Operators and Matrices
Volume 11, Number 1 (2017), 241–244 doi:10.7153/oam-11-16

A GENERALIZED MATHARU–AUJLA INEQUALITY

SIYUAN SHEN, JUNMIN HAN AND JIAN SHI

(*Communicated by T. Ando*)

Abstract. In this paper, we will show a generalized Matharu-Aujla log majorization inequality via an operator order preserving inequality, which extends the related results.

1. Introduction and main results

Thoughout this paper, a capital letter, such as T, stands for an $n\times n$ matrix.

DEFINITION 1.1. ([1]) For two positive semidefinite matrices A and B, if

$$\prod_{i=1}^{k}\lambda_i(A)\geqslant\prod_{i=1}^{k}\lambda_i(B),\quad k=1,2,\cdots,n-1;$$

and

$$\prod_{i=1}^{n}\lambda_i(A)=\prod_{i=1}^{n}\lambda_i(B),\quad i.e.\quad det(A)=det(B),$$

we call the relationship log majorization (denoted by $A\underset{(\log)}{\succ}B$), where $\lambda_1(A)\geqslant\lambda_2(A)\geqslant\cdots\geqslant\lambda_n(A)$ and $\lambda_1(B)\geqslant\lambda_2(B)\geqslant\cdots\geqslant\lambda_n(B)$ are the eigenvalues of A and B, respectively.

DEFINITION 1.2. ([4]) For two positive semidefinite matrices A and B, if $\alpha\in[0,1]$, α-power mean of A and B is defined by

$$A\sharp_\alpha B=\begin{cases}A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\alpha}A^{\frac{1}{2}}, & A,B>0;\\ \lim\limits_{\varepsilon\to0^+}(A+\varepsilon I)\sharp_\alpha(B+\varepsilon I), & A,B\geqslant0.\end{cases}$$

Similarly, if $s\notin[0,1]$, $A\natural_s B$ is defined by

$$A\natural_s B=\begin{cases}A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{s}A^{\frac{1}{2}}, & A,B>0;\\ \lim\limits_{\varepsilon\to0^+}(A+\varepsilon I)\natural_s(B+\varepsilon I), & A,B\geqslant0.\end{cases}$$

In 2012, J. S. Matharu and J. S. Aujla obtained the following log majorization inequality.

Mathematics subject classification (2010): 47A63.
Keywords and phrases: Log majorization, generalized Furuta inequality, Matharu-Aujla inequality.

Paper OaM-11-16

THEOREM 1.1. ([5]) *If* $A, B > 0$, *then*

$$A^{\frac{1-\alpha}{2}} B^{\alpha} A^{\frac{1-\alpha}{2}} \underset{(\log)}{\succ} A \sharp_{\alpha} B \tag{1.1}$$

holds for $\alpha \in [0,1]$.

Immediately after, T. Furuta extended Matharu and Aujla's result and proved the following inequality.

THEOREM 1.2. ([3]) *If* $A > 0$ *and* $B \geqslant 0$, *then for* $0 \leqslant \alpha \leqslant 1$, $t \in [0,1]$ *and* $r \geqslant t$,

$$[A^{\frac{1-t}{2}} (A^t \sharp_{\alpha} B) A^{\frac{1-t}{2}}]^s \underset{(\log)}{\succ} A^{\frac{w}{2}} (A^r \sharp_{\alpha} B^s) A^{\frac{w}{2}} \tag{1.2}$$

holds for $\frac{(1-\alpha)(r-t)}{1-\alpha t} + 1 \geqslant s \geqslant 1$, *where* $w = (1-\alpha)(s-r) + \alpha(1-t)s$.

As a continuation, in this paper, we will prove the following generalized Matharu and Aujla's log majorization inequality.

THEOREM 1.3. *If* $A > 0$ *and* $B \geqslant 0$, $p, q, s \geqslant 1$, $0 \leqslant \alpha \leqslant 1$, $t \in [0,1]$ *and* $r \geqslant t$, *then*

$$[A^{\frac{1-t}{2}} (A^t \sharp_{\alpha} B) A^{\frac{1-t}{2}}]^{psq} \underset{(\log)}{\succ} A^{\frac{w}{2}} [A^r \sharp_{\alpha} (A^t \natural_s B^p)^q] A^{\frac{w}{2}} \tag{1.3}$$

holds for $1 - t + r \geqslant \{[(1-\alpha t)p + \alpha t]s - \alpha t\}q + \alpha r$, *where* $w = \{[(1-\alpha t)p + \alpha t]s - \alpha t\}q + \alpha r - r$.

Furthermore, we shall prove the equivalence between the log majorization inequality above and an operator order preserving inequality as follows.

THEOREM 1.4. *If* $A \geqslant B \geqslant 0$ *with* $A > 0$, $p, q, s \geqslant 1$, $\alpha \in (0,1]$ *then*

$$A^{\{[(1-\alpha t)p+\alpha t]s-\alpha t\}q+\alpha r} \geqslant \left\{ A^{\frac{r}{2}} \left[A^{-\frac{t}{2}} \{ A^{\frac{t}{2}} (A^{-\frac{t}{2}} B^{\frac{1}{\alpha}} A^{-\frac{t}{2}})^p A^{\frac{t}{2}} \}^s A^{-\frac{t}{2}} \right]^q A^{\frac{r}{2}} \right\}^{\alpha} \tag{1.4}$$

holds for $t \in [0,1]$, $r \geqslant t$ *and* $1 - t + r \geqslant \{[(1-\alpha t)p + \alpha t]s - \alpha t\}q + \alpha r$.

In order to prove the results above, first, let us list a useful theorem, which is called generalized Furuta inequality.

THEOREM 1.5. (Generalized Furuta inequality, [2]) *If* $A \geqslant B \geqslant 0$ *with* $A > 0$, $p_1, p_2, p_3, p_4 \geqslant 1$, *then*

$$A^{1-t+r} \geqslant \left\{ A^{\frac{r}{2}} \left[A^{-\frac{t}{2}} \{ A^{\frac{t}{2}} (A^{-\frac{t}{2}} B^{p_1} A^{-\frac{t}{2}})^{p_2} A^{\frac{t}{2}} \}^{p_3} A^{-\frac{t}{2}} \right]^{p_4} A^{\frac{r}{2}} \right\}^{\frac{1-t+r}{\{[(p_1-t)p_2+t]p_3-t\}p_4+r}} \tag{1.5}$$

holds for $t \in [0,1]$ *and* $r \geqslant t$.

REMARK 1.1. Theorem 1.4 and Theorem 1.5 also hold if both A and B are bounded linear operators on a Hilbert space. See [2] for details.

2. Proofs of the main results

In this section, we shall prove our main results.

Proof of Theorem 1.4. Replacing p_1 by $\frac{1}{\alpha}$, p_2 by p, p_3 by s, p_4 by q in Theorem 1.5, respectively, then we have

$$A^{1-t+r} \geqslant \left\{A^{\frac{r}{2}}\left[A^{-\frac{t}{2}}\{A^{\frac{t}{2}}(A^{-\frac{t}{2}}B^{\frac{1}{\alpha}}A^{-\frac{t}{2}})^p A^{\frac{t}{2}}\}^s A^{-\frac{t}{2}}\right]^q A^{\frac{r}{2}}\right\}^{\frac{(1-t+r)\alpha}{\{[(1-\alpha t)p+\alpha t]s-\alpha t\}q+\alpha r}}. \tag{2.1}$$

Notice that $\frac{\{[(1-\alpha t)p+\alpha t]s-\alpha t\}q+\alpha r}{1-t+r} \in [0,1]$. Applying Löwner-Heinz inequality to (2.1), then we can obtain (1.4). □

Next, we shall prove that Theorem 1.3 can be derived from Theorem 1.4.

Proof of Theorem 1.3. We only need to prove that

$$I \geqslant A^{\frac{1-t}{2}}(A^t \sharp_\alpha B)A^{\frac{1-t}{2}} \tag{2.2}$$

ensures

$$I \geqslant A^{\frac{w}{2}}[A^r \sharp_\alpha (A^t \natural_s B^p)^q]A^{\frac{w}{2}}. \tag{2.3}$$

By the Definition1.2, (2.2) is equivalent to

$$A^{-1} \geqslant (A^{-\frac{t}{2}}BA^{-\frac{t}{2}})^\alpha \tag{2.4}$$

and (2.3) is equivalent to

$$A^{-w-r} \geqslant \left[A^{-\frac{r}{2}}\{A^{\frac{t}{2}}(A^{-\frac{t}{2}}B^p A^{-\frac{t}{2}})^s A^{\frac{t}{2}}\}^q A^{-\frac{r}{2}}\right]^\alpha. \tag{2.5}$$

Replacing A by A_1^{-1} and B by $A_1^{-\frac{t}{2}}B_1^{\frac{1}{\alpha}}A_1^{-\frac{t}{2}}$ in (2.4) and (2.5), respectively. (2.4) is just $A_1 \geqslant B_1$ and (2.5) is

$$A_1^{w+r} \geqslant \left\{A_1^{\frac{r}{2}}\left[A_1^{-\frac{t}{2}}\{A_1^{\frac{t}{2}}(A_1^{-\frac{t}{2}}B_1^{\frac{1}{\alpha}}A_1^{-\frac{t}{2}})^p A_1^{\frac{t}{2}}\}^s A_1^{-\frac{t}{2}}\right]^q A_1^{\frac{r}{2}}\right\}^\alpha. \tag{2.6}$$

$A_1 \geqslant B_1 \geqslant 0$ with $A_1 > 0$ ensures (2.6) is obvious by Theorem 1.4. □

Next, we shall show that Theorem 1.4 can also be obtained by Theorem 1.3.

Proof of Theorem 1.4. (via Theorem 1.3) We only need to prove that $A \geqslant B$ ensures (1.4). By Definition 1.2, (1.4) is equivalent to

$$I \geqslant A^{-\frac{w}{2}}\left\{A^{-r}\sharp_\alpha\left[A^{-t}\natural_s(A^{-\frac{t}{2}}B^{\frac{1}{\alpha}}A^{-\frac{t}{2}})^p\right]^q\right\}A^{-\frac{w}{2}}. \tag{2.7}$$

Put $A_1 = A^{-1}$ and $B_1 = (A^{-\frac{t}{2}}B^{\frac{1}{\alpha}}A^{-\frac{t}{2}})$, then $A \geqslant B$ is equivalent to $A_1^{-1} \geqslant (A_1^{-\frac{t}{2}}B_1A_1^{-\frac{t}{2}})^\alpha$, i.e.

$$I \geqslant A_1^{\frac{1-t}{2}}(A_1^t \sharp_\alpha B_1)A_1^{\frac{1-t}{2}}, \tag{2.8}$$

and (2.7) is equivalent to

$$I \geqslant A_1^{\frac{w}{2}}[A_1^r \sharp_\alpha (A_1^t \natural_s B_1^p)^q] A_1^{\frac{w}{2}}. \tag{2.9}$$

(2.8) ensures (2.9) is obvious by Theorem 1.3. □

Acknowledgements. J. Han is supported by Natural Science Foundation of Shandong Province (No. BS2015SF006). J. Shi (corresponding author) is supported by Hebei Education Department (No. ZC2016009), Hebei University Funds for Distinguished Young Scientists and Natural Science Foundation of Shandong Province (No. BS2015SF006).

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(Received July 27, 2016)

Siyuan Shen
Department of Basic Courses
Shijiazhuang Tiedao University, Sifang College
Shijiazhuang, 051132, P. R. China

Junmin Han
School of Mathematics and Information Science
Weifang University
Weifang, 261061, P. R. China

Jian Shi
College of Mathematics and Information Science
Hebei University
Baoding, 071002, P. R. China
e-mail: mathematic@126.com