setminus\operatorname{der}(\lambda)$,

(iii) the series (9) converges locally uniformly on $\mathbb{C}_0^{\lambda}$ and
$$F_J(z):=\mathfrak{F}\left(\left\{\frac{\gamma_n^2}{\lambda_n-z}\right\}_{n=1}^{\infty}\right) \tag{10}$$
is a well defined analytic function on $\mathbb{C}_0^{\lambda}$,

(iv) $F_J(z)$ is meromorphic on $\mathbb{C}\setminus\operatorname{der}(\lambda)$, the order of a pole at $z\in\mathbb{C}\setminus\operatorname{der}(\lambda)$ is less than or equal to the number $r(z)$ of occurrences of z in the sequence λ,

(v) $z\in\mathbb{C}\setminus\operatorname{der}(\lambda)$ belongs to $\operatorname{spec}(J)$ if and only if
$$\lim_{u\to z}(z-u)^{r(z)}F_J(u)=0$$
and, in particular, $\operatorname{spec}(J)\cap\mathbb{C}_0^{\lambda}=\operatorname{spec}_p(J)\cap\mathbb{C}_0^{\lambda}=F_J^{-1}(\{0\})$.

We will mostly focus on real Jacobi matrices, except in Section 5. For our purposes the following particular case, a direct consequence of a more general result derived in [33, Subsec. 3.3], will be sufficient.

THEOREM 2.2. *Let J be a complex Jacobi matrix of the form (6) obeying $\lambda_n=0$, $\forall n$, and $\{w_n\}\in\ell^2(\mathbb{N})$. Then J represents a Hilbert-Schmidt operator, $F_J(z)$ is analytic on $\mathbb{C}\setminus\{0\}$ and*
$$\operatorname{spec}(J)\setminus\{0\}=\operatorname{spec}_p(J)\setminus\{0\}=F_J^{-1}(\{0\}).$$

REMARK 2.1. Although this is not the principal focus of the current paper let us remark, for later reference, that there exists a relation of the function $\mathfrak{F}(x)$ to continued fractions. Most conveniently, this is demonstrated in the framework of formal power series. To this end, let us rewrite (1) in terms of formal variables a_j standing instead of x_jx_{j+1}, $j=1,2,\ldots$. Then instead of $\mathfrak{F}(x)$ we obtain the formal power series
$$\mathfrak{G}(a_1,a_2,\ldots):=1+\sum_{m=1}^{\infty}(-1)^m\sum_{k_1=1}^{\infty}\sum_{k_2=k_1+2}^{\infty}\ldots\sum_{k_m=k_{m-1}+2}^{\infty}a_{k_1}a_{k_2}\ldots a_{k_m}. \tag{11}$$
Here again, $\mathfrak{G}(a_1,a_2,\ldots,a_n)$, for some $n\in\mathbb{N}$, means a polynomial in the indicated variables obtained as a truncation of (11) by letting $a_{n+1}=a_{n+2}=\cdots=0$, and $\mathfrak{G}(\emptyset):=$

1 by definition. As observed in [32, Rem. 6], the continued fraction

$$\mathscr{X}(a) := \cfrac{1}{1-\cfrac{a_1}{1-\cfrac{a_2}{1-\cfrac{a_3}{1-\ddots}}}}. \tag{12}$$

is expressible as

$$\mathscr{X}(a) = \frac{\mathfrak{G}(a_2,a_3,\dots)}{\mathfrak{G}(a_1,a_2,\dots)}. \tag{13}$$

3. The logarithm of $\mathfrak{F}(x)$

$\mathfrak{F}(x_1,\dots,x_n)$ is a polynomial function in n complex variables, with $\mathfrak{F}(0)=1$, and therefore $\log\mathfrak{F}(x_1,\dots,x_n)$ is a well defined analytic function in some neighborhood of the origin. The goal of the current section is to derive an explicit formula for the coefficients of the corresponding power series.

For a multi-index $m\in\mathbb{N}^\ell$ denote by $|m|=\sum_{j=1}^{\ell}m_j$ its order and by $d(m)=\ell$ its length. For $N\in\mathbb{N}$, $\mathscr{M}(N)$ denotes the set of all multi-indices of order N, i.e.

$$\mathscr{M}(N) = \left\{ m\in\bigcup_{\ell=1}^{N}\mathbb{N}^\ell;\ |m|=N \right\}. \tag{14}$$

Obviously, $\cup_{\ell=1}^{\infty}\mathbb{N}^\ell = \cup_{N=1}^{\infty}\mathscr{M}(N)$. One has $\mathscr{M}(1)=\{(1)\}$ and

$$\begin{aligned}\mathscr{M}(N) = {} & \left\{ \left(1,m_1,m_2,\dots,m_{d(m)}\right);\, m\in\mathscr{M}(N-1)\right\} \\ & \cup \left\{ \left(m_1+1,m_2,\dots,m_{d(m)}\right);\, m\in\mathscr{M}(N-1)\right\}.\end{aligned}$$

Hence $|\mathscr{M}(N)|=2^{N-1}$ ($|\cdot|$ standing for the number of elements). Furthermore, for an multi-index $m\in\mathbb{N}^\ell$ put

$$\beta(m) := \prod_{j=1}^{\ell-1}\binom{m_j+m_{j+1}-1}{m_{j+1}},\ \alpha(m) := \frac{\beta(m)}{m_1}. \tag{15}$$

THEOREM 3.1. *In the ring of formal power series in the variables $t_1,\dots,t_n$, one has*

$$\log\mathfrak{F}(t_1,\dots,t_n) = -\sum_{\ell=1}^{n-1}\sum_{m\in\mathbb{N}^\ell}\alpha(m)\sum_{k=1}^{n-\ell}\prod_{j=1}^{\ell}\left(t_{k+j-1}t_{k+j}\right)^{m_j}. \tag{16}$$

For a complex sequence $x=\{x_k\}_{k=1}^{\infty}$ such that $\sum_{k=1}^{\infty}|x_kx_{k+1}|<\log 2$ one has

$$\log\mathfrak{F}(x) = -\sum_{\ell=1}^{\infty}\sum_{m\in\mathbb{N}^\ell}\alpha(m)\sum_{k=1}^{\infty}\prod_{j=1}^{\ell}\left(x_{k+j-1}x_{k+j}\right)^{m_j}.$$

The proof of Theorem 3.1 is based on some combinatorial notions, and among them that of a Dyck path is quite substantial. For $n \in \mathbb{N}$, $n \geqslant 2$, we may regard the set

$$\Lambda_n = \{1, 2, \ldots, n\}$$

as a finite one-dimensional lattice. We shall say that a mapping

$$\pi : \{0, 1, 2, \ldots, 2N\} \to \Lambda_n$$

is a loop of length $2N$ in Λ_n, $N \in \mathbb{N}$, if $\pi(0) = \pi(2N)$ and $|\pi(j+1) - \pi(j)| = 1$ for $1 \leqslant j \leqslant 2N$. The vertex $\pi(0)$ is called the base point of a loop. The loops in Λ_n with the base point $\pi(0) = 1$ are commonly known as Dyck paths of height not exceeding $n-1$. Indeed, if π is such a loop then its graph shifted by 1,

$$\{(j, \pi(j) - 1);\, j = 0, 1, \ldots, 2N\},$$

represents a lattice path in the first quadrant leading from $(0,0)$ to $(2N, 0)$ whose all steps are solely $(1,1)$ and $(1,-1)$. Such a path is called a Dyck path.

For $m \in \mathbb{N}^\ell$ denote by $\Omega(m)$ the set of all loops of length $2|m|$ in $\Lambda_{\ell+1}$ which encounter each edge $(j, j+1)$ exactly $2m_j$ times, $1 \leqslant j \leqslant \ell$ (counting both directions). Let $\Omega_1(m)$ designate the subset of $\Omega(m)$ formed by those loops which are based at the vertex 1. In other words, $\Omega_1(m)$ is the set of Dyck paths with the prescribed numbers $2m_j$ counting the steps at each level $j = 1, 2, \ldots, \ell$. One can call m the specification of a Dyck path. If $\pi \in \Omega_1(m)$ then the sequence $(\pi(0), \pi(1), \ldots, \pi(2N-1))$, with $N = |m|$, contains the vertex 1 exactly m_1 times, the vertices j, $2 \leqslant j \leqslant \ell$, are contained $(m_{j-1} + m_j)$ times in the sequence, and the number of occurrences of the vertex $\ell+1$ equals m_ℓ.

REMARK 3.1. It can be deduced from Theorem 3B in [14] that $|\Omega_1(m)| = \beta(m)$. Let us recall the well known fact that there exists a bijection between the set of Dyck paths of length $2N$ and the set of rooted plane trees with N edges (one can consult, for instance, §§ I.5 and I.6 in [16]). A rooted plane tree is said to have the specification $m \in \mathbb{N}^\ell$ if it has $|m|$ edges and the number of its vertices of height j equals m_j, $j = 1, 2, \ldots, \ell$. Using the mentioned bijection one finds that $\beta(m)$ also equals the number of rooted plane trees with the specification m [14, 29]. More recently, this result was rediscovered and described in [7]. For the reader's convenience we nevertheless include this identity in the following lemma along with a short proof. The other identity in the lemma providing a combinatorial interpretation of the number $\alpha(m)$ seems to be, to the authors' best knowledge, new.

LEMMA 3.2. *For every* $\ell \in \mathbb{N}$ *and* $m \in \mathbb{N}^\ell$, $|\Omega_1(m)| = \beta(m)$ *and* $|\Omega(m)| = 2|m|\alpha(m)$.

Proof. To show the first equality one can proceed by induction in ℓ. For $\ell = 1$ and any $m \in \mathbb{N}$ one clearly has $|\Omega_1(m)| = 1$. Suppose now that $\ell \geqslant 2$ and fix $m \in \mathbb{N}^\ell$. Denote $m' = (m_2, \ldots, m_\ell) \in \mathbb{N}^{\ell-1}$. For any $\pi' \in \Omega_1(m')$ put

$$\tilde{\pi} = (1, \pi'(0) + 1, \pi'(1) + 1, \ldots, \pi'(2N') + 1, 1)$$

where $N' = |m'| = |m| - m_1$. The vertex 2 occurs in $\tilde{\pi}$ exactly (m_2+1) times. After any such occurrence of 2 one may insert none or several copies of the two-letter chain $(1,2)$. Do it so while requiring that the total number of inserted couples equals $m_1 - 1$. This way one generates all Dyck paths from $\Omega_1(m)$, and each exactly once. This implies the recurrence rule

$$|\Omega_1(m_1, m_2, \ldots, m_\ell)| = \binom{m_1 - 1 + m_2}{m_2} |\Omega_1(m_2, \ldots, m_\ell)|,$$

thus proving that $|\Omega_1(m)| = \beta(m)$.

Let us proceed to the second equality. Put $N = |m|$. Consider the cyclic group $G = \langle g \rangle$, $g^{2N} = 1$. G acts on $\Omega(m)$ according to the rule

$$g \cdot \pi = (\pi(1), \pi(2), \ldots, \pi(2N), \pi(0)), \ \forall \pi \in \Omega(m).$$

Clearly, $G \cdot \Omega_1(m) = \Omega(m)$. Let us write $\Omega(m)$ as a disjoint union of orbits,

$$\Omega(m) = \bigcup_{s=1}^{M} \mathscr{O}_s.$$

For each orbit choose $\pi_s \in \mathscr{O}_s \cap \Omega_1(m)$. Let $H_s \subset G$ be the stabilizer of π_s. Then

$$|\Omega(m)| = \sum_{s=1}^{M} \frac{2N}{|H_s|}.$$

Denote further by G_s^1 the subset of G formed by those elements a obeying $a \cdot \pi_s \in \Omega_1(m)$ (i.e. the vertex 1 is still the base point). Then $|G_s^1| = m_1$ and $\mathscr{O}_s \cap \Omega_1(m) = G_s^1 \cdot \pi_s$. Moreover, $G_s^1 \cdot H_s = G_s^1$, i.e. H_s acts freely from the right on G_s^1, with orbits of this action being in one-to-one correspondence with elements of $\mathscr{O}_s \cap \Omega_1(m)$. Hence $|\mathscr{O}_s \cap \Omega_1(m)| = |G_s^1|/|H_s|$ and

$$|\Omega_1(m)| = \sum_{s=1}^{M} |\mathscr{O}_s \cap \Omega_1(m)| = \sum_{s=1}^{M} \frac{m_1}{|H_s|}.$$

This shows that $|\Omega(m)| = (2N/m_1)|\Omega_1(m)|$. In view of the first equality of the proposition and (15), the proof is complete. $\square$

LEMMA 3.3. *For $N \in \mathbb{N}$,*

$$\sum_{m \in \mathscr{M}(N)} \alpha(m) = \frac{1}{2N} \binom{2N}{N}.$$

Proof. According to Lemma 3.2, the sum

$$2N \sum_{m \in \mathscr{M}(N)} \alpha(m) = \sum_{m \in \mathscr{M}(N)} |\Omega(m)|$$

equals the number of equivalence classes of loops of length $2N$ in the one-dimensional lattice $\mathbb{Z}$ assuming that loops differing by translations are identified. These classes are generated by making $2N$ choices, in all possible ways, each time choosing either the sign plus or minus (moving to the right or to the left on the lattice) while the total number of occurrences of each sign being equal to N. $\square$

REMARK 3.2. The sum $C_N := \sum_{m\in\mathscr{M}(N)} \beta(m)$ can readily be evaluated, too, since this is nothing but the total number of Dyck paths of length $2N$. As is well known (see, for instance [11]), this number equals the Catalan number,

$$C_N = \frac{1}{N+1}\binom{2N}{N}. \tag{17}$$

Let us note that, as observed in Example 2.1 in [37], this is also an easy corollary of the Viennot theory which deals with weighted Motzkin or, more particularly, Dyck paths and which was originally designed to treat orthogonal polynomials on a unified abstract level [38, 36]. Giving each individual step in a Dyck path the weight 1 we have

$$\sum_{N=0}^{\infty} C_N t^N \equiv 1 + \sum_{\ell=1}^{\infty}\sum_{m\in\mathbb{N}^\ell} \beta(m)\, t^{|m|} = \cfrac{1}{1-\cfrac{t}{1-\cfrac{t}{1-\cfrac{t}{1-\ddots}}}},$$

see [38, §2] and [14]. But the continued fraction equals $(1-\sqrt{1-4t})/(2t)$ and (17) follows immediately.

For $m\in\mathbb{N}^\ell$ let

$$\binom{|m|}{m} := \frac{|m|!}{m_1!\,m_2!\cdots m_\ell!}.$$

LEMMA 3.4. *For every $\ell\in\mathbb{N}$ and $m\in\mathbb{N}^\ell$,*

$$\alpha(m) \leqslant \frac{1}{|m|}\binom{|m|}{m},$$

and equality holds if and only if $\ell = 1$ or 2.

Proof. Put $\gamma(m) = \alpha(m)/\binom{|m|}{m}$. To show that $\gamma(m) \leqslant 1/|m|$ one can proceed by induction in ℓ. It is immediate to check the equality to be true for $\ell = 1$ and 2. For $\ell \geqslant 3$ and $m_1 > 1$ one readily verifies that

$$\gamma(m_1,m_2,m_3,\ldots,m_\ell) < \gamma(m_1-1,m_2+1,m_3,\ldots,m_\ell).$$

Furthermore, if $\ell \geqslant 3$, $m_1 = 1$ and the inequality is known to be valid for $\ell-1$, one has

$$\gamma(m_1,m_2,m_3,\ldots,m_\ell) = \frac{m_2\,\gamma(m_2,m_3,\ldots,m_\ell)}{1+m_2+m_3+\cdots+m_\ell} < \frac{1}{|m|}.$$

The lemma follows. $\square$

Proof of Theorem 3.1. The coefficients in the power series expansion of the function $\log\mathfrak{F}(t_1,\dots,t_n)$ at the origin can be calculated in the ring of formal power series. As shown in [33], one has

$$\mathfrak{F}(t_1,\dots,t_n)=\det(I+T)$$

where

$$T=\begin{bmatrix} 0 & t_1 & & & & \\ t_2 & 0 & t_2 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & t_{n-1} & 0 & t_{n-1} \\ & & & & t_n & 0 \end{bmatrix}. \tag{18}$$

Since $\det\exp(A)=\exp(\operatorname{Tr}A)$ and $\log\det(I+T)=\operatorname{Tr}\log(I+T)$, and noticing that $\operatorname{Tr}T^{2k+1}=0$, one gets

$$\log\mathfrak{F}(t_1,\dots,t_n)=\operatorname{Tr}\log(I+T)=-\sum_{N=1}^{\infty}\frac{1}{2N}\operatorname{Tr}T^{2N}.$$

From (18) one deduces that

$$\operatorname{Tr}T^{2N}=\sum_{\pi\in\mathscr{L}(N)}\prod_{j=0}^{2N-1}t_{\pi(j)} \tag{19}$$

where $\mathscr{L}(N)$ stands for the set of all loops of length $2N$ in Λ_n. Let

$$k=\min\{\pi(j);\ 1\leqslant j\leqslant 2N\}$$

and put $\tilde{\pi}(j)=\pi(j)-k+1$ for $0\leqslant j\leqslant 2N$. Then $\tilde{\pi}\in\Omega(m)$ for certain (unambiguous) multi-index $m\in\mathscr{M}(N)$ of length $d(m)\leqslant n-k$. Conversely, given $m\in\mathscr{M}(N)$ of length $d(m)\leqslant n-1$ and k, $1\leqslant k\leqslant n-d(m)$, one defines $\pi\in\mathscr{L}(N)$ by $\pi(j)=k+\tilde{\pi}(j)-1$, $0\leqslant j\leqslant 2N$. Hence the RHS of (19) equals

$$\sum_{\substack{m\in\mathscr{M}(N)\\ d(m)<n}}\ \sum_{k=1}^{n-d(m)}|\Omega(m)|\prod_{j=1}^{d(m)}\left(t_{k+j-1}t_{k+j}\right)^{m_j}.$$

To verify (16) it suffices to apply Lemma 3.2.

Suppose now x is a complex sequence. If $\sum_k|x_kx_{k+1}|<\log 2$ one has, by (3), $|\mathfrak{F}(x)-1|<1$ and so $\log\mathfrak{F}(x)$ is well defined. Moreover, according to (4),

$$\log\mathfrak{F}(x)=\lim_{n\to\infty}\log\mathfrak{F}(x_1,\dots,x_n).$$

If $\sum_k |x_k x_{k+1}| < 1$ then the RHS of (16) admits the limit procedure, too, as demonstrated by the simple estimate (replacing t_j s by x_j s)

$$|\text{the RHS of (16)}| \leqslant \sum_{N=1}^{\infty} \left(\max_{m\in\mathscr{M}(N)} \frac{\alpha(m)}{\binom{N}{m}} \right) \sum_{m\in\mathscr{M}(N)} \binom{N}{m} \sum_{k=1}^{\infty} \prod_{j=1}^{d(m)} |x_{k+j-1}x_{k+j}|^{m_j}$$

$$\leqslant \sum_{N=1}^{\infty} \frac{1}{N} \left(\sum_{k=1}^{\infty} |x_k x_{k+1}| \right)^N = -\log\left(1 - \sum_{k=1}^{\infty} |x_k x_{k+1}| \right).$$

Here we have used Lemma 3.4. □

REMARK 3.3. Digressing a bit from the main topic of the present paper let us briefly comment on how the used formalism and the obtained results, and Theorem 3.1 in particular, fit into the theory of continued fractions. The continued fraction $\mathscr{X}(a)$ introduced in (12) equals the formal power series [1]

$$\mathscr{X}(a) = 1 + \sum_{n=1}^{\infty} S_n(a_1, a_2, \ldots, a_n) \tag{20}$$

where the summands are the so called genetic sums (of the lowest level $p = 1$) whose definition somewhat resembles that of (1) or (11),

$$S_n(a_1, a_2, \ldots, a_n) := \sum_{k_2=1}^{2} \sum_{k_3=1}^{k_2+1} \cdots \sum_{k_n=1}^{k_{n-1}+1} a_1 a_{k_2} a_{k_3} \ldots a_{k_n}.$$

This can be compared to Euler's continued fraction formula stating that the continued fraction

$$\mathscr{Y}(b) := \cfrac{1}{1 - \cfrac{b_1}{1 + b_1 - \cfrac{b_2}{1 + b_2 - \cfrac{b_3}{1 + b_3 - \ddots}}}} \tag{21}$$

equals the formal power series

$$\mathscr{Y}(b) = 1 + \sum_{n=1}^{\infty} \left(\prod_{j=1}^{n} b_j \right) \tag{22}$$

where $\{b_j\}$ is another set of formal variables; see, for instance, [39, Thm. 2.1]. It is immediate to transform (21) into (12), and then (20), (22), after substitution

$$a_j = \frac{b_j}{(1 + b_{j-1})(1 + b_j)} \quad (\text{with } b_0 = 0), \tag{23}$$

imply the identity

$$\sum_{n=1}^{\infty} S_n\left(\frac{b_1}{1+b_1}, \frac{b_2}{(1+b_1)(1+b_2)}, \ldots, \frac{b_n}{(1+b_{n-1})(1+b_n)}\right) = \sum_{n=1}^{\infty}\left(\prod_{j=1}^{n} b_j\right).$$

Without going into details we remark that relation (23) can be inverted (see (11)),

$$b_1 = \frac{a_1}{1-a_1}, \quad b_j = \frac{\mathfrak{S}(a_1,a_2,\ldots,a_{j-2})\,a_j}{\mathfrak{S}(a_1,a_2,\ldots,a_j)} \ \text{ for } j \geqslant 2,$$

and this way we arrive at the identity

$$\sum_{n=1}^{\infty} \frac{\prod_{j=1}^{n} a_j}{\mathfrak{S}(a_1,a_2,\ldots,a_{n-1})\mathfrak{S}(a_1,a_2,\ldots,a_n)} = \sum_{n=1}^{\infty} S_n(a_1,a_2,\ldots,a_n).$$

As far as Theorem 3.1 is concerned, it is straightforward to check that in combination with (13) it implies an exponential formula for the continued fraction (12),

$$\mathscr{K}(a) = \exp\left(\sum_{\ell=1}^{\infty}\sum_{m\in\mathbb{N}^{\ell}} \alpha(m)\left(\prod_{j=1}^{\ell} a_j^{m_j}\right)\right).$$

4. Factorization in the spectral parameter

In this section, we introduce a regularized characteristic function of a Jacobi matrix and show that it can be expressed as a Hadamard infinite product.

Let $\lambda = \{\lambda_n\}_{n=1}^{\infty}$, $\{w_n\}_{n=1}^{\infty}$ be real sequences such that $\lim_{n\to\infty}\lambda_n = +\infty$ and $w_n \neq 0$, $\forall n$. In addition, without loss of generality, $\{\lambda_n\}_{n=1}^{\infty}$ is assumed to be positive. Moreover, suppose that

$$\sum_{n=1}^{\infty} \frac{w_n^2}{\lambda_n\lambda_{n+1}} < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} < \infty. \tag{24}$$

Under these assumptions, by Theorem 2.1, J defined in (6) may be regarded as a self-adjoint operator on $\ell^2(\mathbb{N})$. Moreover, $\operatorname{der}(\lambda)$ is clearly empty and the characteristic function $F_J(z)$ is meromorphic on $\mathbb{C}$ with possible poles lying in the range of λ. To remove the poles let us define the function

$$\Phi_\lambda(z) := \prod_{n=1}^{\infty}\left(1-\frac{z}{\lambda_n}\right)e^{z/\lambda_n}.$$

Since $\sum_n \lambda_n^{-2} < \infty$, Φ_λ is a well defined entire function. Moreover, Φ_λ has zeros at the points $z = \lambda_n$, with multiplicity being equal to the number of repetitions of λ_n in the sequence λ, and no zeros otherwise.

Finally we define (see (10))

$$H_J(z) := \Phi_\lambda(z)F_J(z),$$

and call $H_J(z)$ the regularized characteristic function of the Jacobi operator J. Note that for $\varepsilon \geqslant 0$, $F_{J+\varepsilon I}(z) = F_J(z-\varepsilon)$ and so

$$H_{J+\varepsilon I}(z) = H_J(z-\varepsilon)\Phi_\lambda(-\varepsilon)^{-1}\exp\left(-z\sum_{n=1}^{\infty}\frac{\varepsilon}{\lambda_n(\lambda_n+\varepsilon)}\right). \tag{25}$$

According to Theorem 2.1, the spectrum of J is discrete, simple and real. Moreover,

$$\operatorname{spec}(J) = \operatorname{spec}_p(J) = H_J^{-1}(\{0\}).$$

As is well known, the determinant of an operator $I+A$ on a Hilbert space can be defined provided A belongs to the trace class. The definition, in a modified form, can be extended to other Schatten classes $\mathscr{I}_p$ as well, in particular to Hilbert-Schmidt operators; see [30] for a detailed survey of the theory. Let us denote, as usual, the trace class and the Hilbert-Schmidt class by $\mathscr{I}_1$ and $\mathscr{I}_2$, respectively. If $A \in \mathscr{I}_2$ then

$$(I+A)\exp(-A) - I \in \mathscr{I}_1,$$

and one defines

$$\det{}_2(I+A) := \det\left((I+A)\exp(-A)\right).$$

We shall need the following formulas [30, Chp. 9]. For $A, B \in \mathscr{I}_2$ one has

$$\det{}_2(I+A+B+AB) = \det{}_2(I+A)\det{}_2(I+B)\exp\left(-\operatorname{Tr}(AB)\right). \tag{26}$$

A factorization formula holds for $A \in \mathscr{I}_2$ and $z \in \mathbb{C}$,

$$\det{}_2(I+zA) = \prod_{n=1}^{N(A)}\left(1+z\mu_n(A)\right)\exp\left(-z\mu_n(A)\right), \tag{27}$$

where $\mu_n(A)$ are all (nonzero) eigenvalues of A counted up to their algebraic multiplicity (see Theorem 9.2 in [30] and also Theorem 1.1 ibidem introducing the algebraic multiplicity of a nonzero eigenvalue of a compact operator). In particular, $I+zA$ is invertible if and only if $\det_2(I+zA) \neq 0$. Moreover, the Plemelj-Smithies formula tells us that for $A \in \mathscr{I}_2$,

$$\det{}_2(I+zA) = \sum_{m=0}^{\infty} a_m(A)\frac{z^m}{m!}, \tag{28}$$

where

$$a_m(A) = \det\begin{bmatrix} 0 & m-1 & 0 & \dots & 0 & 0 \\ \operatorname{Tr}A^2 & 0 & m-2 & \dots & 0 & 0 \\ \operatorname{Tr}A^3 & \operatorname{Tr}A^2 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \operatorname{Tr}A^{m-1} & \operatorname{Tr}A^{m-2} & \operatorname{Tr}A^{m-3} & \dots & 0 & 1 \\ \operatorname{Tr}A^m & \operatorname{Tr}A^{m-1} & \operatorname{Tr}A^{m-2} & \dots & \operatorname{Tr}A^2 & 0 \end{bmatrix} \tag{29}$$

for $m \geqslant 1$, and $a_0(A) = 1$ [30, Thm. 5.4]. Finally, there exists a constant C_2 such that for all $A, B \in \mathscr{I}_2$,

$$|\det{}_2(I+A) - \det{}_2(I+B)| \leqslant \|A-B\|_2 \exp\left(C_2(\|A\|_2+\|B\|_2+1)^2\right), \tag{30}$$

where $\|\cdot\|_2$ stands for the Hilbert-Schmidt norm.

We write the Jacobi matrix in the form

$$J = L + W + W^*$$

where L is a diagonal matrix while W is lower triangular. By assumption (24), the operators L^{-1} and

$$K := L^{-1/2}(W + W^*)L^{-1/2} \tag{31}$$

are Hilbert-Schmidt. Hence for every $z \in \mathbb{C}$, the operator $L^{-1/2}(W + W^* - z)L^{-1/2}$ belongs to the Hilbert-Schmidt class.

LEMMA 4.1. *For every $z \in \mathbb{C}$,*

$$H_J(z) = \det{}_2\left(I + L^{-1/2}(W + W^* - z)L^{-1/2}\right).$$

In particular,

$$H_J(0) = F_J(0) = \det{}_2(I + K).$$

Proof. We first verify the formula for the truncated finite rank operator $J_N = P_N J P_N$, where P_N is the orthogonal projection onto the subspace spanned by the first N vectors of the canonical basis in $\ell^2(\mathbb{N})$. Using formula (8) one derives

$$\begin{aligned}
&\det\left[(I + P_N L^{-1/2}(W + W^* - z)L^{-1/2}P_N)\exp\left(-P_N L^{-1/2}(W + W^* - z)L^{-1/2}P_N\right)\right] \\
&= \det(P_N L^{-1} P_N)\det(J_N - zI_N)\exp\left(z\operatorname{Tr}(P_N L^{-1} P_N)\right) \\
&= \left(\prod_{n=1}^{N}\left(1 - \frac{z}{\lambda_n}\right)e^{z/\lambda_n}\right)\mathfrak{F}\left(\left\{\frac{\gamma_n^2}{\lambda_n - z}\right\}_{n=1}^{N}\right).
\end{aligned}$$

Sending N to infinity it is clear, by (4) and (24), that the RHS tends to $H_J(z)$. Moreover, one knows that $\det_2(I + A)$ is continuous in A in the Hilbert-Schmidt norm, as it follows from (30). Thus to complete the proof it suffices to notice that $A \in \mathscr{I}_2$ implies $\|P_N A P_N - A\|_2 \to 0$ as $N \to \infty$. $\square$

We intend to apply the Hadamard factorization theorem to $H_J(z)$; see, for example, [9, Thm. XI.3.4]. For simplicity we assume that $F_J(0) \neq 0$ and so J is invertible. Otherwise one could replace J by $J + \varepsilon I$ for some $\varepsilon > 0$ and make use of (25).

As already mentioned, the operator K defined in (31) is Hilbert-Schmidt. At the same time, this is a Jacobi matrix operator with zero diagonal admitting application of Theorem 2.1. One readily finds that

$$F_K(z) = \mathfrak{F}\left(\left\{-\frac{\gamma_n^2}{z\lambda_n}\right\}_{n=1}^{\infty}\right).$$

Hence $F_K(-1) = F_J(0)$, and J is invertible if and only if the same is true for $(I + K)$. In that case, again by Theorem 2.1, 0 belongs to the resolvent set of J, and

$$J^{-1} = L^{-1/2}(I + K)^{-1}L^{-1/2}. \tag{32}$$

LEMMA 4.2. *If J is invertible then J^{-1} is a Hilbert-Schmidt operator and*

$$\det_2\left(I - z(I+K)^{-1}L^{-1}\right) = \det_2\left(I - zJ^{-1}\right) \tag{33}$$

for all $z \in \mathbb{C}$.

Proof. By assumption (24), $L^{-1/2}$ belongs to the Schatten class $\mathscr{I}_4$. Since the Schatten classes are norm ideals and fulfill $\mathscr{I}_p\mathscr{I}_q \subset \mathscr{I}_r$ whenever $r^{-1} = p^{-1} + q^{-1}$ [30, Thm. 2.8], one deduces from (32) that $J^{-1} \in \mathscr{I}_2$.

Furthermore, one knows that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ provided $A \in \mathscr{I}_p$, $B \in \mathscr{I}_q$ and $p^{-1} + q^{-1} = 1$ [30, Cor. 3.8]. Hence

$$\mathrm{Tr}\left((I+K)^{-1}L^{-1}\right)^k = \mathrm{Tr}\left(L^{-1/2}(I+K)^{-1}L^{-1/2}\right)^k = \mathrm{Tr}(J^{-k}),\ \forall k \in \mathbb{N},\ k \geqslant 2.$$

It follows that the coefficients a_m defined in (29) fulfill

$$a_m((I+K)^{-1}L^{-1}) = a_m(J^{-1}) \ \text{ for } \ m = 0, 1, 2, \ldots.$$

The Plemelj-Smithies formula (28) then implies (33). □

THEOREM 4.3. *Using the notation introduced in (6), suppose a real Jacobi matrix J obeys (24) and is invertible. Denote by $\lambda_n(J)$, $n \in \mathbb{N}$, the eigenvalues of J (all of them are real and simple). Then $L^{-1} - J^{-1} \in \mathscr{I}_1$,*

$$\sum_{n=1}^{\infty} \lambda_n(J)^{-2} < \infty, \tag{34}$$

and for the regularized characteristic function of J one has

$$H_J(z) = F_J(0)\, e^{bz} \prod_{n=1}^{\infty}\left(1 - \frac{z}{\lambda_n(J)}\right) e^{z/\lambda_n(J)} \tag{35}$$

where

$$b = \mathrm{Tr}\left(L^{-1} - J^{-1}\right) = \sum_{n=1}^{\infty}\left(\frac{1}{\lambda_n} - \frac{1}{\lambda_n(J)}\right).$$

Proof. Recall equation (32). Since $L^{-1/2} \in \mathscr{I}_4$ and $K \in \mathscr{I}_2$ one has, after some straightforward manipulations,

$$L^{-1} - J^{-1} = L^{-1/2}K(I+K)^{-1}L^{-1/2} \in \mathscr{I}_1. \tag{36}$$

By Lemma 4.2, the operator J^{-1} is Hermitian and Hilbert-Schmidt. This implies (34). Furthermore, by Lemma 4.1, formula (26) and Lemma 4.2,

$$\begin{aligned} H_J(z) &= \det_2(I + K - zL^{-1}) \\ &= \det_2(I+K)\det_2\left(I - z(I+K)^{-1}L^{-1}\right)\exp\left[z\,\mathrm{Tr}\left(K(I+K)^{-1}L^{-1}\right)\right] \\ &= F_J(0)\, e^{bz}\det_2\left(I - zJ^{-1}\right). \end{aligned}$$

Here we have used (36) implying

$$\operatorname{Tr}\left(K(I+K)^{-1}L^{-1}\right)=\operatorname{Tr}\left(L^{-1/2}K(I+K)^{-1}L^{-1/2}\right)=\operatorname{Tr}\left(L^{-1}-J^{-1}\right)=b.$$

Finally, by formula (27),

$$\det\nolimits_2\left(I-zJ^{-1}\right)=\prod_{n=1}^{\infty}\left(1-\frac{z}{\lambda_n(J)}\right)e^{z/\lambda_n(J)}.$$

This completes the proof. □

COROLLARY 4.4. *For each $\varepsilon>0$ there is $R_\varepsilon>0$ such that for $|z|>R_\varepsilon$,*

$$|H_J(z)|<\exp\left(\varepsilon|z|^2\right). \tag{37}$$

Proof. Theorem 4.3, and particularly the product formula (35) implies that $H_J(z)$ is an entire function of genus one. In that case the growth property (37) is known to be valid; see, for example, Theorem XI.2.6 in [9]. □

EXAMPLE 4.1. Put $\lambda_n=n$ and $w_n=w\neq 0$, $\forall n\in\mathbb{N}$. As shown in [32], the Bessel function of the first kind can be expressed as

$$J_\nu(2w)=\frac{w^\nu}{\Gamma(\nu+1)}\mathfrak{F}\left(\left\{\frac{w}{\nu+k}\right\}_{k=1}^{\infty}\right), \tag{38}$$

as long as $w,\nu\in\mathbb{C}$, $\nu\notin-\mathbb{N}$. Using (38) and that

$$\Gamma(z)=\frac{e^{-\gamma z}}{z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)^{-1}e^{z/n},$$

where γ is the Euler constant, one gets $H_J(z)=e^{\gamma z}w^zJ_{-z}(2w)$. In particular, the zeros of $J_{-z}(2w)$ in z are exactly the eigenvalues of the corresponding Jacobi matrix J. Applying Theorem 4.3 one reveals an infinite product formula for the Bessel function considered as a function of its order. Assuming $J_0(2w)\neq 0$, the formula reads

$$\frac{w^zJ_{-z}(2w)}{J_0(2w)}=e^{c(w)z}\prod_{n=1}^{\infty}\left(1-\frac{z}{\lambda_n(J)}\right)e^{z/\lambda_n(J)} \tag{39}$$

where

$$c(w)=\frac{1}{J_0(2w)}\sum_{k=0}^{\infty}(-1)^k\psi(k+1)\frac{w^{2k}}{(k!)^2},$$

$\psi(z)=\Gamma'(z)/\Gamma(z)$ is the digamma function, and the expression for $c(w)$ is obtained by comparison of the coefficients at z on both sides.

5. Factorization in the coupling constant

Let $x = \{x_n\}_{n=1}^{\infty}$ be a sequence of nonzero complex numbers belonging to the domain D defined in (2). Our goal in this section is to prove a factorization formula for the entire function

$$f(w) := \mathfrak{F}(wx), \ w \in \mathbb{C}.$$

Let us remark that $f(w)$ is even.

To this end, let us put $v_k = \sqrt{x_k}$, $\forall k$, (any branch of the square root is suitable) and introduce the auxiliary Jacobi matrix

$$A = \begin{bmatrix} 0 & a_1 & 0 & 0 & \cdots \\ a_1 & 0 & a_2 & 0 & \cdots \\ 0 & a_2 & 0 & a_3 & \cdots \\ 0 & 0 & a_3 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \ \text{with } a_k = v_k v_{k+1}, \ k \in \mathbb{N}. \tag{40}$$

Then A represents a Hilbert-Schmidt operator on $\ell^2(\mathbb{N})$ with the Hilbert-Schmidt norm

$$\|A\|_2^2 = 2\sum_{k=1}^{\infty} |a_k|^2 = 2\sum_{k=1}^{\infty} |x_k x_{k+1}|.$$

The relevance of A to our problem comes from the equality

$$F_A(z) = \mathfrak{F}\left(\left\{\frac{x_k}{z}\right\}_{k=1}^{\infty}\right) = f\left(z^{-1}\right),$$

which can be verified with the aid of (5) and (10). Hence $F_A(z)$ is analytic on $\mathbb{C}\setminus\{0\}$. By Theorem 2.2, the set of nonzero eigenvalues of A coincides with the zero set of $F_A(z)$. It even turns out that the algebraic multiplicity of a nonzero eigenvalue ζ of A equals the multiplicity of ζ as a root of the function $F_A(z)$, as one infers from the following theorem extending the conclusions of Theorem 2.2. For the definition of the algebraic multiplicity of a nonzero eigenvalue of a compact operator see, for instance, [30, Thm. 1.1].

THEOREM 5.1. *Assume that a complex Jacobi matrix J of the form (6) fulfills $\lambda_n = 0$, $\forall n$, and $\{w_n\} \in \ell^2(\mathbb{N})$. Then the algebraic multiplicity of any nonzero eigenvalue ζ of J is equal to the multiplicity of the root ζ^{-1} of the entire function $\varphi(z) = F_J(z^{-1}) = \mathfrak{F}\left(\{z\gamma_n^2\}_{n=1}^{\infty}\right)$ (see (7) and (10)).*

Proof. Recall that $\gamma_n\gamma_{n+1} = w_n$ and therefore, by our assumptions, $\{\gamma_n^2\} \in D$. Denote again by P_N, $N \in \mathbb{N}$, the orthogonal projection onto the subspace spanned by the first N vectors of the canonical basis in $\ell^2(\mathbb{N})$. From formula (8) we deduce that

$$\mathfrak{F}\left(\{z\gamma_n^2\}_{n=1}^{N}\right) = \det(I - zJ_N) = \det\left((I - zJ_N)e^{zJ_N}\right),$$

where $J_N = P_N J P_N$. Since $P_N J P_N$ tends to J in the Hilbert-Schmidt norm, as $N \to \infty$, and by continuity of the generalized determinant as a functional on the space of Hilbert-Schmidt operators (see (30)) one immediately gets

$$\varphi(z) = \mathfrak{F}\big(\{z\gamma_n^2\}_{n=1}^{\infty}\big) = \det\big((I - zJ)e^{zJ}\big) = \det_2(I - zJ).$$

From (27) it follows that $\varphi(z) = (1 - \zeta z)^m \tilde{\varphi}(z)$ where m is the algebraic multiplicity of ζ, $\tilde{\varphi}(z)$ is an entire function and $\tilde{\varphi}(\zeta^{-1}) \neq 0$. $\square$

The zero set of $f(w)$ is at most countable and symmetric with respect to the origin. One can split $\mathbb{C}$ into two half-planes so that the border line passes through the origin and contains no nonzero root of f. Fix one of the half-planes and enumerate all nonzero roots in it as $\{\zeta_k\}_{k=1}^{N(f)}$, with each root being repeated in the sequence according to its multiplicity. The number $N(f)$ may be either a non-negative integer or infinity. Then

$$\operatorname{spec}_p(A) \setminus \{0\} = \big\{\pm\zeta_k^{-1};\ k \in \mathbb{N},\ k \leqslant N(f)\big\}.$$

Since A^2 is a trace class operator one has, by Theorem 5.1 and Lidskii's theorem,

$$\sum_{k=1}^{N(f)} \frac{1}{\zeta_k^2} = \frac{1}{2}\operatorname{Tr} A^2 = \sum_{k=1}^{\infty} x_k x_{k+1}. \tag{41}$$

Moreover, the sum on the LHS converges absolutely, as it follows from Weyl's inequality [30, Thm. 1.15].

THEOREM 5.2. *Let $x = \{x_k\}_{k=1}^{\infty}$ be a sequence of nonzero complex numbers such that*

$$\sum_{k=1}^{\infty} |x_k x_{k+1}| < \infty.$$

Then the zeros of the entire even function $f(w) = \mathfrak{F}(wx)$ can be arranged into sequences

$$\{\zeta_k\}_{k=1}^{N(f)} \cup \{-\zeta_k\}_{k=1}^{N(f)},$$

with each zero being repeated according to its multiplicity, and

$$f(w) = \prod_{k=1}^{N(f)} \left(1 - \frac{w^2}{\zeta_k^2}\right). \tag{42}$$

Proof. Equation (42) can be deduced from Hadamard's factorization theorem; see, for instance, [9, Chp. XI]. In fact, the absolute convergence of the series $\sum \zeta_k^{-2}$ in (41) means that the rank of f is at most 1. Furthermore, (3) implies

$$|f(w)| \leqslant \exp\left(|w|^2 \sum_{k=1}^{\infty} |x_k x_{k+1}|\right),$$

and therefore the order of f is less than or equal to 2. Hadamard's factorization theorem tells us that the genus of f is at most 2. Taking into account that f is even and $f(0)=1$, this means nothing but

$$f(w)=\exp(cw^2)\prod_{k=1}^{N(f)}\left(1-\frac{w^2}{\zeta_k^2}\right)$$

for some $c\in\mathbb{C}$. Equating the coefficients at w^2 one gets

$$-\sum_{k=1}^{\infty}x_kx_{k+1}=c-\sum_{k=1}^{N(f)}\frac{1}{\zeta_k^2}.$$

According to (41), $c=0$. $\square$

COROLLARY 5.3. *For any $n\in\mathbb{N}$ (and recalling (14), (15)),*

$$\sum_{k=1}^{N(f)}\frac{1}{\zeta_k^{2n}}=n\sum_{m\in\mathscr{M}(n)}\alpha(m)\sum_{k=1}^{\infty}\prod_{j=1}^{d(m)}\left(x_{k+j-1}x_{k+j}\right)^{m_j}. \tag{43}$$

Proof. Using Theorem 3.1, one can expand $\log f(w)$ into a power series at $w=0$. Applying log to (42) and equating the coefficients at w^{2n} yields (43). $\square$

If the sequence $\{x_k\}$ in Theorem 5.2 is positive one has some additional information about the zeros of $f(w)$. In that case the v_ks in (40) can be chosen positive, and so A is a self-adjoint Hilbert-Schmidt operator. The zero set of f is countable and all roots are real, simple and have no finite accumulation points. Enumerating positive zeros in ascending order as ζ_k, $k\in\mathbb{N}$, factorization (42) and identities (43) hold true. Since the first positive root ζ_1 is strictly smaller than all other positive roots, one has

$$\zeta_1=\lim_{N\to\infty}\left(\sum_{m\in\mathscr{M}(N)}\alpha(m)\sum_{k=1}^{\infty}\prod_{j=1}^{d(m)}\left(x_{k+j-1}x_{k+j}\right)^{m_j}\right)^{-1/(2N)}.$$

REMARK 5.1. Still assuming the sequence $\{x_k\}$ to be positive let $g(z)$ be an entire function defined by

$$g(z)=1+\sum_{n=1}^{\infty}g_nz^n=\prod_{k=1}^{\infty}\left(1-\frac{z}{\zeta_k^2}\right),$$

i.e. $g(w^2)=f(w)$. Put

$$\sigma(2n)=\sum_{k=1}^{\infty}\frac{1}{\zeta_k^{2n}},\ n\in\mathbb{N}.$$

These are particular values of a function σ which is usually called the zeta function associated with f. If A is invertible in $\ell^2(\mathbb{N})$, and this happens if and only if

$$\sum_{k=1}^{\infty}\frac{1}{x_{2k-1}}<\infty$$

(as observed in [33, Eq. (51)]), σ is also called the spectral zeta function of A^{-1}. In some cases of interest the coefficients g_n are known explicitly and then these values of the spectral zeta function can be evaluated recursively. Taking the logarithmic derivative of $g(z)$ and equating coefficients at the same powers of z leads to the recurrence rule

$$\sigma(2) = -g_1,\ \sigma(2n) = -ng_n - \sum_{k=1}^{n-1} g_{n-k}\,\sigma(2k) \ \text{ for } \ n > 1. \tag{44}$$

EXAMPLE 5.1. Put $x_k = (\nu+k)^{-1}$, with $\nu > -1$. Recalling (38) and letting $z = w/2$, the factorization of the Bessel function [40],

$$\left(\frac{z}{2}\right)^{-\nu}\Gamma(\nu+1)J_\nu(z) = \prod_{k=1}^{\infty}\left(1-\frac{z^2}{j_{\nu,k}^2}\right),$$

is obtained as a particular case of Theorem 5.2. The positive zeros of $J_\nu(z)$, called $j_{\nu,k}$, occur also in the definition of the so called Rayleigh function [24],

$$\sigma_\nu(s) = \sum_{k=1}^{\infty}\frac{1}{j_{\nu,k}^s},\ \ \operatorname{Re} s > 1.$$

Corollary 5.3 implies the formula

$$\sigma_\nu(2N) = 2^{-2N}N\sum_{k=1}^{\infty}\sum_{m\in\mathscr{M}(N)}\alpha(m)\prod_{j=1}^{d(m)}\left(\frac{1}{(j+k+\nu-1)(j+k+\nu)}\right)^{m_j},\ \ N\in\mathbb{N}.$$

According to (44) one has the linear recurrence [24, Eq. (14)]

$$\sigma_\nu(2n) = \frac{(-1)^{n+1}2^{-2n}}{(n-1)!\,(\nu+1)_n} - \sum_{k=1}^{n-1}\frac{(-1)^k 2^{-2k}}{k!\,(\nu+1)_k}\,\sigma_\nu(2n-2k),\ \ n = 1,2,3,\ldots.$$

EXAMPLE 5.2. This examples is perhaps less commonly known and concerns the Ramanujan function, also interpreted as the q-Airy function by some authors [21, 41], and defined by

$$A_q(z) := {}_0\phi_1(;0;q,-qz) = \sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n}(-z)^n, \tag{45}$$

where ${}_0\phi_1(;b;q,z)$ is the basic hypergeometric series (q-hypergeometric series) and $(a;q)_k$ is the q-Pochhammer symbol (see, for instance, [18]). In (45), we suppose that $0 < q < 1$ and $z\in\mathbb{C}$. It has been shown in [32] that

$$A_q(w^2) = q\mathfrak{F}\left(\left\{wq^{(2k-1)/4}\right\}_{k=1}^{\infty}\right).$$

Denote by $0 < \zeta_1(q) < \zeta_2(q) < \zeta_3(q) < \ldots$ the positive zeros of $w\mapsto A_q(w^2)$ and put $\iota_k(q) = \zeta_k(q)^2$, $k\in\mathbb{N}$. Then Theorem 5.2 tells us that the zeros of $A_q(z)$ are exactly $0 < \iota_1(q) < \iota_2(q) < \iota_3(q) < \ldots$, all of them are simple and

$$A_q(z) = \prod_{k=1}^{\infty}\left(1-\frac{z}{\iota_k(q)}\right).$$

One has $\{\iota_k(q)^{-1/2};\ k\in\mathbb{N}\}=\operatorname{spec}(\boldsymbol{A}(q))\backslash\{0\}$ where $\boldsymbol{A}(q)$ is the Hilbert-Schmidt operator in $\ell^2(\mathbb{N})$ whose matrix is of the form (40), with $a_k=q^{k/2}$. Corollary 5.3 yields a formula for the spectral zeta function $D_N(q)$ associated with $A_q(z)$, namely

$$D_N(q):=\sum_{k=1}^{\infty}\frac{1}{\iota_k(q)^N}=\frac{Nq^N}{1-q^N}\sum_{m\in\mathscr{M}(N)}\alpha(m)\,q^{\varepsilon_1(m)},\ N\in\mathbb{N},$$

where, $\forall m\in\mathbb{N}^\ell$, $\varepsilon_1(m)=\sum_{j=1}^{\ell}(j-1)\,m_j$. In accordance with (44), from the power series expansion of $A_q(z)$ one derives the recurrence rule

$$D_n(q)=(-1)^{n+1}\frac{nq^{n^2}}{(q;q)_n}-\sum_{k=1}^{n-1}(-1)^k\frac{q^{k^2}}{(q;q)_k}D_{n-k}(q),\ n=1,2,3,\ldots.$$

Consider now a real Jacobi matrix J of the form (6) such that the diagonal sequence $\{\lambda_n\}$ is semibounded. Suppose further that the off-diagonal elements w_n depend on a real parameter w as $w_n=w\omega_n$, $n\in\mathbb{N}$, with $\{\omega_n\}$ being a fixed sequence of positive numbers. Following physical terminology one may call w the coupling constant. Denote $\lambda_{\inf}=\inf\lambda_n$. Assume that

$$\sum_{n=1}^{\infty}\frac{\omega_n^2}{(\lambda_n-z)(\lambda_{n+1}-z)}<\infty$$

for some and hence any $z<\lambda_{\inf}$. For $z<\lambda_{\inf}$, Theorem 5.2 can be applied to the sequence $x_n(z)=\kappa_n^2/(\lambda_n-z)$, $n\in\mathbb{N}$, where $\{\kappa_n\}$ is defined recursively by $\kappa_1=1$, $\kappa_n\kappa_{n+1}=\omega_n$. Comparing to (7) one has $\kappa_{2k-1}=\gamma_{2k-1}$, $\kappa_{2k}=\gamma_{2k}/w$. Let

$$F_J(z;w)=\mathfrak{F}\left(\left\{\frac{\gamma_n^2}{\lambda_n-z}\right\}_{n=1}^{\infty}\right)=\mathfrak{F}\left(\left\{\frac{w\,\kappa_n^2}{\lambda_n-z}\right\}_{n=1}^{\infty}\right)$$

be the characteristic function of $J=J(w)$. We conclude that for every $z<\lambda_{\inf}$ fixed, the equation $F_J(z;w)=0$ in the variable w has countably many positive simple roots $\zeta_k(z)$, $k\in\mathbb{N}$, enumerated in ascending order, and

$$F_J(z;w)=\prod_{k=1}^{\infty}\left(1-\frac{w^2}{\zeta_k(z)^2}\right).$$

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