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ASYMMETRIC TRUNCATED TOEPLITZ OPERATORS ON FINITE–DIMENSIONAL SPACES

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Abstract. In this paper we consider asymmetric truncated Toeplitz operators acting between two finite-dimensional model spaces. We compute the dimension of the space of all such operators. We also describe the matrix representations of asymmetric truncated Toeplitz operators acting between two finite-dimensional model spaces. Our results are generalizations of the results known for truncated Toeplitz operators.

1. Introduction

Let H^2 be the classical Hardy space of the unit disk $\mathbb{D} = \{z\colon |z| < 1\}$ and let H^∞ be the algebra of bounded analytic functions on $\mathbb{D}$. As usual, H^2 will be identified via boundary values with a closed subspace of $L^2(\partial\mathbb{D})$.

The classical Toeplitz operator T_φ with symbol $\varphi \in L^2(\partial\mathbb{D})$ is defined on H^2 by

$$T_\varphi f = P(\varphi f),$$

where P is the orthogonal projection from $L^2(\partial\mathbb{D})$ onto H^2. The operator T_φ is densely defined and it is bounded if and only if $\varphi \in L^\infty(\partial\mathbb{D})$. Two important examples of classical Toeplitz operators are the unilateral shift $S = T_z$ and its Hilbert space adjoint, the backward shift $S^* = T_{\overline{z}}$.

Let α be an arbitrary inner function, that is, $\alpha \in H^\infty$ and $|\alpha| = 1$ a.e. on $\partial\mathbb{D}$. The model space corresponding to α is the closed subspace K_α of H^2 of the form

$$K_\alpha = H^2 \ominus \alpha H^2.$$

The theorem of A. Beurling (see for instance [7, Thm. 8.1.1]) implies that every nontrivial S^*-invariant subspace of H^2 is a model space K_α corresponding to some inner function α. Denote by P_α the orthogonal projection from $L^2(\partial\mathbb{D})$ onto K_α.

The model space K_α is a reproducing kernel Hilbert space with the kernel function given by

$$k_w^\alpha(z) = \frac{1-\overline{\alpha(w)}\alpha(z)}{1-\overline{w}z}, \quad w,\ z \in \mathbb{D}. \tag{1.1}$$

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In other words, $f(w) = \langle f, k_w^\alpha \rangle$ for every $f \in K_\alpha$ and $w \in \mathbb{D}$ (here $\langle \cdot, \cdot \rangle$ is the usual integral inner product). Note that k_w^α is bounded, and so the subspace K_α^∞ of all bounded functions in K_α is dense in K_α. If $\alpha(w) = 0$, then k_w^α is equal to the Szegö kernel $k_w(z) = (1 - \overline{w}z)^{-1}$.

If α is an inner function, then the formula

$$C_\alpha f(z) = \alpha(z)\overline{z}\overline{f(z)}, \quad |z| = 1, \tag{1.2}$$

defines a conjugation (an antilinear, isometric involution) on $L^2(\partial\mathbb{D})$ which preserves K_α (see [16, Subection 2.3]). A simple calculation reveals that the conjugate kernel $\widetilde{k}_w^\alpha = C_\alpha k_w^\alpha$ is given by

$$\widetilde{k}_w^\alpha(z) = \frac{\alpha(z) - \alpha(w)}{z - w}, \quad w,\ z \in \mathbb{D}.$$

A truncated Toeplitz operator A_φ^α with a symbol $\varphi \in L^2(\partial\mathbb{D})$ is the compression of T_φ to the model space K_α. More precisely, A_φ^α is defined on K_α by

$$A_\varphi^\alpha f = P_\alpha(\varphi f).$$

An extensive study of truncated Toeplitz operators began in 2007 with D. Sarason's paper [16]. Despite similar definitions, truncated Toeplitz operators differ from the classical ones in many ways. For example, $T_\varphi = 0$ if and only if $\varphi = 0$, but $A_\varphi^\alpha = 0$ if and only if $\varphi \in \overline{\alpha H^2} + \alpha H^2$ (see [16, Thm. 3.1]). Moreover, unlike in the classical case, unbounded symbols can produce bounded truncated Toeplitz operators and there are bounded truncated Toeplitz operators for which no bounded symbol exists (see [3] for more details). More interesting results about truncated Toeplitz operators can be found in [6, 10, 11, 12, 13].

Recently, the authors in [4] and [5] introduced the so-called asymmetric truncated Toeplitz operators, which are generalizations of truncated Toeplitz operators. Let α, β be two inner functions and let $\varphi \in L^2(\partial\mathbb{D})$. An asymmetric truncated Toeplitz operator $A_\varphi^{\alpha,\beta}$ with a symbol $\varphi \in L^2(\partial\mathbb{D})$ is the operator from K_α into K_β defined by

$$A_\varphi^{\alpha,\beta} f = P_\beta(\varphi f), \quad f \in K_\alpha.$$

The asymmetric truncated Toeplitz operator $A_\varphi^{\alpha,\beta}$ is closed and densely defined. Obviously, $A_\varphi^{\alpha,\alpha} = A_\varphi^\alpha$.

Let

$$\mathscr{T}(\alpha,\beta) = \{A_\varphi^{\alpha,\beta} \ :\ \varphi \in L^2(\partial\mathbb{D}) \text{ and } A_\varphi^{\alpha,\beta} \text{ is bounded}\}$$

and $\mathscr{T}(\alpha) = \mathscr{T}(\alpha,\alpha)$.

In 2008 [8] J.A. Cima, W.T. Ross and W.R. Wogen considered truncated Toeplitz operators on finite-dimensional model spaces. It is known that K_α has finite dimension m if and only if α is a finite Blaschke product of degree m. In that case every $f \in K_\alpha$ is analytic in a domain containing the closed unit disk (see [10, Prop. 5.7.6]). If α has m distinct zeros $a_1, \ldots, a_m$, then the sets $\{k_{a_1}^\alpha, \ldots, k_{a_m}^\alpha\}$ and $\{\widetilde{k}_{a_1}^\alpha, \ldots, \widetilde{k}_{a_m}^\alpha\}$ are two (non-orthonormal) bases for K_α. The authors in [8] characterized the operators from $\mathscr{T}(\alpha)$

in terms of the matrix representations with respect to each of these bases. They showed that a matrix representing a truncated Toeplitz operator on m-dimensional model space is completely determined by $2m-1$ of its entries, those along the main diagonal and the first row (and the first row can be replaced by any other row or column). They also proved a similar result for the so-called Clark bases.

Matrix representations of truncated Toeplitz operators on infinite-dimensional model spaces were considered in [15]. In particular, it was proved in [15] that if α is an infinite Blaschke product with uniformly separated zeros, then the operators from $\mathscr{T}(\alpha)$ can be described in terms of their matrix representations with respect to the kernel basis.

The main purpose of this paper is to generalize the results from [8] to the case of asymmetric truncated Toeplitz operators.

In Section 2 we compute the dimension of $\mathscr{T}(\alpha,\beta)$ for two finite Blaschke products α, β. D. Sarason [16, Thm. 3.1] proved that if α is a finite Blaschke product of degree $m>0$, then the dimension of $\mathscr{T}(\alpha)$ is $2m-1$. We show that if α and β are finite Blachke products of degree $m>0$ and $n>0$, respectively, then the dimension of $\mathscr{T}(\alpha,\beta)$ is $m+n-1$.

In Section 3 we generalize the results from [8] concerning matrix representations. We characterize matrix representations of asymmetric truncated Toeplitz operators acting between finite-dimensional model spaces. We consider matrix representations with respect to kernel bases, conjugate kernel bases, Clark bases and modified Clark bases. In each of these cases we show how the matrix representing an asymmetric truncated Toeplitz operator is completely determined by $m+n-1$ of its entries.

2. The dimension of $\mathscr{T}(\alpha,\beta)$

Here we compute the dimension of the space of all asymmetric truncated Toeplitz operators acting between finite-dimensional model spaces. We also give examples of bases for $\mathscr{T}(\alpha,\beta)$ in this case. The proofs given here are analogous to those from [16, Thm. 7.1].

As mentioned in the Introduction, if K_α has finite dimension $m>0$, then the dimension of $\mathscr{T}(\alpha)$ is $2m-1$ ([16, Thm. 7.1(a)]). Here we prove the following.

PROPOSITION 2.1. *Let K_α have finite dimension $m>0$ and let K_β have finite dimension $n>0$. The dimension of $\mathscr{T}(\alpha,\beta)$ is $m+n-1$.*

In the proof of Proposition 2.1 we use the fact that if α, β are two nonconstant inner functions, then $A_\varphi^{\alpha,\beta}=0$ if and only if $\varphi\in\overline{\alpha H^2}+\beta H^2$ (see [14, Thm 2.1] for proof). We also use the following simple lemma from [14].

LEMMA 2.2. ([14], Lem. 2.2) *Let α, β be two arbitrary inner functions. If*

$$K_\alpha\subset\beta H^2,$$

then both α and β have no zeros in $\mathbb{D}$, or at least one of the functions α or β is a constant function.

Proof of Proposition 2.1. By [14, Cor. 2.6], every operator A in $\mathscr{T}(\alpha,\beta)$ can be written as a sum $A = A_{\overline{\chi}} + A_{\psi}$ with $\chi \in K_\alpha$ and $\psi \in K_\beta$. Since K_α and K_β have finite dimension, it follows that α and β are finite Blaschke products and $K_\alpha \subset H^\infty$, $K_\beta \subset H^\infty$. Consequently, $\mathscr{T}(\alpha,\beta)$ is spanned by its subspaces

$$\mathscr{T}_\infty(\alpha,\beta) = \{A_\varphi^{\alpha,\beta} : \varphi \in H^\infty\} \quad \text{and} \quad \mathscr{T}_{\overline{\infty}}(\alpha,\beta) = \{A_\varphi^{\alpha,\beta} : \varphi \in \overline{H^\infty}\}.$$

We first compute the dimension of $\mathscr{T}_\infty(\alpha,\beta)$ and the dimension of $\mathscr{T}_{\overline{\infty}}(\alpha,\beta)$. To this end, we consider the linear mapping $\varphi \mapsto A_\varphi^{\alpha,\beta}$ acting from H^∞ onto $\mathscr{T}_\infty(\alpha,\beta)$. By [14, Thm 2.1], its kernel is equal to βH^∞. Indeed, if $\varphi \in \beta H^\infty$, then $A_\varphi^{\alpha,\beta} = 0$. On the other hand, if $\varphi \in H^\infty$ and $A_\varphi^{\alpha,\beta} = 0$, then $\varphi = \overline{\alpha h_1} + \beta h_2$ for some $h_1, h_2 \in H^2$. Hence $\varphi - \beta h_2 = \overline{\alpha h_1}$ is a constant function, $\varphi = \beta h_2 + c$ for some complex number c, and

$$0 = A_\varphi^{\alpha,\beta} = A_{\beta h_2 + c}^{\alpha,\beta} = c P_{\beta | K_\alpha}.$$

If $c \neq 0$, then the above implies that $K_\alpha \subset \beta H^2$, which, by Lemma 2.2, never happens for nonconstant Blaschke products α, β. Therefore $c = 0$ and $\varphi \in \beta H^\infty$. From this

$$\dim \mathscr{T}_\infty(\alpha,\beta) = \dim (H^\infty / \beta H^\infty) = n.$$

Similarly, the mapping $\varphi \mapsto A_\varphi^{\alpha,\beta}$ acting from $\overline{H^\infty}$ onto $\mathscr{T}_{\overline{\infty}}(\alpha,\beta)$ has kernel equal to $\overline{\alpha H^\infty}$ and

$$\dim \mathscr{T}_{\overline{\infty}}(\alpha,\beta) = \dim \left(\overline{H^\infty} / \overline{\alpha H^\infty}\right) = m.$$

To complete the proof, we only need to show that

$$\dim(\mathscr{T}_\infty(\alpha,\beta) \cap \mathscr{T}_{\overline{\infty}}(\alpha,\beta)) = 1, \tag{2.1}$$

for then

$$\begin{aligned}\dim \mathscr{T}(\alpha,\beta) &= \dim \mathscr{T}_\infty(\alpha,\beta) + \dim \mathscr{T}_{\overline{\infty}}(\alpha,\beta) - \dim(\mathscr{T}_\infty(\alpha,\beta) \cap \mathscr{T}_{\overline{\infty}}(\alpha,\beta)) \\ &= m + n - 1.\end{aligned}$$

Note that here $A_1^{\alpha,\beta} \neq 0$. Otherwise, we would have $K_\alpha \subset \beta H^2$, which, by Lemma 2.2 again, is impossible for nonconstant Blaschke products α, β. Clearly, $A_1^{\alpha,\beta} \in \mathscr{T}_\infty(\alpha,\beta) \cap \mathscr{T}_{\overline{\infty}}(\alpha,\beta)$, so $\mathscr{T}_\infty(\alpha,\beta) \cap \mathscr{T}_{\overline{\infty}}(\alpha,\beta) \neq \{0\}$.

Assume now that $A \in \mathscr{T}_\infty(\alpha,\beta) \cap \mathscr{T}_{\overline{\infty}}(\alpha,\beta)$, say $A = A_{\varphi_1}^{\alpha,\beta} = A_{\overline{\varphi}_2}^{\alpha,\beta}$, $\varphi_1, \varphi_2 \in H^\infty$. Then

$$A_{\varphi_1 - \overline{\varphi}_2}^{\alpha,\beta} = 0,$$

and $\varphi_1 - \overline{\varphi}_2 \in \overline{\alpha H^2} + \beta H^2$ by [14, Thm 2.1]. In other words, there exist $h_1, h_2 \in H^2$ such that

$$\varphi_1 - \beta h_2 = \overline{\varphi_2 + \alpha h_1}.$$

This implies that there exists a complex number c such that $\varphi_1 = c + \beta h_2$ and $A = A_{c+\beta h_2}^{\alpha,\beta} = c A_1^{\alpha,\beta}$. Thus every operator in $\mathscr{T}_\infty(\alpha,\beta) \cap \mathscr{T}_{\overline{\infty}}(\alpha,\beta)$ is a scalar multiple of $A_1^{\alpha,\beta}$ and (2.1) holds. □

It was proved by P. R. Ahern and D. N. Clark in [1, 2] that if α has an angular derivative in the sense of Carathéodory (an ADC) at some point $\eta \in \partial\mathbb{D}$, then the function k_η^α defined by (1.1) with η in place of w, belongs to K_α (for more details see [10, pp. 33–37]). Recall the following.

PROPOSITION 2.3. ([14], Prop. 3.1) *Let α, β be two nonconstant inner functions.*

(a) For $w \in \mathbb{D}$, the operators $\widetilde{k}_w^\beta \otimes k_w^\alpha$ and $k_w^\beta \otimes \widetilde{k}_w^\alpha$ belong to $\mathscr{T}(\alpha,\beta)$,

$$\widetilde{k}_w^\beta \otimes k_w^\alpha = A^{\alpha,\beta}_{\frac{\beta(z)}{z-w}} \quad \text{and} \quad k_w^\beta \otimes \widetilde{k}_w^\alpha = A^{\alpha,\beta}_{\frac{\overline{\alpha(z)}}{\overline{z}-\overline{w}}}.$$

(b) If both α and β have an ADC at the point η of $\partial\mathbb{D}$, then the operator $k_\eta^\beta \otimes k_\eta^\alpha$ belongs to $\mathscr{T}(\alpha,\beta)$,

$$k_\eta^\beta \otimes k_\eta^\alpha = A^{\alpha,\beta}_{k_\eta^\beta + \overline{k}_\eta^\alpha - 1}.$$

Consequently, if α and β are two finite Blaschke products, then $k_\eta^\alpha \in K_\alpha$, $k_\eta^\beta \in K_\beta$ and $k_\eta^\beta \otimes k_\eta^\alpha$ belongs to $\mathscr{T}(\alpha,\beta)$ for all $\eta \in \partial\mathbb{D}$. Moreover, it is easy to verify that

$$k_\eta^\beta \otimes \widetilde{k}_\eta^\alpha = \overline{\alpha(\eta)}\eta k_\eta^\beta \otimes k_\eta^\alpha \quad \text{and} \quad \widetilde{k}_\eta^\beta \otimes k_\eta^\alpha = \alpha(\eta)\overline{\eta} k_\eta^\beta \otimes k_\eta^\alpha.$$

COROLLARY 2.4. *Let K_α have finite dimension $m > 0$ and let K_β have finite dimension $n > 0$. If $w_1, \ldots, w_{m+n-1}$ are distinct points in the closed unit disk $\overline{\mathbb{D}}$, then:*

(a) the operators $\widetilde{k}_{w_j}^\beta \otimes k_{w_j}^\alpha$, $j = 1, \ldots, m+n-1$, form a basis for $\mathscr{T}(\alpha,\beta)$;

(b) the operators $k_{w_j}^\beta \otimes \widetilde{k}_{w_j}^\alpha$, $j = 1, \ldots, m+n-1$, form a basis for $\mathscr{T}(\alpha,\beta)$.

Proof. We only prove part (a) of the corollary. Proof of part (b) is similar (compare with [16, Thm. 7.1(b)] and [8, Lem. 3.1]).

Let $w_1, \ldots, w_{m+n-1}$ be distinct points in $\overline{\mathbb{D}}$. By Proposition 2.3, the operators $\widetilde{k}_{w_j}^\beta \otimes k_{w_j}^\alpha$, $j = 1, \ldots, m+n-1$, belong to $\mathscr{T}(\alpha,\beta)$. Since the dimension of $\mathscr{T}(\alpha,\beta)$ is $m+n-1$, it is enough to prove that these operators are linearly independent.

Assume that

$$\sum_{j=1}^{m+n-1} c_j \widetilde{k}_{w_j}^\beta \otimes k_{w_j}^\alpha = 0$$

for some scalars $c_1, \ldots, c_{m+n-1}$. We first show that $c_1 = 0$.

Since the functions $k_{w_1}^\alpha, \ldots, k_{w_m}^\alpha$ are linearly independent (see [16, p. 509]), there exists $f \in K_\alpha$ such that

$$\langle f, k_{w_j}^\alpha \rangle = \begin{cases} 1 & \text{for } j = 1, \\ 0 & \text{for } 1 < j \leqslant m, \end{cases}$$

and

$$0=\sum_{j=1}^{m+n-1} c_j\widetilde{k}_{w_j}^{\beta}\otimes k_{w_j}^{\alpha}(f)=c_1\widetilde{k}_{w_1}^{\beta}+\sum_{j=m+1}^{m+n-1} c_j f(w_j)\widetilde{k}_{w_j}^{\beta}.$$

But $\widetilde{k}_{w_1}^{\beta},\widetilde{k}_{w_{m+1}}^{\beta},\ldots,\widetilde{k}_{w_{m+n-1}}^{\beta}$ also are linearly independent, so $c_1=0$.

A similar reasoning shows that $c_j=0$ for every $j=1,\ldots,m+n-1$, which completes the proof. □

3. Matrix representations

For the reminder of the paper we assume that α and β are two finite Blaschke products with zeros $a_1,\ldots,a_m$ and $b_1,\ldots,b_n$, respectively, that is,

$$\alpha(z)=\prod_{i=1}^{m}\frac{a_i-z}{1-\overline{a}_i z},\qquad \beta(z)=\prod_{j=1}^{n}\frac{b_j-z}{1-\overline{b}_j z}. \tag{3.1}$$

3.1. Kernel bases and conjugate kernel bases

Let α and β be given by (3.1). Here we assume that the zeros $a_1,\ldots,a_m$ are distinct and that the zeros $b_1,\ldots,b_n$ are distinct. Then the kernel functions $\{k_{a_1}^{\alpha},\ldots,k_{a_m}^{\alpha}\}$ form a basis for K_α and so do the conjugate kernel functions $\{\widetilde{k}_{a_1}^{\alpha},\ldots,\widetilde{k}_{a_m}^{\alpha}\}$. Similarly, $\{k_{b_1}^{\beta},\ldots,k_{b_n}^{\beta}\}$ and $\{\widetilde{k}_{b_1}^{\beta},\ldots,\widetilde{k}_{b_n}^{\beta}\}$ are bases for K_β.

Of course, it is possible that α and β have some zeros in common. In this subsection we assume that α and β have precisely l zeros in common ($l=0$ if there are no zeros in common), those zeros being $a_i=b_i$ for $i\leqslant l$.

Let A be any linear operator from K_α into K_β. It can be verified using

$$\langle k_{b_j}^{\beta},\widetilde{k}_{b_s}^{\beta}\rangle=\begin{cases}\overline{\beta'(b_s)} & \text{for } j=s,\\ 0 & \text{for } j\neq s,\end{cases}$$

that the matrix representation $M_A=(r_{s,p})$ of A with respect to the kernel bases $\{k_{a_1}^{\alpha},\ldots,k_{a_m}^{\alpha}\}$ and $\{k_{b_1}^{\beta},\ldots,k_{b_n}^{\beta}\}$ is given by

$$r_{s,p}=(\overline{\beta'(b_s)})^{-1}\langle Ak_{a_p}^{\alpha},\widetilde{k}_{b_s}^{\beta}\rangle,$$

and the matrix representation $\widetilde{M}_A=(t_{s,p})$ of A with respect to the conjugate kernel bases $\{\widetilde{k}_{a_1}^{\alpha},\ldots,\widetilde{k}_{a_m}^{\alpha}\}$ and $\{\widetilde{k}_{b_1}^{\beta},\ldots,\widetilde{k}_{b_n}^{\beta}\}$ is given by

$$t_{s,p}=\beta'(b_s)^{-1}\langle A\widetilde{k}_{a_p}^{\alpha},k_{b_s}^{\beta}\rangle.$$

THEOREM 3.1. *Let the function α be a finite Blaschke product with m distinct zeros $a_1,\ldots,a_m$, let β be a finite Blaschke product with n distinct zeros $b_1,\ldots,b_n$ and assume that α and β have precisely l zeros in common: $a_i=b_i$ for $i\leqslant l$ ($l=0$ if there are no zeros in common). Let A be any linear transformation from K_α into K_β.*

If $M_A = (r_{s,p})$ *is the matrix representation of* A *with respect to the bases* $\{k^\alpha_{a_1}, \dots, k^\alpha_{a_m}\}$ *and* $\{k^\beta_{b_1}, \dots, k^\beta_{b_n}\}$*, and*

(a) $l = 0$*, then* $A \in \mathscr{T}(\alpha, \beta)$ *if and only if*

$$r_{s,p} = \frac{\overline{\beta'(b_s)}(\overline{a}_1 - \overline{b}_s)r_{s,1} + \overline{\beta'(b_1)}(\overline{b}_1 - \overline{a}_1)r_{1,1} + \overline{\beta'(b_1)}(\overline{a}_p - \overline{b}_1)r_{1,p}}{\overline{\beta'(b_s)}(\overline{a}_p - \overline{b}_s)} \tag{3.2}$$

for all $1 \leqslant p \leqslant m$ *and* $1 \leqslant s \leqslant n$*;*

(b) $l > 0$*, then* $A \in \mathscr{T}(\alpha, \beta)$ *if and only if*

$$r_{s,p} = \frac{\overline{\beta'(b_1)}(\overline{a}_1 - \overline{b}_s)r_{1,s} + \overline{\beta'(b_1)}(\overline{a}_p - \overline{b}_1)r_{1,p}}{\overline{\beta'(b_s)}(\overline{a}_p - \overline{b}_s)} \tag{3.3}$$

for all p, s *such that* $1 \leqslant p \leqslant m$*,* $1 \leqslant s \leqslant l$*,* $s \neq p$*, and*

$$r_{s,p} = \frac{\overline{\beta'(b_s)}(\overline{a}_1 - \overline{b}_s)r_{s,1} + \overline{\beta'(b_1)}(\overline{a}_p - \overline{b}_1)r_{1,p}}{\overline{\beta'(b_s)}(\overline{a}_p - \overline{b}_s)} \tag{3.4}$$

for all p, s *such that* $1 \leqslant p \leqslant m$*,* $l < s \leqslant n$*.*

Proof. We first prove the necessity of the conditions given in the theorem. Let $A = A^{\alpha,\beta}_\varphi$ be an asymmetric truncated Toeplitz operator with symbol $\varphi \in L^2(\partial\mathbb{D})$. Recall that the matrix representation $M_{A^{\alpha,\beta}_\varphi} = (r_{s,p})$ of $A^{\alpha,\beta}_\varphi$ with respect to the bases $\{k^\alpha_{a_1}, \dots, k^\alpha_{a_m}\}$ and $\{k^\beta_{b_1}, \dots, k^\beta_{b_n}\}$ is given by

$$r_{s,p} = (\overline{\beta'(b_s)})^{-1}\langle A^{\alpha,\beta}_\varphi k^\alpha_{a_p}, \widetilde{k}^\beta_{b_s}\rangle.$$

By [14, Cor. 2.6], $A^{\alpha,\beta}_\varphi$ can be written as

$$A^{\alpha,\beta}_\varphi = A^{\alpha,\beta}_{\overline{\chi}+\psi},$$

for some $\chi \in K_\alpha$, $\psi \in K_\beta$. Since the functions $\widetilde{k}^\alpha_{a_i}$, $i = 1, \dots, m$, form a basis for K_α and the functions $\widetilde{k}^\beta_{b_j}$, $j = 1, \dots, n$, form a basis for K_β, we can write

$$\chi = \sum_{i=1}^m c_i \widetilde{k}^\alpha_{a_i}, \quad \psi = \sum_{j=1}^n d_j \widetilde{k}^\beta_{b_j}.$$

We now compute $r_{s,p}$ in terms of the scalars $c_1, \dots, c_m$ and $d_1, \dots, d_n$.

Since $\alpha(a_i) = 0$ and $\beta(b_j) = 0$, we have

$$\widetilde{k}^\alpha_{a_i}(z) = \frac{\alpha(z)}{z - a_i}, \qquad \widetilde{k}^\beta_{b_j}(z) = \frac{\beta(z)}{z - b_j}$$

and

$$A_{\varphi}^{\alpha,\beta} = \sum_{i=1}^{m} \overline{c_i} A^{\alpha,\beta}_{\frac{\overline{\alpha(z)}}{\overline{z}-\overline{a}_i}} + \sum_{j=1}^{n} d_j A^{\alpha,\beta}_{\frac{\beta(z)}{z-b_j}}.$$

By Proposition 2.3 (a),

$$A^{\alpha,\beta}_{\frac{\overline{\alpha(z)}}{\overline{z}-\overline{a}_i}} = k^{\beta}_{a_i} \otimes \widetilde{k}^{\alpha}_{a_i} \quad \text{and} \quad A^{\alpha,\beta}_{\frac{\beta(z)}{z-b_j}} = \widetilde{k}^{\beta}_{b_j} \otimes k^{\alpha}_{b_j},$$

and so

$$\begin{aligned} A_{\varphi}^{\alpha,\beta} k^{\alpha}_{a_p} &= \sum_{i=1}^{m} \overline{c}_i \langle k^{\alpha}_{a_p}, \widetilde{k}^{\alpha}_{a_i} \rangle k^{\beta}_{a_i} + \sum_{j=1}^{n} d_j \langle k^{\alpha}_{a_p}, k^{\alpha}_{b_j} \rangle \widetilde{k}^{\beta}_{b_j} \\ &= \overline{c}_p \overline{\alpha'(a_p)} k^{\beta}_{a_p} + \sum_{j=1}^{n} \frac{d_j}{1-\overline{a}_p b_j} \widetilde{k}^{\beta}_{b_j}. \end{aligned}$$

The last equality follows from the fact that

$$\langle k^{\alpha}_{a_p}, \widetilde{k}^{\alpha}_{a_i} \rangle = \begin{cases} \overline{\alpha'(a_p)} & \text{for } i = p, \\ 0 & \text{for } i \neq p. \end{cases}$$

Consequently,

$$\begin{aligned} r_{s,p} &= (\overline{\beta'(b_s)})^{-1} \langle A_{\varphi}^{\alpha,\beta} k^{\alpha}_{a_p}, \widetilde{k}^{\beta}_{b_s} \rangle \\ &= \overline{c}_p \frac{\overline{\alpha'(a_p)}}{\overline{\beta'(b_s)}} \langle k^{\beta}_{a_p}, \widetilde{k}^{\beta}_{b_s} \rangle + \frac{1}{\overline{\beta'(b_s)}} \sum_{j=1}^{n} \frac{d_j}{1-\overline{a}_p b_j} \langle \widetilde{k}^{\beta}_{b_j}, \widetilde{k}^{\beta}_{b_s} \rangle \\ &= \overline{c}_p \frac{\overline{\alpha'(a_p)}}{\overline{\beta'(b_s)}} \langle k^{\beta}_{a_p}, \widetilde{k}^{\beta}_{b_s} \rangle + \frac{1}{\overline{\beta'(b_s)}} \sum_{j=1}^{n} \frac{d_j}{(1-\overline{a}_p b_j)(1-\overline{b}_s b_j)}. \end{aligned}$$

(a) $l = 0$.

In this case $a_p \neq b_s$ for all $1 \leqslant p \leqslant m$ and $1 \leqslant s \leqslant n$. Therefore

$$\langle k^{\beta}_{a_p}, \widetilde{k}^{\beta}_{b_s} \rangle = \frac{\overline{\beta(a_p)}}{\overline{a}_p - \overline{b}_s} \neq 0$$

and

$$r_{s,p} = \frac{\overline{c}_p}{\overline{a}_p - \overline{b}_s} \frac{\overline{\alpha'(a_p)}}{\overline{\beta'(b_s)}} \overline{\beta(a_p)} + \frac{1}{\overline{\beta'(b_s)}} \sum_{j=1}^{n} \frac{d_j}{(1-\overline{a}_p b_j)(1-\overline{b}_s b_j)}$$

for all $1 \leqslant p \leqslant m$, $1 \leqslant s \leqslant n$.

We now show that $r_{s,p}$ satisfies (3.2) for all $1 \leqslant s \leqslant n$ and $1 \leqslant p \leqslant m$. Clearly, (3.2) holds for $s = 1$. Assume that $s \neq 1$. Using the equality

$$\frac{\overline{a}_p - \overline{b}_s}{(1-\overline{a}_p b_j)(1-\overline{b}_s b_j)} = \frac{\overline{a}_p}{1-\overline{a}_p b_j} - \frac{\overline{b}_s}{1-\overline{b}_s b_j},$$

we get

$$
\begin{aligned}
\sum_{j=1}^{n}\frac{d_j}{(1-\overline{a}_p b_j)(1-\overline{b}_s b_j)} &= \sum_{j=1}^{n}\frac{d_j}{\overline{a}_p-\overline{b}_s}\left(\frac{\overline{a}_p}{1-\overline{a}_p b_j}-\frac{\overline{b}_s}{1-\overline{b}_s b_j}\right)\\
&=\sum_{j=1}^{n}\frac{d_j}{\overline{a}_p-\overline{b}_s}\left(\frac{\overline{a}_p-\overline{b}_1}{(1-\overline{a}_p b_j)(1-\overline{b}_1 b_j)}+\frac{\overline{b}_1-\overline{a}_1}{(1-\overline{a}_1 b_j)(1-\overline{b}_1 b_j)}\right.\\
&\quad\left.+\frac{\overline{a}_1-\overline{b}_s}{(1-\overline{a}_1 b_j)(1-\overline{b}_s b_j)}\right)\\
&=\frac{\overline{a}_p-\overline{b}_1}{\overline{a}_p-\overline{b}_s}\sum_{j=1}^{n}\frac{d_j}{(1-\overline{a}_p b_j)(1-\overline{b}_1 b_j)}+\frac{\overline{b}_1-\overline{a}_1}{\overline{a}_p-\overline{b}_s}\sum_{j=1}^{n}\frac{d_j}{(1-\overline{a}_1 b_j)(1-\overline{b}_1 b_j)}\\
&\quad+\frac{\overline{a}_1-\overline{b}_s}{\overline{a}_p-\overline{b}_s}\sum_{j=1}^{n}\frac{d_j}{(1-\overline{a}_1 b_j)(1-\overline{b}_s b_j)}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
r_{s,p} &= \frac{\overline{c}_p}{\overline{a}_p-\overline{b}_s}\frac{\overline{\alpha'(a_p)}}{\overline{\beta'(b_s)}}\overline{\beta(a_p)}+\frac{1}{\overline{\beta'(b_s)}}\sum_{j=1}^{n}\frac{d_j}{(1-\overline{a}_p b_j)(1-\overline{b}_s b_j)}\\
&=\frac{\overline{c}_p}{\overline{a}_p-\overline{b}_s}\frac{\overline{\alpha'(a_p)}}{\overline{\beta'(b_s)}}\overline{\beta(a_p)}+\frac{\overline{a}_p-\overline{b}_1}{\overline{a}_p-\overline{b}_s}\frac{1}{\overline{\beta'(b_s)}}\sum_{j=1}^{n}\frac{d_j}{(1-\overline{a}_p b_j)(1-\overline{b}_1 b_j)}\\
&\quad-\frac{\overline{c}_1}{\overline{a}_p-\overline{b}_s}\frac{\overline{\alpha'(a_1)}}{\overline{\beta'(b_s)}}\overline{\beta(a_1)}+\frac{\overline{b}_1-\overline{a}_1}{\overline{a}_p-\overline{b}_s}\frac{1}{\overline{\beta'(b_s)}}\sum_{j=1}^{n}\frac{d_j}{(1-\overline{a}_1 b_j)(1-\overline{b}_1 b_j)}\\
&\quad+\frac{\overline{c}_1}{\overline{a}_p-\overline{b}_s}\frac{\overline{\alpha'(a_1)}}{\overline{\beta'(b_s)}}\overline{\beta(a_1)}+\frac{\overline{a}_1-\overline{b}_s}{\overline{a}_p-\overline{b}_s}\frac{1}{\overline{\beta'(b_s)}}\sum_{j=1}^{n}\frac{d_j}{(1-\overline{a}_1 b_j)(1-\overline{b}_s b_j)}\\
&=\frac{\overline{a}_p-\overline{b}_1}{\overline{a}_p-\overline{b}_s}\frac{\overline{\beta'(b_1)}}{\overline{\beta'(b_s)}}r_{1,p}+\frac{\overline{b}_1-\overline{a}_1}{\overline{a}_p-\overline{b}_s}\frac{\overline{\beta'(b_1)}}{\overline{\beta'(b_s)}}r_{1,1}+\frac{\overline{a}_1-\overline{b}_s}{\overline{a}_p-\overline{b}_s}r_{s,1}\\
&=\frac{\overline{\beta'(b_s)}(\overline{a}_1-\overline{b}_s)r_{s,1}+\overline{\beta'(b_1)}(\overline{b}_1-\overline{a}_1)r_{1,1}+\overline{\beta'(b_1)}(\overline{a}_p-\overline{b}_1)r_{1,p}}{\overline{\beta'(b_s)}(\overline{a}_p-\overline{b}_s)}.
\end{aligned}
$$

(b) $l>0$.

In this case $a_p=b_p$ for $p\leqslant l$ and $a_p\neq b_s$ for $p>l$ and every $1\leqslant s\leqslant n$. Hence

$$
\langle k^{\beta}_{a_p},\widetilde{k}^{\beta}_{b_s}\rangle=\begin{cases}\overline{\beta'(b_s)}\delta_{s,p} & \text{for } p\leqslant l,\\[2mm] \frac{\overline{\beta(a_p)}}{\overline{a}_p-\overline{b}_s} & \text{for } p>l,\end{cases}
$$

and

$$
r_{s,p}=\overline{c}_p\overline{\alpha'(a_p)}\delta_{s,p}+\frac{1}{\overline{\beta'(b_s)}}\sum_{j=1}^{n}\frac{d_j}{(1-\overline{a}_p b_j)(1-\overline{b}_s b_j)}
$$

for $p \leqslant l$, $1 \leqslant s \leqslant n$, and

$$r_{s,p} = \frac{\overline{c}_p}{\overline{a}_p - \overline{b}_s} \frac{\overline{\alpha'(a_p)}}{\overline{\beta'(b_s)}} \overline{\beta(a_p)} + \frac{1}{\overline{\beta'(b_s)}} \sum_{j=1}^{n} \frac{d_j}{(1-\overline{a}_p b_j)(1-\overline{b}_s b_j)}$$

for $p > l$, $1 \leqslant s \leqslant n$.

We now show that $r_{s,p}$ satisfies (3.4) for all p,s such that $1 \leqslant s \leqslant n$, $p > l$ or $1 \leqslant s \leqslant n$, $1 \leqslant p \leqslant l$, $p \neq s$.

Clearly, (3.4) holds for all p,s with $p \neq s$ and such that $s = 1$ or $p = 1$. Assume that $s \neq 1$ and $p \neq 1$. If $p > l$, then using the fact that $a_1 = b_1$, we get

$$\begin{aligned}
r_{s,p} &= \frac{\overline{c}_p}{\overline{a}_p - \overline{b}_s} \frac{\overline{\alpha'(a_p)}}{\overline{\beta'(b_s)}} \overline{\beta(a_p)} + \frac{1}{\overline{\beta'(b_s)}} \sum_{j=1}^{n} \frac{d_j}{(1-\overline{a}_p b_j)(1-\overline{b}_s b_j)} \\
&= \frac{\overline{c}_p}{\overline{a}_p - \overline{b}_s} \frac{\overline{\alpha'(a_p)}}{\overline{\beta'(b_s)}} \overline{\beta(a_p)} + \frac{1}{\overline{\beta'(b_s)}} \sum_{j=1}^{n} \frac{d_j}{\overline{a}_p - \overline{b}_s} \left(\frac{\overline{a}_p - \overline{b}_1}{(1-\overline{a}_p b_j)(1-\overline{b}_1 b_j)} \right. \\
&\quad \left. + \frac{\overline{a}_1 - \overline{b}_s}{(1-\overline{a}_1 b_j)(1-\overline{b}_s b_j)} \right) \\
&= \frac{\overline{a}_p - \overline{b}_1}{\overline{a}_p - \overline{b}_s} \frac{\overline{\beta'(b_1)}}{\overline{\beta'(b_s)}} \left(\frac{\overline{c}_p}{\overline{a}_p - \overline{b}_1} \frac{\overline{\alpha'(a_p)}}{\overline{\beta'(b_1)}} \overline{\beta(a_p)} + \frac{1}{\overline{\beta'(b_1)}} \sum_{j=1}^{n} \frac{d_j}{(1-\overline{a}_p b_j)(1-\overline{b}_1 b_j)} \right) \\
&\quad + \frac{\overline{a}_1 - \overline{b}_s}{\overline{a}_p - \overline{b}_s} \frac{1}{\overline{\beta'(b_s)}} \sum_{j=1}^{n} \frac{d_j}{(1-\overline{a}_1 b_j)(1-\overline{b}_s b_j)} \\
&= \frac{\overline{a}_p - \overline{b}_1}{\overline{a}_p - \overline{b}_s} \frac{\overline{\beta'(b_1)}}{\overline{\beta'(b_s)}} r_{1,p} + \frac{\overline{a}_1 - \overline{b}_s}{\overline{a}_p - \overline{b}_s} r_{s,1} = \frac{\overline{\beta'(b_s)}(\overline{a}_1 - \overline{b}_s) r_{s,1} + \overline{\beta'(b_1)}(\overline{a}_p - \overline{b}_1) r_{1,p}}{\overline{\beta'(b_s)}(\overline{a}_p - \overline{b}_s)}.
\end{aligned}$$

Similarly, if $p \leqslant l$, $s \neq p$, then

$$\begin{aligned}
r_{s,p} &= \frac{1}{\overline{\beta'(b_s)}} \sum_{j=1}^{n} \frac{d_j}{(1-\overline{a}_p b_j)(1-\overline{b}_s b_j)} \\
&= \frac{\overline{a}_p - \overline{b}_1}{\overline{a}_p - \overline{b}_s} \frac{\overline{\beta'(b_1)}}{\overline{\beta'(b_s)}} \frac{1}{\overline{\beta'(b_1)}} \sum_{j=1}^{n} \frac{d_j}{(1-\overline{a}_p b_j)(1-\overline{b}_1 b_j)} \\
&\quad + \frac{\overline{a}_1 - \overline{b}_s}{\overline{a}_p - \overline{b}_s} \frac{1}{\overline{\beta'(b_s)}} \sum_{j=1}^{n} \frac{d_j}{(1-\overline{a}_1 b_j)(1-\overline{b}_s b_j)} \\
&= \frac{\overline{a}_p - \overline{b}_1}{\overline{a}_p - \overline{b}_s} \frac{\overline{\beta'(b_1)}}{\overline{\beta'(b_s)}} r_{1,p} + \frac{\overline{a}_1 - \overline{b}_s}{\overline{a}_p - \overline{b}_s} r_{s,1} \\
&= \frac{\overline{\beta'(b_s)}(\overline{a}_1 - \overline{b}_s) r_{s,1} + \overline{\beta'(b_1)}(\overline{a}_p - \overline{b}_1) r_{1,p}}{\overline{\beta'(b_s)}(\overline{a}_p - \overline{b}_s)}.
\end{aligned}$$

In particular, (3.4) holds for all p,s such that $l < s \leqslant n$, $1 \leqslant p \leqslant m$.

To complete this part of the proof we need to show that $r_{s,p}$ satisfies (3.3) for all p,s such that $1 \leqslant s \leqslant l$, $1 \leqslant p \leqslant m$, $s \neq p$. Again, this is obvious for $s = 1$. If $1 < s \leqslant l$, then

$$\begin{aligned} r_{s,1} &= \frac{1}{\overline{\beta'(b_s)}} \sum_{j=1}^{n} \frac{d_j}{(1-\overline{a}_1 b_j)(1-\overline{b}_s b_j)} \\ &= \frac{1}{\overline{\beta'(b_s)}} \sum_{j=1}^{n} \frac{d_j}{(1-\overline{a}_s b_j)(1-\overline{b}_1 b_j)} = \frac{\overline{\beta'(b_1)}}{\overline{\beta'(b_s)}} r_{1,s} \end{aligned}$$

and (3.3) follows from (3.4).

The proof of necessity of the given conditions (for $l = 0$ and for $l > 0$) is now complete. We now prove sufficiency.

Assume that $l = 0$ and note that the linear space V of all the matrices satisfying (3.2) has dimension $m+n-1$. By the first part of the proof, the set V_0 of all the matrices representing operators from $\mathscr{T}(\alpha,\beta)$ is a subspace of V,

$$V_0 = \{M_{A_\varphi^{\alpha,\beta}} \colon\ A_\varphi^{\alpha,\beta} \in \mathscr{T}(\alpha,\beta)\} \subset V.$$

However, by Proposition 2.1 we know that V_0 also has dimension $m+n-1$, and so $V_0 = V$.

The proof for $l > 0$ is analogous. □

REMARK 3.2.

(a) Theorem 3.1 states that if α and β have no common zeros ($l = 0$), then the matrix representing an operator from $\mathscr{T}(\alpha,\beta)$ is determined by the entries along the first row and the first column. A slight modification of the proof shows that one can in fact take any other row and any other column.

(b) Note that in the proof of part (b) of Theorem 3.1 we actually showed that the elements $r_{s,p}$ satisfy

$$r_{s,p} = \frac{\overline{\beta'(b_s)}(\overline{a}_1 - \overline{b}_s) r_{s,1} + \overline{\beta'(b_1)}(\overline{a}_p - \overline{b}_1) r_{1,p}}{\overline{\beta'(b_s)}(\overline{a}_p - \overline{b}_s)}$$

for all p,s such that $1 \leqslant s \leqslant n$, $p > l$ or $1 \leqslant s \leqslant n$, $1 \leqslant p \leqslant l$, $p \neq s$. However, this only says that the matrix representing an asymmetric truncated Toeplitz operator is determined by $m+n+l-2$ of its entries, which is more that $m+n-1$ for $l > 1$. To reduce the number of the determining entries we consider two equations: (3.3) and (3.4). These equations say that the matrix is determined by entries along the first row, first l entries along the main diagonal and last $n-l$ entries along the first column.

(c) A modification of the proof of part (b) of Theorem 3.1 shows that the first row and column can be replaced by any other row and column that intersect at one of the first l elements of the main diagonal. The theorem can also be formulated with rows in place of the columns and vice versa.

(d) Note that if $\alpha = \beta$ is a Blaschke product with m distinct zeros, then $l = m = n$ and part (b) of Theorem 3.1 is precisely the result obtained in [8, Thm. 1.4].

Theorem 3.1 can also be formulated in terms of the matrix representation with respect to $\{\widetilde{k}^{\alpha}_{a_1}, \dots, \widetilde{k}^{\alpha}_{a_m}\}$ and $\{\widetilde{k}^{\beta}_{b_1}, \dots, \widetilde{k}^{\beta}_{b_n}\}$.

THEOREM 3.3. *Let the function α be a finite Blaschke product with m distinct zeros $a_1, \dots, a_m$, let β be a finite Blaschke product with n distinct zeros $b_1, \dots, b_n$ and assume that α and β have precisely l zeros in common: $a_i = b_i$ for $i \leqslant l$ ($l = 0$ if there are no zeros in common). Let A be any linear transformation from K_α into K_β. If $\widetilde{M}_A = (t_{s,p})$ is the matrix representation of A with respect to the bases $\{\widetilde{k}^{\alpha}_{a_1}, \dots, \widetilde{k}^{\alpha}_{a_m}\}$ and $\{\widetilde{k}^{\beta}_{b_1}, \dots, \widetilde{k}^{\beta}_{b_n}\}$, and*

(a) $l = 0$, then $A \in \mathscr{T}(\alpha, \beta)$ if and only if

$$t_{s,p} = \frac{\beta'(b_s)(a_1 - b_s)t_{s,1} + \beta'(b_1)(b_1 - a_1)t_{1,1} + \beta'(b_1)(a_p - b_1)t_{1,p}}{\beta'(b_s)(a_p - b_s)}$$

for all $1 \leqslant p \leqslant m$ and $1 \leqslant s \leqslant n$;

(b) $l > 0$, then $A \in \mathscr{T}(\alpha, \beta)$ if and only if

$$t_{s,p} = \frac{\beta'(b_1)(a_1 - b_s)t_{1,s} + \beta'(b_1)(a_p - b_1)t_{1,p}}{\beta'(b_s)(a_p - b_s)}$$

for all p, s such that $1 \leqslant p \leqslant m$, $1 \leqslant s \leqslant l$, $s \neq p$, and

$$t_{s,p} = \frac{\beta'(b_s)(a_1 - b_s)t_{s,1} + \beta'(b_1)(a_p - b_1)t_{1,p}}{\beta'(b_s)(a_p - b_s)}$$

for all p, s such that $1 \leqslant p \leqslant m$, $l < s \leqslant n$.

Proof. Let A be any linear transformation from K_α into K_β. The proof is based on the fact that the matrix representation of A with respect to the conjugate kernel basis satisfies the conditions from Theorem 3.3 if and only if the matrix representation of A^* with respect to the reproducing kernel basis satisfies the conditions from Theorem 3.1. Since $A \in \mathscr{T}(\alpha, \beta)$ if and only if $A^* \in \mathscr{T}(\beta, \alpha)$ (see [4, Lem. 3.2]), this proves the theorem. The details are left to the reader. □

3.2. Clark bases and modified Clark bases

Now let α and β be as in (3.1) but do not assume that the zeros are distinct. For any $\lambda_1 \in \partial\mathbb{D}$ define

$$\alpha_{\lambda_1} = \frac{\lambda_1 + \alpha(0)}{1 + \overline{\alpha(0)}\lambda_1}.$$

Then $\alpha_{\lambda_1} \in \partial\mathbb{D}$ and, since $|\alpha| = 1$ and $|\alpha'| > 0$ on $\partial\mathbb{D}$ (see [11, p. 6]), the equation

$$\alpha(\eta) = \alpha_{\lambda_1} \tag{3.5}$$

has precisely m distinct solutions $\eta_1, \dots, \eta_m$, which lie on the unit circle $\partial\mathbb{D}$. The corresponding kernel functions $k^{\alpha}_{\eta_1}, \dots, k^{\alpha}_{\eta_m}$ belong to K_α. Moreover,

$$\langle k^{\alpha}_{\eta_i}, k^{\alpha}_{\eta_j} \rangle = \begin{cases} \|k^{\alpha}_{\eta_i}\|^2 & \text{for } i = j, \\ 0 & \text{for } i \neq j. \end{cases}$$

Therefore the functions $k^{\alpha}_{\eta_1}, \dots, k^{\alpha}_{\eta_m}$ form an orthogonal basis for K_α and the normalized kernel functions

$$v^{\alpha}_{\eta_j} = \|k^{\alpha}_{\eta_j}\|^{-1} k^{\alpha}_{\eta_j}, \quad j = 1, \dots, m,$$

form an orthonormal basis for K_α. The basis $\{v^{\alpha}_{\eta_1}, \dots, v^{\alpha}_{\eta_m}\}$ is called the Clark basis corresponding to λ_1 (see [9] and [11] for more details).

We can also define the so-called modified Clark basis corresponding to λ_1 by

$$e^{\alpha}_{\eta_j} = \omega^{\alpha}_j v^{\alpha}_{\eta_j}, \quad j = 1, \dots, m,$$

where

$$\omega^{\alpha}_j = e^{-\frac{i}{2}(\arg\eta_j - \arg\lambda_1)}, \quad j = 1, \dots, m.$$

Then the basis $\{e^{\alpha}_{\eta_1}, \dots, e^{\alpha}_{\eta_m}\}$ is an orthonormal basis for K_α and such that

$$C_\alpha e^{\alpha}_{\eta_j} = e^{\alpha}_{\eta_j}, \quad j = 1, \dots, m,$$

where C_α is the conjugation given by (1.2).

Similarly, for any $\lambda_2 \in \partial\mathbb{D}$ there are precisely n distinct solutions $\zeta_1, \dots, \zeta_n$ on $\partial\mathbb{D}$ of the equation

$$\beta(\zeta) = \beta_{\lambda_2} = \frac{\lambda_2 + \beta(0)}{1 + \overline{\beta(0)}\lambda_2}. \tag{3.6}$$

The Clark basis $\{v^{\beta}_{\zeta_1}, \dots, v^{\beta}_{\zeta_n}\}$ and modified Clark basis $\{e^{\beta}_{\zeta_1}, \dots, e^{\beta}_{\zeta_n}\}$ corresponding to λ_2 are defined as above by

$$v^{\beta}_{\zeta_j} = \|k^{\beta}_{\zeta_j}\|^{-1} k^{\beta}_{\zeta_j}, \quad j = 1, \dots, n,$$

and

$$e^{\beta}_{\zeta_j} = \omega^{\beta}_j v^{\beta}_{\zeta_j}, \quad j = 1, \dots, n,$$

where

$$\omega^{\beta}_j = e^{-\frac{i}{2}(\arg\zeta_j - \arg\lambda_2)}, \quad j = 1, \dots, n.$$

Of course, it may happen that the equations (3.5) and (3.6) have some solutions in common. Here we assume that (3.5) and (3.6) have precisely l solutions in common ($l = 0$ if there are no solutions in common), these solutions being $\eta_j = \zeta_j$ for $j \leqslant l$.

THEOREM 3.4. *Let* α *and* β *be two finite Blaschke products of degree* $m>0$ *and* $n>0$*, respectively. Let* $\{v^{\alpha}_{\eta_1},\dots,v^{\alpha}_{\eta_m}\}$ *be the Clark basis for* K_α *corresponding to* $\lambda_1\in\partial\mathbb{D}$*, let* $\{v^{\beta}_{\zeta_1},\dots,v^{\beta}_{\zeta_n}\}$ *be the Clark basis for* K_β *corresponding to* $\lambda_2\in\partial\mathbb{D}$ *and assume that the sets* $\{\eta_1,\dots,\eta_m\}$, $\{\zeta_1,\dots,\zeta_n\}$ *have precisely* l *elements in common:* $\eta_j=\zeta_j$ *for* $j\leqslant l$ *(*$l=0$ *if there are no elements in common). Finally, let* A *be any linear transformation from* K_α *into* K_β*. If* $M_A=(r_{s,p})$ *is the matrix representation of* A *with respect to the bases* $\{v^{\alpha}_{\eta_1},\dots,v^{\alpha}_{\eta_m}\}$ *and* $\{v^{\beta}_{\zeta_1},\dots,v^{\beta}_{\zeta_n}\}$*, and*

(a) $l=0$*, then* $A\in\mathscr{T}(\alpha,\beta)$ *if and only if*

$$r_{s,p}=\left(\frac{\sqrt{|\alpha'(\eta_1)|}}{\sqrt{|\alpha'(\eta_p)|}}\frac{\eta_p}{\eta_1}\frac{\eta_1-\zeta_s}{\eta_p-\zeta_s}r_{s,1}+\frac{\sqrt{|\alpha'(\eta_1)|}\sqrt{|\beta'(\zeta_1)|}}{\sqrt{|\alpha'(\eta_p)|}\sqrt{|\beta'(\zeta_s)|}}\frac{\eta_p}{\eta_1}\frac{\zeta_1-\eta_1}{\eta_p-\zeta_s}r_{1,1}\right.$$
$$\left.+\frac{\sqrt{|\beta'(\zeta_1)|}}{\sqrt{|\beta'(\zeta_s)|}}\frac{\eta_p-\zeta_1}{\eta_p-\zeta_s}r_{1,p}\right) \tag{3.7}$$

for all $1\leqslant p\leqslant m$ *and* $1\leqslant s\leqslant n$*;*

(b) $l>0$*, then* $A\in\mathscr{T}(\alpha,\beta)$ *if and only if*

$$r_{s,p}=\left(\frac{\sqrt{|\alpha'(\eta_s)|}\sqrt{|\beta'(\zeta_1)|}}{\sqrt{|\alpha'(\eta_p)|}\sqrt{|\beta'(\zeta_s)|}}\frac{\eta_p}{\eta_s}\frac{\eta_1-\zeta_s}{\eta_p-\zeta_s}r_{1,s}+\frac{\sqrt{|\beta'(\zeta_1)|}}{\sqrt{|\beta'(\zeta_s)|}}\frac{\eta_p-\zeta_1}{\eta_p-\zeta_s}r_{1,p}\right) \tag{3.8}$$

for all p,s *such that* $1\leqslant p\leqslant m$, $1\leqslant s\leqslant l$, $s\neq p$*, and*

$$r_{s,p}=\left(\frac{\sqrt{|\alpha'(\eta_1)|}}{\sqrt{|\alpha'(\eta_p)|}}\frac{\eta_p}{\eta_1}\frac{\eta_1-\zeta_s}{\eta_p-\zeta_s}r_{s,1}+\frac{\sqrt{|\beta'(\zeta_1)|}}{\sqrt{|\beta'(\zeta_s)|}}\frac{\eta_p-\zeta_1}{\eta_p-\zeta_s}r_{1,p}\right) \tag{3.9}$$

for all p,s *such that* $1\leqslant p\leqslant m$, $l<s\leqslant n$*.*

Proof. Let A be any linear transformation from K_α into K_β and let $M_A=(r_{s,p})$ be its matrix representation with respect to the bases $\{v^{\alpha}_{\eta_1},\dots,v^{\alpha}_{\eta_m}\}$ and $\{v^{\beta}_{\zeta_1},\dots,v^{\beta}_{\zeta_n}\}$.

We first show that if A belongs to $\mathscr{T}(\alpha,\beta)$, then M_A has the desired properties.

Assume that $A=A^{\alpha,\beta}_{\varphi}$ for $\varphi\in L^2(\partial\mathbb{D})$. To compute $r_{s,p}$ pick $m+n-1$ distinct points $\xi_1,\dots,\xi_{m+n-1}$ from $\partial\mathbb{D}$, different from η_i, $i=1,\dots,m$, and from ζ_j, $j=1,\dots,n$. It follows form Corollary 2.4 that the operators $k^{\beta}_{\xi_i}\otimes k^{\alpha}_{\xi_i}$, $i=1,\dots,m+n-1$, form a basis for $\mathscr{T}(\alpha,\beta)$. Hence there exist scalars $c_1,\dots,c_{m+n-1}$ such that

$$A^{\alpha,\beta}_{\varphi}=\sum_{i=1}^{m+n-1}c_ik^{\beta}_{\xi_i}\otimes k^{\alpha}_{\xi_i}. \tag{3.10}$$

Since the Clark bases are orthonormal,

$$r_{s,p} = \langle A_\varphi^{\alpha,\beta} v_{\eta_p}^\alpha, v_{\zeta_s}^\beta \rangle$$

for $1 \leqslant p \leqslant m$ and $1 \leqslant s \leqslant n$. We now compute $r_{s,p}$ in terms of c_i, $i = 1, \ldots, m+n-1$. By (3.10) we have

$$A_\varphi^{\alpha,\beta} v_{\eta_p}^\alpha = \sum_{i=1}^{m+n-1} c_i k_{\xi_i}^\beta \otimes k_{\xi_i}^\alpha (v_{\eta_p}^\alpha) = \sum_{i=1}^{m+n-1} c_i v_{\eta_p}^\alpha(\xi_i) k_{\xi_i}^\beta,$$

and

$$\begin{aligned} r_{s,p} &= \sum_{i=1}^{m+n-1} c_i v_{\eta_p}^\alpha(\xi_i)\overline{v_{\zeta_s}^\beta(\xi_i)} \\ &= \frac{1}{\|k_{\eta_p}^\alpha\|\|k_{\zeta_s}^\beta\|} \sum_{i=1}^{m+n-1} c_i \frac{1-\overline{\alpha(\eta_p)}\alpha(\xi_i)}{1-\overline{\eta}_p\xi_i} \frac{1-\beta(\zeta_s)\overline{\beta(\xi_i)}}{1-\zeta_s\overline{\xi}_i} \\ &= \frac{\eta_p}{\|k_{\eta_p}^\alpha\|\|k_{\zeta_s}^\beta\|} \sum_{i=1}^{m+n-1} c_i\xi_i \frac{(\overline{\alpha(\eta_p)}\alpha(\xi_i)-1)(1-\beta(\zeta_s)\overline{\beta(\xi_i)})}{(\xi_i-\eta_p)(\xi_i-\zeta_s)}. \end{aligned}$$

Note that $\|k_{\eta_p}^\alpha\| = \sqrt{|\alpha'(\eta_p)|}$ and $\|k_{\zeta_s}^\beta\| = \sqrt{|\beta'(\zeta_s)|}$. Moreover, η_p and ζ_s are solutions of (3.5) and (3.6), respectively. Consequently, $\alpha(\eta_p) = \alpha_{\lambda_1}$, $p = 1, \ldots, m$, $\beta(\zeta_s) = \beta_{\lambda_2}$, $s = 1, \ldots, n$, and

$$r_{s,p} = \frac{\eta_p}{\sqrt{|\alpha'(\eta_p)|}\sqrt{|\beta'(\zeta_s)|}} \sum_{i=1}^{m+n-1} \frac{d_i}{(\xi_i-\eta_p)(\xi_i-\zeta_s)}, \tag{3.11}$$

where

$$\begin{aligned} d_i &= c_i\xi_i(\overline{\alpha(\eta_p)}\alpha(\xi_i)-1)(1-\beta(\zeta_s)\overline{\beta(\xi_i)}) \\ &= c_i\xi_i(\overline{\alpha_{\lambda_1}}\alpha(\xi_i)-1)(1-\beta_{\lambda_2}\overline{\beta(\xi_i)}) \end{aligned}$$

is independent of p and s.

(a) $l = 0$.

In this case $\eta_p \neq \zeta_s$ for all $1 \leqslant p \leqslant m$ and $1 \leqslant s \leqslant n$. Using (3.11) and the equality

$$\frac{\eta_p-\zeta_s}{(\xi_i-\eta_p)(\xi_i-\zeta_s)} = \frac{1}{\xi_i-\eta_p} - \frac{1}{\xi_i-\zeta_s}$$

we get

$$\begin{aligned}
r_{s,p} &= \frac{\eta_p}{\sqrt{|\alpha'(\eta_p)|}\sqrt{|\beta'(\zeta_s)|}} \sum_{i=1}^{m+n-1} \frac{d_i}{(\xi_i-\eta_p)(\xi_i-\zeta_s)} \\
&= \frac{1}{\sqrt{|\alpha'(\eta_p)|}\sqrt{|\beta'(\zeta_s)|}} \frac{\eta_p}{\eta_p-\zeta_s} \sum_{i=1}^{m+n-1} d_i \left(\frac{1}{\xi_i-\eta_p} - \frac{1}{\xi_i-\zeta_s} \right) \\
&= \frac{1}{\sqrt{|\alpha'(\eta_p)|}\sqrt{|\beta'(\zeta_s)|}} \frac{\eta_p}{\eta_p-\zeta_s} \sum_{i=1}^{m+n-1} d_i \left(\frac{\eta_p-\zeta_1}{(\xi_i-\eta_p)(\xi_i-\zeta_1)} \right. \\
&\quad \left. + \frac{\zeta_1-\eta_1}{(\xi_i-\zeta_1)(\xi_i-\eta_1)} + \frac{\eta_1-\zeta_s}{(\xi_i-\eta_1)(\xi_i-\zeta_s)} \right) \\
&= \frac{1}{\sqrt{|\alpha'(\eta_p)|}\sqrt{|\beta'(\zeta_s)|}} \frac{\eta_p}{\eta_p-\zeta_s} \left(\sqrt{|\alpha'(\eta_p)|}\sqrt{|\beta'(\zeta_1)|} \frac{\eta_p-\zeta_1}{\eta_p} r_{1,p} \right. \\
&\quad \left. + \sqrt{|\alpha'(\eta_1)|}\sqrt{|\beta'(\zeta_1)|} \frac{\zeta_1-\eta_1}{\eta_1} r_{1,1} + \sqrt{|\alpha'(\eta_1)|}\sqrt{|\beta'(\zeta_s)|} \frac{\eta_1-\zeta_s}{\eta_1} r_{s,1} \right) \\
&= \left(\frac{\sqrt{|\alpha'(\eta_1)|}}{\sqrt{|\alpha'(\eta_p)|}} \frac{\eta_p}{\eta_1} \frac{\eta_1-\zeta_s}{\eta_p-\zeta_s} r_{s,1} + \frac{\sqrt{|\alpha'(\eta_1)|}\sqrt{|\beta'(\zeta_1)|}}{\sqrt{|\alpha'(\eta_p)|}\sqrt{|\beta'(\zeta_s)|}} \frac{\eta_p}{\eta_1} \frac{\zeta_1-\eta_1}{\eta_p-\zeta_s} r_{1,1} \right. \\
&\quad \left. + \frac{\sqrt{|\beta'(\zeta_1)|}}{\sqrt{|\beta'(\zeta_s)|}} \frac{\eta_p-\zeta_1}{\eta_p-\zeta_s} r_{1,p} \right).
\end{aligned}$$

Hence (3.7) holds.

(b) $l > 0$.

In this case the proof of part (a) can be repeated to show that (3.7) holds for all p, s such that $1 \leqslant p \leqslant m$, $s > l$, and for all p, s such that $1 \leqslant p \leqslant m$, $1 \leqslant s \leqslant l$, $s \neq p$. Since here $\zeta_1 = \eta_1$, we get (3.9).

We now show that (3.8) holds for all p, s such that $1 \leqslant p \leqslant m$, $1 \leqslant s \leqslant l$, $p \neq s$. Clearly, (3.8) holds for $s = 1$, $p \neq s$. Since here $\eta_s = \zeta_s$, it follows that for $1 < s \leqslant l$,

$$\begin{aligned}
r_{s,1} &= \frac{\eta_1}{\sqrt{|\alpha'(\eta_1)|}\sqrt{|\beta'(\zeta_s)|}} \sum_{i=1}^{m+n-1} \frac{d_i}{(\xi_i-\eta_1)(\xi_i-\zeta_s)} \\
&= \frac{\eta_1}{\sqrt{|\alpha'(\eta_1)|}\sqrt{|\beta'(\zeta_s)|}} \sum_{i=1}^{m+n-1} \frac{d_i}{(\xi_i-\eta_s)(\xi_i-\zeta_1)} \\
&= \frac{\sqrt{|\alpha'(\eta_s)|}\sqrt{|\beta'(\zeta_1)|}}{\sqrt{|\alpha'(\eta_1)|}\sqrt{|\beta'(\zeta_s)|}} \frac{\eta_1}{\eta_s} r_{1,s}.
\end{aligned}$$

Hence, for $1 < s \leqslant l$, $p \neq s$, the equation (3.8) follows form (3.9).

The rest of the proof is similar to that of Theorem 3.1. □

REMARK 3.5.

(a) Part (a) of Theorem 3.4 states that the matrix representing an operator from $\mathscr{T}(\alpha,\beta)$ is determined by entries along the first row and the first column. The proof can be modified to show that one may replace the first row and the first column by any other row and any other column.

(b) Proof of part (b) of Theorem 3.4 can also be modified to show that the first row and column can be replaced by any other row and column that intersect at one of the first l elements of the main diagonal. The theorem can be formulated with rows in place of the columns and vice versa.

(c) If $\alpha=\beta$ is a Blaschke product of degeree m, then $l=m=n$. Moreover, if $\lambda_1=\lambda_2$, then $\eta_j=\zeta_j$ for all $j=1,\dots,m$, and part (b) of Theorem 3.4 is precisely the result form [8, Thm. 1.11].

THEOREM 3.6. *Let α and β be two finite Blaschke products of degree $m>0$ and $n>0$, respectively. Let $\{e^{\alpha}_{\eta_1},\dots,e^{\alpha}_{\eta_m}\}$ be the modified Clark basis for K_α corresponding to $\lambda_1\in\partial\mathbb{D}$, let $\{e^{\beta}_{\zeta_1},\dots,e^{\beta}_{\zeta_n}\}$ be the modified Clark basis for K_β corresponding to $\lambda_2\in\partial\mathbb{D}$ and assume that the sets $\{\eta_1,\dots,\eta_m\}$, $\{\zeta_1,\dots,\zeta_n\}$ have precisely l elements in common: $\eta_j=\zeta_j$ for $j\leqslant l$ ($l=0$ if there are no elements in common). Finally, let A be any linear transformation from K_α into K_β. If $\widetilde{M}_A=(t_{s,p})$ is the matrix representation of A with respect to the bases $\{e^{\alpha}_{\eta_1},\dots,e^{\alpha}_{\eta_m}\}$ and $\{e^{\beta}_{\zeta_1},\dots,e^{\beta}_{\zeta_n}\}$, and*

(a) $l=0$, then $A\in\mathscr{T}(\alpha,\beta)$ if and only if

$$t_{s,p}=\left(\frac{\sqrt{|\alpha'(\eta_1)|}}{\sqrt{|\alpha'(\eta_p)|}}\frac{\omega_1^\alpha}{\omega_p^\alpha}\frac{\eta_1-\zeta_s}{\eta_p-\zeta_s}t_{s,1}+\frac{\sqrt{|\alpha'(\eta_1)|}\sqrt{|\beta'(\zeta_1)|}}{\sqrt{|\alpha'(\eta_p)|}\sqrt{|\beta'(\zeta_s)|}}\frac{\omega_1^\alpha\omega_1^\beta}{\omega_p^\alpha\omega_s^\beta}\frac{\zeta_1-\eta_1}{\eta_p-\zeta_s}t_{1,1}\right.$$
$$\left.+\frac{\sqrt{|\beta'(\zeta_1)|}}{\sqrt{|\beta'(\zeta_s)|}}\frac{\omega_1^\beta}{\omega_s^\beta}\frac{\eta_p-\zeta_1}{\eta_p-\zeta_s}t_{1,p}\right)$$

for all $1\leqslant p\leqslant m$ and $1\leqslant s\leqslant n$;

(b) $l>0$, then $A\in\mathscr{T}(\alpha,\beta)$ if and only if

$$t_{s,p}=\left(\frac{\sqrt{|\alpha'(\eta_s)|}\sqrt{|\beta'(\zeta_1)|}}{\sqrt{|\alpha'(\eta_p)|}\sqrt{|\beta'(\zeta_s)|}}\frac{\omega_s^\alpha\omega_1^\beta}{\omega_p^\alpha\omega_s^\beta}\frac{\eta_1-\zeta_s}{\eta_p-\zeta_s}t_{1,s}+\frac{\sqrt{|\beta'(\zeta_1)|}}{\sqrt{|\beta'(\zeta_s)|}}\frac{\omega_1^\beta}{\omega_s^\beta}\frac{\eta_p-\zeta_1}{\eta_p-\zeta_s}t_{1,p}\right)$$

for all p,s such that $1\leqslant p\leqslant m$, $1\leqslant s\leqslant l$, $s\neq p$, and

$$t_{s,p}=\left(\frac{\sqrt{|\alpha'(\eta_1)|}}{\sqrt{|\alpha'(\eta_p)|}}\frac{\omega_1^\alpha}{\omega_p^\alpha}\frac{\eta_1-\zeta_s}{\eta_p-\zeta_s}t_{s,1}+\frac{\sqrt{|\beta'(\zeta_1)|}}{\sqrt{|\beta'(\zeta_s)|}}\frac{\omega_1^\beta}{\omega_s^\beta}\frac{\eta_p-\zeta_1}{\eta_p-\zeta_s}t_{1,p}\right)$$

for all p,s such that $1\leqslant p\leqslant m$, $l<s\leqslant n$.

Proof. Let A be any linear transformation from K_α into K_β. Here it is enough to compare the matrix representation of A with respect to the Clark basis with its matrix representation with respect to the modified Clark basis and use Theorem 3.4. The straightforward computations are left to the reader. □

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