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NUMERICAL RANGES OF THE PRODUCT OF OPERATORS

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Abstract. We study containment regions of the numerical range of the product of operators A and B such that $W(A)$ and $W(B)$ are line segments. It is shown that the containment region is equal to the convex hull of elliptical disks determined by the spectrum of AB, and conditions on A and B for the set equality holding are obtained. The results cover the case when A and B are self-adjoint operators extending the previous results on the numerical range of the product of two orthogonal projections.

1. Introduction

Let $B(H)$ be the algebra of bounded linear operators on a complex Hilbert space H. We identify $B(H)$ with M_n, the algebra of n-by-n complex matrices, if H has finite dimension n. The spectrum $\sigma(A)$, and the numerical range $W(A)$ of an operator $A \in B(H)$ are defined by

$$\sigma(A) = \{\lambda : A - \lambda I \text{ is not invertible}\} \quad \text{and} \quad W(A) = \{\langle Ax, x\rangle : x \in H, \|x\| = 1\},$$

respectively. Here $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ are the standard inner product and its associated norm on H, respectively. The spectrum and the numerical range are useful tools in the study of matrices and operators; for example, see [4, 5, 6]. It is known that $W(A)$ is a bounded convex subset of $\mathbf{C}$. When H is finite dimensional, it is compact. In general, the closure of the numerical range satisfies $\sigma(A) \subseteq \overline{W(A)}$. Especially, for $A \in M_2$, $W(A)$ is an elliptical disk with λ_1 and λ_2 as foci and $\{\mathrm{tr}(A^*A) - |\lambda_1|^2 - |\lambda_2|^2\}^{1/2}$ as minor axis, where λ_1 and λ_2 are eigenvalues of A.

An operator $A \in B(H)$ is an orthogonal projection if $A^2 = A = A^*$, contraction if $\|A\| \equiv \sup_{\|x\|=1} \|Ax\| \leqslant 1$, and positive if $\langle Ax, x\rangle \geqslant 0$ for all $x \in H$. In [1], it was shown that if $P, Q \in B(H)$ are orthogonal projections and $0 \in \sigma(P) \cup \sigma(Q)$, then

$$\overline{W(PQ)} = \overline{\mathrm{conv}\{\cup_{\lambda \in \sigma(PQ)} \mathscr{E}(\lambda)\}},$$

where $\mathrm{conv}\{\mathscr{S}\}$ is the convex hull of the set $\mathscr{S}$ and $\mathscr{E}(\lambda)$ is the ellipse disc with foci 0 and λ, and length of minor axis $\sqrt{\lambda(1-\lambda)}$. In general, the following example

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shows that the above equality may not hold for positive contractions $A,B \in B(H)$. Let $A = B = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}$. Then $W(AB) = [1/4,1] \neq \overline{\text{conv}\{\mathscr{E}(1) \cup \mathscr{E}(1/4)\}}$.

In this paper, we consider the containment regions for the numerical range of the product of positive contractions, and extend the result to more general operators, namely, those operators with numerical ranges equal to line segments.

First of all, for two positive contraction operators $A,B \in B(H)$, it is known that

$$\overline{W(AB)} \subseteq \{x+iy : -1/8 \leqslant x \leqslant 1, -1/4 \leqslant y \leqslant 1/4\};$$

see [1], [2], [3]. To see this[1], we use the positive definite ordering that $X \geqslant Y$, and show that

(a) $-I/8 \leqslant (AB+BA)/2 \leqslant I$ and (b) $-I/4 \leqslant (AB-BA)/(2i) \leqslant I/4$.

For (a), it is clear that $\|AB+BA\| \leqslant 2$ so that $-2I \leqslant AB+BA \leqslant 2I$ for two positive contractions A and B. Furthermore, note that $A+B-A^2-B^2 \geqslant 0$, and hence

$$\begin{aligned} 0 &\leqslant (A+B-I/2)^2 = A^2+B^2+(AB+BA)+I/4-A-B \\ &= (AB+BA)+I/4+(A^2-A)+(B^2-B) \leqslant (AB+BA)+I/4. \end{aligned}$$

For (b), since $\|A-I/2\| \leqslant 1/2$ and $\|B-I/2\| \leqslant 1/2$, we have

$$\begin{aligned} \|i(AB-BA)\| &= \|(A-I/2)(B-I/2)-(B-I/2)(A-I/2)\| \\ &\leqslant 2\|(A-I/2)(B-I/2)\| \leqslant 1/2. \end{aligned}$$

Moreover, $i(AB-BA)$ is self-adjoint and then condition (b) holds.

Suppose $\lambda \in [0,1]$. Denote by $\mathscr{E}(\lambda)$ the elliptical disk with foci $0,\lambda$, minor axis with end points $(\lambda \pm i\sqrt{\lambda(1-\lambda)})/2$, and major axis with end points $(\lambda \pm \sqrt{\lambda})/2$. Then $W\left(\begin{pmatrix} \lambda & 0 \\ \sqrt{\lambda(1-\lambda)} & 0 \end{pmatrix}\right) = \mathscr{E}(\lambda)$. We have the following result in [4].

THEOREM 1.1. *Let $P,Q \in M_n$ be non-scalar orthogonal projections. Then*

$$W(PQ) = \text{conv}\{\cup_{\lambda \in \sigma(PQ)} \mathscr{E}(\lambda)\}.$$

One can obtain the above result using the following canonical for a product of projections $P,Q \in M_n$; see [1, 7] and its references.

PROPOSITION 1.2. *Suppose $P,Q \in M_n$ are non-scalar projections and $U \in M_n$ is unitary such that $U^*(P+iQ)U$ is a direct sum of $(I_p+iI_p) \oplus I_q \oplus iI_r \oplus 0_s$, and*

$$C_j = \begin{pmatrix} c_j^2 + i & c_j s_j \\ c_j s_j & s_j^2 \end{pmatrix}, \quad j = 1, \ldots, k,$$

where $c_j \in (0,1), s_j = \sqrt{1-c_j^2}$. Then U^*PQU will be a direct sum of $I_p \oplus 0_{q+r+s}$ and

$$\hat{C}_j = \begin{pmatrix} c_j^2 & 0 \\ c_j s_j & 0 \end{pmatrix}, \quad j = 1, \ldots, k.$$

In the next two sections, we will consider containment regions for the numerical range of the product of a pair of positive contractions, and extend the results to a more general class of matrices, namely, those matrices with numerical ranges contained in line segments. We will consider the infinite dimensional version of Theorem 1.1 and its generalization in Section 4.

[1] Li would like to thank Professor Fuzheng Zhang for showing him this proof.

2. Positive contractions

In this section, we extend Theorem 1.1 to obtain a containment region $\mathscr{S}$ of $W(AB)$ for two positive contractions $A,B \in M_n$, and determine the conditions for $\mathscr{S} = W(AB)$. We begin with some technical lemmas. We will denote by $\lambda_1(X) \geqslant \cdots \geqslant \lambda_n(X)$ the eigenvalues of a Hermitian matrix $X \in M_n$.

LEMMA 2.1. *Suppose $T = T_1 \oplus \ldots \oplus T_m \oplus T_0 \in M_n$ and $\mu = \mu_1 + i\mu_2 \in \mathbf{C}$ satisfying $T_1 = \cdots = T_m \in M_2$ are non-scalar matrices,*

$$2\mu_1 = \lambda_1(T_1 + T_1^*) > \lambda_1(T_0).$$

Then up to a unit multiple, there is a unique unit vector $\tilde{x} \in \mathbf{C}^2$ such that $\tilde{x}^* T_1 \tilde{x} = \mu_1 + i\mu_2$. Moreover, if $x \in \mathbf{C}^n$ such that $x^*Tx = \mu_1 + i\mu_2$, then

$$x = v \otimes \tilde{x} \oplus 0_{n-2m} = (v_1\tilde{x}^t, \ldots, v_m\tilde{x}^t, 0, \ldots, 0)^t$$

where $v = (v_1, \ldots, v_m)^t \in \mathbf{C}^m$ is a unit vector.

Proof. Obviously, $\mu \in \partial W(T_1)$. Hence there is a unit vector $\tilde{x} \in \mathbf{C}^2$ such that $\tilde{x}^* T_1 \tilde{x} = \mu_1 + i\mu_2$.

Now, suppose $x = x_1 \oplus \cdots \oplus x_m \oplus x_0$, where, $x_1, \ldots, x_m \in \mathbf{C}^2$ and $x_0 \in \mathbf{C}^{n-2m}$, is a unit vector such that $x^*Tx = \mu_1 + i\mu_2$. Then

$$2\mu_1 = x^*(T+T^*)x = \sum_{j=0}^{m} x_j^*(T_j + T_j^*)x_j \leqslant 2\mu_1 \sum_{j=1}^{m} \|x_j\|^2 + \lambda_1(T_0 + T_0^*)\|x_0\|^2.$$

Thus, $x_0 = 0$ and $(T_j + T_j^*)x_j = 2\mu_1 x_j$ for $j = 1, \ldots, m$. and $x_j = v_j \tilde{x}$ with $v_j \in \mathbf{C}$ for $j = 1, \ldots, m$. Let $v = (v_1, \ldots, v_m)^t$. Then $x = v \otimes \tilde{x} \oplus 0_{n-2m}$, and $\|v\| = \|x\|/\|\tilde{x}\| = 1$ as asserted. $\square$

LEMMA 2.2. *Let $P, Q \in M_n$ be non-scalar orthogonal projections. Suppose that there is a supporting line L of $W(PQ)$ satisfying $L \cap W(PQ) = \{\mu\} \subseteq \mathscr{E}(\hat{\lambda})$ with $\hat{\lambda} \in \sigma(PQ)$ and $\hat{\lambda} \in (0,1)$. Suppose that $\mu \notin \mathscr{E}(\lambda)$ for all other $\lambda \in \sigma(PQ)$ and that $\langle PQx, x\rangle = \mu$ for some unit vector x. Then there is a unitary matrix $V \in M_n$ with the first two columns v_1, v_2 such that* $\operatorname{span}\{v_1, v_2\} = \operatorname{span}\{x, PQx\}$, *and*

$$V^*PV = \begin{pmatrix} \hat{\lambda} & \sqrt{\hat{\lambda} - \hat{\lambda}^2} \\ \sqrt{\hat{\lambda} - \hat{\lambda}^2} & 1 - \hat{\lambda} \end{pmatrix} \oplus P' \qquad \text{and} \qquad V^*QV = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \oplus Q'.$$

Proof. Suppose PQ has the canonical form described in Proposition 1.2, and U is the unitary such that $U^*PQU = C_1 \oplus \cdots \oplus C_k \oplus I_r \oplus 0_s$, $P = P_1 \oplus \cdots \oplus P_k \oplus I_r \oplus P'$ and $Q = Q_1 \oplus \cdots \oplus Q_k \oplus I_r \oplus Q'$, where $P'Q' = 0_s$ and $C_i = P_iQ_i$ for $1 \leqslant i \leqslant k$. We may further assume that $C_1, \ldots, C_m$ satisfy $W(C_1) = \cdots = W(C_m) = \mathscr{E}(\hat{\lambda})$ such that $\mu \notin W(C_j)$ for all other $j \in \{m+1, \ldots, k\}$. Thus,

$$C_1 = \cdots = C_m = \begin{pmatrix} c^2 & 0 \\ cs & 0 \end{pmatrix}, \quad P_1 = \cdots = P_m = \begin{pmatrix} c^2 & cs \\ cs & s^2 \end{pmatrix}$$

and

$$Q_1 = \cdots = Q_m = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

with $\hat{\lambda} = c^2$ and $s^2 = \sqrt{1-c^2}$. Because $L \cap W(PQ) = \{\mu\}$, there is $t \in [0, 2\pi)$ such that $e^{it}\mu + e^{-it}\overline{\mu}$ is the largest eigenvalue of $e^{it}PQ + e^{-it}QP$; e.g., see [5, Chapter 1]. Now, set $\hat{P} + i\hat{Q} = U^*(P+iQ)U$ and $\hat{x} = U^*x$ so that $\hat{x}^*\hat{P}\hat{Q}\hat{x} = \mu$. Then $T = e^{it}\hat{P}\hat{Q}$ will satisfy the hypothesis of Lemma 2.1. It follows that

$$\hat{x} = U^*x = v_0 \otimes \tilde{x} \oplus 0_{n-2m} \quad \text{and} \quad \hat{P}\hat{Q}\hat{x} = (U^*PQU)(U^*x) = v_0 \otimes \tilde{y} \oplus 0_{n-2m} \equiv \hat{y},$$

where $v_0 \in \mathbf{C}^m$ with $\|v_0\| = 1$ and $\tilde{y} = C_1\tilde{x}$. For any $y \in \operatorname{span}\{\hat{x}, \hat{y}\}$, we have $y = v_0 \otimes y_0 \oplus 0_{n-2m}$ for some $y_0 \in \operatorname{span}\{\tilde{x}, \tilde{y}\}$, and then

$$\hat{P}\hat{Q}y = v_0 \otimes C_1 y_0 \oplus 0_{n-2m},\ \hat{P}y = v_0 \otimes P_1 y_0 \oplus 0_{n-2m},\ \hat{Q}y = v_0 \otimes Q_1 y_0 \oplus 0_{n-2m}. \tag{1}$$

Let $V' \in M_n$ be a unitary such that the span of the first two columns of V' contains the set $\{x, PQx\}$, and let $\hat{V} = (\hat{v}_1, \ldots, \hat{v}_n) = U^*V'$. Then $\hat{V}$ is a unitary and $\operatorname{span}\{\hat{v}_1, \hat{v}_2\} = \operatorname{span}\{\hat{x}, \hat{y}\}$. From (1), we obtain that

$$\hat{V}^*\hat{P}\hat{Q}\hat{V} = C_1' \oplus C',\ \hat{V}^*\hat{P}\hat{V} = P_1' \oplus P', \quad \text{and} \quad \hat{V}^*\hat{Q}\hat{V} = Q_1' \oplus Q',$$

where $C_1' \cong C_1$ and P_1', Q_1' are two 2-by-2 orthogonal projections. Since $c^2 = \hat{\lambda} \in (0,1)$ and $P_1'Q_1' = C_1'$, $Q_1' \neq 0_2, I_2$. There is a unitary $\hat{R} \in M_2$ such that

$$\hat{R}^*Q_1'\hat{R} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \hat{R}^*P_1'\hat{R} = \begin{pmatrix} p_{11} & p_{12} \\ \overline{p}_{12} & p_{22} \end{pmatrix}.$$

Hence $\begin{pmatrix} p_{11} & 0 \\ \overline{p}_{12} & 0 \end{pmatrix} = \hat{R}^*C_1'\hat{R} \cong C_1$, and then $c^2 = p_{11}, \overline{p}_{12} = e^{i\theta}cs$ for some $\theta \in [0, 2\pi)$. Let $R = \left(\hat{R}\begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}\right) \oplus I_{n-2}$, and $V = U\hat{V}R$. Then

$$V^*PV = R^*\hat{V}^*\hat{P}\hat{V}R = \begin{pmatrix} \hat{\lambda} & \sqrt{\hat{\lambda} - \hat{\lambda}^2} \\ \sqrt{\hat{\lambda} - \hat{\lambda}^2} & 1 - \hat{\lambda} \end{pmatrix} \oplus P' \text{ and } V^*QV = R^*\hat{V}^*\hat{Q}\hat{V}R = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \oplus Q'$$

as asserted. □

THEOREM 2.3. *Let $A, B \in M_n$ be two non-scalar positive contractions. Then*

$$W(AB) \subseteq \operatorname{conv}\{\cup_{\lambda \in \sigma(AB)} \mathscr{E}(\lambda)\}.$$

The set equality holds if and only if there is a unitary matrix U such that $U^*AU = A' \oplus A''$, $U^*BU = B' \oplus B''$ such that A', B' are orthogonal projections such that $W(A''B'') \subseteq W(A'B') = W(AB)$.

Proof. Let

$$\hat{A} = \begin{bmatrix} A & \sqrt{A - A^2} & 0 \\ \sqrt{A - A^2} & I_n - A & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \hat{B} = \begin{bmatrix} B & 0 & \sqrt{B - B^2} \\ 0 & 0 & 0 \\ \sqrt{B - B^2} & 0 & I_n - B \end{bmatrix}.$$

Then

$$T=\hat{A}\hat{B}=\begin{bmatrix} AB & 0 & A\sqrt{B-B^2} \\ \sqrt{A-A^2}B & 0 & \sqrt{(A-A^2)(B-B^2)} \\ 0 & 0 & 0 \end{bmatrix}$$

satisfies $\sigma(\hat{A}\hat{B})=\sigma(AB)\cup\{0\}$ and

$$W(AB)\subseteq W(T)=\operatorname{conv}\{\cup_{\lambda\in\sigma(\hat{A}\hat{B})}\mathscr{E}(\lambda)\}.$$

Now, suppose that $W(AB)=\operatorname{conv}\{\cup_{\lambda\in\sigma(AB)}\mathscr{E}(\lambda)\}=W(T)$. Then

$$W(AB)=\operatorname{conv}\{S\cup_{\lambda\in\sigma(AB)\setminus S}\mathscr{E}(\lambda)\},$$

where $S=\sigma(AB)\cap\{0,1\}$. Obviously, $\sigma(AB)\setminus S=\emptyset$ if and only if $W(AB)\subseteq[0,1]$.

If $W(AB)\subseteq[0,1]$, then AB is a Hermitian matrix so that $AB=B^*A^*=BA$. Hence A and B commute, and there is a unitary U such that $A=U^*\Lambda_1U$ and $B=U^*\Lambda_2U$, where $\Lambda_1=\operatorname{diag}(a_1,\ldots,a_n)$ and $\Lambda_2=\operatorname{diag}(b_1,\ldots,b_n)$. Then $W(AB)=W(\Lambda_1\Lambda_2)=[\alpha_0,\alpha_1]$, where $\alpha_0=\min_{1\leqslant i\leqslant n}a_ib_i$ and $\alpha_1=\max_{1\leqslant i\leqslant n}a_ib_i$. Hence we have the desired conclusion.

Next, suppose that $W(AB)$ is not in $[0,1]$. This is $\sigma(AB)\setminus S\neq\emptyset$. Let $\lambda_1\in\sigma(AB)\setminus S$ be such that $\partial\mathscr{E}(\lambda_1)\cap\partial W(AB)$ contains an arc. Then there exists $\mu\in\partial\mathscr{E}(\lambda_1)\cap\partial W(AB)$ with $\mu\notin\mathscr{E}(\lambda)$ for all other $\lambda\in\sigma(AB)$. Let $x_1\in\mathbf{C}^n$ be a unit vector with $x_1^*ABx_1=\mu$. Since $\partial W(AB)=\partial W(T)$, there is $\theta_1\in[0,2\pi)$ satisfying $2\operatorname{Re}(e^{i\theta_1}\mu)=\max\sigma(e^{i\theta_1}T+e^{-i\theta_1}T^*)$. Let $\hat{x}_1=x_1\oplus 0_{2n}$. Then $\hat{x}_1$ is an eigenvector of $e^{i\theta_1}T+e^{-i\theta_1}T^*$ corresponding to $e^{i\theta_1}T$ so that

$$(e^{i\theta_1}AB+e^{-i\theta_1}BA)x_1=2\operatorname{Re}(e^{i\theta_1}\mu)x_1,\ e^{i\theta_1}\sqrt{A-A^2}Bx_1=0,\text{ and } e^{-i\theta_1}\sqrt{B-B^2}Ax_1=0.$$

Hence $T\hat{x}_1=ABx_1\oplus 0_{2n}$. By Lemma 2.2, there is a unitary $\hat{U}_1$ and $\hat{U}_1=U_1\oplus I_{2n}$ such that the span of the first two columns of U_1 contains the set $\{x_1,ABx_1\}$,

$$\hat{U}_1^*\hat{A}\hat{U}_1=\begin{pmatrix}\lambda_1 & \sqrt{\lambda_1(1-\lambda_1)}\\ \sqrt{\lambda_1(1-\lambda_1)} & 1-\lambda_1\end{pmatrix}\oplus\hat{A}_1\quad\text{and}\quad \hat{U}_1^*\hat{B}\hat{U}_1=\begin{pmatrix}1&0\\0&0\end{pmatrix}\oplus\hat{B}_1,$$

where

$$\hat{A}_1=\begin{bmatrix}A_1' & C_1^* & 0\\ C_1 & I-A & 0\\ 0&0&0\end{bmatrix}\quad\text{and}\quad \hat{B}_1=\begin{bmatrix}B_1' & 0 & D_1^*\\ 0&0&0\\ D_1 & 0 & I-B\end{bmatrix}.$$

Thus,

$$U_1^*AU_1=\begin{pmatrix}\lambda_1 & \sqrt{\lambda_1(1-\lambda_1)}\\ \sqrt{\lambda_1(1-\lambda_1)} & 1-\lambda_1\end{pmatrix}\oplus A_1',\quad U_1^*BU_1=\begin{pmatrix}1&0\\0&0\end{pmatrix}\oplus B_1',$$

and $U_1^*ABU_1=C_1\oplus A_1'B_1'$, where $C_1=\begin{pmatrix}\lambda_1 & 0\\ \sqrt{\lambda_1(1-\lambda_1)} & 0\end{pmatrix}$. Then A_1',B_1' are positive contractions, $\hat{A}_1,\hat{B}_1$ are orthogonal projections, and

$$\operatorname{conv}\{\cup_{\lambda\in\sigma(AB)\setminus S}\mathscr{E}(\lambda)\}=\operatorname{conv}\{\mathscr{E}(\lambda_1)\cup_{\lambda\in\sigma(A_1B_1)\setminus S}\mathscr{E}(\lambda)\}.$$

Now, suppose $W(AB)=W(T)=\operatorname{conv}\{S\cup_{j=1}^k W(C_j)\}$ for k distinct matrices $C_1,\ldots,C_k\in M_2$ such that for $j=1,\ldots,k$, $\lambda_j\in\sigma(AB)\setminus S$, $\partial\mathscr{E}(\lambda_j)\cap\partial W(AB)$ contains an arc, and $W(C_j) = \mathscr{E}(\lambda_j)$. Since the argument in the preceding paragraph is true for any $\mathscr{E}(\lambda_j)$ for $j = 1,\ldots,k$, there is an orthonormal set $\{v_1,\ldots,v_{2k}\} \subseteq \mathbf{C}^n$ and a unitary $V = V_1 \oplus V_2 \in M_{3n}$, where the first $2k$ columns of V_1 equals $v_1,\ldots,v_{2k}$, such that $V^*TV = C_1 \oplus \cdots \oplus C_k \oplus T_0$. Thus, $V^*\hat{A}V = A_1 \oplus \cdots \oplus A_k \oplus \tilde{A}_0$, $V^*\hat{B}V = B_1 \oplus \cdots \oplus B_k \oplus \tilde{B}_0$. Consequently, $V_1^*AV_1 = A_1 \oplus \cdots \oplus A_k \oplus A_0$, and $V^*BV = B_1 \oplus \cdots \oplus B_k \oplus B_0$.

Evidently, $0 \in \sigma(A_1B_1) \cap \{0,1\} \subseteq S$. If $1 \notin S$, then $W(AB) = W(A_1B_1)$ so that the conclusion of the theorem holds with $(A',B') = (A_1,B_1)$. Suppose that $1 \in S$. Then $1 \in \sigma(A_0B_0)$ because $\sigma(C_j) = \{0,\lambda_j\}$ with $\lambda_j \in (0,1)$. Because A_0, B_0 are positive contractions, there is a unitary U_0 satisfying $U_0^*A_0U_0 = [1] \oplus A_0'$ and $U_0^*B_0U_0 = [1] \oplus B_0'$. Let $U = (V_1(I_{2k} \oplus U_0)) \oplus V_2$. Then $(U^*AU, U^*BU) = (A' \oplus A'', B' \oplus A'')$ with $(A',B') = (A_1 \oplus [1], A_2 \oplus [1])$, and the desired conclusion follows. $\square$

3. Essentially Hermitian matrices

Recall that a matrix $A \in M_n$ is an essentially Hermitian matrix if $e^{it}(A - (\mathrm{tr}A)I_n/n)$ is Hermitian for some $t \in [0,2\pi)$.

It is known and not hard to show that the following conditions are equivalent for $A \in M_n$.

(a) A is essentially Hermitian

(b) $W(A)$ is a line segment in $\mathbf{C}$ joining two complex numbers a_1, a_2.

(c) A is normal and all its eigenvalues lie on a straight line.

The results in the previous section can be extended to essentially Hermitian matrices. We begin with the following result which follows readily from Proposition 1.2.

PROPOSITION 3.1. *Suppose $A, B \in M_n$ are normal matrices with $\sigma(A) = \{a_1,a_2\}$ and $\sigma(B) = \{b_1,b_2\}$. Then $A = (a_1-a_2)P + a_2I_n$ and $B = (b_1-b_2)Q + b_2I_n$, where P and Q are orthogonal projections, and there is a unitary matrix U such that $U^*(P+iQ)U$ is a direct sum of $(I_p+iI_p) \oplus I_q \oplus iI_r \oplus 0_s$, and*

$$\begin{pmatrix} c_j^2 + i\, c_js_j \\ c_js_j & s_j^2 \end{pmatrix}, \quad j = 1,\ldots,k,$$

where $c_j \in (0,1), s_j = \sqrt{1-c_j^2}$. Consequently, U^*ABU is a direct sum of a diagonal matrix D with $\sigma(D) \subseteq \{a_1b_1, a_1b_2, a_2b_1, a_2b_2\}$ and

$$C_j = \begin{pmatrix} a_1c_j^2 + a_2s_j^2 & (a_1-a_2)c_js_j \\ (a_1-a_2)c_js_j & a_1s_j^2 + a_2c_j^2 \end{pmatrix} \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}, \quad j = 1,\ldots,k.$$

Suppose $A, B \in M_n$ satisfy the hypotheses of the above proposition. Then

$$W(AB) = \mathrm{conv}\{\cup_{j=1}^k W(C_j) \cup W(D)\}.$$

Evidently, $W(D)$ is the convex hull of the diagonal entries of D. Here note that some or all of the entries $a_1b_1, a_1b_2, a_2b_1, a_2b_2$ may absent in D. By the result on the numerical range of 2×2 matrix, we have the following proposition.

PROPOSITION 3.2. *Let $a_1, a_2, b_1, b_2 \in \mathbf{C}$ with $a_1 \neq a_2, b_1 \neq b_2$, $c \in (0,1)$, $s = \sqrt{1-c^2}$, and*

$$C = \begin{pmatrix} a_1c^2 + a_2s^2 & (a_1-a_2)cs \\ (a_1-a_2)cs & a_1s^2 + a_2c^2 \end{pmatrix} \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}.$$

Then $W(C)$ is the elliptical disk $\mathscr{E}(a_1,a_2,b_1,b_2;\gamma)$ with foci $\gamma \pm \sqrt{\gamma^2 - a_1a_2b_1b_2}$ and length of minor axis

$$\{2|\hat{\gamma}|^2 + (|b_1|^2 + |b_2|^2)|a_1 - a_2|^2c^2s^2 - 2|\hat{\gamma}^2 + b_1b_2(a_1-a_2)^2c^2s^2|\}^{1/2},$$

where $\gamma = \operatorname{tr} C = [(a_1b_1 + a_2b_2)c^2 + (a_1b_2 + a_2b_1)s^2]$ and $\hat{\gamma} = [(a_1b_1 - a_2b_2)c^2 + (a_2b_1 - a_1b_2)s^2]/2$.

Several remarks in connection to Proposition 3.2 are in order.

1. If $(a_1,a_2) = (b_1,b_2) = (1,0)$, then $\mathscr{E}(a_1,a_2,b_2,b_2;\gamma) = \mathscr{E}(\gamma)$ defined in Section 2.

2. The center of $W(C)$ in Proposition 3.2 always lies in the line segment with end points $a_1b_1 + a_2b_2$ and $a_1b_2 + a_2b_1$, and these two points are different if $a_1 \neq a_2$ and $b_1 \neq b_2$.

3. Suppose a_1, a_2, b_1, b_2 are given such that $a_1 \neq a_2, b_1 \neq b_2$. Every γ in the interior of the line segment with end points $a_1b_1 + a_2b_2$ and $a_1b_2 + a_2b_1$ uniquely determine $c \in (0,1)$ and $s = \sqrt{1-c^2}$ so that one can construct the matrix C (based on a_1,a_2,b_1,b_2,γ) such that $W(C) = \mathscr{E}(a_1,a_2,b_1,b_2;\gamma)$.

4. Let $A, B \in M_n$ satisfy the hypothesis of Proposition 3.1. Then for every $\lambda \in (\sigma(AB) \setminus S)$ with $S = \sigma(AB) \cap \{a_1b_1, a_1b_2, a_2b_1, a_2b_2\}$, there is $\tilde{\lambda} \in \sigma(AB)$ such that $\lambda\tilde{\lambda} = a_1a_2b_1b_2$ and $\lambda + \tilde{\lambda} = (a_1b_1 + a_2b_2)c_j^2 + (a_1b_2 + a_2b_1)s_j^2$. Such a pair of eigenvalues correspond to the eigenvalues of C_j. If $a_1a_2b_1b_2 = 0$, then $\lambda \in (\sigma(C_j) \setminus S)$ will ensure that $\lambda \neq 0$ so that $\hat{\lambda} = 0$. Otherwise, $\hat{\lambda} = a_1a_2b_1b_2/\lambda$. As a result, we can always assume that $\hat{\lambda} = a_1a_2b_1b_2/\lambda$ and $\gamma = \lambda + a_1a_2b_1b_2/\lambda$.

By the above remarks and Propositions 3.1, 3.2, we have the following.

THEOREM 3.3. *Suppose $A, B \in M_n$ are non-scalar normal matrices with $\sigma(A) = \{a_1, a_2\}$ and $\sigma(B) = \{b_1, b_2\}$. Let $S = \sigma(AB) \cap \{a_1b_1, a_1b_2, a_2b_1, a_2b_2\}$. Then*

$$W(AB) = \operatorname{conv}\{\cup_{\lambda \in (\sigma(AB)\setminus S)} \mathscr{E}(a_1,a_2,b_1,b_2;\lambda + (a_1a_2b_1b_2)/\lambda) \cup S\}.$$

We can use the dilation technique to study the numerical range of the product of essentially Hermitian matrices. Let $\tilde{A}$ be an essentially Hermitian matrix such that $W(\tilde{A})$ is a line segment joining $a_1, a_2 \in \mathbf{C}$. Then $\tilde{A} = a_2I_n + (a_1 - a_2)A$ for a positive contraction A. Then $\tilde{A}$ has a dilation of the form $\tilde{P} = a_2I_{2n} + (a_1 - a_2)P$ with

$$P = \begin{bmatrix} A & \sqrt{A - A^2} \\ \sqrt{A - A^2} & I - A \end{bmatrix}.$$

Then $\tilde{P}$ is normal with $\sigma(\tilde{P}) = \{a_1, a_2\}$ so that $W(\tilde{P}) = W(A)$. Now, if $\tilde{B} = b_2 I_n + (b_1 - b_2)B$ is another essentially Hermitian matrix, then $\tilde{B}$ has a dilation $\tilde{Q} = b_2 I_{2n} + (b_1 - b_2)Q$, where

$$Q = \begin{bmatrix} B & \sqrt{B-B^2} \\ \sqrt{B-B^2} & I-B \end{bmatrix},$$

such that $\tilde{Q}$ is normal with $\sigma(\tilde{Q}) = \{b_1, b_2\}$ and $W(\tilde{B}) = W(\tilde{Q})$.

Using this observation and arguments similar to those in the proof of Theorem 2.3, we have the following.

THEOREM 3.4. *Suppose $A, B \in M_n$ are essentially Hermitian matrices such that $A = a_2 I_n + (a_1 - a_2)A_1$ and $B = b_2 I_n + (b_1 - b_2)B_1$ for two positive contractions A_1, B_1. Let $\tilde{A} = a_2 I_{3n} + (a_1 - a_2)P$ and $\tilde{B} = b_2 I_{3n} + (b_1 - b_2)Q$*

$$P = \begin{bmatrix} A_1 & \sqrt{A_1 - A_1^2} & 0 \\ \sqrt{A_1 - A_1^2} & I_n - A_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} B_1 & 0 & \sqrt{B_1 - B_1^2} \\ 0 & 0 & 0 \\ \sqrt{B_1 - B_1^2} & 0 & I_n - B_1 \end{bmatrix}.$$

Then $W(AB) \subseteq W(\tilde{A}\tilde{B})$, where $W(\tilde{A}\tilde{B})$ can be determined by Theorem 3.3. The set equality holds if and only if there is a unitary U such that $UAU^* = A_1 \oplus A_2, UBU^* = B_1 \oplus B_2$ satisfying $\sigma(A_1) = \{a_1, a_2\}$, $\sigma(B_1) = \{b_1, b_2\}$, and $W(A_2B_2) \subseteq W(A_1B_1) = W(AB)$.

4. Extension to infinite dimensional spaces

We can extend the results in the previous sections to $B(H)$, where H is infinite dimensional. Note that for a pair of non-scalar orthogonal projections $P, Q \in B(H)$, there is a unitary U such that $U^*(P + iQ)U$ is a direct sum of $(1+i)I \oplus I \oplus iI \oplus 0$, and

$$\begin{bmatrix} C^2 + iI & C\sqrt{I-C^2} \\ C\sqrt{I-C^2} & I-C^2 \end{bmatrix},$$

where C is a positive contraction; see [1, 7] and their references. Consequently, PQ is a direct sum of $I \oplus 0$ and

$$T = \begin{bmatrix} C^2 & 0 \\ C\sqrt{I-C^2} & 0 \end{bmatrix}.$$

Note that T can be approximated by a sequence of operators of the form

$$T_m = \begin{bmatrix} C_m^2 & 0 \\ C_m\sqrt{I-C_m^2} & 0 \end{bmatrix}, \qquad m = 1, 2, \ldots,$$

where C_m has finite spectrum and therefore can be assumed to be the direct sum of $c_j^2 I_{H_j}$ on some subspace H_j for $j = 1, \ldots, c_{k_m}^2$ with $c_j \in (0,1)$. It is easy to see that

$$W(T_m) = \operatorname{conv}\{\cup_{\lambda \in \sigma(C_m)} \mathscr{E}(\lambda)\}.$$

In fact, the same conclusion holds if $\sigma(C) = \sigma(T)$ is a discrete set, equivalently, $\sigma(PQ)$ is a discrete set. In other words, if $P, Q \in B(H)$ are non-scalar orthogonal projections such that $\sigma(PQ)$ is a finite or countably infinite, then

$$W(PQ) = \operatorname{conv}\{\cup_{\lambda\in\sigma(PQ)}\mathscr{E}(\lambda)\}.$$

This result was proved in [1, Theorem 1.3], for separable Hilbert spaces. One readily sees that the proof works for general Hilbert space. In general, one can approximate T by T_m and obtain the following result concerning the closure of $W(PQ)$; see [1, Theorem 1.2].

PROPOSITION 4.1. *Let $P, Q \in B(H)$ be non-scalar orthogonal projections. Then*

$$\overline{W(PQ)} = \overline{\operatorname{conv}\{\cup_{\lambda\in\sigma(PQ)}\mathscr{E}(\lambda)\}}.$$

The closure signs on both sides can be removed if $\sigma(PQ)$ is a discrete set.

One can show that the results in Sections 2-3 hold in the infinite dimension setting.

THEOREM 4.2. *Let $A, B \in B(H)$ be non-scalar positive semi-definite contractions. Then*

$$\overline{W(AB)} \subseteq \overline{\operatorname{conv}\{\cup_{\lambda\in\sigma(AB)}\mathscr{E}(\lambda)\}}.$$

The closure signs can be removed if $\sigma(AB)$ is a discrete set. The set equality holds if A, B are orthogonal projections.

THEOREM 4.3. *Let $A, B \in B(H)$ be such that $\overline{W(A)}$ is the line segment joining a_1, a_2, and $\overline{W(B)}$ is the line segment joining b_1, b_2, where $a_1 \neq a_2$ and $b_1 \neq b_2$. Then*

$$\overline{W(AB)} \subseteq \overline{\operatorname{conv}\{\cup_{\lambda\in(\sigma(AB)\setminus S)}\mathscr{E}(a_1,a_2,b_1,b_2;\lambda+\hat{\lambda})\cup S\}},$$

where $S = \sigma(AB)\cap\{a_1b_1, a_1b_2, a_2b_1, a_2b_2\}$, $\hat{\lambda} = a_1a_2b_1b_2/\lambda$, and $\mathscr{E}(a_1,a_2,b_1,b_2;\lambda+\hat{\lambda})$ is defined as in Theorem 3.3. The closure signs can be removed if $\sigma(AB)$ is a discrete set. The set equality holds if $\sigma(A) = \{a_1, a_2\}$ and $\sigma(B) = \{b_1, b_2\}$.

In Theorems 4.2 and 4.3, we only have sufficient conditions for the set inclusions become set equalities. The problems of characterizing the set equality cases are open.

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